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New Black Hole Solutions in $\mathcal{N} = 2$ and $\mathcal{N} = 8$ Gauged Supergravity †

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Abstract: We review a special class of $\mathcal{N} = 2$ supergravity model that interpolates all the single-dilaton truncations of the maximal SO(8) gauged supergravity. We also provide explicit non-extremal, charged black hole solutions and their supersymmetric limits, asymptotic charges, thermodynamics and boundary conditions. We also discuss a suitable Hamilton–Jacobi formulation and related BPS flow equations for the supersymmetric configurations, with an explicit form for the superpotential function. Finally, we briefly analyze certain models within the class under consideration as consistent truncations of the maximal, $\mathcal{N} = 8$ gauged supergravity in four dimensions.

Keywords: supergravity; black hole solutions; black holes thermodynamics; BPS black holes

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1. Introduction

Anti-de Sitter (AdS) black hole configurations are an intriguing field of research due to the role they play in high energy theory as well as in the phenomenology of the AdS/CFT conjecture [1]. The latter exploits a surprising geometric picture ('holographic duality') for $D-1$ dimensional conformal quantum field theories living at the boundary of a dual (super)gravity model in D dimensions, also providing a concrete 'recipe' for relating the bulk gravity model with the dual field theory at its boundary.[1] The correspondence can then be exploited to gain new insights into both of the dual theories since stable AdS solutions would describe conformal critical points of the gauge theory at the boundary. This turns out to be a realization of the holographic principle [2], being the description of the bulk AdS spacetime encoded on the dual conformal gauge field theory. Moreover, the energy scale of the boundary theory is related to the radial direction of the bulk spacetime [3,4] so that the geometric radial flow can be (holographically) interpreted as the renormalization group flow of the dual quantum field theory [5,6], the UV perturbation appearing as boundary conditions on the supergravity fields at large radius.

In light of the above discussion, AdS black hole solutions can provide a powerful framework to reveal specific features about a dual, strongly coupled gauge theory, providing then a possible description of many condensed matter phenomena. The thermodynamic properties of AdS black holes were first analyzed in [7] and subsequently extended to various other AdS black hole configurations [8–12]. These studies describe how black holes feature specific phase structures, giving rise to critical phenomena analogous to other common thermodynamic systems. Of particular interest are black hole solutions preserving a certain amount of supersymmetry, since they allow mapping a weak (string) coupling description of the system thermodynamics to the strong-coupling regime, where

1 The boundary theory is located at the UV critical point and is it possible to employ a UV/IR connection, which relates gravity degrees of freedom at large (small) radius with the corresponding counterparts in the dual field theory at high (low) energy regime.

a formulation in terms of a black hole configuration is valid [13]. Moreover, these solutions can be exploited to study the BPS attractor flows in AdS spacetime [14–27].

Gauged Supergravities and Black Holes

The analysis of stationary black hole configurations was motivated by the study of classical general relativity solutions as well as by a deeper understanding of the mentioned AdS/CFT duality. For instance, in general relativity, these kind of solutions are important for clarifying relevant aspects of no-hair theorems [28], the role of scalar charges for black hole thermodynamics [29–32], and other specific features related to the stability of the theory [33–35].

In general, the construction of black hole solutions in gauged supergravity theories is a fundamental groundwork for (phenomenologically realistic) cosmological models, supporting an effective cosmological constant term as well as a suitable scalar potential and related scalar mass contributions. In this regard, gauged supergravity theories in four dimensions provide realistic quantum field models featuring a non-trivial scalar potential: the latter could lift the moduli degeneracy and select a consistent vacuum state for our universe. Moreover, an embedding of the scalar potential itself in a supergravity model is important, since many physical aspects of the theory can be better understood.

On the other hand, from the perspective of the AdS/CFT duality, the discovery of one-parameter family of $\mathrm{SO}(8)$ maximal four-dimensional supergravity theories [36] led to significant progresses toward understanding the vacuum properties and the structure of the dual field theories [37–43]. Together with the original $\mathrm{SO}(8)$ model [44], other gauged supergravities have been developed, exploiting the presence of a dyonic embedding tensor [45–47]; these models feature a richer vacuum structure and scalar field dynamics with respect to the original theory.

Several procedures have been proposed in order to obtain exact, regular hairy [23,47–52] and supersymmetric black hole solutions [25,53,54] for a general scalar potential. The mentioned studies also seem to suggest that an important condition for the existence of hairy black hole solutions is the presence of suitable scalar field self-interaction properties, together with an appropriate gravitational interaction determining the near-horizon behavior as well as the far-region hair physics.[2] This also implies a probable connection between the integrability of the equations of motion and the explicit form of the scalar potential.

In the following we will review exact charged hairy black hole solutions in gauged $\mathcal{N} = 2$, $D = 4$ supergravity of [55].[3] These hairy black hole configurations feature a scalar potential whose specific form is connected with the existence of exact solutions and, in turn, with a black hole solution generating technique. The dilatonic scalar field features generalized boundary conditions preserving the conformal isometries of Anti-de Sitter spacetime: these conditions can be interpreted, according to the holographic dictionary, as perturbing the boundary (UV critical point) with multi-trace deformations of the dual conformal field theory [58]. In particular, the mixed boundary conditions for the scalar would correspond to adding a triple-trace operator to the dual field theory action.

Below, we will provide the explicit expressions for two different families of non-extremal black hole solutions, analyzing the duality relation between them. We will then consider the thermodynamic properties of the solutions together with the analysis of the related boundary conditions. We will also analyze which conditions for the parameters give rise to (non-singular) BPS configurations. In a separate section, we will exploit an Hamilton–Jacobi formalism for our new class of black holes, characterizing their dynamics in terms of a first-order description. Finally, we will briefly discuss specific models (within the new general class under consideration) as consistent truncations of the maximal $\mathcal{N} = 8$, $D = 4$ gauged supergravity.

[2] Scalar self-interactions could be relevant for dynamic and thermodynamic stability of the configuration: naively, we can imagine the condition for the existence of hairy solutions as if the self-interaction of the scalar and the gravitational interaction were to combine such that the near-horizon hair did not collapse into the black hole, while the far-region hair did not escape to infinity.

[3] These new solutions generalize the uncharged configurations of [47,51,56,57].

2. The Model

Let us consider an extended $\mathcal{N}=2$ supergravity theory in $D=4$, coupled to n_v vector multiplets and no hypermultiplets, in the presence of Fayet–Iliopoulos terms (FI-terms).

The model describes $n_\mathrm{v}+1$ vector fields A^Λ_μ, ($\Lambda=0,\ldots,n_\mathrm{v}$) and $n_\mathrm{s}=n_\mathrm{v}$ complex scalar fields z^i ($i=1,\ldots,n_\mathrm{s}$). The bosonic gauged Lagrangian has the standard form

$$\frac{1}{e_\mathrm{D}}\mathscr{L}_{\scriptscriptstyle\mathrm{BOS}} = -\frac{R}{2} + g_{i\bar{\jmath}}\,\partial_\mu z^i\,\partial^\mu\bar{z}^{\bar{\jmath}} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(z,\bar{z})\,F^\Lambda_{\mu\nu}\,F^{\Sigma\,\mu\nu} + \frac{1}{8\,e_\mathrm{D}}\mathcal{R}_{\Lambda\Sigma}(z,\bar{z})\,\varepsilon^{\mu\nu\rho\sigma}\,F^\Lambda_{\mu\nu}\,F^\Sigma_{\rho\sigma} - V(z,\bar{z})\,, \tag{1}$$

and is defined in terms of the $n_\mathrm{v}+1$ vector field strengths $F^\Lambda_{\mu\nu}=\partial_\mu A^\Lambda_\nu-\partial_\nu A^\Lambda_\mu$. The n_s complex scalars z^i span a special Kähler manifold $\mathscr{M}_\mathrm{SK}$ [59–61] and feature non-minimal coupling to the vector fields through the matrices $\mathcal{I}_{\Lambda\Sigma}(z,\bar{z})$, $\mathcal{R}_{\Lambda\Sigma}(z,\bar{z})$, while the form of the scalar potential $V(z,\bar{z})$ explicitly depends on the FI-terms.

The presence of a non-trivial scalar potential comes from the gauging of a U(1)-symmetry of the corresponding ungauged model (without FI-terms) and implies the existence of minimal couplings of the vectors to the fermion fields only.

2.1. Single-Dilaton Truncation

Let us now focus on a $\mathcal{N}=2$ model with no hypermultiplets and a single vector multiplet ($n_\mathrm{v}=1$) with a single complex scalar field z ($n_\mathrm{s}=1$) [47,55]. The geometry of the special Kähler manifold is characterized by a prepotential function with explicit form:

$$\mathcal{F}(\mathcal{X}^\Lambda) = -\frac{i}{4}\,(\mathcal{X}^0)^n\,(\mathcal{X}^1)^{2-n}\,. \tag{2}$$

The coordinate z is identified with the ratio $\mathcal{X}^1/\mathcal{X}^0$, in a local patch in which $\mathcal{X}^0\neq 0$. For special values of n, this model can be reduced to a consistent truncation of the STU model[4].

If we set $\mathcal{X}^0=1$, the holomorphic section Ω^M of the model reads:

$$\Omega^M = \begin{pmatrix}\mathcal{X}^\Lambda\\ \mathscr{F}_\Lambda\end{pmatrix} = \begin{pmatrix}1\\ z\\ -\frac{i}{4}\,n\,z^{2-n}\\ -\frac{i}{4}\,(2-n)\,z^{1-n}\end{pmatrix}, \tag{3}$$

and, in the *special coordinate frame*, the lower components $\mathscr{F}_\Lambda$ can be expressed as the gradient with respect to the upper entries of the prepotential function, $\mathscr{F}_\Lambda=\frac{\partial\mathcal{F}}{\partial\mathcal{X}^\Lambda}$. The Kähler potential $\mathcal{K}$ is expressed as

$$e^{-\mathcal{K}} = \frac{1}{4}\,z^{1-n}\,\big(n\,z-(n-2)\,\bar{z}\big) + \text{c.c.} \tag{4}$$

The theory is then deformed with the introduction of dyonic, electric–magnetic FI-terms. The latter are defined by a constant symplectic vector $\theta_M=(\theta_1,\,\theta_2,\,\theta_3,\,\theta_4)$, encoding the gauge parameters of the model. The scalar potential $V(z,\bar{z})$ can be then obtained from [47,55]

$$V = \Big(g^{i\bar{\jmath}}\,\mathcal{U}^M_i\,\bar{\mathcal{U}}^N_{\bar{\jmath}} - 3\,\mathcal{V}^M\,\bar{\mathcal{V}}^N\Big)\theta_M\,\theta_N = -\frac{1}{2}\,\theta_M\,\mathcal{M}^{MN}\,\theta_N - 4\,\mathcal{V}^M\,\bar{\mathcal{V}}^N\theta_M\,\theta_N\,, \tag{5}$$

where

[4] The STU model [62–64] is a $\mathcal{N}=2$ supergravity coupled to $n_\mathrm{v}=3$ vector multiplets and characterized, in a suitable symplectic frame, by the prepotential $\mathcal{F}_\mathrm{STU}(\mathcal{X}^\Lambda)=-\frac{i}{4}\sqrt{\mathcal{X}^0\,\mathcal{X}^1\,\mathcal{X}^2\,\mathcal{X}^3}$, together with symmetric scalar manifold of the form $\mathscr{M}_\mathrm{STU}=(\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2))^3$ spanned by the three complex scalars $z^i=\mathcal{X}^i/\mathcal{X}^0$ ($i=1,2,3$); this model is in turn a consistent truncation of the maximal $\mathcal{N}=8$ theory in four dimensions with SO(8) gauge group [65–68].

$$\mathcal{V}^M = e^{\frac{\mathcal{K}}{2}}\,\Omega^M = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix}, \tag{6}$$

$$\mathcal{U}_i^M \equiv \mathcal{D}_i\,\mathcal{V}^M = \left(\partial_i + \frac{1}{2}\,\partial_i\mathcal{K}\right)\mathcal{V}^M = \begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix}, \tag{7}$$

The matrix $\mathcal{M}(z,\bar{z})$ is a symplectic, symmetric, negative definite matrix, expressed in terms of $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ as

$$\mathcal{M}_{MN}(z,\bar{z}) \equiv \begin{pmatrix} \mathcal{I}_{\Lambda\Sigma} + (\mathcal{R}\,\mathcal{I}^{-1}\,\mathcal{R})_{\Lambda\Sigma} & -(\mathcal{R}\,\mathcal{I}^{-1})_\Lambda{}^\Gamma \\ -(\mathcal{I}^{-1}\,\mathcal{R})^\Delta{}_\Sigma & (\mathcal{I}^{-1})^{\Delta\Gamma} \end{pmatrix}, \tag{8}$$

and satisfies [69,70]

$$\mathcal{M}^{MN} = -\,\mathbb{C}^{MP}\,\mathcal{M}_{PQ}\,\mathbb{C}^{QN}, \qquad \mathbb{C}_{MN} \equiv \mathbb{C}^{MN} \equiv \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix}. \tag{9}$$

The scalar manifold is endowed with a flat symplectic bundle and it is possible to associate with each point of this space the matrix $\mathcal{M}_{MN}$, thus determining a metric on the symplectic fiber. Being defined in terms of the coupling matrices $\mathcal{I}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma}$ in the bosonic Lagrangian (1), the matrix $\mathcal{M}_{MN}$ also encodes the information about the non-minimal couplings between the scalars and the vectors of the theory.

If we express the scalar field as

$$z = e^{\lambda\,\phi} + i\,\chi\,, \tag{10}$$

the truncation to the dilaton field ϕ (i.e., setting the axion $\chi = 0$) is consistent provided:[5]

$$(2-n)\,\theta_1\,\theta_3 - n\,\theta_2\,\theta_4 = 0\,, \tag{11}$$

$$F^\Lambda_{\mu\nu}\,\partial_\chi\mathcal{R}_{\Lambda\Sigma}\,F^\Sigma_{\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma}\big|_{\chi=0} = 0\,. \tag{12}$$

After the restriction to the dilaton, the scalar metric reads:

$$ds^2 = 2\,g_{z\bar{z}}\,dz\,d\bar{z}\,\big|_{\substack{\chi=0\\ d\chi=0}} = \frac{1}{2}\lambda^2\,n\,(2-n)\,d\phi^2\,, \tag{13}$$

and is positive for $0 < n < 2$. We then express λ in terms of n as

$$\lambda = \sqrt{\frac{2}{n\,(2-n)}}\,, \tag{14}$$

so that the kinetic term for ϕ is canonically normalized.

As a function of the dilaton field only, the scalar potential has the following explicit form [47,55]:

$$\begin{aligned} V(\phi) = &-2\,e^{\lambda\,\phi\,(n-2)}\left(\frac{2\,n-1}{n}\,\theta_1^2 + 4\,\theta_1\,\theta_2\,e^{\lambda\,\phi} + \frac{2\,n-3}{n-2}\,\theta_2^2\,e^{2\,\lambda\,\phi}\right) - \\ &-\frac{1}{8}\,e^{-\lambda\,\phi\,(n-2)}\left((2\,n-1)\,n\,\theta_3^2 - 4\,\theta_3\,\theta_4\,n\,(n-2)\,e^{-\lambda\,\phi} + (n-2)\,(2\,n-3)\,\theta_4^2\,e^{-2\,\lambda\,\phi}\right). \end{aligned} \tag{15}$$

2.1.1. Vacua

Let us make the change of variable

[5] The conditions come from the consistency of the axion field equations after the $\chi = 0$ truncation.

$$\phi \;\rightarrow\; \frac{1}{\lambda}\,\log y\;, \tag{16}$$

in the potential (15), using also the consistent truncation condition (11)

$$\theta_2 \;=\; \frac{2-n}{n}\,\frac{\theta_1\,\theta_3}{\theta_4}\;. \tag{17}$$

The resulting potential has the form:

$$V_0(y) \;=\; \frac{1}{8\,n^2\,\theta_4^2}\,y^{-n}\,\big(F(y)-G(y)\big)\;, \tag{18}$$

where

$$\begin{aligned} F(y) \;&=\; -16\,\theta_1^2\,y^{2n-2}\Big((n-2)\,(2n-3)\,\theta_3^2\,y^2 - 4\,n\,(n-2)\,\theta_3\,\theta_4\,y + n\,(2\,n-1)\,\theta_4^2\Big)\;,\\ G(y) \;&=\; n^2\,\theta_4^2\,\Big(n\,(2n-1)\,\theta_3^2\,y^2 + 4\,n\,(2-n)\,\theta_3\,\theta_4\,y + (n-2)\,(2n-3)\,\theta_4^2\Big)\;, \end{aligned} \tag{19}$$

with the condition $y > 0$, to ensure the reality of the solution.

If we want to find the vacua of the theory, we have to solve the equation

$$\frac{dV_0(y)}{dy} \;=\; 0\;, \tag{20}$$

where the derivative of the potential has the form:

$$\frac{dV_0(y)}{dy} \;=\; \frac{n\,(n-2)\,\theta_3}{8\,y^{n+1}}\,\big(\theta_3\,y-\theta_4\big)\,\big(h(y)-g(y)\big)\;, \tag{21}$$

with

$$\begin{aligned} h(y) \;&=\; \left(\frac{4\,\theta_1}{n\,\theta_4}\right)^2\left((3-2\,n)\,y^{2\,n-1} + (2\,n-1)\,\frac{\theta_4}{\theta_3}\,y^{2\,n-2}\right)\;,\\ g(y) \;&=\; -(2\,n-1)\,y - (3-2\,n)\,\frac{\theta_4}{\theta_3}\;. \end{aligned} \tag{22}$$

A first vacuum solution is given by

$$y_1 \;=\; \frac{\theta_4}{\theta_3} \qquad\quad \text{if} \;\; \frac{\theta_4}{\theta_3} > 0\;, \tag{23}$$

while other vacuum configurations come from the solutions of the equation

$$h(y) \;=\; g(y)\;. \tag{24}$$

The above Equation (24) has a variable number of solutions, depending on the range of values to which the number n belongs. Table 1 summarizes the number, types and mass of vacuum solutions for each possible interval of values of n [47].

Table 1. Vacuum types.

$0 < n < \frac{1}{2}$	$\theta_4/\theta_3 < 0$ $\theta_4/\theta_3 > 0$	1 Anti-de Sitter 3 Anti-de Sitter	$m^2\,L = 6$ $m^2\,L = -2, 6$	Figure 1a Figure 1b
$n = \frac{1}{2}$	$\theta_4/\theta_3 > 0$	1 Anti-de Sitter	$m^2\,L = -2$	Figure 1c
$\frac{1}{2} < n < 1$	$\theta_4/\theta_3 < 0$ $\theta_4/\theta_3 > 0$	1 de Sitter 1 Anti-de Sitter	$m^2\,L = 6$ $m^2\,L = -2$	Figure 1d Figure 1c
$n = 1$	$\theta_4/\theta_3 > 0$	1 Anti-de Sitter	$m^2\,L = -2$	Figure 1c
$1 < n < \frac{3}{2}$	$\theta_4/\theta_3 < 0$ $\theta_4/\theta_3 > 0$	1 de Sitter 1 Anti-de Sitter	$m^2\,L = 6$ $m^2\,L = -2$	Figure 1d Figure 1c
$n = \frac{3}{2}$	$\theta_4/\theta_3 > 0$	1 Anti-de Sitter	$m^2\,L = -2$	Figure 1c
$\frac{3}{2} < n < 2$	$\theta_4/\theta_3 < 0$ $\theta_4/\theta_3 > 0$	1 Anti-de Sitter 3 Anti-de Sitter	$m^2\,L = 6$ $m^2\,L = -2, 6$	Figure 1e Figure 1f

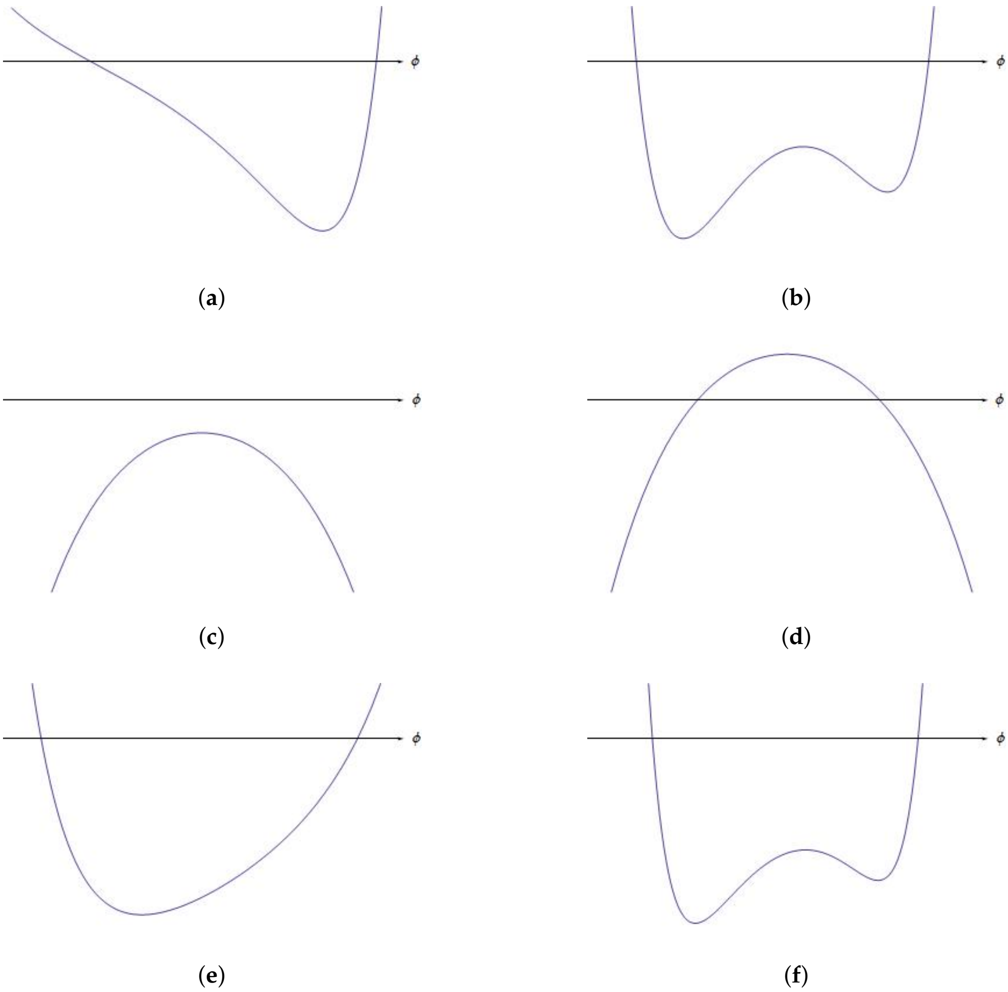

Figure 1. Vacua potential graphics.

2.1.2. Redefinitions

We now consider the shift

$$\phi \;\rightarrow\; \phi - \frac{2\,\nu}{\lambda\,(\nu+1)}\,\log(\theta_2\,\xi)\,, \tag{25}$$

and redefine the FI-terms as:

$$\theta_1 = \frac{\nu+1}{\nu-1}\,\theta_2^{-\frac{\nu-1}{\nu+1}}\,\xi^{-\frac{2\nu}{\nu+1}}\,, \qquad \theta_3 = 2\,\alpha(\xi\,\theta_2)^{\frac{\nu-1}{\nu+1}}\,s\,, \qquad \theta_4 = \frac{2\,\alpha}{\theta_2\,\xi\,s}\,, \tag{26}$$

having also introduced the quantity

$$\nu = \frac{1}{n-1}\,, \tag{27}$$

together with the parameters α, s and

$$\xi \;=\; \frac{2\,L\,\nu}{\nu-1}\,\frac{1}{\sqrt{1-\alpha^2\,L^2}}\,, \tag{28}$$

expressed in terms of the AdS radius L. The above definitions ensure the existence of an AdS vacuum without further constrains on the original FI-terms components [55].

The truncation to the dilaton is consistent provided Equations (11), (12) are satisfied. The former, in the new parametrization (26), is rewritten as

$$(s^2-1)\,(\nu^2-1)\,\alpha\,\sqrt{1-L^2\,\alpha^2} \;=\; 0\,, \tag{29}$$

which is solved, excluding $n=0$ and $n=2$, either for $\alpha=0$ (pure electric FI-terms) or for $s=\pm1$: since we are interested in dyonic FI-terms configurations, we will restrict ourselves to the latter case.

After the shift (25), the scalar field z is expressed as

$$z \;=\; (\theta_2\,\xi)^{-\frac{2\nu}{\nu+1}}\,e^{\lambda\,\phi}\,, \tag{30}$$

and the same redefinition in the potential (with $s=\pm1$) yields

$$\begin{aligned} V(\phi) \;=\; & -\frac{\alpha^2}{\nu^2}\left(\frac{(\nu-1)(\nu-2)}{2}\,e^{-\phi\,\ell\,(\nu+1)}+2(\nu^2-1)\,e^{-\phi\,\ell}+\frac{(\nu+1)(\nu+2)}{2}\,e^{\phi\,\ell\,(\nu-1)}\right)+ \\ & +\frac{\alpha^2-L^{-2}}{\nu^2}\left(\frac{(\nu-1)(\nu-2)}{2}\,e^{\phi\,\ell\,(\nu+1)}+2(\nu^2-1)\,e^{\phi\,\ell}+\frac{(\nu+1)(\nu+2)}{2}\,e^{-\phi\,\ell\,(\nu-1)}\right), \end{aligned} \tag{31}$$

where $\ell = \dfrac{\lambda}{\nu}$.

After the restriction to the dilaton, the matrix $\mathcal{R}_{\Lambda\Sigma}$ vanishes, while the $\mathcal{I}_{\Lambda\Sigma}$ takes a diagonal form and, if we define the canonically normalized gauge fields

$$\bar{F}^1 = \frac{1}{2}\sqrt{\frac{1+\nu}{\nu}}\,(\theta_2\,\xi)^{\frac{1-\nu}{1+\nu}}\,F^1\,, \qquad \bar{F}^2 = \frac{1}{2}\sqrt{\frac{-1+\nu}{\nu}}\,(\theta_2\,\xi)\,F^2\,, \tag{32}$$

the action takes the explicit form

$$\mathscr{S} \;=\; \frac{1}{8\pi G}\int d^4x\,\sqrt{-g}\Big(-\frac{R}{2}+\frac{\partial_\mu\phi\,\partial^\mu\phi}{2}-\frac{1}{4}\,e^{(-1+\nu)\,\ell\,\phi}\left(\bar{F}^1\right)^2-\frac{1}{4}\,e^{-(1+\nu)\,\ell\,\phi}\left(\bar{F}^2\right)^2-V(\phi)\Big)\,. \tag{33}$$

3. Results

3.1. Hairy Black Hole Solutions

Now we consider two distinct families of solutions, which we refer to as electric and magnetic, respectively [55].

3.1.1. Family 1 – Electric Solutions

A first family of solutions is given by

$$\phi = -\ell^{-1}\ln(x)\,, \qquad \bar{F}^1_{tx} = Q_1\,x^{-1+\nu}, \qquad \bar{F}^2_{tx} = Q_2\,x^{-1-\nu}, \qquad \Upsilon(x) = \frac{x^{\nu-1}\,\nu^2 L^2}{\eta^2\,(x^\nu-1)^2}\,, \tag{34}$$

$$\begin{aligned} f(x) \;=\; & \frac{x^{2-\nu}\,\eta^2\,(x^\nu-1)^2}{\nu^2}\,\frac{k}{L^2} + \alpha^2 L^2\left(-1+\frac{x^2}{\nu^2}\left(\,(\nu+2)\,x^{-\nu}-(\nu-2)\,x^\nu+\nu^2-4\,\right)\right)+ \\ & +1+\frac{x^{2-\nu}\,\eta^2\,(x^\nu-1)^3}{\nu^3 L^2}\left(\frac{Q_1^2}{(\nu+1)}-\frac{Q_2^2}{(\nu-1)}\,x^{-\nu}\right), \end{aligned} \tag{35}$$

$$ds^2 \;=\; \Upsilon(x)\left(f(x)\,dt^2 - \frac{\eta^2}{f(x)}\,dx^2 - L^2\,d\Sigma_k\right), \tag{36}$$

where η is an integration constant and $d\Sigma_k^2 = d\vartheta^2 + \frac{\sin^2(\sqrt{k}\,\vartheta)}{k}\,d\varphi^2$ is the metric on the $2D$-surfaces $\Sigma_k = \{\mathbb{S}^2, \mathbb{H}^2, \mathbb{R}^2\}$ with constant scalar curvature $R = 2k$. [6]

Boundary conditions, mass and thermodynamics for the electric solutions.

We now consider the change of coordinates [55]

$$x \;=\; 1 \pm \left(\frac{L^2}{\eta\,r} + L^6\,\frac{1-\nu^2}{24\,(\eta\,r)^3}\right) + L^8\,\frac{\nu^2-1}{24\,(\eta\,r)^4}\,, \tag{37}$$

to make contact with AdS canonical coordinates in the asymptotic limit, the $+\,(-)$ sign holding for $x > 1$ $(x < 1)$. The corresponding asymptotic behavior of the scalar field is

$$\phi \;=\; L^2\,\frac{\phi_0}{r} + L^4\,\frac{\phi_1}{r^2} + O\!\left(r^{-3}\right) \;=\; \mp L^2\,\frac{1}{\ell\,\eta\,r} + L^4\,\frac{1}{2\,\ell\,\eta^2\,r^2} + O\!\left(r^{-3}\right). \tag{38}$$

The coefficients of the leading and subleading coefficients read

$$\phi_0 = \mp\frac{1}{\ell\,\eta}\,, \qquad \phi_1 = \frac{\ell}{2}\,\phi_0^2\,, \tag{39}$$

and correspond to AdS invariant boundary conditions, namely a triple-trace deformation in the boundary theory preserving the boundary conformal symmetry. The asymptotic expansion of the spacetime (36) in the new coordinates reads [55]

$$g_{tt} \;=\; \frac{r^2}{L^2} + k - \frac{\mu_{\rm E}\,L^4}{r} + O\!\left(r^{-2}\right), \qquad g_{rr} \;=\; -\frac{L^2}{r^2} - L^6\,\frac{k\,L^{-2}+\frac{1}{2}\,\phi_0^2}{r^4} + O\!\left(r^{-5}\right), \tag{40}$$

where

$$\mu_{\rm E} = \pm\left(\frac{\nu^2-4}{3\,\eta^3}\,\alpha^2\,L^2 - \frac{k}{\eta\,L^2} + \frac{Q_2^2}{\eta\,(\nu-1)\,L^2} - \frac{Q_1^2}{\eta\,(\nu+1)\,L^2}\right). \tag{41}$$

The black hole mass can be read-off from the above expansion and reads[7]

$$M_{\rm E} \;=\; \frac{\sigma_k}{8\pi G}\,L^4\,\mu_{\rm E}\,, \tag{42}$$

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[6] The explicit form of the solution makes the uncharged limit well-defined, giving the hairy black hole configurations of [47]; the result should be not taken for granted, since in the standard literature the uncharged limit gives either Schwarzschild or Schwarzschild–AdS spacetime.

[7] Since the solution preserves conformal symmetry at the boundary, the mass can be directly extracted from the metric [30,31,71].

where $\sigma_k = \int d\Sigma_k$. The energy density of the system [72] is expressed as

$$\rho_{\rm E} \;=\; \frac{1}{8\pi G}\, L^2\, \mu_{\rm E}\,, \tag{43}$$

the dual boundary field theory living then on a manifold of radius L. The temperature and the entropy are given by

$$T \;=\; \frac{|f(x)'|}{4\pi\,\eta}\bigg|_{x=x_+}\,, \qquad\qquad S \;=\; \frac{\sigma_k}{4\,G}\, L^2\, \Upsilon(x_+)\,, \tag{44}$$

where $f(x_+) = 0$. Finally, the physical charges and electric potentials are

$$q_1 = \frac{\sigma_k}{8\pi G}\,\frac{L^2\, Q_1}{\eta}\,, \qquad \Phi^{\rm E}_1 = Q_1\,\frac{x_+^\nu - 1}{\nu}\,; \qquad\qquad q_2 = \frac{\sigma_k}{8\pi G}\,\frac{L^2\, Q_1}{\eta}\,, \qquad \Phi^{\rm E}_2 = Q_2\,\frac{1 - x_+^{-\nu}}{\nu}\,, \tag{45}$$

and it is possible to verify that these quantities satisfy the first law of thermodynamics:[8]

$$dM_{\rm E} \;=\; T\, dS + \Phi^{\rm E}_1\, dq_1 + \Phi^{\rm E}_2\, dq_2\,. \tag{46}$$

3.1.2. Family 2 – Magnetic Solutions

A second family of solutions is given by

$$\phi = \ell^{-1}\ln(x)\,, \qquad \bar F^1_{\vartheta\varphi} = P_1\,\frac{\sin(\sqrt{k}\,\vartheta)}{\sqrt{k}}\,, \qquad \bar F^2_{\vartheta\varphi} = P_2\,\frac{\sin(\sqrt{k}\,\vartheta)}{\sqrt{k}}\,, \qquad \Upsilon(x) = \frac{x^{\nu-1}\,\nu^2\, L^2}{\eta^2\,(x^\nu - 1)^2}\,, \tag{47}$$

$$\begin{aligned} f(x) \;=\; &\frac{x^{2-\nu}\,\eta^2\,(x^\nu-1)^2}{\nu^2}\,\frac{k}{L^2} + \left(1 - \alpha^2 L^2\right)\left(-1 + \frac{x^2}{\nu^2}\left(\,(\nu+2)\,x^{-\nu} - (\nu-2)\,x^\nu + \nu^2 - 4\,\right)\right) + \\ &+ 1 + \frac{x^{2-\nu}\,\eta^4\,(x^\nu-1)^3}{\nu^3\, L^6}\left(\frac{P_1^2}{(\nu+1)} - \frac{P_2^2}{(\nu-1)}\,x^{-\nu}\right)\,, \end{aligned} \tag{48}$$

$$ds^2 \;=\; \Upsilon(x)\left(f(x)\, dt^2 - \frac{\eta^2}{f(x)}\, dx^2 - L^2\, d\Sigma_k\right)\,. \tag{49}$$

The electric and magnetic solutions are related to each other by means of electromagnetic duality

$$\phi \;\to\; -\phi\,, \qquad\quad \alpha^2 \;\to\; L^{-2} - \alpha^2\,, \tag{50}$$

and the corresponding transformation of the electromagnetic fields (see also Section 4.1). In each of the two families, the asymptotic region is found for $x = 1$. The geometry and scalar field are singular at $x = 0$ and $x = \infty$ and, therefore, the configuration features two inequivalent, disjoint geometries for $x \in (1, \infty)$ or $x \in (0, 1)$.

Boundary conditions, mass and thermodynamics for the magnetic solutions.

The metric of the magnetic family can be obtained from the electric solutions by means of the transformation

$$Q_i \;\to\; \frac{\eta}{L^2}\, P_i\,, \qquad\quad \alpha^2 \;\to\; L^{-2} - \alpha^2\,, \tag{51}$$

and, therefore, the same analysis performed for the electric family can be exploited in the magnetic frame. In the latter case, the scalar asymptotic leading and subleading coefficients satisfy

$$\phi_1 = -\frac{\ell}{2}\,\phi_0^2\,, \tag{52}$$

which correspond again to AdS invariant boundary conditions (triple-trace deformation in the boundary theory). We introduce the quantity [55]

8 We do not have to modify the 1st law with the contribution of scalar charges having preserved the boundary conformal symmetry [30,31,73].

$$\mu_{\mathrm{M}} = \pm\left(\frac{\nu^2-4}{3\,\eta^3}\,(1-\alpha^2\,L^2) - \frac{k}{\eta\,L^2} + \frac{\eta\,P_2^2}{(\nu-1)\,L^6} - \frac{\eta\,P_1^2}{(\nu+1)\,L^6}\right), \tag{53}$$

where the $+$ $(-)$ corresponds to $x > 1$ $(x < 1)$ and express the black hole mass as

$$M_{\mathrm{M}} \;=\; \frac{\sigma_k}{8\pi G}\,L^4\,\mu_{\mathrm{M}}\,. \tag{54}$$

The magnetic charges and potentials are given by

$$p_1 = \frac{\sigma_k}{8\pi G}\,P_1\,, \qquad \Phi_1^{\mathrm{M}} = \frac{P_1\,\eta}{L^2}\,\frac{x_+^{\nu}-1}{\nu}\,; \qquad\qquad p_2 = \frac{\sigma_k}{8\pi G}\,P_2\,, \qquad \Phi_2^{\mathrm{M}} = \frac{P_2\,\eta}{L^2}\,\frac{1-x_+^{-\nu}}{\nu}\,, \tag{55}$$

and it is possible to verify that the above quantities satisfy the first law of thermodynamics:

$$dM_{\mathrm{M}} \;=\; T\,dS + \Phi_1^{\mathrm{M}}\,dp_1 + \Phi_2^{\mathrm{M}}\,dp_2\,. \tag{56}$$

3.1.3. Case $n = 1$ or $\nu = \infty$

The action in this special case[9] has the explicit form

$$\mathscr{S} \;=\; \frac{1}{8\pi G}\int d^4x\,\sqrt{-g}\left(-\frac{R}{2} + \frac{\partial_\mu\phi\,\partial^\mu\phi}{2} - \frac{1}{4}\,e^{\sqrt{2}\,\phi}\left(\bar{F}^1\right)^2 - \frac{1}{4}\,e^{-\sqrt{2}\,\phi}\left(\bar{F}^2\right)^2 + \frac{1}{L^2}\left(2+\cosh(\sqrt{2}\,\phi)\right)\right). \tag{57}$$

The scalar potential coming from (5) in the $n = 1$ limit is redefined through the shift

$$\phi \;\to\; \phi - \sqrt{2}\,\log\!\left(\frac{\rho\,L}{2}\right), \tag{58}$$

and writing the FI-terms as:

$$\theta_1 = \frac{\cos(\zeta)}{\rho\,L^2}\,, \qquad \theta_2 = \frac{\rho}{4}\,\cos(\zeta)\,, \qquad \theta_3 = \rho\,\sin(\zeta)\,, \qquad \theta_4 = \frac{4\,\sin(\zeta)}{\rho\,L^2}\,, \tag{59}$$

while the field strengths have the form

$$\bar{F}^1 \;=\; (\rho\,L)^{-1}\,F^1\,, \qquad\qquad \bar{F}^2 \;=\; \frac{\rho\,L}{4}\,F^2\,. \tag{60}$$

The above Lagrangian yields a consistent truncation of the $\mathcal{N} = 2$ minimal coupling supergravity [74] (corresponding to the $n = 1$ case) if the constraints (11), (12) are satisfied. In particular, the latter explicitly reads

$$\bar{F}^1 \wedge \bar{F}^1 - e^{-2\sqrt{2}\,\phi}\,\bar{F}^2 \wedge \bar{F}^2 \;=\; 0\,. \tag{61}$$

The theory features an explicit dyonic solution of the form [55]

[9] This particular class of solutions is noteworthy as it can be embedded in gauged $\mathcal{N} = 8$ supergravity [74].

$$\phi \,=\, -\,\ln(x)/\sqrt{2}\,, \tag{62}$$

$$\bar{F}^1_{tx} \,=\, Q_1\,, \qquad \bar{F}^1_{\vartheta\varphi} \,=\, P_1\,\frac{\sin(\sqrt{k}\,\vartheta)}{\sqrt{k}}\,, \tag{63}$$

$$\bar{F}^2_{tx} \,=\, \frac{Q_2}{x^2}\,, \qquad \bar{F}^2_{\vartheta\varphi} \,=\, P_2\,\frac{\sin(\sqrt{k}\,\vartheta)}{\sqrt{k}}\,, \tag{64}$$

$$f(x) \,=\, \frac{k\,\eta^2\,(x-1)^2}{L^2\,x} + 1 + \frac{\eta^2\,(x-1)^3}{L^2\,x}\left(Q_1^2 - \frac{Q_2^2}{x} - \frac{\eta^2}{L^4}\,\frac{P_1^2}{x} + \frac{\eta^2}{L^4}\,P_2^2\right), \tag{65}$$

$$\Upsilon(x) \,=\, \frac{x\,L^2}{\eta^2\,(x-1)^2}\,, \tag{66}$$

$$ds^2 \,=\, \Upsilon(x)\left(f(x)\,dt^2 - \frac{\eta^2}{x^2\,f(x)}\,dx^2 - L^2\,d\Sigma_k\right), \tag{67}$$

and the above constraint (61) requires

$$P_1\,Q_1 - P_2\,Q_2 \,=\, 0\,. \tag{68}$$

4. Discussion

4.1. Duality Relation between the Two Families of Solutions

The two families of solutions can be related by an electric–magnetic duality symmetry.[10] The transformation of the non-dynamical of θ_M amounts to a change in the theory, the duality being then interpreted as an equivalence between different models. Consider a generic $\mathcal{N}=2$ theory described by the Lagrangian (1), the scalar potential $V(\theta,\phi^s)$ given by Equation (5). Let us introduce the magnetic field strengths

$$G_{\Lambda\,\mu\nu} \,=\, -\,\epsilon_{\mu\nu\rho\sigma}\,\frac{\delta\mathscr{L}_{\text{BOS}}}{\delta F^{\Lambda}_{\rho\sigma}} \,=\, \mathcal{R}_{\Lambda\Sigma}\,F^{\Sigma}_{\mu\nu} - \mathcal{I}_{\Lambda\Sigma}\,{}^*F^{\Sigma}_{\mu\nu}\,, \tag{69}$$

and the symplectic field strength vector

$$\mathbb{F}^M = \begin{pmatrix} F^{\Lambda}_{\mu\nu} \\ G_{\Lambda\,\mu\nu} \end{pmatrix}. \tag{70}$$

The special Kähler manifold spanned by the scalars has a flat symplectic bundle structure, the matrix $\mathcal{M}_{MN}$ acting as a metric on the symplectic fiber. With each isometry $\mathbf{g}$ of the scalar manifold, it can be associated a constant symplectic matrix $\mathscr{R}[\mathbf{g}]$ such that, if $\phi'^s = \phi'^s(\phi^t)$ is the action of $\mathbf{g}$ on the scalars, $\mathcal{M}_{MN}(\phi^s)$ transforms as [69,70]:

$$\mathcal{M}(\phi'^s) \,=\, (\mathscr{R}[\mathbf{g}]^{-1})^T\mathcal{M}(\phi^s)\,\mathscr{R}[\mathbf{g}]^{-1}\,, \tag{71}$$

having suppressed symplectic indices. Then, if we transform correspondingly the Maxwell fields $\mathbb{F}^M$ and the FI-terms θ_M,

$$\mathbb{F}^M_{\mu\nu} \,\rightarrow\, \mathbb{F}'^M_{\mu\nu} = \mathscr{R}[\mathbf{g}]^M{}_N\,\mathbb{F}^N_{\mu\nu}\,, \qquad \theta_M \,\rightarrow\, \theta'_M = (\mathscr{R}[\mathbf{g}]^{-1})^N{}_M\,\theta_N\,, \tag{72}$$

the equations of motion are left invariant.

[10] The latter is a non-perturbative (global) symmetry of the ungauged theory and is extended to the gauged one if the constant tensor θ_M is made to transform under it as well [69,70].

Aside from the above duality, it is possible to redefine $\mathbb{F}^M$ and θ_M as [55]:

$$\mathbb{F}_{\mu\nu} \to \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}^{-1})^T \end{pmatrix} \mathbb{F}_{\mu\nu}\,, \qquad \theta \to \begin{pmatrix} (\mathbf{A}^{-1})^T & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \theta\,, \tag{73}$$

where $\mathbf{A} = (A^\Lambda{}_\Sigma)$ is a generic real, invertible matrix. This amounts to a change of basis in the symplectic fiber and does not alter the physics of the model. We can apply this mechanism to our truncated single-dilaton model, taking care that the transformations do not turn on the truncated axion χ. The absence of the latter implies $\mathcal{R}_{\Lambda\Sigma} = 0$, while the isometries of the dilaton can be expressed as

$$\phi \to \phi' = \phi + \beta\,, \tag{74}$$

$$\phi \to \phi' = -\phi\,, \tag{75}$$

β being a constant. The transformation (74) was used in Section 2.1.2 to reabsorb the θ_2 dependence of the FI-terms, the latter then depending only on α. This was then followed by the redefinition (73) of the field strengths and θ_M terms. If we now focus on the duality transformation (75), it is possible to verify that the associated transformation $\mathscr{R}[\mathbf{g}]$ is given by matrix $\mathbb{C}$. In particular, we find:

$$\begin{gathered} \mathcal{M}(\phi') \;=\; \mathcal{M}(-\phi) \;=\; (\mathbb{C}^{-1})^T \mathcal{M}(\phi)\, \mathbb{C}^{-1}\,. \\ \begin{pmatrix} \bar{F}'^\Lambda_{\mu\nu} \\ \bar{G}'_{\Lambda\,\mu\nu} \end{pmatrix} \;=\; \mathbb{C} \begin{pmatrix} \bar{F}^\Lambda_{\mu\nu} \\ \bar{G}_{\Lambda\,\mu\nu} \end{pmatrix} \;=\; \begin{pmatrix} \bar{G}_{\Lambda\,\mu\nu} \\ -\bar{F}^\Lambda_{\mu\nu} \end{pmatrix}. \end{gathered} \tag{76}$$

and, applying shift symmetry (74) and the above redefinition, the new FI-terms $\bar{\theta}_M$ have the form

$$\bar{\theta}_M \;=\; \left(\sqrt{\frac{\nu+1}{\nu}}\sqrt{L^{-2}-\alpha^2}\,,\; \sqrt{\frac{\nu-1}{\nu}}\sqrt{L^{-2}-\alpha^2}\,,\; \alpha\, s\sqrt{\frac{\nu+1}{\nu}}\,,\; \frac{\alpha}{s}\sqrt{\frac{\nu-1}{\nu}} \right). \tag{77}$$

Now it is possible to prove that the scalar potential satisfies

$$V(\bar{\theta},\phi) \;=\; V(\bar{\theta}',\phi')\,, \qquad \bar{\theta}' \;=\; \mathbb{C}\,\bar{\theta}\,, \tag{78}$$

or, equivalently

$$V(\alpha,\phi) \;=\; V(\alpha',\phi')\,, \qquad \alpha'^2 = L^{-2} - \alpha^2\,, \tag{79}$$

$\bar{\theta}$ only depending on α. Then, if the electric solution

$$\phi(x)\,,\;\; \bar{F}^\Lambda_{\mu\nu}(x)\,,\;\; g_{\mu\nu}(x)\,, \tag{80}$$

solves the field equations with parameter α, the magnetic configuration

$$\phi'(x) = -\phi(x)\,,\quad \bar{G}'_{\Lambda\,\mu\nu}(x) = -\bar{F}^\Lambda_{\mu\nu}(x)\,,\quad g'_{\mu\nu}(x) = g_{\mu\nu}(x)\,, \tag{81}$$

is a solution with parameter α'.

4.2. Supersymmetric Solutions

Now we want to study supersymmetryc configurations for our model, imposing the vanishing of the SUSY variations [75] (see Appendix A). First, it is useful to make a change of coordinates that puts metric (36) in the standard form

$$ds^2 \;=\; e^{2\,U(r)}\,dt^2 - e^{-2\,U(r)}\left(dr^2 + e^{2\,\Psi(r)}\,d\Sigma_k^2\right). \tag{82}$$

This can be achieved through the change of coordinate

$$x(r) \;=\; \left(1+\frac{L^2\,\nu}{\eta\,(r-c)}\right)^{\frac{1}{\nu}}, \tag{83}$$

c being an integration constant.

4.2.1. Family 1

The scalar field z in the new parametrization has the form

$$z \;=\; (\theta_2\,\xi)^{-\frac{2\,\nu}{1+\nu}}\left(1+\frac{L^2\,\nu}{\eta\,(r-c)}\right)^{-1}, \tag{84}$$

while the electric–magnetic charges explicitly read

$$\Gamma^M \;=\; \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ \dfrac{L^2}{2\,\eta}\,Q_1\,\sqrt{\frac{1+\nu}{\nu}}\,(\theta_2\,\xi)^{\frac{1-\nu}{1+\nu}} \\ \dfrac{L^2}{2\,\eta}\,Q_2\,\sqrt{\frac{-1+\nu}{\nu}}\,\theta_2\,\xi \end{pmatrix}. \tag{85}$$

The solution is supersymmetric if the following relations are satisfied:

$$\begin{aligned} Q_1 &= -\,Q_2\,\sqrt{\frac{-1+\nu}{1+\nu}} \;+\; \frac{k\,\eta}{\alpha\,L^2}\,\sqrt{\frac{\nu}{1+\nu}}\,, \\ Q_2 &= \left(\frac{k\,\eta}{2\,\alpha\,L^2}+\frac{\alpha\,L^2\,(1+\nu)}{2\,\eta}\right)\sqrt{\frac{-1+\nu}{\nu}}\,. \end{aligned} \tag{86}$$

4.2.2. Family 2

The scalar field z reads

$$z \;=\; (\theta_2\,\xi)^{-\frac{2\,\nu}{1+\nu}}\left(1+\frac{L^2\,\nu}{\eta\,(r-c)}\right), \tag{87}$$

while the electric–magnetic charges are

$$\Gamma^M \;=\; \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} \;=\; \begin{pmatrix} 2\,P_1\,\sqrt{\frac{\nu}{1+\nu}}\,(\theta_2\,\xi)^{\frac{-1+\nu}{1+\nu}} \\ 2\,P_2\,\sqrt{\frac{\nu}{-1+\nu}}\,(\theta_2\,\xi)^{-1} \\ 0 \\ 0 \end{pmatrix}. \tag{88}$$

The solution is supersymmetric if

$$\begin{aligned} P_1 &= -\,P_2\,\sqrt{\frac{-1+\nu}{1+\nu}} \;+\; \frac{k\,L}{\sqrt{1-\alpha^2\,L^2}}\,\sqrt{\frac{\nu}{1+\nu}}\,, \\ P_2 &= \left(\frac{k\,L}{2\,\sqrt{1-\alpha^2\,L^2}}+\frac{L^3\,(1+\nu)}{2\,\eta^2}\,\sqrt{1-\alpha^2\,L^2}\right)\sqrt{\frac{-1+\nu}{\nu}}\,. \end{aligned} \tag{89}$$

The supersymmetric magnetic condition can be obtained from the supersymmetric electric condition by means of the duality transformation

$$Q_i \to \frac{\eta}{L^2} P_i \,, \qquad \alpha^2 \to L^{-2} - \alpha^2 \,. \tag{90}$$

4.3. Old and New Solutions

Here we briefly mention already known configurations as particular cases of our new general solutions [55].

4.3.1. Duff–Liu

The Duff and Liu solution [65] features singular supersymmetric configurations, while some of their non-extremal solutions can be put in relation with our model for special values of the ν parameter. The theory under consideration is the well known STU model of gauged $\mathcal{N} = 8$ supergravity with three scalar fields φ_1, φ_2, φ_3.

The T^3 model. If we equal the three scalars, $\varphi_1 = \varphi_2 = \varphi_3$, we obtain the so-called T^3 model. The latter coincides with our formulation (33) for $\nu = -2$.

The $n = 1$ or $\nu = \infty$ model. In this case, we set $\varphi_1 = \varphi_3 = 0$. The STU model then reduces to the $n = 1$ model, coinciding with our (57). The Duff–Liu configuration can be recovered from our solution provided either the magnetic or the electric charges are set to zero.

4.3.2. Cacciatori–Klemm

This model was important since it provided the first non singular supersymmetric black holes in AdS_4 [14].

The T^3 model. The spherically symmetric, purely magnetic supersymmetric solution can be obtained from our prepotential (2) with $n = \frac{1}{2}$, corresponding to $\nu = -2$, and pure magnetic $\theta_M = (\xi_0, \xi_1, 0, 0)$ FI-terms.

4.4. New BPS Black Holes with Finite Area

4.4.1. Family 1: BPS Electric Black Holes

The electric family has BPS black holes of finite area for $\alpha^2 = L^{-2}$. In this case, we find [55]:

$$f(x_+) = 0 \quad \Longrightarrow \quad x_\pm^\nu = \frac{\nu^2 - 1 - k\,\eta^2 L^{-2}}{\nu - 1 - k\,\eta^2 L^{-2}} \pm \nu\, \frac{\sqrt{\nu^2 - 1 - 2\,k\,\eta^2 L^{-2}}}{\nu - 1 - k\,\eta^2 L^{-2}} \,, \tag{91}$$

and it is possible to verify that $x_\pm$ is an even function of ν, $x_\pm(\nu) = x_\pm(-\nu)$, so that we can restrict our analysis to the $\nu > 1$ interval.

Keeping in mind that the spacetime has disjoint and inequivalent geometries for $x \in (0,1)$ and $x \in (1, \infty)$ (the asymptotic region being located at $x = 1$), we can characterize the location of the horizon as follows

- $k = 0$: in the flat case the location of the horizon is very simple (see the above (91)) and it follows that $x_+^\nu > 0$ and $x_-^\nu < 0$, so we conclude that only x_+ exists;
- $k = -1$: in the hyperbolic case, $x_+ > 1$ always exists while the solution $0 < x_- < 1$ exists provided $\eta^2 L^{-2} > \nu + 1$;
- $k = +1$: for spherical black hole only x_+ exists, provided $\nu - 1 > \eta^2 L^{-2} > 0$.

Using BPS conditions (86), we can write the explicit form for the physical charges and electric potentials for the supersymmetric solutions as [55]:

$$\begin{aligned}
q_1^{\mathrm{BPS}} &= \pm \frac{L^2 \sigma_k}{8\pi G}\left(\frac{k}{2\,L} + \frac{L\,(1-\nu)}{2\,\eta^2}\right)\sqrt{\frac{\nu+1}{\nu}}\,, \qquad & \Phi_1^{\mathrm{BPS}} &= \pm\left(\frac{k\,\eta}{2\,L} + \frac{L\,(1-\nu)}{2\,\eta}\right)\frac{x_+^\nu - 1}{\nu}\sqrt{\frac{\nu+1}{\nu}}\,, \\
q_2^{\mathrm{BPS}} &= \pm \frac{L^2 \sigma_k}{8\pi G}\left(\frac{k}{2\,L} + \frac{L\,(1+\nu)}{2\,\eta^2}\right)\sqrt{\frac{\nu-1}{\nu}}\,, \qquad & \Phi_2^{\mathrm{BPS}} &= \pm\left(\frac{k\,\eta}{2\,L} + \frac{L\,(1+\nu)}{2\,\eta}\right)\frac{1 - x_+^{-\nu}}{\nu}\sqrt{\frac{\nu-1}{\nu}}\,,
\end{aligned} \tag{92}$$

having fixed $\alpha = \pm L^{-1}$. The mass of the electric BPS black hole configuration is given by

$$M_{\rm E}^{\rm BPS} = \frac{L^4\,\sigma_k}{8\pi G}\,\frac{\nu^2-1}{3\,\eta^3}\,, \tag{93}$$

and it is then verified the extremality condition

$$\delta M_{\rm E}^{\rm BPS} = \Phi_1^{\rm BPS}\,\delta q_1^{\rm BPS} + \Phi_2^{\rm BPS}\,\delta q_2^{\rm BPS}\,. \tag{94}$$

4.4.2. Family 2: BPS Magnetic Black Holes

We can again restrict to the $\nu > 1$ case, and verify that this family has BPS black holes of finite area for $\alpha^2 = 0$ (electric gauging). In this case, the metric for the magnetic solution is the same than for the electric one, implying a coinciding analysis about the location of the horizons. The extremality of the magnetic solutions is the same as for the electric solutions when the electric charges and potentials are interchanged by their magnetic counterparts.

4.5. Hamilton–Jacobi Formulation

If we consider a stationary dilatonic black hole configuration with radial dependence for the scalar field, $\phi = \phi(r)$, the most general metric ansatz, with spherical or hyperbolic symmetry, has the form (82),

$$ds^2 = e^{2\,U(r)}\,dt^2 - e^{-2\,U(r)}\left(dr^2 + e^{2\,\Psi(r)}\,d\Sigma_k^2\right). \tag{95}$$

We can obtain the equations of motion coming from the bosonic gauged Lagrangian (1), with the above metric ansatz, from a one-dimensional effective action that, apart from total derivative terms, has the form

$$\mathscr{S}_{\rm eff} = \int dr\,\mathscr{L}_{\rm eff} = \int dr\left[e^{2\,\Psi}\left(\dot U^2 - \dot\Psi^2 + \frac{1}{2}\,\dot\phi^2\right) - V_{\rm eff}\right], \tag{96}$$

where the prime stands for derivative with respect to r and where we can define an effective potential

$$V_{\rm eff} = -\,e^{2(U-\Psi)}\,V_{\rm BH} - e^{-2(U-\Psi)}\,V + k\,, \tag{97}$$

in terms of the scalar potential V and the (charge-dependent) black hole potential $V_{\rm BH}$. The latter can be written in the symplectically covariant form [75–77]

$$V_{\rm BH} = -\frac{1}{2}\,\Gamma^T\mathcal{M}\,\Gamma\,, \tag{98}$$

in terms of the magnetic and electric charges and scalar-dependent matrix $\mathcal{M}$. The black hole potential (98) can also be schematically rewritten in terms of the central charges as [75–77]

$$V_{\rm BH} = |\mathcal{D}\mathscr{Z}| - |\mathscr{Z}|^2\,. \tag{99}$$

Once given the effective action, one can make use of the Hamilton–Jacobi formalism and derive a system of first-order equations (*flow equations*) for the warp factors $U(r)$, $\Psi(r)$ and scalar fields $\phi(r)$.

Flow equations

If we collectively denote the metric and scalar degrees of freedom characterizing the solution by $q^I = \{U(r),\,\Psi(r),\,\phi(r)\}$, we can rewrite the effective Lagrangian of Equation (96) as

$$\mathscr{L}_{\rm eff} = \mathcal{G}_{IJ}\,\dot q^I\,\dot q^J - V_{\rm eff}\,, \tag{100}$$

with

$$\mathcal{G}_{IJ} = \begin{pmatrix} e^{2\Psi} & & \\ & -e^{2\Psi} & \\ & & \frac{1}{2}e^{2\Psi} \end{pmatrix} . \tag{101}$$

The conjugate momenta are obtained from

$$p_I = \frac{\delta \mathscr{L}_{\text{eff}}}{\delta \dot{q}^I} = 2\,\mathcal{G}_{IJ}\,\dot{q}^J . \tag{102}$$

while the Hamiltonian can be written as

$$\mathscr{H} = p_I\,\dot{q}^I - \mathscr{L} = \mathcal{G}_{IJ}\,\dot{q}^I\,\dot{q}^J + V_{\text{eff}} , \tag{103}$$

together with the energy conservation constraint $\mathscr{H} = k$.

Let us now consider a BPS configuration. If a superpotential function $W(q^I)$ can be defined for the latter, then the radial evolution of the system can be described in terms of equations of the form

$$\dot{q}^I = \mathcal{G}^{IJ}\,\frac{\partial W}{\partial q^J} . \tag{104}$$

BPS hairy black hole. The superpotential function for our electric solution has the explicit form

$$W_\phi = -e^{2\Psi - U}\,|\mathcal{P}(\phi)| , \tag{105}$$

where $\mathcal{P}(\phi) = e^{2(U-\Psi)}\,\mathscr{Z}(\phi) + i\,\mathcal{W}(\phi)$ is written in terms of the central charge $\mathscr{Z}$ and complex superpotential $\mathcal{W}$ obtained from the geometry of the supergravity Kähler manifold [75]. The above superpotential correctly satisfies equations (104) for the hairy black hole configuration, namely

$$\begin{aligned} \dot{U} &= e^{-2\Psi}\,\frac{\partial W_\phi}{\partial U} , \\ \dot{\Psi} &= -e^{-2\Psi}\,\frac{\partial W_\phi}{\partial \Psi} , \\ \dot{\phi} &= 2\,e^{-2\Psi}\,\frac{\partial W_\phi}{\partial \phi} , \end{aligned} \tag{106}$$

when evaluated on the solution.

4.6. $\mathcal{N} = 8$ *Truncations*

4.6.1. Uncharged Case

The infinitely many theories defined by the model under consideration comprise all the possible one-dilaton consistent truncations of the ω-deformed, SO(8) gauged maximal supergravities [36,42,45,46,69,70,78,79][11].

Let us consider the following truncation to gravity and scalar field sector of the maximal supergravity [81,82]:

$$\mathscr{S}(g_{\mu\nu},\vec{\phi}) = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\left[-\frac{R}{2} + \frac{1}{2}\,(\partial\vec{\phi})^2 - V(\vec{\phi})\right] , \tag{107}$$

with 7 independent scalars $\vec{\phi} \equiv \phi_i$ ($i = 1, \ldots, 8,\ \ \sum_{i=1}^{8} \phi_i = 0$) and scalar potential

$$V(\vec{\phi}) = -\frac{g^2}{32}\left[\cos^2(\omega)\left(\left(\sum_{i=1}^{8} X_i\right)^2 - 2\sum_{i=1}^{8} X_i^2\right) + \sin^2(\omega)\left(\left(\sum_{i=1}^{8} X_i^{-1}\right)^2 - 2\sum_{i=1}^{8} X_i^{-2}\right)\right] , \tag{108}$$

[11] The construction of these models was carried out by exploiting the freedom in the initial choice of the symplectic frame of the maximal theory, that is, gauging a group in different symplectic frames by rotating the original one [44,80] making use of a suitable symplectic matrix, thus obtaining a one-parameter class of inequivalent theories (ω-deformed models).

where

$$X_i = e^{2\,\phi_i}\,,\qquad \prod_{i=1}^{8} X_i = 1\,. \tag{109}$$

A single-scalar field reduction preserving $\mathrm{SO}(p)\times\mathrm{SO}(8-p)$ can be obtained through [55]

$$\phi_1 = \cdots = \phi_p = \frac{1}{2\sqrt{2}}\,\sigma\,\phi\,,\qquad \phi_{p+1} = \cdots = \phi_8 = -\frac{1}{2\sqrt{2}}\,\frac{\phi}{\sigma}\,, \tag{110}$$

having defined:

$$\sigma = \sqrt{\frac{8-p}{p}} = \sqrt{\frac{\nu-1}{\nu+1}}\,,\qquad p = \frac{4\,(\nu+1)}{\nu}\,, \tag{111}$$

that also implies

$$X_1 = \cdots = X_p = X := e^{\frac{1}{\sqrt{2}}\,\sigma\,\phi}\,,\qquad X_{p+1} = \cdots = X_8 = Y := e^{-\frac{1}{\sqrt{2}}\,\frac{\phi}{\sigma}}\,. \tag{112}$$

With the above choices, we can obtain a consistent truncation of the ω-rotated $\mathrm{SO}(8)$ gauged maximal supergravity. The corresponding action is invariant under $\sigma\to 1/\sigma$, $\phi\to-\phi$ and $p\to 8-p$ and reduces, in the absence of vector fields, to the action (33) upon identifications:

$$g = \frac{\sqrt{2}}{L}\,,\qquad \cos(\omega) = L\,\alpha\,,\qquad \sin(\omega) = \sqrt{1-L^2\,\alpha^2}\,, \tag{113}$$

or, changing ϕ into $-\phi$ in (112),

$$g = \frac{\sqrt{2}}{L}\,,\qquad \sin(\omega) = L\,\alpha\,,\qquad \cos(\omega) = \sqrt{1-L^2\,\alpha^2}\,. \tag{114}$$

We conclude that our new single-scalar models with vanishing vector fields coincide with gauged ω-deformed maximal supergravity truncations to a singlet sector, with respect to the following subgroups of the $\mathrm{SO}(8)$ gauging [55]:

$$\begin{aligned} \nu = \tfrac{4}{3} &\quad\rightarrow\quad \mathrm{SO}(7)\ , \\ \nu = 2 &\quad\rightarrow\quad \mathrm{SO}(6)\times\mathrm{SO}(2)\ , \\ \nu = 4 &\quad\rightarrow\quad \mathrm{SO}(5)\times\mathrm{SO}(3)\ , \\ \nu = \infty &\quad\rightarrow\quad \mathrm{SO}(4)\times\mathrm{SO}(4)\ . \end{aligned} \tag{115}$$

In particular, $\nu=\infty$ or $\nu=\pm2$ correspond to models which can be embedded in the STU truncation of the $\mathrm{SO}(8)$ gauged $\mathcal{N}=8$ supergravity. Therefore, the new black hole solutions, in the absence of electric and magnetic charges, can all be embedded in the maximal theory.

4.6.2. Charged Case

Now we consider charged solutions and their possible embedding in the dyonic, $\mathrm{SO}(8)$ gauged maximal supergravity [55].

$\nu = 4$ case.

Let us consider first the $\omega=0$ case. These solutions describe gravity coupled to one scalar and two vector fields. The latter, when identified with their counterparts in the maximally supersymmetric theory, should not excite other fields in the model: this requires the condition $F^\lambda\wedge F^\sigma = 0$ in our solution[12], again with the presence of seven independent

[12] λ, σ label the 28 vectors of the maximal theory, while m, n are the symplectic indices of the 56 electric and magnetic charges.

scalar fields $\vec{\phi} = \phi_i$ $(i = 1, \dots, 8,\ \sum_{i=1}^{8} \phi_i = 0)$ parameterizing the Cartan subalgebra of $\mathfrak{e}_{7(7)}$.

Then, we want to further truncate the model to a single scalar ϕ, singlet with respect to the subgroup $\mathrm{SO}(5) \times \mathrm{SO}(3)$ of $\mathrm{SO}(8)$. The two vector fields of the charged solution are identified with two of the 28 vectors of the maximal theory that do not source the six scalars we want to vanish. We then express $\vec{\phi}$ as [55]:

$$
\begin{aligned}
&\phi_1 = -\frac{1}{2}\sqrt{\frac{3}{10}}\,\phi + \varphi_1\,, \quad && \phi_2 = -\frac{1}{2}\sqrt{\frac{3}{10}}\,\phi + \varphi_2\,, \quad && \phi_3 = -\frac{1}{2}\sqrt{\frac{3}{10}}\,\phi + \varphi_3\,, \\
&\phi_4 = -\frac{1}{2}\sqrt{\frac{3}{10}}\,\phi + \varphi_4\,, && \phi_5 = -\frac{1}{2}\sqrt{\frac{3}{10}}\,\phi - \sum_{k=1}^{4}\varphi_k\,, && \phi_6 = \frac{1}{2}\sqrt{\frac{5}{6}}\phi + \varphi_5\,, \\
&\phi_7 = \frac{1}{2}\sqrt{\frac{5}{6}}\phi + \varphi_6\,, && \phi_8 = \frac{1}{2}\sqrt{\frac{5}{6}}\phi - \varphi_5 - \varphi_6\,.
\end{aligned}
\tag{116}
$$

The equations for the six scalars φ_ℓ $(\ell = 1, \dots 6)$ are satisfied when $\varphi_\ell \equiv 0$, while ϕ enters the kinetic terms of the vector fields as in (33), with $\nu = 4$, after identifying the two vectors A^1_μ and A^2_μ as:

$$
\frac{1}{2}\,A^{IJ}_\mu\,T_{IJ} \;=\; A^1_\mu\,J_1 + A^2_\mu\,J_2\,, \tag{117}
$$

in terms of the A^{IJ}_μ of the maximal theory and the $\mathrm{SO}(8)$ generators $T_{IJ} = -T_{JI}$ $(I, J = 1, \dots, 8)$.[13] The two field strengths $\bar{F}^1_{\mu\nu}$, $\bar{F}^2_{\mu\nu}$ in (33) can be in turn identified with the $F^{IJ}_{\mu\nu}$ of the maximal theory as [55]:

$$
\begin{aligned}
&F^{12}_{\mu\nu} = \sqrt{\frac{2}{5}}\,\bar{F}^1_{\mu\nu}\,, \quad F^{34}_{\mu\nu} = \frac{\varepsilon_1}{\sqrt{5}}\,\bar{F}^1_{\mu\nu}\,, \quad F^{35}_{\mu\nu} = \frac{\varepsilon_2}{\sqrt{5}}\,\bar{F}^1_{\mu\nu}\,, \quad F^{45}_{\mu\nu} = \frac{\varepsilon_3}{\sqrt{5}}\,\bar{F}^1_{\mu\nu}\,, \\
&F^{67}_{\mu\nu} = \frac{1}{\sqrt{3}}\,\bar{F}^2_{\mu\nu}\,, \quad F^{68}_{\mu\nu} = \frac{\varepsilon_4}{\sqrt{3}}\,\bar{F}^2_{\mu\nu}\,, \quad F^{78}_{\mu\nu} = \frac{\varepsilon_5}{\sqrt{3}}\,\bar{F}^2_{\mu\nu}\,.
\end{aligned}
\tag{118}
$$

In the $\omega \neq 0$ case, the same generators T_{IJ} are gauged by linear combinations of A^{IJ}_μ and $A_{IJ\,\mu}$ of the form

$$
\hat{A}^{IJ}_\mu \;=\; \cos(\omega)\,A^{IJ}_\mu - \sin(\omega)\,A_{IJ\,\mu}\,, \tag{119}
$$

which amounts to gauging $\mathrm{SO}(8)$ in a different symplectic frame, with $\hat{A}^{IJ}_\mu$ electric vector fields (the hat denoting indices in the new ω-rotated symplectic frame). The related symplectic vector of electric and magnetic field strengths in the new frame is obtained from the relation [55]:

$$
\hat{\mathbb{F}}^{\hat{m}}_{\mu\nu} \;=\; E_m{}^{\hat{m}}\,\hat{\mathbb{F}}^{m}_{\mu\nu}\,, \qquad E_m{}^{\hat{m}} = \begin{pmatrix} \cos(\omega)\,\mathbb{1}_{28\times 28} & \sin(\omega)\,\mathbb{1}_{28\times 28} \\ -\sin(\omega)\,\mathbb{1}_{28\times 28} & \cos(\omega)\,\mathbb{1}_{28\times 28} \end{pmatrix}, \tag{120}
$$

while the deformed kinetic matrix $\mathcal{M}_{\hat{m}\hat{n}}(\phi,\omega)$ is expressed in terms of the $\mathcal{M}_{mn}(\phi)$ in the original frame as:

$$
\mathcal{M}_{\hat{m}\hat{n}}(\phi,\omega) \;=\; E^{-1}(\omega)_{\hat{m}}{}^{m}\,E^{-1}(\omega)_{\hat{n}}{}^{n}\,\mathcal{M}_{mn}(\phi)\,. \tag{121}
$$

After the truncation, the vector kinetic terms in the new symplectic frame will depend on ω through the restrictions to $\hat{A}^{\hat{\Lambda}}_\mu = (\hat{A}^1_\mu, \hat{A}^2_\mu)$. The dependence can be disposed of at the level of field equations and Bianchi identities, since the latter depend on $\hat{\mathbb{F}}^{\hat{M}}_{\mu\nu}$ only in symplectic-invariant contractions with the matrix $\mathcal{M}_{\hat{M}\hat{N}}(\phi,\omega)$ and its derivatives. The ω dependence of the terms involving the vector field strengths can be undone redefining

[13] We explicitly have $J_1 = \sqrt{\frac{2}{5}}\left(T_{12} + \frac{\varepsilon_1}{\sqrt{2}}\,T_{34} + \frac{\varepsilon_2}{\sqrt{2}}\,T_{35} + \frac{\varepsilon_3}{\sqrt{2}}\,T_{45}\right)$, $J_2 = \frac{1}{\sqrt{3}}\,(T_{67} + \varepsilon_4\,J_{68} + \varepsilon_5\,J_{78})$, $\varepsilon_\ell^2 = 1$ [55].

the latter, which amounts to express them in terms of their counterparts in the original frame ($\bar{F}^1_{\mu\nu}$, $\bar{F}^2_{\mu\nu}$ and their magnetic duals) through the matrix E. Upon these redefinitions, the bosonic field equations of the truncated model coincide with those obtained from the action (33), with $\nu = 4$, provided we identify [55]:

$$g = \frac{\sqrt{2}}{L}\,, \qquad \sin(\omega) = L\,\alpha\,, \qquad \cos(\omega) = \sqrt{1 - L^2\alpha^2}\,. \tag{122}$$

The embedding of the $\nu = 4/3$ model in the maximal theory will be studied in the future. In the remaining cases $\nu = \infty$ or $\nu = \pm 2$, our solutions can be extended to charged solutions within SO(8) gauged $\mathcal{N} = 8$ supergravity within the bosonic part of the STU truncation of it.

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Appendix A. Supersymmetric Black Hole Solutions

In order to obtain supersymmetric configurations, we have to impose the vanishing of the SUSY transformations in addition to solving the equations of motion.

The relevant supersymmetry variations can be expressed as:

$$\begin{aligned} \delta\psi_{\mu A} &= D_\mu\epsilon_A + i\,T^-_{\mu\nu}\,\gamma^\nu\,\varepsilon_{AB}\,\epsilon^B + i\,\mathbb{S}_{AB}\,\gamma_\mu\,\epsilon^B\,, \\ \delta\lambda^{iA} &= i\,\partial_\mu z^i\,\gamma^\mu\,\epsilon^A - \frac{1}{2}\,g^{i\bar{\jmath}}\,\bar{f}^\Lambda_{\bar{\jmath}}\,\mathcal{I}_{\Lambda\Sigma}\,F^{-\Sigma}_{\mu\nu}\,\gamma^{\mu\nu}\,\varepsilon^{AB}\,\epsilon_B + W^{iAB}\,\epsilon_B\,, \end{aligned} \tag{A1}$$

with $\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}$.[14] The covariant derivatives are written as

$$D_\mu\epsilon_A = \partial_\mu\epsilon_A + \frac{1}{4}\,\omega_\mu{}^{ab}\,\gamma_{ab}\,\epsilon_A + \frac{i}{2}\left(\sigma^2\right)_A{}^B\,\mathbb{A}^M_\mu\,\theta_M\,\epsilon_B + \frac{i}{2}\,\mathcal{Q}_\mu\,\epsilon_A\,, \tag{A2}$$

with

$$\mathcal{Q}_\mu = \frac{i}{2}\left(\partial_{\bar{\imath}}\mathcal{K}\,\partial_\mu\bar{z}^{\bar{\imath}} - \partial_i\mathcal{K}\,\partial_\mu z^i\right)\,, \tag{A3}$$

and, in the chosen parametrization, we also have [75]:

[14] For the gamma-matrices we can use the conventions of App. A of [83].

$$
\begin{aligned}
F^{\pm}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\nu} \pm F_{\mu\nu}\right)\,, \qquad G^{\pm}_{\mu\nu} \;=\; \frac{1}{2}\left(G_{\mu\nu} \pm G_{\mu\nu}\right)\,,\\
T_{\mu\nu} &= L^{\Lambda}\,\mathcal{I}_{\Lambda\Sigma}\,F^{\Sigma}_{\mu\nu} \;=\; \frac{1}{2i}\,L^{\Lambda}\,(\mathfrak{N}-\overline{\mathfrak{N}})_{\Lambda\Sigma}\,F^{\Sigma}_{\mu\nu} \;=\; -\frac{i}{2}\Big(M_{\Sigma}\,F^{\Sigma}_{\mu\nu} - L^{\Lambda}\,G_{\Lambda\mu\nu}\Big) \;=\; \frac{i}{2}\,\mathcal{V}^{M}\,\mathbb{C}_{MN}\,\mathbb{F}^{N}_{\mu\nu}\,,\\
T^{-}_{\mu\nu} &= L^{\Lambda}\,\mathcal{I}_{\Lambda\Sigma}\,F^{-\Sigma}_{\mu\nu} \;=\; \frac{i}{2}\,\mathcal{V}^{M}\,\mathbb{C}_{MN}\,\mathbb{F}^{-N}_{\mu\nu}\,,\\
T_{i\,\mu\nu} &= \mathcal{D}_i T_{\mu\nu} \;=\; f^{\Lambda}_i\,\mathcal{I}_{\Lambda\Sigma}\,F^{\Sigma}_{\mu\nu} \;=\; -\frac{i}{2}\Big(h_{i\Sigma}\,F^{\Sigma}_{\mu\nu} - f^{\Lambda}_i\,G_{\Lambda\mu\nu}\Big) \;=\; \frac{i}{2}\,\mathcal{U}^{M}_i\,\mathbb{C}_{MN}\,\mathbb{F}^{N}_{\mu\nu}\,,\\
\mathbb{S}_{AB} &= \frac{i}{2}\left(\sigma^2\right)_A{}^C\,\varepsilon_{BC}\,\theta_M\,\mathcal{V}^{M} \;=\; \frac{i}{2}\left(\sigma^2\right)_A{}^C\,\varepsilon_{BC}\,\mathcal{W}\,,\\
W^{i\,AB} &= i\left(\sigma^2\right)_C{}^B\,\varepsilon^{CA}\,\theta_M\,g^{i\bar{\jmath}}\,\overline{\mathcal{U}}^{M}_{\bar{\jmath}}\,,
\end{aligned}
\tag{A4}
$$

having used properties

$$
\overline{\mathfrak{N}}_{\Lambda\Sigma}\,F^{-\Sigma} = G^{-}_{\Lambda}\,, \qquad L^{\Lambda}\,\mathfrak{N}_{\Lambda\Sigma} = M_{\Sigma}\,. \tag{A5}
$$

The kinetic matrix $\mathfrak{N}$ is defined as

$$
\mathfrak{N} = \mathcal{R} + i\,\mathcal{I}\,, \tag{A6}
$$

and can be expressed in terms of the prepotential as [84]

$$
\mathfrak{N}_{\Lambda\Sigma} = \partial_{\bar{\Lambda}}\partial_{\bar{\Sigma}}\overline{\mathcal{F}} + 2\,i\,\frac{\mathrm{Im}[\partial_{\Lambda}\partial_{\Gamma}\mathcal{F}]\;\mathrm{Im}[\partial_{\Sigma}\partial_{\Delta}\mathcal{F}]\;L^{\Gamma}\,L^{\Delta}}{\mathrm{Im}[\partial_{\Delta}\partial_{\Gamma}\mathcal{F}]\;L^{\Delta}\,L^{\Gamma}}\,, \tag{A7}
$$

with $\partial_{\Lambda} = \frac{\partial}{\partial\mathfrak{X}^{\Lambda}}$, $\partial_{\bar{\Lambda}} = \frac{\partial}{\partial\bar{\mathfrak{X}}^{\Lambda}}$. We emphasize that, in the special coordinate frame, the whole $\mathcal{N} = 2$ Lagrangian can be written in terms of the holomorphic prepotential function $\mathcal{F}(\mathfrak{X})$ (if any) and its derivatives.

From an explicit computation of the supersymmetry variations (A1), we find the following relations for the warp factors [75]

$$
\begin{aligned}
U' &= e^{U-2\Psi}\,\mathrm{Re}\Big[e^{-i\alpha}\,\mathscr{Z}\Big] + e^{-U}\,\mathrm{Im}\Big[e^{-i\alpha}\,\mathcal{W}\Big]\,,\\
\Psi' &= 2\,e^{-U}\,\mathrm{Im}\Big[e^{-i\alpha}\,\mathcal{W}\Big]\,,
\end{aligned}
\tag{A8}
$$

and for the scalars

$$
z'^{\,i} = e^{-U}\,e^{i\alpha}\,g^{i\bar{\jmath}}\,\mathcal{D}_{\bar{\jmath}}\Big(e^{2U-2\Psi}\,\overline{\mathscr{Z}} - i\,\overline{\mathcal{W}}\Big)\,, \tag{A9}
$$

the above covariant derivative acting on objects with weight $p = -1$, and having introduced two projectors relating the spinor components as [15,16,75,85]

$$
\begin{aligned}
\gamma^0\,\epsilon_A &= i\,e^{i\alpha}\,\varepsilon_{AB}\,\epsilon^B\,,\\
\gamma^1\,\epsilon_A &= e^{i\alpha}\,\delta_{AB}\,\epsilon^B\,.
\end{aligned}
\tag{A10}
$$

The choice of phase α turns out to be irrelevant from the physical point of view (due to the presence of the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry), while it will amount to putting the symplectic sections of the vector multiplet moduli space in a particular frame [15,16].

The Killing spinors must satisfy the relations [75]

$$\begin{aligned} \epsilon_A &= \chi_A\, e^{\frac{1}{2}(U - i\int dr\, \mathcal{B})}\,, \\ \epsilon^A &= i\, e^{-i\alpha}\, \varepsilon^{AB}\, \gamma^0\, \epsilon_B\,, \end{aligned} \tag{A11}$$

where we have

$$\begin{aligned} \partial_r \chi_A &= 0\,, \\ \mathcal{B} &= \mathcal{Q}_r + 2\, e^{-U}\, \mathrm{Re}\Big[e^{-i\alpha}\, \mathcal{W}\Big]\,, \end{aligned} \tag{A12}$$

and the following expression for the phase α holds [75]:

$$\partial_r \alpha = -\mathcal{B}\,. \tag{A13}$$

The chosen type of Killing spinors explicitly break $3/4$ of the supersymmetry, i.e., the solution is $1/4$ BPS, meaning that only 2 supercharges are preserved, corresponding to the 2 degrees of freedom of the complex $\mathcal{B}(r)$ function parameterizing the ϵ_1, ϵ_2 Killing spinors [15,16].

Finally, from the SUSY variations we obtain the property [75]

$$\mathrm{Im}\Big[e^{-i\alpha}\, \mathscr{Z}\Big] = -e^{2\Psi - 2U}\mathrm{Re}\Big[e^{-i\alpha}\, \mathcal{W}\Big]\,, \tag{A14}$$

and, using also the ansatz

$$\mathbb{F}^M = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = e^{2(U-\Psi)}\, \mathbb{C}^{MP}\, \mathcal{M}_{PN}\, \Gamma^N\, dt \wedge dr + \Gamma^M\, f_k(\vartheta)\, d\vartheta \wedge d\varphi = d\mathbb{A}^M\,, \tag{A15}$$

for the $\mathbb{F}^M$ symplectic vector, we find for the $\mathbb{A}^M_\mu$ components:

$$\begin{aligned} \mathbb{A}^M_t\, \theta_M &= 2\, e^{U}\, \mathrm{Re}\Big[e^{-i\alpha}\, \mathcal{W}\Big]\,, \\ \mathbb{A}^M_r &= 0\,, \\ \mathbb{A}^M_\vartheta &= 0\,, \\ \mathbb{A}^M_\varphi &= -\frac{\Gamma^M}{k}\, \cos\Big(\sqrt{k}\, \vartheta\Big)\,, \end{aligned} \tag{A16}$$

together with the relation

$$\Gamma^M\, \theta_M = k\,. \tag{A17}$$

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