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Towards a Geometrization of Renormalization Group Histories in Asymptotic Safety

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Abstract: Considering the scale-dependent effective spacetimes implied by the functional renormalization group in d-dimensional quantum Einstein gravity, we discuss the representation of entire evolution histories by means of a single, $(d+1)$-dimensional manifold furnished with a fixed (pseudo-) Riemannian structure. This "scale-spacetime" carries a natural foliation whose leaves are the ordinary spacetimes seen at a given resolution. We propose a universal form of the higher dimensional metric and discuss its properties. We show that, under precise conditions, this metric is always Ricci flat and admits a homothetic Killing vector field; if the evolving spacetimes are maximally symmetric, their $(d+1)$-dimensional representative has a vanishing Riemann tensor even. The non-degeneracy of the higher dimensional metric that "geometrizes" a given RG trajectory is linked to a monotonicity requirement for the running of the cosmological constant, which we test in the case of asymptotic safety.

Keywords: quantum gravity; functional renormalization group; geometric flows; scale-spacetime; asymptotic safety

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1. Introduction

The familiar renormalization group (RG) equations of quantum field theory are formulated in a mathematical setting that is rather simple and, in a way, structureless from the geometric point of view. The only ingredients involved are a manifold $\mathscr{T}$, often referred to as the theory space, and a vector field $\boldsymbol{\beta}$ thereon. The data $(\mathscr{T}, \boldsymbol{\beta})$ suffice to describe what is called the RG flow and to define the integral curves of $\boldsymbol{\beta}$ on $\mathscr{T}$. Since the components of $\boldsymbol{\beta}$ are given by the ordinary beta functions, the first-order differential equations that govern these integral curves, also known as RG trajectories, are nothing but the standard renormalization group equations. Even for the more general functional RG equations, the situation is essentially the same, except for the infinite dimensionality of the manifold $\mathscr{T}$ whose points represent full-fledged effective action functionals.

There is, however, a longstanding conjecture that beyond $\boldsymbol{\beta}$, further natural geometric objects might be "living" on the manifold $\mathscr{T}$. For example, after the advent of Zamolodchikov's c-theorem [1,2], related investigations in more than two dimensions focused on searching for a scalar "c-function" and a metric on $\mathscr{T}$ by means of which the RG flow could be promoted to a gradient flow. Even though this program was not fully successful in the generality originally hoped for, it ultimately led to important developments such as the proof of the a-theorem [3].

Furthermore, various authors, guided by different motivations, have tried to furnish the manifold $\mathscr{T}$ with a connection [4–6]. Recently, significant progress has been made along these lines after, in [7], a powerful functional RG framework for the analysis of composite operators had been introduced. In this setting, the connection that was proposed [8] is related to the operator product expansion coefficients.

The conjectured AdS/CFT correspondence "geometrizes" RG flows by a different approach that identifies the scale variable of the RG equations with a specific coordinate on

a higher dimensional (bulk) spacetime [9–11]. In this way the "RG time" acquires a status similar to the ordinary spacetime coordinates.

Along a different line of research, the fundamental idea of dimensionally extending spacetime by scale variables was developed in considerable generality in the work of L. Nottale [12]. In his approach, the RG time is on a par with the usual spacetime coordinates, both conceptually and geometrically.

(1) The present paper is devoted to a different notion of geometrized RG flows. While it does have certain traits in common with the various theoretical settings mentioned above, it is more conservative, however, in that its starting point does not involve any unproven assumptions. This starting point consists of nothing but the standard RG trajectories supplied by a functional renormalization group equation (FRGE). We proposed to exploit those RG-derived data, and only those, to initiate a systematic search for natural geometric structures, which can help in efficiently structuring those data and/or facilitate their physical interpretation or application.[1]

(2) Specifically, we dealt in this paper with the nonperturbative functional RG flows of quantum Einstein gravity (QEG), i.e., quantum gravity in a metric-based formulation. We assumed that it is described by an effective action functional $\Gamma_k[\cdot]$ that depends both on a 4D spacetime metric, $g_{\mu\nu}$, and on some kind of RG scale, $k \in \mathbb{R}^+$, implemented as an infrared cutoff, for example. Furthermore, we supposed that we managed to solve the corresponding FRGE for (partial) trajectories in theory space, i.e., maps $k \mapsto \Gamma_k[\cdot]$, whereby the curve parameter k does not necessarily cover all scales $k \in \mathbb{R}^+$.

For every given value of k, the running effective action Γ_k implies an effective field equation for the expectation value of the metric, typically a generalization of Einstein's equation. Solutions to those effective Einstein equations inherit a k-dependence from $\Gamma_k[\cdot]$, and we shall denote them $g^k_{\mu\nu}(x^\rho)$ in the following. More precisely, in this paper, we analyzed a situation where the solutions at differing scales are selected such that $g^k_{\mu\nu}$ depends on k smoothly. Therefore, we may regard the map $k \mapsto g^k_{\mu\nu}(x^\rho)$ as a smooth trajectory in the space of all metrics that are compatible with a given differentiable manifold, $\mathscr{M}_4$. Thus, technically speaking, the output of the functional RG—and effective Einstein equations—amounts to a family of Riemannian structures on one and the same spacetime manifold:

$$\left\{ \left(\mathscr{M}_4,\ g^k_{\mu\nu} \right) \ \middle|\ k \in \mathbb{R}^+ \right\} \tag{1}$$

(3) In this paper, we proposed a new way of thinking about the infinitely many metrics $g^k_{\mu\nu}$ that furnish the same 4D spacetime manifold $\mathscr{M}$. Namely, we shall interpret the family $\left(\mathscr{M}_4, g^k_{\mu\nu} \right)$ as different 4D slices through a single five-dimensional Riemannian or pseudo-Riemannian manifold:

$$\left(\mathscr{M}_5,\ {}^{(5)}g_{IJ} \right) \tag{2}$$

Hereby, all $g^k_{\mu\nu}$'s arise from only one 5D metric ${}^{(5)}g_{IJ}$ by isometrically embedding the slices into $\mathscr{M}_5$.

If k has the interpretation of an (inverse) coarse graining scale on $\mathscr{M}_4$, then $\mathscr{M}_5$ naturally comes close to a "scale-spacetime" manifold [12]. In addition to the usual event coordinates x^μ, its points involve a certain value of the scale or coarse graining parameter: (k, x^μ).

(4) Actually, $\mathscr{M}_5$, equipped with some metric ${}^{(5)}g_{IJ}$, can encode more information than is contained in the underlying family $\left(\mathscr{M}_4, g^k_{\mu\nu} \right)$. This is most obvious if we use local coordinates that are adapted to the foliation by the surfaces of equal scale. The scale parameter (or an appropriate function thereof) plays the role of a fifth coordinate then, and the basic trajectory of 4D metrics $g^k_{\mu\nu}(x^\rho) \equiv g_{\mu\nu}(k, x^\rho)$ is reinterpreted as 10 out of the 15 independent components that ${}^{(5)}g_{IJ}(k, x^\rho)$ possesses.

[1] A first analysis along these lines can be found in [13] where contact was made with the Randall–Sundrum model.

Our main interest was in its additional components, ${}^{(5)}g_{\mu k}(k, x^\rho)$ and ${}^{(5)}g_{kk}(k, x^\rho)$, respectively. The question we addressed is whether those functions can be determined in a mathematically or physically interesting way such that a single 5D geometry not only encapsulates or "visualizes" a trajectory of 4D geometries, but also enriches it by additional information. Schematically,

$$\text{trajectory of 4D geometries} \quad + \quad ? \quad = \quad \text{unique 5D geometry} \tag{3}$$

Loosely speaking, what we proposed here is a bottom-up approach that starts out from the safe harbor of a well-understood and fully general RG framework and only in a second step tries to assess whether, and under what conditions, there exist natural options for geometrizing the RG flows.

This approach must be contrasted with top-down approaches like the one based on the AdS/CFT conjecture, for instance. They would rather begin by postulating the geometrization and ask about its relation to standard RG flows in the second stage only.

(5) The discussions in this paper are largely independent of the precise details concerning the underlying RG technique and the concrete trajectories $g^k_{\mu\nu}$. An exception is Section 6 below, which makes essential use of the gravitational effective average action [14,15]. Implementing a background-independent [16] coarse graining procedure in the presence of dynamical gravity, it is ideally suited for the description of self-gravitating quantum systems like the ones we shall consider [17–19]. While not restricted to this application [20–24], the effective average action has been used extensively in the asymptotic safety program[2] [15,26–31]. There, its background independence is likely to be the essential ingredient responsible for the formation of a nontrivial RG fixed point [32].

(6) The rest of this paper is organized as follows. In Section 2, we set up a convenient framework, based on a generalization of the ADM construction, for the embedding of a given family $g^k_{\mu\nu}$ in a higher dimensional geometry. In Section 3, we then provide a simple, yet fully explicit and sufficiently general class of such families $g^k_{\mu\nu}$. They correspond to running Einstein metrics, and all subsequent demonstrations refer to this class of solutions. A priori, our goal of searching for "interesting" 5D geometries is an extremely broad one; to be able to make practical progress, we therefore narrowed its scope to a particular class of ADM metrics, which we introduce and discuss in Section 4. Then, in Section 5, we derive our main results. We show that, under precise conditions, the running 4D Einstein spaces can always be embedded in a 5D geometry that admits a homothetic Killing vector field and is Ricci flat; should the Einstein spaces be maximally symmetric, it is even strictly, i.e., Riemann flat.

The important point about these options for a geometrization of RG flows is that they neither follow from pure geometry alone, nor are they "for free" as concerns the properties of the RG trajectory. Rather, they are a global geometric manifestation of a specific general feature of the RG trajectory. In the present example, this sine qua non is that the running cosmological constant $\Lambda(k)$ is a strictly increasing function of the scale. In Section 6, we show that for the asymptotically safe trajectories of QEG, this is indeed the case. Finally, Section 7 contains a summary and the conclusions.

2. From Trajectories of Metrics to Higher Dimensions

Let us suppose that we employed some sort of functional renormalization group (FRG) framework, whose specifics do not matter here, in order to derive a scale-dependent effective field equation, i.e., a generalization of Einstein's equation. We assumed that we furthermore managed to solve this one-parameter family of differential equations, thus obtaining families of metrics $g^k_{\mu\nu}$ labeled by the RG scale k.

According to the standard interpretation outlined in [33,34], the set $\{g^k_{\mu\nu}\}_{k\geq 0}$ gives rise to a family of different Riemannian structures, all of which furnish one and the same

[2] A fairly comprehensive and up-to-date list of publications on this subject can be found in [25].

d-dimensional manifold $\mathscr{M}_d$. Correspondingly, one formally regards $k \mapsto (\mathscr{M}_d, g^k_{\mu\nu})$ for $k \in \mathbb{R}^+$ as a "trajectory" in the space of d-dimensional (Euclidean) spacetimes. In local coordinates, we write their line elements as:

$$\mathrm{d}s^2_d = g^k_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \qquad \mu,\nu,\cdots = 1,2,\cdots,d. \tag{4}$$

For generality, we switched here from four to d spacetime dimensions.

(1) The key idea of the present work was to re-interpret the RG parameter k, possibly after a convenient reparametrization $\tau = \tau(k)$, as an additional coordinate that, together with x^μ, coordinatizes a $(d+1)$-dimensional manifold $\mathscr{M}_{d+1}$. The original manifold $\mathscr{M}_d$ is isometrically embedded in $\mathscr{M}_{d+1}$ in a k-dependent way, and so, $\mathscr{M}_{d+1}$ comes into being equipped with a natural foliation.

According to this re-interpretation, the entire RG trajectory of ordinary spacetimes is described by a single Riemannian structure on the higher dimensional manifold. We denote it $\left(\mathscr{M}_{d+1}, {}^{(d+1)}g_{IJ}\right)$ and write the corresponding line element as:

$$\mathrm{d}s^2_{d+1} = {}^{(d+1)}g_{IJ}(y^K)\,\mathrm{d}y^I\,\mathrm{d}y^J \tag{5}$$

where $y^I \equiv (y^0, y^\mu)$ are generic local coordinates on $\mathscr{M}_{d+1}$.

Here and in the following, indices $I, J, K, \cdots$ always assume values in $\{0, 1, 2, \cdots, d\}$, while Greek indices run from one to d only.

(2) Since $\mathscr{M}_{d+1}$ is endowed with a natural foliation, it is convenient to employ a variant of the ADM formalism [15]. To prepare the stage, let us assume we are given an arbitrary manifold $\left(\mathscr{M}_{d+1}, {}^{(d+1)}g_{IJ}\right)$. We start by defining a scalar function $y \mapsto \tau(y)$ that assigns a specific scale to each of its points. Then, we construct the level sets of this "RG time function", $\Sigma_\tau \equiv \left\{y \in \mathscr{M}_{d+1} | \tau(y) = \tau\right\}$, and interpret them as a stack of standard, d-dimensional spacetimes that differ with respect to their resolution scale $\tau \equiv \tau(k)$.

The gradient $n_I \equiv N\partial_I\tau$ defines a vector n^I that is everywhere normal to the slices Σ_τ. By choosing the lapse function $N(y)$ appropriately, we normalize it such that:

$${}^{(d+1)}g_{IJ}\, n^I n^J = \varepsilon \tag{6}$$

where $\varepsilon = \pm 1$ depends on whether the normal vector is space- or time-like.[3]

Next, we transform from the generic coordinates $y^I = y^I(x^J)$ to new ones, $x^I \equiv (x^0, x^\mu) \equiv (\tau, x^\mu)$, which are adapted to the foliation: τ labels different "RG time slices", and the x^μ's are coordinates on a given Σ_τ. Defining a vector field t^I by the condition $t^I\partial_I\tau = 1$, we relate the coordinate systems on neighboring slices by requiring that the coordinates x^μ are constant along the integral curves of t^I.

The tangent space at any point of $\mathscr{M}_{d+1}$ can be decomposed into a subspace spanned by vectors tangent to Σ_τ and its complement. The corresponding basis vectors are given by derivatives of the functions $y^I = y^I(x^J) = y^I(\tau, x^\mu)$ that describe the embedding of Σ_τ into $\mathscr{M}_{d+1}$:

$$e_\mu{}^I = \frac{\partial}{\partial x^\mu} y^I(\tau, x^\alpha), \qquad t^I = \frac{\partial}{\partial \tau} y^I(\tau, x^\alpha) \tag{7}$$

As a result, the e_μ's are orthogonal to n:

$${}^{(d+1)}g_{IJ}\, n^I e_\mu{}^J = 0 \tag{8}$$

Furthermore, on the slices Σ_τ, the embedding induces the following metric from the ambient metric ${}^{(d+1)}g_{IJ}$:

[3] We allow ${}^{(d+1)}g_{IJ}$ to be a Lorentzian metric of any signature. However, $g^k_{\mu\nu}$ is assumed to have a Euclidean signature, unless stated otherwise.

$$ {}^{(d)}g_{IJ} = e_\mu{}^I e_\nu{}^J \ {}^{(d+1)}g_{IJ} \tag{9} $$

In general, the vector t^I has nonvanishing components in the directions of both n^I and $e_\mu{}^I$. Its expansion:

$$ t^I = N n^I + N^\mu e_\mu{}^I \tag{10} $$

involves the lapse function $N(\tau, x^\mu)$ and the shift vector $N^\mu(\tau, x^\mu)$. The definitions (7) also entail that the coordinate one-forms in the two coordinate systems are related by:

$$ \mathrm{d}y^I = t^I \,\mathrm{d}\tau + e_\mu{}^I \,\mathrm{d}x^\mu = N n^I \,\mathrm{d}\tau + e_\mu{}^I (\mathrm{d}x^\mu + N^\mu \,\mathrm{d}\tau) \tag{11} $$

Upon inserting (11) into $\mathrm{d}s^2_{d+1} = {}^{(d+1)}g_{IJ}\,\mathrm{d}y^I\,\mathrm{d}y^J$, we obtain the line element recast in terms of the ADM variables $\{N, N^\mu, {}^{(d)}g_{\mu\nu}\}$:

$$ \mathrm{d}s^2_{d+1} = \varepsilon N(x^I)^2 \,\mathrm{d}\tau^2 + {}^{(d)}g_{\mu\nu}(x^I)\Big[\mathrm{d}x^\mu + N^\mu(x^I)\,\mathrm{d}\tau\Big]\Big[\mathrm{d}x^\nu + N^\nu(x^I)\,\mathrm{d}\tau\Big] \tag{12} $$

The sign $\varepsilon = \pm 1$ that determines the signature of the higher dimensional metric is left open at this point. Later on, we shall encounter criteria that determine whether the RG time τ really turns into a time coordinate ($\varepsilon = -1$) and describes a Lorentzian metric on $\mathscr{M}_{d+1}$, or whether it amounts to a further spatial dimension ($\varepsilon = +1$).

(3) To make contact with the RG approach, we assumed that the higher dimensional metric has the ADM form (12), and we then identified ${}^{(d)}g_{\mu\nu}$ with the output of the computations based on the FRGE and the effective field equations:

$$ {}^{(d)}g_{\mu\nu}(\tau, x^\rho) = g^k_{\mu\nu}(x^\rho)\Big|_{k=k(\tau)} \tag{13} $$

The (invertible) function $k(\tau)$ amounts to an optional and physically irrelevant redefinition of the original scale parameter in terms of a convenient RG time τ. A typical example is $\tau = \ln(k/\kappa)$, or even simpler, $\tau = k/\kappa$.[4] In the following, we shall assume that both $g^k_{\mu\nu}(x^\rho)$ and $k(\tau)$ are known, externally prescribed functions.

Thus, knowing ${}^{(d)}g_{\mu\nu}(x^I)$, what is still lacking in order to fully specify the higher dimensional line element (12) are the lapse and shift functions N and N^μ, respectively, as well as the sign ε. These are properties of the metric on $\mathscr{M}_{d+1}$ that do not follow from the flow equations.

(4) This leads us back to our main question: Is it conceivable that there exist general reasons or principles, over and above those inherent in the RG framework, that determine those missing ingredients in a meaningful and physically relevant way?

Inspired by the familiar applications of the ADM formalism in general relativity, one might be tempted to argue that there can be little physics in N and N^μ, since, by a $\mathscr{D}iff(\mathscr{M}_{d+1})$ transformation, we can change them in an almost arbitrary way. It is important though to emphasize that this argument does not apply in the present context.

The reason is that here, the possibility of performing coordinate transformations has been exhausted already in solving the k-dependent effective field equations. The ADM framework in $d+1$ dimensions imports concrete functions $g^k_{\mu\nu}(x^\rho)$ from the RG side, and they refer to a specific set of coordinates. Since we do not want those functions to be changed by a $\mathscr{D}iff(\mathscr{M}_{d+1})$ transformation and we insist (for the time being) that they occupy the μ-ν sub-matrix of ${}^{(d+1)}g_{IJ}$, we have to allow functions N and N^μ of any form in this gauge picked by $g^k_{\mu\nu}$.

[4] Unless stated otherwise, all coordinates are dimensionless in our conventions, while all metric coefficients have mass dimension -2. Since $[k] = +1$, the constant κ must have $[\kappa] = +1$.

As a consequence, we first must arrive at a certain triple $\{N, N^\mu, \varepsilon\}$ that completes the specification of ds^2_{d+1}, and only then, we are free to perform general coordinate transformations, if we desire to do so.

3. Solutions of the Rescaling Type: Running Einstein Spaces

To make the later discussion as explicit as possible, let us pause here for a moment and introduce a technically particularly convenient class of running metrics whose k-dependence resides entirely in their conformal factor:

$$g^k_{\mu\nu} = f(k)\; g^{k=k_0}_{\mu\nu} \tag{14}$$

(1) To this end, we assume that we are dealing with pure quantum gravity (no matter fields) and that the Einstein–Hilbert truncation is employed, meaning that the effective field equations are[5] $G_{\mu\nu}[g^k_{\alpha\beta}] = -\Lambda(k) g^k_{\mu\nu}$, or equivalently, with $R_\mu{}^\nu[g^k_{\alpha\beta}] = (g^k)^{\nu\rho} R_{\mu\rho}[g^k]$,

$$R_\mu{}^\nu[g^k_{\alpha\beta}] = \frac{2}{d-2}\;\Lambda(k)\;\delta_\mu{}^\nu \tag{15}$$

(2) In this setting, the only input from the RG equations is the k-dependence of the running cosmological constant, $\Lambda(k)$. The latter can be of either sign, and it also might vanish at isolated scales. It will turn out to be convenient to express it in the form:

$$\Lambda(k) = \sigma\;|\Lambda(k)|$$

with the piecewise constant sign function $\sigma = \pm 1$, and to introduce the quantity

$$H(k) = \left[\frac{2|\Lambda(k)|}{(d-1)(d-2)}\right]^{1/2} \tag{16}$$

in order to write the absolute value of the cosmological constant as:

$$|\Lambda(k)| = \frac{1}{2}(d-1)(d-2)\;H(k)^2. \tag{17}$$

For every fixed value of k, the solutions to the effective field equation:

$$R_\mu{}^\nu[g^k_{\alpha\beta}] = \sigma(d-1)\;H(k)^2\;\delta_\mu{}^\nu \tag{18}$$

are arbitrary Einstein manifolds [35] with scalar curvature:

$$R[g^k_{\alpha\beta}] = \sigma\; d(d-1)\;H(k)^2. \tag{19}$$

Among them, there are the distinguished ones that possess a maximum number of Killing vectors, namely the spheres S^d, pseudo-spheres H^d, and the flat space R^d. They exist for $\sigma = +1$ and $\sigma = -1$ when $H(k) \neq 0$, and for $H(k) = 0$, respectively.

The motivation for the d-dependent factors in the definition (16) is as follows. Comparing (19) with the standard result for the curvature scalar of maximally symmetric spaces reveals that, for the special case when $g^k_{\mu\nu}$ is maximally symmetric, $1/H(k)$ is nothing but the radius of curvature of the corresponding sphere or pseudo-sphere. Thus, $H(k)$ can be identified with the conventionally defined Hubble parameter. Hence, the Riemann tensor is normalized as follows in the case of maximal symmetry:

$$R_{\mu\nu\rho\sigma}[g^k_{\alpha\beta}] = \sigma\;H(k)^2\;[g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}] \tag{20}$$

[5] In this paper, we denote higher dimensional geometric objects (e.g., the curvature scalar ${}^{(d+1)}R$, etc.) by the prepended label $(d+1)$, while all those without are the original ones referring to $\mathscr{M}_d$. In particular, $R_{\mu\nu}$ denotes the Ricci tensor related to $g_{\mu\nu}$ in d dimensions, while ${}^{(d+1)}R_{\mu\nu}$ are the μ-ν components of the tensor ${}^{(d+1)}R_{IJ}$ built from ${}^{(d+1)}g_{IJ}$.

We emphasize however that while we are going to employ the quantity $H(k)$ defined by (16) as a convenient way of representing the cosmological constant, we are not confining our attention to maximal symmetry in what follows.
(3) Coming back to the problem of finding solutions to (15), let us fix some convenient reference scale k_0 at which:

$$\Lambda(k_0) \equiv \Lambda_0 \equiv \frac{1}{2}\sigma\ (d-1)(d-2)\ H_0^2 \tag{21}$$

and let us pick an arbitrary solution $g^{(0)}_{\mu\nu}(x^\rho)$ of the classical vacuum Einstein equation involving this particular value of the cosmological constant:

$$R_\mu{}^\nu[g^{(0)}_{\alpha\beta}] = \sigma(d-1)\ H_0^2\ \delta_\mu{}^\nu \tag{22}$$

It then follows that the "running metric" given by:

$$g^k_{\mu\nu}(x^\rho) = Y(k)^{-1}\ g^{(0)}_{\mu\nu}(x^\rho) \tag{23}$$

with $g^k_{\mu\nu}|_{k=k_0} = g^{(0)}_{\mu\nu}$, and:

$$Y(k) \equiv \frac{|\Lambda(k)|}{|\Lambda_0|} \equiv \frac{H(k)^2}{H_0^2} \tag{24}$$

solves the effective field Equation (15) on all scales k that are sufficiently close to k_0. This is to say that Λ must not have any zero between k and k_0 so that $\sigma = \mathrm{sign}(\Lambda(k)) = \mathrm{sign}(\Lambda_0)$ is a constant function. Equation (23) is easily proven by noting that the Ricci tensor, with mixed indices, behaves as:

$$R_\mu{}^\nu[c^{-2}g_{\alpha\beta}] = c^2 R_\mu{}^\nu[g_{\alpha\beta}] \tag{25}$$

under global Weyl transformations with an arbitrary real c.
(4) We now have a simple, but, as we shall see, instructive example of a trajectory made of generic, i.e., not necessarily maximal symmetric, Einstein spaces at our disposal. Upon inserting (23) into (13), the spacetime part of the ADM metric reads:

$${}^{(d)}g_{\mu\nu}(\tau, x^\rho) = Y(k(\tau))^{-1}\ g^{(0)}_{\mu\nu}(x^\rho) \tag{26}$$

with the externally prescribed function $\tau \mapsto Y(k(\tau))$ coming from the RG machinery. We emphasize that all results in this paper refer specifically to the Einstein–Hilbert truncation and to solutions of the rescaling type. For the time being, nothing is known about generalizations involving more general actions or solutions.

4. Focusing on the Lapse Function

Let us recall that it is our aim to explore the theoretical possibilities of fixing the missing ingredients of the higher dimensional metric, $\{N, N^\mu, \varepsilon\}$, in a way that is physically or mathematically distinguished, for one reason or another.
(1) As it stands, the scope of this investigation was extremely broad. In a first attempt, it can help therefore to narrow down the setting in order to make the problem technically more clear-cut and its physics interpretation more transparent.

For this reason, we focused in the sequel on a vanishing shift vector and on lapse functions that depend on τ only: $N^\mu(x^I) = 0$, $N(x^I) = N(\tau)$. As a consequence, Equation (12) boils down to:

$$\mathrm{d}s^2_{d+1} = \varepsilon\ N(\tau)^2\ \mathrm{d}\tau^2 + {}^{(d)}g_{\mu\nu}(\tau, x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{27}$$

As we shall see, this truncated form of the ADM metric is still sufficiently rich, and yet simple enough to allow for practical progress.

(2) Regarding the RG input, we now insert the explicit trajectory of Einstein metrics found in the previous section:

$$\mathrm{d}s^2_{d+1} = \varepsilon\, N(\tau)^2\ \mathrm{d}\tau^2 + Y(k(\tau))^{-1}\ \ g^{(0)}_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{28}$$

Thus, should there exist a yet-to-be-discovered general principle that endows the metric on $\mathscr{M}_{d+1}$ with information that goes beyond the input data provided by the RG equations, this information must reside in the lapse function $N(\tau)$.

It is important at this point to remember that the $g^{(0)}_{\mu\nu}$'s are externally prescribed coefficient functions, which we do not want to be changed by coordinate transformations. Hence, if some principle when applied to (28) demands that the lapse must have a particular functional form $\tau \mapsto N(\tau)$, this lapse function refers to an already fully gauge-fixed metric, the corresponding gauge being selected in the process of solving the effective field equations.

(3) What is a single Riemannian or pseudo-Riemannian manifold $\left(\mathscr{M}_{d+1}, {}^{(d+1)}g_{IJ}\right)$ capable of doing for us that would not already be possible using the original stack of unrelated manifolds $\left\{\left(\mathscr{M}_d, g^k_{\mu\nu}\right)|k \in \mathbb{R}^+\right\}$?

One answer is that it can ascribe proper lengths to curves in $\mathscr{M}_{d+1}$ that are not confined to a single slice of the foliation. Such curves explore not only different points of spacetime, but also different scales.

As an example, let us consider a curve $\mathcal{C}(P_1, P_2)$ connecting two points $P_{1,2} \in \mathscr{M}_{d+1}$. In the coordinate system of (28), they are assumed to possess the same x^μ-, but different τ-coordinates, namely τ_1 and τ_2, respectively. The curve begins on the RG time slice having $k(\tau_1) = k_1$ and ends on the one with $k(\tau_2) = k_2$. Furthermore, we assumed that $x^\mu = \text{const} \equiv c^\mu$ is constant along $\mathcal{C}(P_1, P_2)$, so that the curve projects onto a single point of $\mathscr{M}_d$.

Then, for $\varepsilon = +1$, say, the metric in Equation (28) tells us that this curve has the proper length:

$$\Delta s_{d+1} \equiv \int_{\mathcal{C}_{(P_1,P_2)}} \sqrt{\mathrm{d}s^2_{d+1}} \ = \ \int_{\tau_1}^{\tau_2} \mathrm{d}\tau N(\tau) \tag{29}$$

Loosely speaking, this integral allows us now to answer questions like: "What is the distance between a high-scale object and a low-scale object at one and the same spacetime event (x^μ)?"

In more realistic examples, P_1 and P_2 may have different x^μ-coordinates so that $\mathcal{C}$ visits more than one point of $\mathscr{M}_d$. Hence, the two "objects" need not to lie on top of one another. The resulting proper length Δs_{d+1} is a mixture then of the familiar distance in spacetime and the separation of the two objects in the scale direction.

If Δs_{d+1} is to have any physical meaning, it must be possible to experimentally connect coordinates (τ, x^μ) to the results of certain measurements. A well-known model for achieving this on ordinary spacetimes equips $\mathscr{M}_d$ with a set of scalar fields ϕ^μ whose observable values represent x^μ then [36]. In the case at hand, we must invoke an additional field that allows a determination of the scale $k(\tau)$. In cosmology, say, one might think of a local temperature field, for instance.

(4) As we mentioned earlier, the function $k(\tau)$ can be chosen freely. It is gratifying to see therefore that the proper time Δs_{d+1} in (29) is indeed independent of this choice. Assume that two such functions belong to the same foliation, i.e., $k(\tau) = k = \bar{k}(\bar{\tau})$, and the respective RG times are related by the coordinate transformation $\tau = \tau(\bar{\tau})$. The latter belongs to the foliation-preserving subgroup of $\mathscr{D}iff(\mathscr{M}_{d+1})$, and it acts on the lapse function according to [15]:

$$\bar{N}(\bar{\tau}) = N(\tau)\left(\frac{\mathrm{d}\tau}{\mathrm{d}\bar{\tau}}\right) \tag{30}$$

As a consequence, $\bar{N}(\bar{\tau})\,\mathrm{d}\bar{\tau} = N(\tau)\,\mathrm{d}\tau$, and (29) is seen to be invariant.

5. Distinguished Higher Dimensional Geometries

The crucial question is what kind of physical or mathematical principle could possibly determine the higher dimensional metric and what are the universal geometric features of the manifold $\left(\mathscr{M}_{d+1}, {}^{(d+1)}g_{IJ}\right)$ that result from it. The information coming from the RG trajectory determines ${}^{(d+1)}g_{IJ}$ only incompletely. Using the prescribed coordinate system of Equation (28), what are left to be determined by this principle are $N(\tau)$ and ε.

In this paper, we postulated that the RG trajectories under consideration possess the following monotonicity property:

(P) The cosmological constant $\Lambda(k)$ is a strictly increasing function of k. (31)

Taking **(P)** for granted, we demonstrated that it is always possible to complete the specification of the $(d+1)$-dimensional (pseudo-) Riemannian geometry in such a way that it enjoys the following features:

(G) The higher dimensional metric ${}^{(d+1)}g_{IJ}$ is *Ricci flat*: ${}^{(d+1)}R_{IJ} = 0$. (32)

This property is universal in the sense that it pertains to arbitrary k-dependent Einstein metrics $g^k_{\mu\nu}(x^\rho)$.

Furthermore, if the d-dimensional Einstein metrics $g^k_{\mu\nu}(x^\rho)$ happen to be maximally symmetric, but still curved in general, then **(G)** can be replaced by the stronger statement:

(G') The higher dimensional metric ${}^{(d+1)}g_{IJ}$ is *Riemann flat*: ${}^{(d+1)}R^I{}_{JKL} = 0$. (33)

In this section, we show that **(G)** and **(G')**, respectively, are indeed made possible by **(P)** since it allows us to postulate a highly distinguished and universal form of ${}^{(d+1)}g_{IJ}$. Thereafter, we shall investigate whether the postulated property **(P)** is actually realized in asymptotic safety.

5.1. The Hubble Length as a Coordinate

We considered k-intervals with different signs of $\Lambda(k)$ separately, should they occur. If $\Lambda(k) > 0$, **(P)** entails that $Y(k)$ and $H(k)$ are monotonically increasing with the scale, while the Hubble length:

$$L_H(k) \equiv \frac{1}{H(k)} = \left[\frac{(d-1)(d-2)}{2|\Lambda(k)|}\right]^{1/2} \tag{34}$$

is a decreasing function of k. If instead, $\Lambda(k) < 0$, the postulate **(P)** requires $Y(k)$ and $H(k)$ to decrease and $L_H(k)$ to increase with k. In either case, the postulated strict monotonicity implies that the function $L_H(k)$ is invertible, i.e., the relationship between k and L_H is one-to-one. A a consequence, we may regard the map $k \mapsto L_H(k)$ given by (34) as a reparametrization of the "scale manifold" $\mathbb{R}^+$ or a subset thereof and L_H as a concrete example of an RG time $\tau = \tau(k)$. Up to now, the τ-k relationship has been an arbitrary convention. Here, now, we made a specific choice for this coordinate, not by hand, but by invoking the RG trajectory itself.

For clarity, we denote this special RG time coordinate by ξ. The corresponding coordinate transformation $k = k(\xi)$ is determined by the implicit condition:

$$\xi \equiv L_H\big(k(\xi)\big), \tag{35}$$

while its inverse is known explicitly:

$$\xi(k) = \left[\frac{(d-1)(d-2)}{2|\Lambda(k)|}\right]^{1/2} \tag{36}$$

When we reexpress $\mathrm{d}s^2_{d+1}$ in terms of ξ, we are led to the conformal factor:

$$Y(k(\xi))^{-1} = H_0^2 \;\; H(k(\xi))^{-2} = H_0^2 \;\; L_H(k(\xi))^2 = H_0^2 \;\xi^2 \tag{37}$$

Hence, in the new system of coordinates, the second term on the RHS of Equation (28) assumes a very simple dependence on the RG time, being proportional to ξ^2.

The sought-for principle that decides about the $(d+1)$-dimensional geometry, after having installed the coordinates[6] $x^I \equiv (x^0, x^\mu) \equiv (\xi, x^\mu)$, must come up with a unique function $\xi \mapsto N(\xi)$. This, then, will allow us to completely specify the line element $\mathrm{d}s^2_{d+1} \equiv {}^{(d+1)}g_{IJ}(x^K)\,\mathrm{d}x^I\,\mathrm{d}x^J$ in Equation (28).

In order to prove in a constructive way that the postulate **(P)** indeed allows us to achieve **(G)** or **(G')**, respectively, we enact the following rule for the completion of ${}^{(d+1)}g_{IJ}$:

(R) In the (ξ, x^μ) system, the lapse function must assume the simplest form possible,

$$\text{namely } N(\xi) = 1. \tag{38}$$

In other words, the coordinates realizing the "proper RG time gauge" are required to coincide with those that employ the Hubble length as the scale coordinate. The rule **(R)** enforces that the higher dimensional metric is unambiguously given by:

$$\mathrm{d}s^2_{d+1} = \varepsilon\,(\mathrm{d}\xi)^2 + \xi^2 \;\; H_0^2 \;\; g^{(0)}_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{39}$$

which is fully determined except for sign ε.

We emphasize that the property **(P)** is crucial for making the rule **(R)** meaningful. Without the strict monotonicity of $\Lambda(k)$, we could not have replaced k with $\xi \propto |\Lambda(k)|^{-1/2}$ in its role as a coordinate.[7]

Note that the metric (39) possesses a remarkable universality property: It has no explicit dependence on the function $\Lambda(k)$. In the (ξ, x^μ) coordinate system, the proposed metric "remembers" $\Lambda(k)$ only via the implicit requirement that $\xi \leftrightarrow |\Lambda(k)|^{-1/2}$ must be one-to-one.

In the (ξ, x^μ) system, the information about the actual RG evolution resides entirely in the "time function" $k = k(x^I) \equiv k(\xi, x^\mu)$ that describes how the slices $\Sigma_\tau \equiv \Sigma_{\bar{\xi}}$ are embedded into $\mathscr{M}_{d+1}$. In the case at hand, the time function has no dependence on x^μ and boils down to $k = k(\xi)$. It is this function that has been adjusted in (35) by imposing $\xi = L_H(k(\xi))$. Since the inverse function $\xi = \xi(k)$ is given by (36), we recover the time function belonging to the line element (39) by solving $\xi \propto |\Lambda(k)|^{-1/2}$ for $k = k(\xi)$.

Equation (39) is our proposal for the single higher dimensional metric that "geometrizes" the entire RG history of the original metrics.[8] In the following subsections, we discuss its detailed properties, which, as a matter of fact, were the actual motivation for this specific proposal.

5.2. Equivalent Forms of the Postulated Metric

The special status of the (ξ, x^μ) system of coordinates resides solely in the fact that in this system, the lapse function is defined to be particularly simple, namely $N = 1$.

[6] The special RG time ξ and its "conformal" analogue η to be introduced below are the only exceptions to our rule that coordinates are dimensionless. While $[x^\mu] = 0$ throughout, ξ and η have the dimension of a length: $[\xi] = [\eta] = -1$.

[7] For a similar discussion of coordinate transformations on the g-λ theory space, see [30].

[8] It is interesting to note that the metric (39) played a prominent role also in the 5D "spacetime-matter theory" in [37–39].

After having set up the metric ${}^{(d+1)}g_{IJ}$, we may transform it to any coordinate system $\bar{x}^I \equiv (\bar{x}^0, \bar{x}^\mu)$ we like. Here, we mention two simple foliation-preserving transformations.
(1) The conformal RG time: In practical computations, it is often convenient to transfer the scale dependence from the conformal factor of $g^{(0)}_{\mu\nu}$ to the overall conformal factor of ${}^{(d+1)}g_{IJ}$. This is achieved by the (x^μ-independent!) transformation trading $\xi \in \mathbb{R}^+$ for $\eta \in \mathbb{R}$ via $\xi = H_0^{-1} e^{H_0 \eta}$, or conversely,

$$\eta = H_0^{-1} \ \ln(H_0 \xi) = L_H^0 \ \ln\Big(\xi / L_H^0\Big), \tag{40}$$

with $L_H^0 \equiv H_0^{-1} \equiv L_H(k_0)$. The new coordinate η is positive (negative) if the length ξ is of super- (sub-) Hubble size, according to the metric at the reference scale k_0. In the $(x^0 \equiv \eta, x^\mu)$ system, the line element (39) assumes the desired form:

$$\mathrm{d}s^2_{d+1} = e^{2H_0\eta} \left[\varepsilon \ (\mathrm{d}\eta)^2 + g^{(0)}_{\mu\nu}(x^\rho) \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu\right] \tag{41}$$

While, in its original form (39), ξ is reminiscent of the cosmological time in a Robertson–Walker metric, the new variable η has the interpretation of the corresponding conformal RG time.
(2) The IR cutoff as a coordinate: Both in the (ξ, x^μ) and the (η, x^μ) system of coordinates, the metric is independent of $\Lambda(k)$, while the time functions $k = k(\xi)$ and $k = k(\eta)$ know about it. We can reverse the situation and make the time function trivial by introducing directly the cutoff k (or the dimensionless $L_H^0 \ k$) as the new coordinate. The change of coordinates $\xi \to k$ defined by (36) brings the metric (39) to the form:

$$\mathrm{d}s^2_{d+1} = \left|\frac{\Lambda_0}{\Lambda(k)}\right| \left\{\varepsilon \left(\frac{1}{2}\partial_k \ln|\Lambda(k)|\right)^2 \left(L_H^0 \ \mathrm{d}k\right)^2 + g^{(0)}_{\mu\nu}(x^\rho) \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu\right\} \tag{42}$$

which is manifestly $\Lambda(k)$-dependent. We see that the metric (42) degenerates at points where $\partial_k \Lambda(k) = 0$, hinting at the importance of **(P)** again. Note also that the proposed metric ascribes a nonzero distance to high- and low-scale objects at the same x^μ only when there is a non-trivial RG running, $\partial_k \Lambda(k) \neq 0$, so that the effective spacetimes acquire fractal properties [40,41].

5.3. Homothetic Killing Vector and Self-Similarity

The $(d+1)$-dimensional geometry described by Equation (39), or equivalently by (41), is a very particular one in that it admits a homothetic Killing vector field $X \equiv X^I \partial_I$. With $\mathscr{L}_X$ denoting the Lie derivative along X, this vector field satisfies the defining equation:

$$\mathscr{L}_X \ {}^{(d+1)}g_{IJ} = 2\,C \ {}^{(d+1)}g_{IJ} \tag{43}$$

for $C = H_0$. Note that (43) differs from the condition for a generic conformal Killing vector field since C is a constant rather than an arbitrary function on $\mathscr{M}_{d+1}$ [42,43]. The homothetic vector field is explicitly given by:

$$X = \frac{\partial}{\partial \eta} = H_0 \, \xi \frac{\partial}{\partial \xi} \tag{44}$$

It is easily checked therefore that it generates x^μ-independent rescalings of the metric. The existence of such a vector field is the hallmark of self-similarity in the general relativistic context [44]. It is a coordinate-independent manifestation of the underlying foliation with self-similar leaves, which may be hidden if inappropriate coordinates are used.

5.4. Ricci Flatness

Finally, we turn to the curvature of the postulated higher dimensional geometry. In order to better appreciate its rather unique character, we consider the following slightly more general class of metrics:

$$ {}^{(d+1)}g_{IJ}(x^K)\,\mathrm{d}x^I\,\mathrm{d}x^J = \Omega^2(\eta)\left[\varepsilon(\mathrm{d}\eta)^2 + g^{(0)}_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right] \tag{45} $$

Here, we employ the same coordinates $x^K \equiv (x^0 = \eta, x^\mu)$ as in Equation (41), but we admit for a moment an arbitrary overall conformal factor $\Omega(\eta)$. Working out the Ricci tensor of (45), one finds[9]:

$$ {}^{(d+1)}R^0{}_0 = -\varepsilon\, d\, \Omega^{-2}\left[\frac{\ddot{\Omega}}{\Omega} - \left(\frac{\dot{\Omega}}{\Omega}\right)^2\right] \tag{46a} $$

$$ {}^{(d+1)}R^0{}_\mu = 0, \qquad {}^{(d+1)}R^\mu{}_0 = 0 \tag{46b} $$

$$ {}^{(d+1)}R^\mu{}_\nu = \Omega^{-2}\left\{R^\mu{}_\nu - \varepsilon\,\delta^\mu{}_\nu\left[\frac{\ddot{\Omega}}{\Omega} + (d-2)\left(\frac{\dot{\Omega}}{\Omega}\right)^2\right]\right\} \tag{46c} $$

Here, $R^\mu{}_\nu$ denotes the Ricci tensor of $g^{(0)}_{\mu\nu}(x^\rho)$, and the dot indicates derivatives with respect to η.

Now let us ask under what circumstances (45) is Ricci flat:

$$ {}^{(d+1)}R^I{}_J = 0 \tag{47} $$

By (46a), the necessary and sufficient condition for ${}^{(d+1)}R^0{}_0 = 0$ is that $\Omega\,\ddot{\Omega} = (\dot{\Omega})^2$. The most general solution to this differential equation is given by:

$$ \Omega(\eta) = e^{B(\eta-\eta_0)} \tag{48} $$

with arbitrary real constants B and η_0. Using this form of Ω in (46c), the condition ${}^{(d+1)}R^\mu{}_\nu = 0$ is found to be equivalent to:

$$ R^\mu{}_\nu - \varepsilon(d-1)\,B^2\,\delta^\mu{}_\nu = 0 \tag{49} $$

Up to this point, $g^{(0)}_{\mu\nu}(x^\rho)$, and so, the Ricci tensor $R^\mu{}_\nu$ has been left unspecified. When we now exploit that $g^{(0)}_{\mu\nu}(x^\rho)$ actually describes an Einstein space complying with Equation (22), the condition (49) boils down to $\sigma H_0^2 - \varepsilon B^2 = 0$. This latter equation has the unique solution $\varepsilon = \sigma$, $B = H_0$. Thus, the conclusion is that there does exist a Ricci flat higher dimensional metric of the form (45) for every d-dimensional metric with $g^{(0)}_{\mu\nu}$ describing a (curved) Einstein space. Furthermore, this metric is essentially unique and is obtained by letting:

$$ \varepsilon = \sigma \qquad \text{and} \qquad \Omega(\eta) = e^{H_0(\eta-\eta_0)} \tag{50} $$

in the family of line elements (45):

$$ \mathrm{d}s^2_{d+1} = e^{2H_0(\eta-\eta_0)}\left[\sigma\,(\mathrm{d}\eta)^2 + g^{(0)}_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right] \tag{51} $$

Choosing $\eta_0 = 0$ brings us back to the metric (41) that we set out to investigate, with an additional piece of information, however. Originally, we had admitted an arbitrary sign $\varepsilon = \pm 1$. However, now, we see that Ricci flatness can be achieved only if we allow the sign of the cosmological constant $\sigma = \Lambda_0/|\Lambda_0|$ to determine the signature of ${}^{(d+1)}g_{IJ}$. If

[9] Our curvature conventions are $R^\sigma{}_{\rho\mu\nu} = +\partial_\mu\Gamma^\sigma_{\nu\rho} - \cdots$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$.

the cosmological constant is positive (negative), the scale parameter becomes a spacelike (timelike) coordinate.

5.5. Strict Flatness

Let us go one step further and ask under what conditions metrics of the form (45) are not only Ricci flat, but even strictly, i.e., Riemann flat:

$$ {}^{(d+1)}R^{I}{}_{JKL} = 0. \tag{52}$$

Modulo the usual symmetries, the Riemann tensor of (45) has only the following nonzero components:

$$ {}^{(d+1)}R^{0\mu}{}_{0\nu} = \varepsilon\, \Omega^{-2}\left[\left(\frac{\dot{\Omega}}{\Omega}\right)^2 - \frac{\ddot{\Omega}}{\Omega}\right]\delta^{\mu}{}_{\nu} \tag{53a}$$

$$ {}^{(d+1)}R^{\mu\nu}{}_{\rho\sigma} = \Omega^{-2}\left\{R^{\mu\nu}{}_{\rho\sigma} - \varepsilon\left(\frac{\dot{\Omega}}{\Omega}\right)^2\left[\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} - \delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\rho}\right]\right\} \tag{53b}$$

Herein, $R^{\mu\nu}{}_{\rho\sigma}$ is the Riemann tensor that belongs to $g^{(0)}_{\mu\nu}(x^{\rho})$.

Imposing ${}^{(d+1)}R^{0\mu}{}_{0\nu} = 0$, Equation (53a) reproduces the requirement $\Omega\,\ddot{\Omega} = (\dot{\Omega})^2$ and (48) as its general solution. Inserting this solution into (53b), the vanishing of the second set of components, ${}^{(d+1)}R^{\mu\nu}{}_{\rho\sigma} = 0$, implies the following condition on the curvature tensor of $g^{(0)}_{\mu\nu}(x^{\rho})$:

$$ R^{\mu\nu}{}_{\rho\sigma} = \varepsilon B^2\left[\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} - \delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\rho}\right] \tag{54}$$

The tensor structure on the RHS of (54) is the hallmark of a maximally symmetric manifold; see Equation (20). We conclude therefore that the metric (45) is strictly flat if and only if, first, $\Omega(\eta)$ and ε are fixed according to (50) and, second, the running Einstein metric at $k = k_0$, i.e., $g^{(0)}_{\mu\nu}$, corresponds to a maximally symmetric d-dimensional space.

This completes our demonstration that, under this symmetry constraint, the geometric feature **(G)** of the higher dimensional manifold can be tightened to **(G')**.

It is indeed quite remarkable that the inclusion of the scale variable has "flattened" the curved spacetime. In the case at hand, the metric (39) specializes to:

$$ \mathrm{d}s^2_{d+1} = (\mathrm{d}\xi)^2 + \xi^2\ \mathrm{d}\Omega^2_d \qquad (\Lambda_0 > 0) \tag{55}$$

$$ \mathrm{d}s^2_{d+1} = -(\mathrm{d}\xi)^2 + \xi^2\ \mathrm{d}H^2_d \qquad (\Lambda_0 < 0) \tag{56}$$

where $\mathrm{d}\Omega^2_d$ and $\mathrm{d}H^2_d$ are the line elements for, respectively, S^d and H^d with the unit length scale. Both of these metrics are well known to be flat: Equation (55) describes the $(d+1)$-dimensional Euclidean space in spherical coordinates. Hereby, ξ plays the role of the radial variable, $\mathscr{M}_{d+1} \equiv R^{d+1}$ being foliated by d-spheres of radius ξ.

Similarly, the metric (56) describes Minkowski space $M^{1,d}$. The RG time has become a genuine time coordinate in this case. Here, Minkowski space is foliated by hyperbolic d-spaces whose radius of curvature is given by ξ. For $d = 3$, Equation (56) is nothing but the metric of Milne's universe.

6. Asymptotic Safety

In the previous section, we saw that the "principle" or "property" **(P)** is a necessary condition for being able to define ${}^{(d+1)}g_{IJ}$ as a (Ricci) flat metric in the higher dimensional sense. In the present section, we are going to discuss the actual situation concerning the monotonicity of $\Lambda(k)$ within the concrete setting of pure quantum gravity (QEG) in $d = 4$ dimensions. We employ the prototypical Einstein–Hilbert truncation of the

effective average action, the first one used to demonstrate asymptotic safety [14,27,45]. The truncation is based on the ansatz:

$$\Gamma_k = \frac{1}{16\pi\, G(k)} \int \mathrm{d}^4x\, \sqrt{g}\, \Big(-R(g) + 2\Lambda(k) \Big) + \cdots \tag{57}$$

where the dots indicate the classical gauge fixing and ghost terms. The resulting RG equations for the running couplings $G(k)$ and $\Lambda(k)$ were obtained in [14] and solved numerically in [45]. In the following, we are particularly interested in the properties of the function $\Lambda(k)$ along typical RG trajectories.

(1) Mode counting functions: It is quite remarkable that considerations about 5D representations of the histories of 4D geometries have led us to scrutinize the monotonicity properties of $\Lambda(k)$. In fact, in [46], a closely related question, the monotonicity of the dimensionless product $G(k)\ \Lambda(k)$, was explored already, for an entirely different reason though. In [46], a c-function-like quantity $\mathscr{C}(k)$ was proposed in 4D quantum gravity, which, when evaluated exactly, should be monotonically decreasing along RG trajectories and be stationary at fixed points. In simple truncations, $\mathscr{C}(k)$ is proportional to $\big[G(k)\ \Lambda(k)\big]^{-1}$. Not unlike Zamolodchikov's c-function, $\mathscr{C}(k)$ can be argued to count the number of the fluctuation modes already integrated out, thus explaining its monotonicity when evaluated exactly. As for approximate calculations, it was found however that the above Einstein–Hilbert truncation is not precise enough to render $\mathscr{C}(k)$ monotonic, while it does turn out monotone if we use more general truncations [47] of the bi-metric type [48–50]. It is not unreasonable to expect that $\Lambda(k)$ might have similar, if not better, mode counting properties. After all, at least in the most naive picture, every bosonic fluctuation mode that is not suppressed by the cutoff contributes a positive zero-point energy to the cosmological constant and therefore should contribute additively to $\Lambda(k)$.

(2) The trajectories simplified: The classification of the RG trajectories implied by the ansatz (57) on the g-λ-plane of the dimensionless Newton constant g and cosmological constant λ is well known [45]. Here, we focus on the three main classes, i.e., trajectories of Type Ia, Type IIa, and Type IIIa, respectively (see Figure 1).

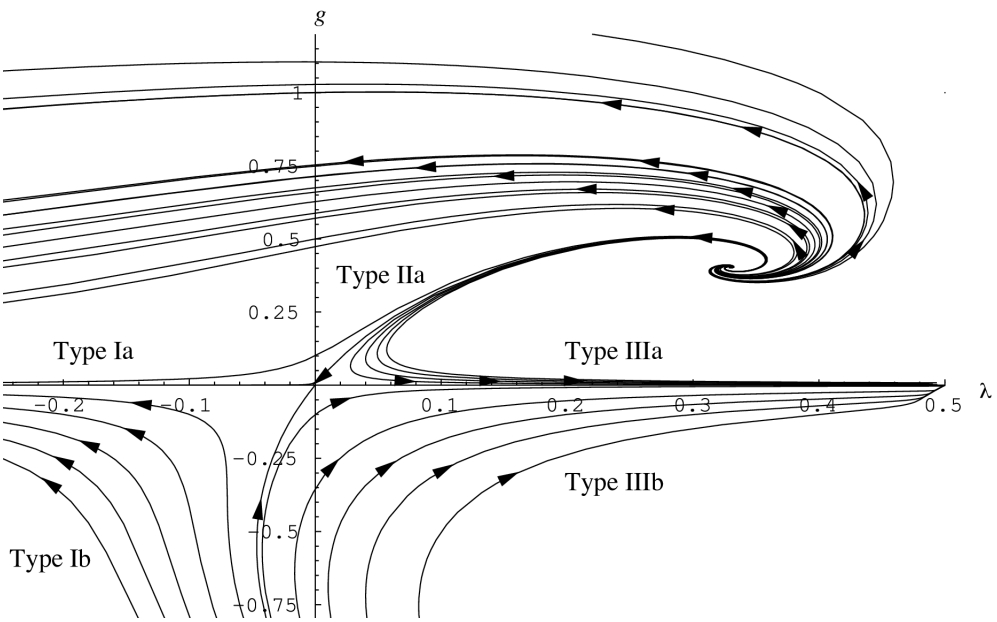

Figure 1. Part of coupling constant space of the Einstein–Hilbert truncation with its RG flow. The arrows point in the direction of decreasing values of k. The flow pattern is dominated by a non-Gaussian fixed point in the first quadrant and a trivial one at the origin. Taken from [45].

(i) All of these trajectories approach a non-Gaussian fixed point (g_*, λ_*) when $k \to \infty$. In particular, the dimensionless cosmological constant behaves as $\lambda(k) \equiv \Lambda(k)/k^2 \to \lambda_*$ in the asymptotic region. Hence:

$$\Lambda(k) = \lambda_* \, k^2 \qquad\qquad (k \gtrsim \hat{k}) \tag{58}$$

is a reliable approximation to the exact trajectory in this regime. It extends from "$k = \infty$" down to a scale $\hat{k}$ that is of the order of the Planck mass $m_{\rm Pl} \equiv G_0^{-1/2}$ typically.
(ii) Below a relatively complicated, but short transition regime near $\hat{k}$, all trajectories of the above three types enter a semiclassical regime within which the behavior of $\Lambda(k)$ is easy to describe again. At least qualitatively, the following simple formula provides a reliable approximation:

$$\Lambda(k) = \Lambda_0 + \nu \, G_0 \, k^4 \tag{59}$$

Here, $\nu > 0$ is a scheme-dependent constant, and the infrared values $\Lambda_0 \equiv \Lambda(k = 0)$ and $G_0 \equiv G(k = 0)$ arise as constants of integration whose values select a specific RG trajectory in the 2D theory space. The three types of trajectories differ with respect to the value of Λ_0. We have $\Lambda_0 < 0$, $\Lambda_0 = 0$, and $\Lambda_0 > 0$ for trajectories of Type Ia, IIa, and IIIa, respectively.

If $\Lambda_0 \neq 0$, it is convenient to introduce the two length scales:

$$\ell \equiv \left(\frac{\nu \, G_0}{|\Lambda_0|} \right)^{1/4} \quad , \qquad L \equiv \left(\frac{\lambda_*}{|\Lambda_0|} \right)^{1/2} \tag{60}$$

Hence, in the semiclassical regime,

$$\Lambda(k) = |\Lambda_0| \, \left(\ell^4 \, k^4 \pm 1 \right) \tag{61}$$

where the plus sign (minus sign) applies to Type IIIa (Type Ia).
(iii) When $\Lambda_0 \neq 0$, the following "caricature" of the function $\Lambda(k)$ is useful:

$$\Lambda(k) = |\Lambda_0| \, \cdot \, \begin{cases} \ell^4 \, k^4 \pm 1 & \text{for} \quad 0 \leq k \lesssim \hat{k} \\ L^2 \, k^2 & \text{for} \quad k \gtrsim \hat{k} \end{cases} \tag{62}$$

It should be a reliable approximation, except possibly during a short interval of scales near $\hat{k}$ where the transition between the two regimes takes place. We shall investigate this transition regime separately below.

In the case $\Lambda_0 = 0$, the corresponding approximation reads instead:

$$\Lambda(k) = |\Lambda_0| \, \cdot \, \begin{cases} \nu \, m_{\rm Pl}^{-2} \, k^4 & \text{for} \quad 0 \leq k \lesssim \hat{k} \\ \lambda_* \, k^2 & \text{for} \quad k \gtrsim \hat{k} \end{cases} \tag{63}$$

Equation (63) applies to the single trajectory of Type IIa, the separatrix [45].
(iv) Regarding the monotonicity, we observe that, whenever (62) and (63) are applicable, the dimensionful cosmological constant $\Lambda(k)$ is indeed a strictly monotonic function of k, and all trajectories of Types Ia, IIa, and IIIa have the crucial property **(P)**.
(3) The signature: A second important piece of information concerning $\Lambda(k)$ is the piecewise constant sign function $\sigma(k) \equiv \Lambda(k)/|\Lambda(k)|$. Equations (62) and (63) yield:

$$\text{for Type Ia:} \qquad \varepsilon(k) = \begin{cases} -1 & \text{for} \quad 0 \leq k < \ell^{-1} \\ +1 & \text{for} \quad k > \ell^{-1} \end{cases} \tag{64}$$

$$\text{for Type IIa:} \qquad \varepsilon(k) = +1 \qquad \text{for all} \qquad k \geq 0 \tag{65}$$

$$\text{for Type IIIa:} \qquad \varepsilon(k) = +1 \qquad \text{for all} \qquad k \geq 0 \tag{66}$$

Thus, we conclude that everywhere along RG trajectories of Types IIa and IIIa, the RG "time" amounts to a spatial coordinate actually. Starting from a Euclidean spacetime

$\mathscr{M}_4$ with signature $(+\,+\,+\,+)$, the proposed geometrization of the RG flow leads us unavoidably to a manifold $\mathscr{M}_5$ having $(+\,+\,+\,+\,+)$.

For trajectories of Type Ia, the situation is more complicated. They display an intermediate scale $k = \ell^{-1}$ at which the cosmological constant vanishes, $\Lambda(\ell^{-1}) = 0$. When k passes this special scale, the solutions to the effective field equations undergo a change of topology. Coming from above, the scalar curvature changes from $R[g^k_{\alpha\beta}] > 0$, via $R[g^{1/\ell}_{\alpha\beta}] = 0$, to $R[g^k_{\alpha\beta}] < 0$. In the maximally symmetric case, for example, this topology change corresponds to a sequence of spaces $S^4 \to R^4 \to H^4$.

We note however that at the present stage of its development, asymptotic safety cannot yet describe topology change processes in a dynamical fashion, neither in physical time, nor in RG time. For this reason, we adopt a conservative attitude here and consider the two branches of the Ia trajectories, having $\Lambda(k) > 0$ and $\Lambda(k) < 0$, respectively, as two unrelated (incomplete) trajectories and, at this stage, study them separately.

The upper branch ($k > \ell^{-1}$) of a Type Ia trajectory augments the Euclidean $\mathscr{M}_4$ to an, again, Euclidean $\mathscr{M}_5$ having signature $(+\,+\,+\,+\,+)$, while its lower branch ($k < \ell^{-1}$) gives rise to a Lorentzian 5D manifold with $(-\,+\,+\,+\,+)$.

It is quite intriguing to speculate that an RG trajectory of this kind could underlie a mechanisms of chronogenesis: the emergence of time in an a priori purely Euclidean system.

(4) The coordinate change: The dimensionless function $Y(k)$ can be written as:

$$Y(k) = \frac{|\Lambda(k)|}{|\Lambda_0|} = \frac{H(k)^2}{H_0^2} = \left(\frac{L_H^0}{L_H(k)}\right)^2 \tag{67}$$

with $\Lambda_0 = 3H_0^2$ in $d = 4$ and $L_H^0 \equiv 1/H_0$. Hence, Equation (62) yields the following "running Hubble length" $L_H(k) = L_H^0 Y(k)^{-1/2}$ along the trajectories of Type IIIa (plus sign) and of Type Ia (minus sign), respectively:

$$L_H(k) = L_H^0 \cdot \begin{cases} \dfrac{1}{\sqrt{|\ell^4\, k^4 \pm 1|}} & \text{for} \quad 0 \le k \lesssim \hat{k} \\ \dfrac{1}{L\, k} & \text{for} \quad k \gtrsim \hat{k} \end{cases} \tag{68}$$

This function is sketched in Figure 2.

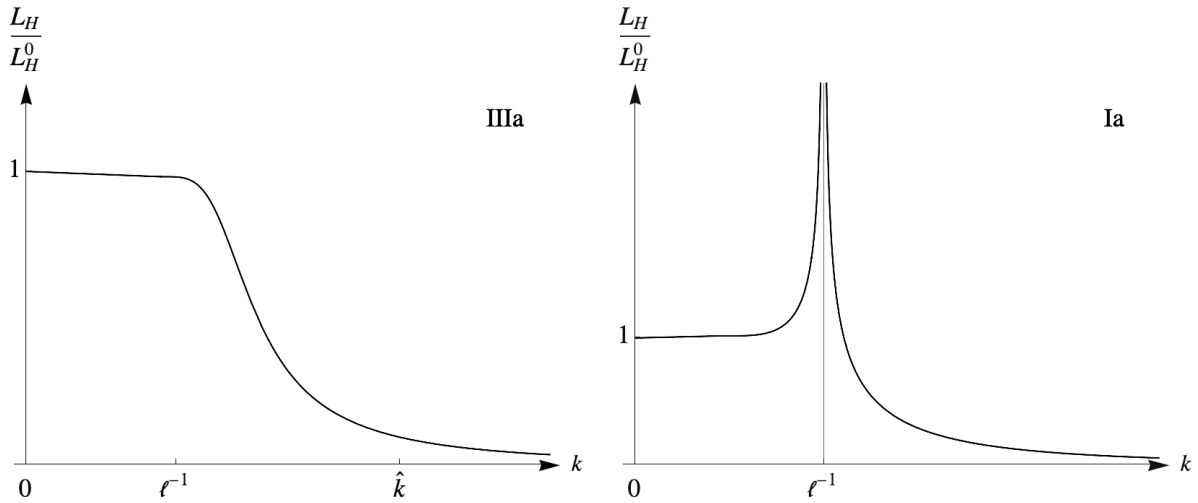

Figure 2. The scale-dependent Hubble length along trajectories of Type IIIa (left) and Type Ia (right), respectively.

For the Type IIIa trajectories, the Hubble length $L_H(k)$ is seen to be a strictly decreasing function of the scale for all $k \in (0, \infty)$. It is one-to-one therefore, and so, $\xi = L_H(k)$ defines

a legitimate change of coordinates on the interval $(0, \infty)$. The same is true in the limiting case $\Lambda_0 \searrow 0$, i.e., for the Type IIa case.

Type Ia trajectories on the other hand decompose into two branches with $k \in (0, \ell^{-1})$ and $k \in (\ell^{-1}, \infty)$, respectively. On each branch separately, setting $\xi = L_H(k)$ is an allowed change of coordinates. On the upper (lower) branch, the RG time ξ becomes a strictly decreasing (increasing) function of k then. However, employing ξ globally would create a 2-1 ambiguity where $L_H > L_H^0$.

(5) The transition region: Finally, let us investigate more carefully the monotonicity question in the transition region near $\hat{k} = O(m_{\rm Pl})$. The RG flow linearized about the fixed point (g_*, λ_*) is useful for a first orientation here. The linearization is governed by a pair of complex conjugate critical exponents $\theta_{1,2} = \theta' \pm i\theta''$, with $\theta', \theta'' \in \mathbb{R}^+$, which are responsible for the spiral-shaped trajectories $k \mapsto (g(k), \lambda(k))$ encircling the fixed point. The latter is located in the first quadrant of the g-λ-plane: $g_* > 0$, $\lambda_* > 0$. In the linear regime, the condition $\partial_k \Lambda(k) > 0$, or equivalently $k\partial_k \lambda(k) + 2\lambda(k) > 0$, assumes the form:

$$\lambda_* + \zeta \left(\frac{k_0}{k}\right)^{\theta'} \cos\left(\theta'' \ \ln(k/k_0) + \alpha\right) > 0 \tag{69}$$

Here, α and ζ are dimensionless parameters that depend on the constants of integration, that is on the trajectory under consideration, as well as on the eigenvectors of the stability matrix.[10] Equation (69) shows that for k sufficiently large, the monotonicity condition can never be violated since the potentially negative cosine is multiplied by too small a coefficient to compete with the positive λ_*. On the other hand, once the scale is low enough for $\zeta(k_0/k)^{\theta'}$ to be of order unity, there exist parameters α, ζ for which (69) could be violated. However, at those low scales, the linear approximation is not necessarily valid any longer. If by then, the trajectory is already in the semiclassical regime, the "caricature" trajectory applies, and monotonicity is guaranteed; but if not, violations could occur.

A detailed numerical analysis revealed however that in reality, there are no such violations of monotonicity in the transition region. For all three types of trajectories, one finds that $\partial_k \Lambda(k) > 0$ on all scales.

Figure 3 displays the numerical result for $\Lambda(k)$ and compares it to the product $G(k)\ \Lambda(k)$ and the anomalous dimension of Newton's constant, $\eta_N(k) = k\partial_k \ln G(k)$, along the same trajectory. The example shown is of Type IIIa, but since the plot focuses on the transition region, it would look basically the same for the other types. It is quite impressive to see that $\Lambda(k)$ is indeed perfectly monotonic even in the transition regime, while this is by no means the case for $G(k)\ \Lambda(k)$ and $\eta_N(k)$. In particular, the anomalous dimension displays significant oscillations in the transition regime.

This completes our demonstration that the asymptotically safe trajectories of QEG in four dimensions do indeed comply with the general property **(P)** and are thus eligible for a geometrization based on the proposed rule **(R)**.

[10] See Equation (5.30) of [27].

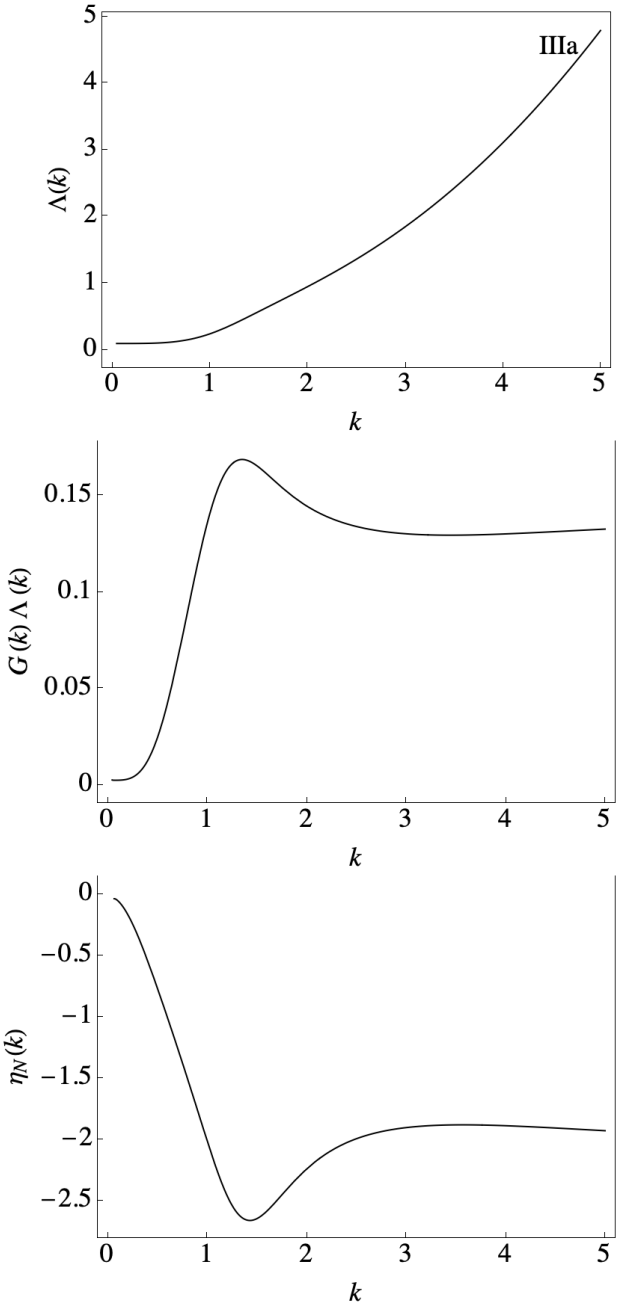

Figure 3. Numerical results for $\Lambda(k)$, the product $G(k)\ \Lambda(k)$, and the anomalous dimension $\eta_N(k)$ along a typical Type IIIa trajectory.

7. Summary and Conclusions

In this paper, we advocated a bottom-up approach towards geometrizing, and thus subsuming and visualizing the entire family of all effective spacetime metrics that occur along a given RG trajectory supplied by the established apparatus of the functional renormalization group for gravity. Different members of this family correspond to coarse-grained 4D spacetimes at different resolutions.

The proposed geometrization was constituted by a single 5D Riemannian or pseudo-Riemannian manifold, $\mathscr{M}_5$. It carries a natural foliation whose leaves are the 4D spacetimes corresponding to a fixed RG scale. The RG trajectory that delivers the "input data" for this construction determines the geometry of $\mathscr{M}_5$ only incompletely. A single metric on $\mathscr{M}_5$ can encode more information than the collection of all metrics on the slices. This raises the question if there exist any distinguished ways of completing the specification of the

5D geometry. Such completions might, for instance, be "natural" from the mathematical perspective, or they could transport additional physics information that is not, or not easily, accessible by the RG methods.

It was one of the motivations for this paper to initiate a survey of the logical possibilities concerning such distinguished higher dimensional geometries that is unbiased with regard to particular geometries or models (AdS, Randall–Sundrum, etc.). Nevertheless, a long-term goal of this search program is to ultimately try making contact with "top-down" formalisms like the AdS/CFT approach, which also invoke scale-spacetimes, but bear no obvious relation to the effective average action and its functional RG flows.

As a proof of principle, we explicitly analyzed the simplified situation where the ADM metric on $\mathscr{M}_5$ has a vanishing shift vector; we also assumed that the RG evolution of the 4D metrics is purely multiplicative and that it is governed by the Einstein–Hilbert truncation of the effective average action. Under these conditions, we proved that it is always possible to complete the specification of the 5D geometry in such a way that it possesses the following distinctive features: first, the metric on $\mathscr{M}_5$ admits a homothetic Killing vector field as an intrinsic characterization of its self-similarity, and second, the metric on $\mathscr{M}_5$ is Ricci flat. In the special case of maximally symmetric 4D spacetimes, it even can be chosen strictly flat.

These results are based on a specific proposal for the general structure of the full 5D metric, Equation (39). Surprisingly enough, in the literature, this class of metrics had already been studied in considerable detail, for quite different reasons though, namely in connection with the "spacetime-matter theory" advocated in [37–39].

From a more physics-oriented point of view, it is remarkable that in order to be well defined, i.e., non-degenerate, the proposed metric requires the cosmological constant $\Lambda(k)$ to be a strictly increasing function of the cutoff k. In other words, the coefficient $\Lambda(k)$ in the effective average action must have properties similar to a c-function that "counts" the number of fluctuation modes that get integrated out when k is changed.

It is intriguing therefore to speculate that, ultimately, the envisaged geometrization encodes global information about the underlying flows that is not easily seen at the FRGE level. Hence, future work will have to focus on drawing a more complete picture by relaxing some, or perhaps all, of our assumptions.

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