Article

Ion-Acoustic Rogue Waves in Double Pair Plasma Having Non-Extensive Particles

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Abstract: The modulational instability (MI) of ion-acoustic (IA) waves (IAWs) and associated IA rogue waves (IARWs) are studied in double-pair plasma containing inertial positive and negative ions, inertialess non-extensive electrons and iso-thermal positrons. A standard nonlinear Schrödinger equation (NLSE) is derived by employing reductive perturbation method. It can be seen from the numerical analysis that the plasma system supports both modulationally stable (unstable) parametric regime in which the dispersive and nonlinear coefficients of the NLSE have opposite (same) sign. It is also found that the basic features of IAWs (viz., MI criteria of IAWs, amplitude, and width of the IARWs, etc.) are rigorously changed by the plasma parameters such as mass, charge state, and number density of the plasma species. The outcomes of our present investigation should be useful in understanding the propagation of nonlinear electrostatic IAWs and associated IARWs in astrophysical and laboratory plasmas.

Keywords: ion-acoustic waves; NLSE; modulational instability; rogue waves

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1. Introduction

Pair-ion (PI) plasma is characterised as fully ionized gas, having electrons, positive and negative ions, and is believed to exist in astrophysical environments such as Van Allen radiation belt and near the polar cap of fast rotation neutron stars [1], solar atmosphere [2], upper regions of Titan's atmosphere [3–10], cometary comae [11], (H^+ , O_2^-) and (H^+ , H^-) plasmas in the D and F-regions of Earth's ionosphere [4–9], and also in laboratory experiments namely, (Ar^+ , F^-) plasma [12], (K^+ , SF_6^-) plasma [13,14], neutral beam sources [15], plasma processing reactors [16], (Ar^+ , SF_6^-) plasma [17–20], combustion products [21], plasma etching [21], (Xe^+ , F^-) plasma [22], (Ar^+ , O_2^-) plasma, Fullerene (C_{60}^+ , C_{60}^-) plasma [23–25], RF glow discharges [16], and Tokamak [26], etc. A number of authors studied ion-acoustic (IA) waves (IAWs) and associated nonlinear electrostatic structures, namely, solitons, shocks, rogue waves, and double layers in the presence of positrons in PI plasma [27–29].

The Maxwellian distribution function is one of the most widely used velocity distribution functions for describing the dynamics of iso-thermal particles. Highly energetic particles have been observed in the galaxy clusters [30], Earth's bow-shock [31], upper ionosphere of Mars [32], vicinity of the Moon [33], and magnetospheres of Jupiter and Saturn [34], etc. So, to describe the dynamics of these highly energetic particles, Renyi [35] first recognized the modification of Maxwellian distribution, and finally, Tsallis [36] generalized the non-extensive q -distribution. It is noted that the parameter q in the non-extensive q -distribution characterizes the degree of non-extensivity of these highly energetic particles [37–40]. Shalini et al. [38] studied IAWs in non-extensive plasma having two-temperature electrons, and observed that the width of the first and second-order IA rogue

waves (IARWs) associated with IAWs decreases with increasing the value of q but the amplitude of the first and second-order IARWs associated with IAWs is remain constant. Tribeche et al. [39] investigated electrostatic solitary waves in the presence of non-extensive electrons, and found that the the amplitude of the potential increases with non-extensive parameter. Hafez and Talukder [40] examined the propagation of nonlinear electrostatic waves in a three-component non-extensive plasma having inertialess non-extensive electrons and positrons, and inertial ions, and reported that the amplitude of the soliton increases with increasing temperature of non-extensive electron.

The investigation of modulational instability (MI) of electrostatic waves is one of the most important research areas for plasma physicists [41,42]. It may be noted that the MI of electrostatic waves is considered to be the primary reason for the formation of massive and gigantic rogue waves (RWs) [43–48]. Rogue wave, which is the rational solution of the standard nonlinear Schrödinger equation (NLSE), is a short-lived phenomenon that emerges from nowhere and disappears without a trace [43–48]. A number of authors have investigated the MI of IAWs by considering the non-extensive particles [49–53]. Bains et al. [51] studied the MI of IAWs in the presence of non-extensive electrons, and demonstrated that the critical wave number (k_c), at which the instability sets in, increases with the increase in the value of q ($q > 0$). Bouzit et al. [52] investigated the stability conditions of IAWs in the presence of non-extensive non-thermal electrons. Eslami et al. [53] investigated the MI of IAWs in electron-positron-ion plasma having non-extensive electrons and positrons, and observed that the k_c decreases with q for $q < 0$ while increases with q for $q > 0$. To the best knowledge of the authors, no attempt has been made to investigate the MI of the IAWs and associated IARWs in a four-component plasma containing inertial positive and negative ions, and inertialess non-extensive electrons, and iso-thermal positrons. Therefore, it is a rational fascination to examine the influence of non-extensive electrons and iso-thermal positrons on the MI of IAWs and associated IARWs in a four-component double pair plasma (DPP).

The manuscript is organized in the following fashion: The model equations are presented in Section 2. The derivation of the NLSE is shown in Section 3. The stability of IAWs is provided in Section 4. The IARWs are demonstrated in Section 5. Finally, a conclusion is given in Section 6.

2. Model Equations

We consider the propagation of IAWs in a collisionless, fully ionized, unmagnetized plasma system consisting of warm negative ions (denoted by n_{-i} ; charge $q_{-i} = -Z_{-i}e$; mass m_{-i}), warm positive ions (denoted by n_{+i} ; charge $q_{+i} = Z_{+i}e$; mass m_{+i}), non-extensive q -distributed electrons (denoted by n_e ; charge $q_e = -e$; mass m_e), and iso-thermal positrons (denoted by n_p ; charge $q_p = +e$; mass m_p); where Z_{-i} (Z_{+i}) is the charge state of the negative (positive) ion, and e being the magnitude of the charge of electron. The charge neutrality condition of our present model can be written as $n_{p0} + Z_{+i}n_{+i0} = n_{e0} + Z_{-i}n_{-i0}$. Now, the normalized governing equations can be written as

$$\frac{\partial n_{-i}}{\partial t} + \frac{\partial}{\partial x}(n_{-i}u_{-i}) = 0, \quad (1)$$

$$\frac{\partial u_{-i}}{\partial t} + u_{-i} \frac{\partial u_{-i}}{\partial x} + \lambda_1 n_{-i} \frac{\partial n_{-i}}{\partial x} = \frac{\partial \phi}{\partial x}, \quad (2)$$

$$\frac{\partial n_{+i}}{\partial t} + \frac{\partial}{\partial x}(n_{+i}u_{+i}) = 0, \quad (3)$$

$$\frac{\partial u_{+i}}{\partial t} + u_{+i} \frac{\partial u_{+i}}{\partial x} + \lambda_3 n_{+i} \frac{\partial n_{+i}}{\partial x} = -\lambda_2 \frac{\partial \phi}{\partial x}, \quad (4)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \lambda_4 n_e + n_{-i} - (1 + \lambda_4 - \lambda_5)n_p - \lambda_5 n_{+i}, \quad (5)$$

where n_{-i} , n_{+i} , n_e , and n_p are normalized by n_{-i0} , n_{+i0} , n_{e0} , and n_{p0} , respectively; u_{-i} (u_{+i}) indicates the negative (positive) ion fluid, and is normalized by the IA wave speed $C_{-i} =$

$(Z_{-i}k_B T_e / m_{-i})^{1/2}$ (with k_B being the Boltzmann constant and T_e being the temperature of the electron); ϕ denotes the electrostatic wave potential, and is normalized by $k_B T_e / e$; the time and space variables are, respectively, normalized by $\omega_{-ip}^{-1} = (m_{-i} / 4\pi e^2 Z_{-i}^2 n_{-i0})^{1/2}$ and $\lambda_{-i} = (k_B T_e / 4\pi e^2 Z_{-i} n_{-i0})^{1/2}$. The pressure term of the negative ion can be represented as $P_{-i} = P_{-i0} (N_{-i} / n_{-i0})^\gamma$ with $P_{-i0} = n_{-i0} k_B T_{-i}$, where P_{-i0} (T_{-i}) being the equilibrium pressure (temperature) of negative ion; and $P_{+i} = P_{+i0} (N_{+i} / n_{+i0})^\gamma$ with $P_{+i0} = n_{+i0} k_B T_{+i}$, where P_{+i0} (T_{+i}) being the equilibrium pressure (temperature) of the positive ion, and $\gamma = (N + 2) / N$ (where N represents the degree of freedom and for one-dimensional case $N = 1$, and then $\gamma = 3$). Other parameters are $\lambda_1 = 3T_{-i} / Z_{-i} T_e$, $\lambda_2 = Z_{+i} m_{-i} / Z_{-i} m_{+i}$, $\lambda_3 = 3T_{+i} m_{-i} / Z_{-i} m_{+i} T_e$, $\lambda_4 = n_{e0} / Z_{-i} n_{-i0}$, and $\lambda_5 = Z_{+i} n_{+i0} / Z_{-i} n_{-i0}$. Now, the number densities of non-extensive electron [54] and iso-thermal positron can be represented, respectively, by the following normalized equations

$$n_e = [1 + (q - 1)\phi]^{q+1}, \quad (6)$$

$$n_p = \exp(-\lambda_6 \phi), \quad (7)$$

where $\lambda_6 = T_e / T_p$ (with T_p being the temperature of the iso-thermal positron and $T_e > T_p$). The parameter q , generally known as entropic index which defines the degree of non-extensivity. It can be noted that when $q = 1$, the entropy reduces to standard Maxwell–Boltzmann distribution. On the other hand, in the limits $q > 0$ ($q < 0$), the entropy shows sub-extensivity (super-extensivity). Now, by substituting Equations (6) and (7) into Equation (5) and expanding up to third order in ϕ , we can write

$$\frac{\partial^2 \phi}{\partial x^2} + 1 + \lambda_5 n_{+i} = \lambda_5 + n_{-i} + M_1 \phi + M_2 \phi^2 + M_3 \phi^3 + \dots, \quad (8)$$

where

$$\begin{aligned} M_1 &= [\lambda_4(q + 1) + 2\lambda_6(1 + \lambda_4 - \lambda_5)] / 2, \\ M_2 &= [\lambda_4(q + 1)(3 - q) - 4\lambda_6^2(1 + \lambda_4 - \lambda_5)] / 8, \\ M_3 &= [\lambda_4(q + 1)(q - 3)(3q - 5) + 8\lambda_6^3(1 + \lambda_4 - \lambda_5)] / 48. \end{aligned}$$

It may be noted here that the terms containing M_1 , M_2 , and M_3 in Equation (8) are due to the contribution of non-extensive electrons and iso-thermal positrons.

3. Derivation of the NLSE

To study the MI of the IAWs, first we want to derive the NLSE by employing the reductive perturbation method. In that case, the stretched co-ordinates can be written in the following form [54,55]

$$\xi = \epsilon(x - v_g t), \quad (9)$$

$$\tau = \epsilon^2 t, \quad (10)$$

where v_g is the group speed and ϵ is a small parameter. After that the dependent variables can be represented as [54,55]

$$\Pi(x, t) = \Pi_0 + \sum_{m=1}^{\infty} \epsilon^m \sum_{l=-\infty}^{\infty} \Pi_l^{(m)}(\xi, \tau) \exp[i l(kx - \omega t)], \quad (11)$$

where $\Pi_{il}^m = [n_{-il}^m, u_{-il}^m, n_{+il}^m, u_{+il}^m, \phi_{il}^m]$, $\Pi_0 = [1, 0, 1, 0, 0]^T$, and k (ω) is real variables representing the carrier wave number (frequency). The derivative operators can be written as [54,55]

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \epsilon v_g \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau}, \quad (12)$$

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi}. \quad (13)$$

Now, by substituting Equations (9)–(13) into Equations (1)–(4) and (8), and collecting the power terms of ϵ , the first order ($m = 1$ with $l = 1$) reduced equations can be written as

$$n_{-i1}^{(1)} = \frac{k^2}{S} \phi_1^{(1)}, \quad (14)$$

$$u_{-i1}^{(1)} = \frac{k\omega}{S} \phi_1^{(1)}, \quad (15)$$

$$n_{+i1}^{(1)} = \frac{\lambda_2 k^2}{A} \phi_1^{(1)}, \quad (16)$$

$$u_{+i1}^{(1)} = \frac{\lambda_2 \omega k}{A} \phi_1^{(1)}, \quad (17)$$

where $S = \lambda_1 k^2 - \omega^2$ and $A = \omega^2 - \lambda_3 k^2$. These equations provide the dispersion relation of IAWs in the following form

$$\omega^2 \equiv \omega_f^2 = \frac{I + \sqrt{I^2 - 4UJ}}{2U}, \quad (18)$$

$$\omega^2 \equiv \omega_s^2 = \frac{I - \sqrt{I^2 - 4UJ}}{2U}, \quad (19)$$

where $I = (\lambda_1 k^2 + \lambda_3 k^2 + \lambda_1 M_1 + \lambda_3 M_1 + \lambda_2 \lambda_5 + 1)$, $U = (k^2 + M_1)/k^2$, and $J = k^2(\lambda_1 \lambda_3 k^2 + \lambda_1 \lambda_3 M_1 + \lambda_3 + \lambda_1 \lambda_2 \lambda_5)$. However, to obtain the real and positive values of ω , the conditions $I^2 > 4UJ$ must be satisfied. It is noted that ω_f (ω_s) represents the fast (slow) IA modes. The second-order ($m = 2$ with $l = 1$) equations and with the compatibility condition, we can write the group velocity of IAWs in the DPP

$$v_g = \frac{(\lambda_2 \lambda_3 \lambda_5 k^2 S^2 + \lambda_2 \lambda_5 \omega^2 S^2 + \lambda_2 \lambda_5 A S^2 + \lambda_1 A^2 k^2 + A^2 \omega^2 - 2A^2 S^2 - S A^2)}{2\omega k(A^2 + \lambda_2 \lambda_5 S^2)}, \quad (20)$$

Now, the co-efficient of ϵ (when $m = 2$ with $l = 2$) yields the second-order harmonic amplitudes, is found to be proportional to $|\phi_1^{(1)}|^2$

$$n_{-i2}^{(2)} = M_4 |\phi_1^{(1)}|^2, \quad (21)$$

$$u_{-i2}^{(2)} = M_5 |\phi_1^{(1)}|^2, \quad (22)$$

$$n_{+i2}^{(2)} = M_6 |\phi_1^{(1)}|^2, \quad (23)$$

$$u_{+i2}^{(2)} = M_7 |\phi_1^{(1)}|^2, \quad (24)$$

$$\phi_2^{(2)} = M_8 |\phi_1^{(1)}|^2, \quad (25)$$

where

$$\begin{aligned} M_4 &= \frac{2M_8 k^2 S^2 - 3\omega^2 k^4 - \lambda_1 k^6}{2S^3}, & M_5 &= \frac{\omega M_4 S^2 - \omega k^4}{k S^2}, \\ M_6 &= \frac{2\lambda_2 M_8 A^2 k^2 + 3\lambda_2^2 \omega^2 k^4 + \lambda_3 \lambda_2^2 k^6}{2A^3}, & M_7 &= \frac{\omega M_6 A^2 - \omega \lambda_2^2 k^4}{k A^2}, \\ M_8 &= \frac{2M_2 A^3 S^3 - 3\lambda_5 \lambda_2^2 \omega^2 S^3 k^4 - 3\omega^2 A^3 k^4 - \lambda_3 \lambda_5 \lambda_2^2 S^3 k^6 - \lambda_1 A^3 k^6}{2A^2 S^2 (S \lambda_2 \lambda_5 k^2 - 4k^2 A S - M_1 A S - A k^2)}, \end{aligned}$$

Now, we consider the expression for ($m = 3$ with $l = 0$) and ($m = 2$ with $l = 0$), which leads the zeroth harmonic modes. Thus, we obtain

$$n_{-i0}^{(2)} = M_9 |\phi_1^{(1)}|^2, \quad (26)$$

$$u_{-i0}^{(2)} = M_{10} |\phi_1^{(1)}|^2, \quad (27)$$

$$n_{+i0}^{(2)} = M_{11} |\phi_1^{(1)}|^2, \quad (28)$$

$$u_{+i0}^{(2)} = M_{12} |\phi_1^{(1)}|^2, \quad (29)$$

$$\phi_0^{(2)} = M_{13} |\phi_1^{(1)}|^2, \quad (30)$$

where

$$\begin{aligned} M_9 &= \frac{2\omega v_g k^3 + \lambda_1 k^4 + k^2 \omega^2 - M_{13} S^2}{S^2 (v_g^2 - \lambda_1)}, \quad M_{10} = \frac{M_9 v_g S^2 - 2\omega k^3}{S^2}, \\ M_{11} &= \frac{\lambda_2 M_{13} A^2 + \lambda_3 \lambda_2^2 k^4 + \lambda_2^2 \omega^2 k^2 + 2\omega v_g \lambda_2^2 k^3}{A^2 (v_g^2 - \lambda_3)}, \\ M_{12} &= \frac{M_{11} v_g A^2 - 2\omega \lambda_2^2 k^3}{A^2}, \\ M_{13} &= \frac{A^2 (v_g^2 - \lambda_3) (2\omega v_g k^3 + \lambda_1 k^4 + \omega^2 k^2) - M_{14}}{A^2 S^2 v_g^2 + A^2 S^2 [\lambda_2 \lambda_5 (v_g^2 - \lambda_1) - \lambda_3 - M_1 (v_g^2 - \lambda_1) (v_g^2 - \lambda_3)]}, \\ M_{14} &= (v_g^2 - \lambda_1) S^2 (2\lambda_5 \omega v_g \lambda_2^2 k^3 + \lambda_3 \lambda_5 \lambda_2^2 k^4 + \lambda_5 \lambda_2^2 \omega^2 k^2) \\ &\quad - 2M_2 A^2 S^2 (v_g^2 - \lambda_1) (v_g^2 - \lambda_3). \end{aligned}$$

Finally, the third harmonic modes ($m = 3$) and ($l = 1$), with the help of Equations (14)–(30), give a set of equations, which can be reduced to the following NLSE:

$$i \frac{\partial \Phi}{\partial \tau} + P \frac{\partial^2 \Phi}{\partial \xi^2} + Q |\Phi|^2 \Phi = 0, \quad (31)$$

where $\Phi = \phi_1^{(1)}$ for simplicity. In Equation (31), P is the dispersion co-efficient, which can be written as

$$P = \frac{4\lambda_2 \lambda_5 \omega^2 k^2 v_g^2 S^3 + 4\lambda_1 \omega v_g A^3 k^3 + 4k v_g \omega^3 A^3 + M_{15}}{2\omega A S k^2 (A^2 + \lambda_2 \lambda_5 S^2)},$$

$$\begin{aligned} M_{15} &= 2\lambda_2 \lambda_3 \lambda_5 \omega^2 k^2 S^3 + \lambda_2 \lambda_5 \lambda_3^2 S^3 k^4 + \lambda_1 S k^2 A^3 + \lambda_2 \lambda_5 S^3 \omega^4 + \lambda_2 \lambda_3 \lambda_5 A k^2 S^3 \\ &\quad - 4\lambda_2 \lambda_5 k v_g \omega^3 S^3 - 4\lambda_2 \lambda_3 \lambda_5 \omega v_g k^3 S^3 - 4\omega^2 k^2 v_g^2 A^3 - \lambda_1^2 A^3 k^4 \\ &\quad - 2\lambda_1 \omega^2 k^2 A^3 - S k^2 v_g^2 A^3 - A^3 \omega^4 - \lambda_2 \lambda_5 A k^2 v_g^2 S^3 - A^3 S^3, \end{aligned}$$

and Q is the nonlinear co-efficient, which can be written as

$$Q = \frac{3M_3 A^2 S^2 + 2M_2 M_8 A^2 S^2 + 2M_2 M_{15} A^2 S^2 - M_{16}}{2\omega k^2 (A^2 + \lambda_2 \lambda_5 S^2)},$$

where

$$\begin{aligned} M_{16} &= 2\omega M_5 A^2 k^3 + 2\lambda_2 \lambda_5 \omega M_7 S^2 k^3 + 2\omega M_{10} A^2 k^3 \\ &\quad + 2\lambda_2 \lambda_5 \omega M_{12} S^2 k^3 + \lambda_1 M_4 A^2 k^4 + M_4 \omega^2 A^2 k^2 \\ &\quad + \lambda_2 \lambda_3 \lambda_5 M_6 S^2 k^4 + \lambda_2 \lambda_5 M_6 \omega^2 k^2 S^2 + M_9 \omega^2 A^2 k^2 \\ &\quad + \lambda_1 M_9 A^2 k^4 + \lambda_2 \lambda_5 M_{11} \omega^2 k^2 S^2 + \lambda_2 \lambda_3 \lambda_5 M_{11} S^2 k^4 \end{aligned}$$

It may be noted here that both P and Q are directly depend on different parameters namely, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, q, k$, and are indirectly depend on the mass, number density, temperature, and charge state of the different plasma species.

4. Modulational Instability of the IAWs

The propagation of IAWs is modulationally stable when P and Q have the opposite sign (i.e., $P/Q < 0$), and is modulationally unstable when both P and Q have same sign (i.e., $P/Q > 0$) [54,55]. The point, at which the transition of P/Q curve intersects with the k -axis in the “ P/Q versus k ” graph, is known as threshold or critical wave number k ($= k_c$) [54,55]. We have depicted the variation of P/Q with k for different values of λ_5 under the consideration of fast (ω_f) and slow (ω_s) IA modes, respectively, in Figures 1 and 2, and it can be seen from these two figures that (a) both stable and unstable parametric regimes are allowed by the plasma system; (b) the k_c increases (decreases) with increasing of positive (negative) ion number density when their charge state remains constant; (c) the stable parametric regime (i.e., $P/Q < 0$) increases (decreases) with positive (negative) ion charge state when their mass remains constant.

The effects of sub-extensivity and super-extensivity of electrons on the stability condition of IAWs can be seen in Figures 3 and 4, respectively. It is obvious from these two figures that the sub-extensive property (i.e., $q > 0$) of the electrons allows the IAWs to be stable for large wave number while the super-extensive property (i.e., $q < 0$) of the electrons allows the IAWs to be stable for small a wavenumber, and this result agrees with the result of Eslami et al. [53].

Figure 1. The variation of P/Q with k for different values of λ_5 when $\lambda_1 = 0.007$, $\lambda_2 = 1.2$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_6 = 1.7$, $q = 1.4$, and $\omega \equiv \omega_f$.

Figure 2. The variation of P/Q with k for different values of λ_5 when $\lambda_1 = 0.007$, $\lambda_2 = 1.2$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_6 = 1.7$, $q = 1.4$, and $\omega \equiv \omega_s$.

Figure 3. The variation of P/Q with k for different values of q when $\lambda_1 = 0.007$, $\lambda_2 = 1.2$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_5 = 1.5$, $\lambda_6 = 1.7$, and $\omega \equiv \omega_f$.

Figure 4. The variation of P/Q with k for different values of q when $\lambda_1 = 0.007$, $\lambda_2 = 1.2$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_5 = 1.5$, $\lambda_6 = 1.7$, and $\omega \equiv \omega_f$.

5. Rogue Waves

The instability induced from a small perturbation on top of a plane wave leads to an increase in the perturbation up to its highest amplitude and then to a decay so that it finally “disappears without a trace” [43–48]. The NLSE has a variety of solutions, from which the rogue wave/rational solution (developed by Darboux Transformation Scheme) is localized in both the ζ and τ [43–48]. The first-order rational rogue wave solution of the NLSE in the unstable parametric regime (i.e., $P/Q > 0$) of the IAWs can be written as [43–48]

$$\Phi(\zeta, \tau) = \sqrt{\frac{2P}{Q}} \left[\frac{4 + 16i\tau P}{1 + 4\zeta^2 + 16\tau^2 P^2} - 1 \right] \exp(2i\tau P). \quad (32)$$

It can be seen from the literature that the PI plasma system can support these conditions: $m_{-i} > m_{+i}$ (i.e., $H^+ - O_2^-$ [4–9], $Ar^+ - SF_6^-$ [17–20], and $Xe^+ - SF_6^-$ [17–20]), $m_{-i} = m_{+i}$ (i.e., $H^+ - H^-$ [4–9] and $C_{60}^+ - C_{60}^-$ [23–25]), and $m_{-i} < m_{+i}$ (i.e., $Ar^+ - F^-$ [5,6]). So, in our present investigation, we have graphically observed the variation of electrostatic potential with λ_2 under the consideration of $m_{-i} > m_{+i}$ (i.e., $\lambda_2 > 1$) in Figure 5, and it is obvious from this figure that (a) the amplitude and width of the IARWs decrease with an increase in the value of the negative ion mass but increase with an increase in the value of the positive ion mass for a fixed value of their charge state; (b) the height of the IARWs increases (decreases) with negative (positive) ion charge state for a constant mass of positive and negative ion species. So, the mass and charge state of the PI play an opposite role to the formation of IARWs in PI plasma. Figure 6 describes the nature of the electrostatic IARWs with the variation of λ_2 under the consideration of $m_{-i} < m_{+i}$ (i.e., $\lambda_2 < 1$). It is clear from this figure that (a) the amplitude and width of IARWs increases (decreases) with negative (positive) ion mass when other plasma parameters are constant; and (b) the existence of

the heavy positive ion change the dynamics of the plasma system. So, the dynamics of the DPP rigourously changes with these conditions $m_{-i} > m_{+i}$ (i.e., $\lambda_2 > 1$) and $m_{-i} < m_{+i}$ (i.e., $\lambda_2 < 1$).

The nature of IARWs may also be affected by the electron and positron temperature which can be observed in Figure 7. This figure reveals that as we increase the value of electron (positron) temperature, then the IARWs associated with IAWs are taller (smaller). The physics behind this result is that the nonlinearity of the plasma medium as well as the amplitude of the IARWs decreases (increases) with positron (electron) temperature.

Figure 5. The variation of $|\Phi|$ with ξ for different values of λ_2 when $\lambda_1 = 0.007$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_5 = 1.5$, $\lambda_6 = 1.7$, $\tau = 0$, $k = 0.7$, and $\omega \equiv \omega_f$.

Figure 6. The variation of $|\Phi|$ with ξ for different values of λ_2 when $\lambda_1 = 0.007$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_5 = 1.5$, $\lambda_6 = 1.7$, $\tau = 0$, $k = 0.7$, and $\omega \equiv \omega_f$.

Figure 7. The variation of $|\Phi|$ with ξ for different values of λ_6 when $\lambda_1 = 0.007$, $\lambda_2 = 1.2$, $\lambda_3 = 0.07$, $\lambda_4 = 1.8$, $\lambda_5 = 1.5$, $\tau = 0$, $k = 0.7$, and $\omega \equiv \omega_f$.

6. Conclusions

We have investigated the MI of IAWs and associated IARWs in a four-component DPP having inertial positive and negative ions, and inertialess non-extensive electrons and iso-thermal positrons by deriving a standard NLSE. The results that have been found from our investigation can be summarized as follows:

- Both stable and unstable parametric regimes of IAWs can be observed.
- The sub-extensive property of the electrons allows the IAWs to be stable for large wave number while the super-extensive property of the electrons allows the IAWs to be stable for small wave number.
- The dynamics of the DPP rigourously changes with these conditions $m_{-i} > m_{+i}$ (i.e., $\lambda_2 > 1$) and $m_{-i} < m_{+i}$ (i.e., $\lambda_2 < 1$).
- The nonlinearity of the plasma medium as well as the amplitude of the IARWs decreases (increases) with positron (electron) temperature.

The results of our present investigation are applicable in understanding the process of MI of IAWs and associated IARWs in both astrophysical environments, viz., Van Allen radiation belt and near the polar cap of fast rotation neutron stars [1], solar atmosphere [2], upper regions of Titan's atmosphere [3–10], cometary comae [11], (H^+, O_2^-) and (H^+, H^-) plasmas in the D and F-regions of Earth's ionosphere [4–9], and also in laboratory experiments namely, (Ar^+, F^-) plasma [12], (K^+, SF_6^-) plasma [13,14], neutral beam sources [15], plasma processing reactors [16], (Ar^+, SF_6^-) plasma [17–20], combustion products [21], plasma etching [21], (Xe^+, F^-) plasma [22], (Ar^+, O_2^-) plasma, Fullerene (C_{60}^+, C_{60}^-) plasma [23–25], RF glow discharges [16], and Tokamak [26], etc.

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References

1. Lightman, A.P. Relativistic thermal plasmas-Pair processes and equilibria. *Astrophys. J.* **1982**, *253*, 842. [\[CrossRef\]](#)
2. Tandberg-Hansen, E.; Emsile, A.G. *The Physics of Solar Flares*; Cambridge University Press: Cambridge, UK, 1988.
3. Coates, A.J.; Wellbrock, A.; Lewis, G.R.; Jones, G.H.; Young, D.T.; Cray, F.J.; Waite, J.H., Jr. Heavy negative ions in Titan's ionosphere: Altitude and latitude dependence. *Geophys. Res. Lett.* **2007**, *34*, L22103.
4. Massey, H. *Negative Ions*, 3rd ed.; Cambridge University Press: Cambridge, UK, 1976.
5. Sabry, R.; Moslem, W.M.; Shukla, P.K. Fully nonlinear ion-acoustic solitary waves in a plasma with positive-negative ions and nonthermal electrons. *Phys. Plasmas* **2009**, *16*, 032302. [\[CrossRef\]](#)
6. Abdelwahed, H.G.; El-Shewy, E.K.; Zahran, M.A.; Elwakil, S.A. On the rogue wave propagation in ion pair superthermal plasma. *Phys. Plasmas* **2016**, *23*, 022102. [\[CrossRef\]](#)
7. Misra, A.P. Dust ion-acoustic shocks in quantum dusty pair-ion plasmas. *Phys. Plasmas* **2009**, *16*, 033702.
8. Mushtaq, A.; Khattak, M.N.; Ahmad, Z.; Qamar, A. Dust ion acoustic soliton in pair-ion plasmas with non-isothermal electrons. *Phys. Plasmas* **2012**, *19*, 042304.
9. Jannat, N.; Ferdousi, M.; Mamun, A.A. Ion-acoustic shock waves in nonextensive multi-Ion plasmas. *Commun. Theor. Phys.* **2015**, *64*, 479. [\[CrossRef\]](#)
10. El-Labany, S.k.; Behery, E.E.; El-Razek, S.N.A.; Abdelrazek, L.A. Shock waves in magnetized electronegative plasma with nonextensive electrons. *Eur. Phys. J. D* **2020**, *74*, 104. [\[CrossRef\]](#)
11. Chaizy, P.H.; Reme, H.; Sauvaud, J.A.; d'Uston, C.; Lin, R.P.; Larson, D.E.; Mitchell, D.L.; Anderson, K.A.; Carlson, C.W.; Korth, A.; et al. Negative ions in the coma of comet Halley. *Nature* **1991**, *349*, 393. [\[CrossRef\]](#)
12. Nakamura, Y.; Tsukabayashi, I. Observation of modified Korteweg—de Vries solitons in a multicomponent plasma with negative Ions. *Phys. Rev. Lett.* **1984**, *52*, 2356. [\[CrossRef\]](#)

13. Song, B.; D'Angelo, N.; Merlino, R.L. Ion-acoustic waves in a plasma with negative ions. *Phys. Fluids B* **1991**, *3*, 284. [[CrossRef](#)]
14. Sato, N.; Production of negative ion plasmas in a Q-machine. *Plasma Sources Sci. Technol.* **1994**, *3*, 395. [[CrossRef](#)]
15. Bacal, M.; Hamilton, G.W. H^- and D^- Production in Plasmas. *Phys. Rev. Lett.* **1979**, *42*, 1538. [[CrossRef](#)]
16. Gottscho, R.A.; Gaebe, C.E. Negative ion kinetics in RF glow discharges. *IEEE Trans. Plasma Sci.* **1986**, *14*, 92. [[CrossRef](#)]
17. Wong, A.Y.; Mamas, D.L.; Arnush, D. Negative ion plasmas. *Phys. Fluids* **1975**, *18*, 1489. [[CrossRef](#)]
18. Nakamura, Y.; Odagiri, T.; Tsukabayashi, I. Ion-acoustic waves in a multicomponent plasma with negative ions. *Plasma Phys. Control. Fusion* **1997**, *39*, 105. [[CrossRef](#)]
19. Cooney, J.L.; Gavin, M.T.; Lonngren, K.E. Experiments on Korteweg–de Vries solitons in a positive ion–negative ion plasma. *Phys. Fluids B* **1991**, *3*, 2758. [[CrossRef](#)]
20. Nakamura, Y.; Bailung, H.; Lonngren, K.E. Oblique collision of modified Korteweg–de Vries ion-acoustic solitons. *Phys. Plasmas* **1999**, *6*, 3466. [[CrossRef](#)]
21. Sheehan, D.P.; Rynn, N. Negative-ion plasma sources. *Rev. Sci. Instrum.* **1988**, *59*, 8. [[CrossRef](#)]
22. Ichiki, R.; Yoshimura, S.; Watanabe, T.; Nakamura, Y.; Kawai, Y. Experimental observation of dominant propagation of the ion-acoustic slow mode in a negative ion plasma and its application. *Phys. Plasmas* **2002**, *9*, 4481.
23. Oohara, W.; Hatakeyama, R. Pair-ion Plasma generation using Fullerenes. *Phys. Rev. Lett.* **2003**, *91*, 205005. [[CrossRef](#)]
24. Hatakeyama, R.; Oohara, W. Properties of Pair-ion plasmas using Fullerenes. *Phys. Scripta* **2005**, *116*, 101. [[CrossRef](#)]
25. Oohara, W.; Date, D.; Hatakeyama, R. Electrostatic waves in a paired Fullerene-ion plasma. *Phys. Rev. Lett.* **2005**, *95*, 175003. [[CrossRef](#)] [[PubMed](#)]
26. Helander, P.; Ward, D.J. Positron Creation and Annihilation in Tokamak Plasmas with Runaway Electrons. *Phys. Rev. Lett.* **2003**, *90*, 135004. [[CrossRef](#)] [[PubMed](#)]
27. Esfandiyari-Kalejahi, A.; Kourakis, I.; Shukla, P.K. Oblique modulation of electrostatic modes and envelope excitations in pair-ion and electron-positron plasmas. *Phys. Plasmas* **2006**, *13*, 122310. [[CrossRef](#)]
28. Abdelsalam, U.M.; Moslem, W.M.; Shukla, P.K. Ion-acoustic solitary waves in a dense pair-ion plasma containing degenerate electrons and positrons. *Phys. Lett. A* **2008**, *372*, 4057. [[CrossRef](#)]
29. Sabry, R. Modulation instability of ion thermal waves in a pair-ion plasma containing charged dust impurities. *Phys. Plasma* **2008**, *15*, 092101. [[CrossRef](#)]
30. Hansen, S.H. Cluster temperatures and non-extensive thermo-statistics. *New Astron.* **2005**, *10*, 371. [[CrossRef](#)]
31. Asbridge, J.R.; Bame, S.J.; Strong, I.B. Outward flow of protons from the Earth's bow shock. *J. Geophys. Res.* **1968**, *73*, 5777. [[CrossRef](#)]
32. Lundlin, R.; Zakharov, A.; Pellinen, R.; Borg, H.; Hultqvist, D.; Pissarenko, N.; Dubinin, E.M.; Barabash, S.W.; Liede, I.; Koskinen, H. First measurements of the ionospheric plasma escape from Mars. *Nature* **1989**, *341*, 609. [[CrossRef](#)]
33. Futaana, Y.; Machida, S.; Saito, Y.; Matsuoka, A.; Hayakawa, H. Moon-related nonthermal ions observed by Nozomi: Species, sources, and generation mechanisms. *J. Geophys. Res.* **2003**, *108*, 1025. [[CrossRef](#)]
34. Krimigis, S.M.; Carbary, J.F.; Keath, E.P.; Armstrong, T.P.; Lanzerotti, L.J.; Gloeckler, G. General characteristics of hot plasma and energetic particles in the Saturnian magnetosphere: Results from the Voyager spacecraft. *J. Geophys. Res.* **1983**, *88*, 8871.
35. Renyi, A. On a new axiomatic theory of probability. *Acta Math. Acad. Sci. Hung.* **1955**, *6*, 285. [[CrossRef](#)]
36. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479. [[CrossRef](#)]
37. Wang, Y.Y.; Li, J.T.; Dai, C.Q.; Chen, V.F. Solitary Waves Rogue Waves A Plasma Nonthermal Electrons Featur. Tsallis Distribution. *Phys. Lett. A* **2013**, *377*, 2097. [[CrossRef](#)]
38. Shalini; Saini, N.S.; Misra, A.P. Modulation of ion-acoustic waves in a nonextensive plasma with two-temperature electrons. *Phys. Plasmas* **2015**, *22*, 092124. [[CrossRef](#)]
39. Tribeche, M.; Djebarni, L.; Amour, R. Ion-acoustic solitary waves in a plasma with a q-nonextensive electron velocity distribution. *Phys. Plasmas* **2010**, *17*, 042114. [[CrossRef](#)]
40. Hafez, M.G.; Talukder, M.R. Ion acoustic solitary waves in plasmas with nonextensive electrons, Boltzmann positrons and relativistic thermal ions. *Astrophys. Space Sci.* **2015**, *359*, 27. [[CrossRef](#)]
41. Lesur, M.; Diamond, P.H.; Kosuga, Y. Nonlinear current-driven ion-acoustic instability driven by phase-space structures. *Plasma Phys. Control. Fusion* **2014**, *56*, 075005.
42. Berman, R.H.; Tetreault, D.J.; Dupree, T.H. Simulation of phase space hole growth and the development of intermittent plasma turbulence. *Phys. Fluids* **1985**, *28*, 155. [[CrossRef](#)]
43. Akhmediev, N.; Ankiewicz, A.; Soto-Crespo, J.M. Rogue waves and rational solutions of the nonlinear Schrödinger equation. *Phys. Rev. E* **2009**, *80*, 026601. [[CrossRef](#)] [[PubMed](#)]
44. Anikiewicz, A.; Devine, N.; Akhmediev, N. Are rogue waves robust against perturbations? *Phys. Lett. A* **2009**, *373*, 3997. [[CrossRef](#)]
45. Akhmediev, N.; Soto-Crespo, J.M.; Ankiewicz, A. Extreme waves that appear from nowhere: On the nature of rogue waves. *Phys. Lett. A* **2009**, *373*, 2137. [[CrossRef](#)]
46. Akhmediev, N.; Soto-Crespo, J.M.; Ankiewicz, A. How to excite a rogue wave. *Phys. Rev. A* **2009**, *80*, 043818. [[CrossRef](#)]
47. Guo, S.; Mei, L. Modulation instability and dissipative rogue waves in ion-beam plasma: Roles of ionization, recombination, and electron attachment. *Phys. Plasmas* **2014**, *21*, 112303. [[CrossRef](#)]
48. Kedziora, D.; Ankiewicz, A.; Akhmediev, N. Circular rogue wave clusters. *Phys. Rev. E* **2011**, *84*, 056611. [[CrossRef](#)] [[PubMed](#)]

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49. Haque, M.N.; Mannan, A.; Mamun, A.A. Three-dimensional Nonlinear Structures in Magnetized Complex Plasmas. *Plasma Phys. Rep.* **2019**, *45*, 1026. [[CrossRef](#)]
 50. Haque, M.N.; Mannan, A.; Mamun, A.A. The (3+1) dimensional dust-acoustic waves in multi-components magneto-plasmas. *Contrib. Plasma Phys.* **2019**, *59*, e201900049. [[CrossRef](#)]
 51. Bains, A.S.; Tribeche, M.; Gill, T.S. Modulational instability of ion-acoustic waves in a plasma with a q-nonextensive electron velocity distribution. *Phys. Plasmas* **2011**, *18*, 022108. [[CrossRef](#)]
 52. Bouzit, O.; Tribeche, M.; Bains, A.S. Modulational instability of ion-acoustic waves in plasma with a q-nonextensive nonthermal electron velocity distribution. *Phys. Plasmas* **2015**, *22*, 084506. [[CrossRef](#)]
 53. Eslami, P.; Mottaghizadeh, M.; Pakzad, H.R. Modulational instability of ion acoustic waves in e-p-i plasmas with electrons and positrons following a q-nonextensive distribution. *Phys. Plasmas* **2011**, *18*, 102313. [[CrossRef](#)]
 54. Chowdhury, N.A.; Mannan, A.A.; Hasan, M.M.; Mamun, A.A. Heavy ion-acoustic rogue waves in electron-positron multi-ion plasmas. *Chaos* **2017**, *27*, 093105.
 55. Banik, S.; Shikha, R.K.; Noman, A.A.; Chowdhury, N.A.; Mannan, A.; Roy, T.S.; Mamun, A.A. First and second-order dust-ion-acoustic rogue waves in non-thermal plasma. *Eur. Phys. J. D* **2021**, *75*, 43. [[CrossRef](#)]