



Article Quantum Potentiality in Inhomogeneous Cosmology

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Abstract: For the Szekeres system which describes inhomogeneous and anisotropic spacetimes we make use of a point-like Lagrangian, which describes the evolution of the physical variables of the Szekeres model, in order to perform a canonical quantization and to study the quantum potentiality of the Szekeres system in the content of de Broglie–Bohm theory. We revise previous results on the subject and we find that for a specific family of trajectories with initial conditions which satisfy a constraint equation, there exists additional conservation laws for the classical Szekeres system which are used to define differential operators and to solve the Wheeler–DeWitt equation. From the new conservation laws we construct a wave function which provides a nonzero quantum potential term that modifies the Szekeres system. The quantum potential corresponds to new terms in the dynamical system such that new asymptotic solutions with a nonzero energy momentum tensor of an anisotropic fluid exist. Therefore, the silent property of the Szekeres spacetimes is violated by quantum correction terms, which results in the quantum potential adding pressure to the solution.

Keywords: Bohmian mechanics; quantum cosmology; inhomogeneous spacetimes; exact solutions; Szekeres universes



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1. Introduction

Quantum gravity is motivated by the idea to have a quantum description of all the matter fields and of their interactions in a gravitational system [1]. The dynamical variables of the spacetime interact with the matter fields as is described by General Relativity. Hence, in quantum gravity the dynamical variables of the spacetimes are described by quantum physics. The study of the quantum properties of the whole universe as a unique gravitational system is part of quantum cosmology. For extended discussions we refer the reader to [1,2]. In this work we are interested in the analytic solutions of the Wheeler–DeWitt Equation [3,4] for quantum cosmology in the case of inhomogeneous spacetimes and in the derivation of quantum corrections on the semiclassical limit as described by de Broglie–Bohm theory [5–9].

Inhomogeneous cosmological models are exact spacetimes which in general do not admit any isometry vector field while the conditions described by the cosmological principle, that is, the limit of the Friedmann–Lemaître–Robertson–Walker universe is provided [10]. Inhomogeneous cosmological models can be used for the description of the universe in the preinflationary era [11–15], as also in the description of the small inhomogeneities which are found by cosmological observations [16–20]. This is an alternative approach based on exact solutions and is different from the cosmological perturbation theory.

In the following we are interested in Szekeres spacetimes [21]. These spacetimes are exact solutions of the gravitational field equations for a diagonal line element with two dynamical variables and zero magnetic component for the Weyl tensor. The matter field is described by an inhomogeneous pressureless fluid. Szekeres spacetimes are characterized as "partially locally rotational" spacetimes [22]. The term "partially" indicates that the exact solutions do not admit any isometry vector field. Furthermore, there is no information dissemination with gravitational or sound waves on these exact solutions which means that they are silent universes [23]. There are various interesting results in the literature

which have shown that Szekeres spacetimes can play a significant role in the description of various epochs of our universe [18,24–26], while the present isotropic and homogeneous on large scales observed universe can be provided by Szekeres spacetimes for specific initial conditions with or without an inflationary era in the cosmological evolution [27–29]. Various generalizations of the Szekeres spacetimes with other kind of matter fields can be found in [30–36].

Szekeres universes are described by (pseudo)-Riemannian geometry with line element $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, $x^{\mu} = (t, x, y, z)$ and

$$g_{\mu\nu} = diag \left(-1, e^{2\alpha(t, x, y, z)}, e^{2\beta(t, x, y, z)}, e^{2\beta(t, x, y, z)} \right), \tag{1}$$

where functions $\alpha(t, x, y, z)$ and $\beta(t, x, y, z)$ satisfy the following algebraic-differential system known as the Szekeres system. The first-order differential equations are

$$\dot{\rho} + \theta \rho = 0, \tag{2}$$

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}\rho = 0,$$
 (3)

$$\dot{\sigma} - \sigma^2 + \frac{2}{3}\theta\sigma + E = 0, \tag{4}$$

$$\dot{E} + 3E\sigma + \theta E + \frac{1}{2}\rho\sigma = 0 \tag{5}$$

with the algebraic constraint

$$\frac{\theta^2}{3} - 3\sigma^2 + \frac{^{(3)}R}{2} = \rho, \tag{6}$$

where $\theta = \left(\frac{\partial \alpha}{\partial t}\right) + 2\left(\frac{\partial \beta}{\partial t}\right)$ and $\sigma^2 = \frac{2}{3}\left(\left(\frac{\partial \alpha}{\partial t}\right) - \left(\frac{\partial \beta}{\partial t}\right)\right)^2$ are the expansion rate and the anisotropic parameter, $\rho = \rho(t, x, y, z)$ is the energy density of the inhomogeneous dust fluid and E = E(t, x, y, z) is the electric component of the Weyl tensor $E_{\nu}^{\mu} = Ee_{\nu}^{\mu}$; and ${}^{(3)}R$ is the spatial curvature of the three-dimensional hypersurface. In addition to the latter system, the dynamical variables satisfy the propagation equations which are $h_{\mu}^{\nu}\sigma_{\nu;\alpha}^{\alpha} = \frac{2}{3}h_{\mu}^{\nu}\theta_{;\nu}$, $h_{\mu}^{\nu}E_{\nu;\alpha}^{\alpha} = \frac{1}{3}h_{\mu}^{\nu}\rho_{;\nu}$ in which $h_{\mu\nu}$ is the decomposable tensor defined by the expression $h_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}$, where u^{μ} is a unitary vector field, $u^{\mu}u_{\mu} = -1$, which defines the physical observer [37].

The integrability properties of the Szekeres system have been widely studied in the literature [38–40], In [38], the Szekeres system has been written as a system of two second-order differential equations and a conservation law quadratic in the derivatives of the dynamical variables was derived. The gravitational field equations of the Szekeres model do not admit a minisuperspace description. However, as was shown in [38], the time-dependent field equations can be described by a point-like Lagrangian with respect to the variables { ρ , E}, while the conservation law quadratic in the derivatives is derived easily with the use of Noether's theorem. In addition in this work we show that the Szekeres system admits as an additional conservation law the Lewis invariant.

The Lagrangian description of the Szekeres system and the Noetherian conservation law were applied in [41] to quantize and write the "time"-independent Wheeler–DeWitt equation for the Szekeres system. The solution of the Wheeler–DeWitt equation which satisfies the quadratic conservation law has been found which does not affect the classical trajectories of the Szekeres system and there is no any quantum potential term provided. Moreover, a probability function was found from which it was found that a stationary surface of the probability function is related with classical exact solutions. A similar study was performed recently for the Szekeres system with the cosmological constant term [42].

In the following we revise the analysis of [41]. Specifically, we find that, in the special case for which the "energy" of the two-dimensional Hamiltonian system which describes the Szekeres system vanishes, new conservation laws exist for the Szekeres

system. These new conservation laws are used to define quantum operators which are used as supplementary conditions on the Wheeler–DeWitt equation and to determine new similarity solutions for the Wheeler–DeWitt Equation [43–45]. From the new wavefunction we are able to construct a nontrivial quantum potential given by the Broglie–Bohm theory. The effects of the quantum correction in the original Szekeres system is investigated as also the effects on the dynamics are studied. The plan of the paper is as follows.

In Section 2, we present the two-dimensional Hamiltonian dynamical system which is equivalent to the Szekeres system (2)-(5) and we derive the new conservation laws. In particular, we show that the Szekeres system admits as conservation law the Lewis invariant as also a family of conservation laws generated by Lie point symmetries when the energy of the Hamiltonian dynamical system is zero. The new conservation laws are applied for the derivation of quantum operators and for the derivation of exact solutions of the Wheeler–DeWitt equation in Section 3. The quantum potential is determined in Section 4, in which we show that a nonzero quantum correction exists for specific trajectories of the Szekeres system satisfying a specific set of initial conditions. In Section 5, we study the effects of the nonzero quantum correction in the Szekeres system. We write the modified Szekeres system for which a contribution in the equation for the expansion rate θ follows by quantum corrections. This new term modifies the dynamics of the Szekeres system and leads to new asymptotic solutions with nonzero matter, pressure and anisotropic component of an energy momentum tensor. These latter fluid components have their origin in the quantum corrections as provided by Bohmian mechanics. Finally, in Section 6, we summarize our results and we draw our conclusions.

2. Hamiltonian Formulation of the Szekeres System

From Equations (2) and (5) we find that

$$\theta = -\frac{\dot{
ho}}{
ho}, \ \sigma = \frac{2(\dot{
ho}E - \rho\dot{E})}{
ho(
ho + 6E)}.$$
(7)

Thus, by substituting (7) into (3) and (4) we obtain an equivalent form for the Szekeres system comprising two second-order differential equations with respect to the variables ρ and *E*. The resulting equations are [38]

$$\ddot{u} + \frac{1}{u^2} = 0,$$
 (8)

$$\ddot{v} - 2\frac{v}{u^3} = 0,\tag{9}$$

where $\{u, v\}$ are new variables defined as [38]

$$\rho(u,v) = \frac{6}{(u-v)u^2}, \ E(u,v) = \frac{v}{u^3(v-u)}$$
(10)

with inverse transformation

$$u(\rho, E) = \left(\frac{\rho}{6} + E\right)^{-\frac{1}{3}}, \ v(\rho, E) = -\frac{6E}{\rho} \left(\frac{\rho}{6} + E\right)^{-\frac{1}{3}}.$$
 (11)

The parameters θ and σ , are expressed in terms of the new variables as

$$\theta(u, v, \dot{u}, \dot{v}) = \frac{(3u - 2v)\dot{u} - u\dot{v}}{u(u - v)},$$
(12)

$$\sigma(u,v,\dot{u},\dot{v}) = \frac{u\dot{v} - v\dot{u}}{3u^2 - 3uv} \,. \tag{13}$$

Equation (8) can be integrated by quadratures, that is, $\int \frac{du}{\sqrt{2u^{-1}+I_0}} = t - t_0$, where I_0 is a constant of integration and a conservation law for the dynamical system (8)–(9). That

is, $I_0 = \dot{u}^2 - 2u^{-1}$. Thus, by writing u = u(t) in (9) we obtain a linear equation of the form $\ddot{v} + \omega(t)v = 0$, $\omega(t) = -2u(t)^{-3}$. This linear second-order equation is known as the time-dependent oscillator [46] and it is a maximally symmetric equation. Moreover, it admits as conservation law the Lewis invariant given by the expression [47–49]

$$\Phi = \frac{1}{2} \left(\left(y \dot{v} - \dot{y} v \right)^2 + \left(\frac{v}{y} \right)^2 \right), \tag{14}$$

where y(t) is any solution of the Ermakov–Pinney differential Equation [50], that is,

$$\ddot{y} + \omega(t)y = \frac{1}{y^3}.$$
(15)

Conservation law (14) is a new conservation law found for the Szekeres system and has not been derived before. It is really a point of interest that the Lewis invariant and the Ermakov–Pinney equation appears in inhomogeneous cosmology. Previously, the Ermakov–Pinney equation has appeared and in other gravitational such in scalar tensor theories and in modified theories of gravity [51,52].

It is easy to observe that the Szekeres Equations (8) and (9) can be derived by the variation of the point-like Lagrangian [38]

$$L(u, \dot{u}, v, \dot{v}) = \dot{u}\dot{v} - \frac{v}{u^2}$$
(16)

with Hamiltonian function

$$\mathcal{H} = p_u p_v + \frac{v}{u^2} \equiv h(u, v, p_u, p_v), \tag{17}$$

where

$$\dot{u} = p_v , \ \dot{v} = p_u \tag{18}$$

and

$$\dot{p}_v = -\frac{1}{u^2}$$
, $\dot{p}_v = 2\frac{v}{u^3}$. (19)

That is a Hamiltonian description for the Szekeres system, for which we can see that (17) is a conservation law with $h(u, v, p_u, p_v) = const.$, which corresponds to the "energy" of the Hamiltonian system, because Equations (8) and (9) are autonomous. In terms of the momentum the conservation law I_0 becomes $I_0 = p_v^2 - 2u^{-1}$, which is in involution and independent of the Hamiltonian h. Hence, as has found before the Szekeres system is Liouville integrable [38]. The conservation law I_0 is related with a generalized symmetry which generates a constant transformation for the dynamical system in which the Action Integral for the Szekeres system remains invariant, that is, I_0 is a Noetherian conservation law. That is not true for the Lewis invariant (14).

The application of Noether's theorem for point and contact transformations of the Hamiltonian system (17), (18) and (19) as was found in [38] does not provide additional conservation laws. However, the existence of conservation laws for the trajectories with initial conditions h = 0 has been investigated before. When h = 0, the trajectories of the Szekeres system can be seen as "null-like" trajectories which are conformally invariant, see the discussion in [53–55].

Therefore, when $h \equiv 0$, we find that the Szekeres system (8), (9) admits the additional conservation laws

$$I_1 = u^2 \dot{v} \text{ or } I_1 = u^2 p_u$$
, (20)

$$I_2 = \frac{u}{v} \text{ or } I_2 = \frac{p_v}{v} , \qquad (21)$$

and

$$I_3 = 2u\dot{v} + v\dot{u} \text{ or } I_3 = 2up_u + vp_v.$$
 (22)

These conservation laws can be derived by the application of Noether's theorem for a conformally related Lagrangian of function (16), while they are generated by conformal transformations of the two-dimensional flat space which defines the kinetic energy for the Lagrangian function (16).

Consider now the conformal transformation $dt = u^2 d\tau$. The conformally equivalent Lagrangian of (16) is given by

$$\bar{L}\left(u,\frac{du}{d\tau},v,\frac{d\dot{u}}{d\tau}\right) = \frac{1}{u^2} \left(\frac{du}{d\tau}\right) \left(\frac{dv}{d\tau}\right) - v,$$
(23)

where now the Szekeres system becomes

$$\ddot{u} - \frac{2}{u}\dot{u}^2 + u^2 = 0, \ \ddot{v} = 0.$$
(24)

Under the conformal transformation the conservation law \bar{I}_1 becomes $\bar{I}_1 = \dot{v}$. Hence, with the use of the constraint equation $\frac{1}{u^2} \left(\frac{du}{d\tau}\right) \left(\frac{dv}{d\tau}\right) + v = 0$, the closed-form solution of the Szekeres system is expressed as

$$u(\tau) = 2\left((\tau - \tau_0)^2 + u_1\right)^{-1}, \ v(t) = \bar{I}_1(\tau - \tau_0).$$
(25)

Hence, we can write the closed-form solution for the original variables

$$\rho(\tau) = \frac{3\left((\tau - \tau_0)^2 + u_1\right)^3}{2\left(I_1(\tau - \tau_0)\left((\tau - \tau_0)^2 + u_1\right) - 2\right)},$$
(26)

$$E(\tau) = \frac{I_1(\tau - \tau_0) \left((\tau - \tau_0)^2 + u_1 \right)^4}{8 \left(I_1(\tau - \tau_0) \left((\tau - \tau_0)^2 + u_1 \right) - 2 \right)},$$
(27)

$$\theta(\tau) = \frac{\left((\tau - \tau_0)^2 + u_1\right) \left(I_1\left((\tau - \tau_0)\left((\tau - \tau_0)^2 + u_1\right)^3 - 4\right) - 12(\tau - \tau_0)\right)}{8 - I_1\left((\tau - \tau_0)^2 + u_1\right)^3}, \quad (28)$$

$$\sigma(\tau) = \frac{I_1 \left((\tau - \tau_0)^2 + u_1 \right) \left(8 + (\tau - \tau_0) \left((\tau - \tau_0)^2 + u_1 \right) \right)}{6 \left(8 - I_1 \left((\tau - \tau_0)^2 + u_1 \right)^3 \right)}.$$
 (29)

In a similar way closed-form solutions can be found for other conformal time $dt \rightarrow d\bar{\tau}$, by using the remaining vector fields. However, they are the same solutions expressed in different coordinates. Moreover, it is important to mention here that the constants of integration and the conservation laws, are constants with respect to the derivative in time, that is, they are functions of the original coordinates of the spacetime, for instance $t_0 = t_0(x, y, z)$ and t_0 is an essential integration parameter that is why we have not omitted it.

We continue our analysis by performing a canonical quantization for the Hamiltonian (17) to write the Wheeler–DeWitt equation for the Szekeres system, while we use the conservation laws related with point and contact symmetries to solve the Wheeler–DeWitt equation and write the wave function.

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3. The Wheeler–Dewitt Equation

In terms of the 3 + 1 decomposition notation of General Relativity, the Wheeler–DeWitt equation follows from the Hamiltonian constraint of the field equations [3]. The Wheeler–DeWitt equation is not a single differential equation, but it defines a family of equations where at every point of the 3-dimensional hypersurfaces a unique equation is defined. However, in the case of the minisuperspace approximation the infinite degrees of freedom of the superspace reduce to a finite number. Hence, instead of having an equation for each point of the hypersurface, there follows a unique equation for all of the points [4].

The Szekeres system (2)–(5) does not admit a minisuperspace description. However, through the dynamical variables it can be written as a two-dimensional Hamiltonian system with constraint (17). By using that property we are able to study the quantization of the Szekeres system. Specifically we perform a canonical quantization by promoting the Poisson brackets to commutators and the variables on the phase space into operators $x^i \rightarrow \hat{x}^i = x^i$, $p_i \rightarrow \hat{p}_i = i \frac{\partial}{\partial x^i}$. Thus, from the Hamiltonian (17) there follows the time-independent Schrödinger equation [41]

$$\left(-\frac{\partial}{\partial u\partial v} + \frac{v}{u^2} - h\right)\Psi(u, v) = 0.$$
(30)

At this point we can use the conservation laws to define operators which keep invariant the Wheeler–DeWitt Equation (30). For arbitrary value of h from I_0 , there follows the quantum operator

$$\left(\frac{\partial}{\partial v^2} + \frac{2}{u} + I_0\right)\Psi(u, v) = 0, \qquad (31)$$

while, when h = 0, the additional operators

$$\left(u^2\frac{\partial}{\partial u}+iI_1\right)\Psi(u,v)=0\,,\tag{32}$$

$$\left(\frac{1}{v}\frac{\partial}{\partial v} + iI_2\right)\Psi(u,v) = 0, \qquad (33)$$

$$\left(2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + iI_3\right)\Psi(u,v) = 0 \tag{34}$$

exist.

With the use of one of these differential operators we can construct a solution for the Wheeler–DeWitt Equation (30). These solutions are called similarity solutions because the differential operators are related with Lie symmetries for the differential Equation (30).

In [41], the differential operator (31) was applied, which provides the wavefunction

$$\Psi_A(h, I_0, u, v) = \frac{\sqrt{u}}{\sqrt{2 + I_0 u}} \Big(\Psi_A^1 \cos f(u, v) + \Psi_A^2 \sin f(u, v) \Big),$$
(35)

where $f(u,v) = (I_0^{3/2}\sqrt{u})^{-1} ((hu + I_0 v)\sqrt{2I_0 + I_0^2 u} - 2h\sqrt{u} \operatorname{arcsinh}(\sqrt{\frac{I_0 u}{2}}))$, for $I_0 \neq 0$, or $f(u,v) = \sqrt{2}(hu^2 + 3v)u^{-\frac{1}{2}}$ for $I_0 = 0$. Coefficients $\Psi^{1/2}$ are constants of integration. In

or $f(u, v) = \frac{\sqrt{2}}{3} (hu^2 + 3v) u^{-\frac{1}{2}}$, for $I_0 = 0$. Coefficients $\Psi_A^{1,2}$ are constants of integration. In a similar way we can construct similarity solutions by using the other differential operators. For more details on the properties of the wavefunction (35) we refer the reader in [41].

For h = 0, the wave function, $\Psi(u, v)$, which satisfies the differential operator (32) is

$$\Psi_B(I_1, u, v) = \Psi_B^1 \exp\left(-\frac{i}{2I_1}\left(v^2 + 2\frac{(I_1)^2}{u}\right)\right),$$
(36)

while from (33) we find that

$$\Psi_C(I_2, u, v) = \Psi_C^1 \exp\left(\frac{i}{2I_2} \left(\frac{2}{u} + (I_2)^2 v^2\right)\right).$$
(37)

Furthermore, from (34) there follows the wave function

$$\Psi_D(\beta, u, v) = \left(v^2 u\right)^{\frac{\beta}{4}} \left(\Psi_D^1 J_{\frac{\beta}{2}}\left(v\sqrt{\frac{2}{u}}\right) + \Psi_D^2 Y_{\frac{\beta}{2}}\left(v\sqrt{\frac{2}{u}}\right)\right)$$
(38)

in which $J_{\alpha}(u, v)$ and $Y_{\alpha}(u, v)$ are the Bessel functions of the first and of the second kinds, respectively, and $I_3 = -i\beta$.

In the limit for which $\sqrt{\frac{2}{u}}v \to \infty$, the wave function (38) is approximated by the functional form

$$\hat{\Psi}_{D}(\beta, u, v) = v^{\frac{\beta-1}{2}} u^{\frac{\beta+1}{4}} \left(\hat{\Psi}_{D}^{1} \exp\left(i\left(\sqrt{\frac{2}{u}}v - (\beta+1)\frac{\pi}{4}\right)\right) + \hat{\Psi}_{D}^{2} \exp\left(-i\left(\sqrt{\frac{2}{u}}v - (\beta+1)\frac{\pi}{4}\right)\right) \right),$$
(39)

where $\hat{\Psi}_D^1$, $\hat{\Psi}_D^2$ are constants. Function $\hat{\Psi}_D(\beta, u, v)$ describes oscillations of the polar form $\Psi(u, v) = \Omega(u, v)e^{iS(u,v)}$ with amplitude $\Omega(u, v) = v^{\frac{\beta-1}{2}}u^{\frac{\beta+1}{4}}$ and radial argument $S(u, v) = \left(\sqrt{\frac{2}{u}}v - (\beta+1)\frac{\pi}{4}\right).$

4. Quantum Potential

In Bohmian quantum theory, the main difference from the classical theory is the quantum Hamilton–Jacobi equation which for a wave function expressed in the Madelung representation $\Psi(\mathbf{y}) = \Omega(\mathbf{y})e^{iS(\mathbf{y})}$ is defined as

$$\frac{1}{2}G^{AB}\partial_A S(\mathbf{y})\partial_B S(\mathbf{y}) + V(\mathbf{y}) + h + Q_V(\mathbf{y}) = 0,$$
(40)

where the term $Q_V(\mathbf{y})$ is known as the quantum potential and is related with the amplitude of the wave function

$$Q_V(\mathbf{y}) = -\frac{\Box \Omega(\mathbf{y})}{2\Omega(\mathbf{y})} \tag{41}$$

and \Box is the Laplacian of the time-independent Schrödinger equation, that is, for our model, $\Box = \frac{\partial}{\partial u \partial v}$. The radial argument play the role of the action, so that the canonical momentum is given as $p_A = \frac{\partial S}{\partial y^A}$. In the WKB approximation the quantum potential it is neglected and the classical limit is recovered. The Madelung representation is an alternative way to write the Schrödinger equation in a real and complex imaginary part [56].

We continue with the calculation of the quantum potential for the wave functions which were found above.

For the wave function $\Psi_A(u, v)$ in [41] it was found that there is no nonzero corresponding quantum potential. Wave functions $\Psi_B(u, v)$, $\Psi_C(u, v)$ are already written in polar form with constant amplitudes. Hence the quantum potential related with these wave functions is zero.

However, from the wave function $\Psi_D(u, v)$ and in the limit $\sqrt{\frac{2}{u}}v \to \infty$ which is expressed in polar form from the amplitude $\Omega(u, v) = v^{\frac{\gamma-1}{2}}u^{\frac{\gamma+1}{4}}$ the quantum potential term is

$$Q_V(u,v) = \frac{Q_0}{uv}$$
 with $Q_0 = -\frac{(\beta^2 - 1)}{8}$. (42)

Therefore, for the trajectories with h = 0, a quantum correction term exists. Below, we continue our analysis with the study of the effects of the quantum corrections on the original variables and how the Szekeres system is modified.

5. The Modified Szekeres System

With the use of the quantum potential the Hamiltonian equivalent of the Szekeres system (17) is modified to be

$$\mathcal{H} = p_u p_v + \frac{v}{u^2} + \frac{Q_0}{uv} \equiv 0.$$
(43)

Thus, the equations of motion are

$$\dot{u} = p_v , \ \dot{v} = p_u \tag{44}$$

and

$$\dot{p}_v = -\frac{1}{u^2} + \frac{Q_0}{uv^2}$$
, $\dot{p}_v = 2\frac{v}{u^3} + \frac{Q_0}{u^2v}$. (45)

Therefore with the use of the inverse transformation from the variables $\{u, v, \dot{u}, \dot{v}\} \rightarrow \{\rho, E, \theta, \sigma\}$ as is given by expressions (11), (12) and (13) we find that the only equation of the Szekeres system which is modified is Equation (3) which becomes

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}\rho + \frac{Q_0}{72}\frac{\rho^2}{E^2}(\rho + 6E)^{\frac{4}{3}} = 0.$$
(46)

Because we are working in the limit where $\sqrt{\frac{2}{u}}v \rightarrow \infty$, it follows that $\sqrt{\frac{2}{u}}v = -\frac{6\sqrt{2}E}{\rho}(\frac{\rho}{6}+E)^{-\frac{1}{6}} \rightarrow \infty$. Hence the last term of (46) is approximated as $\frac{Q_0}{72}\frac{\rho^2}{E^2}(\rho+6E)^{\frac{4}{3}} \rightarrow \varepsilon \frac{Q_0}{2}(\rho+6E)$, where ε is an infinitesimal parameter such that $\varepsilon^{-1} \rightarrow \infty$. Thus, Equation (46) takes the form

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}\rho + \frac{\varepsilon Q_0}{2}(\rho + 6E) = 0.$$
(47)

We proceed with the study of the evolution of the modified Szekeres system and we compare our results with that of the original system.

We use the new dimensionless variables in the θ -normalization

$$\Omega_m = \frac{3\rho}{\theta^2} , \ \Sigma = \frac{\sigma}{\theta} , \ \alpha = \frac{E}{\theta^2} , \ \Omega_R = \frac{3}{2} \frac{{}^{(3)}R}{\theta^2}.$$
(48)

The modified Szekeres system is written as follows

$$\Omega'_{m} = \frac{1}{3}\Omega_{m} \Big(36\Sigma^{2} + \Omega_{m} - 6\Omega_{\Lambda} - 1 + \varepsilon Q_{0}(\Omega_{m} + 18\alpha) \Big), \tag{49}$$

$$\Sigma' = -\alpha + \frac{1}{6}\Sigma(6\Sigma(1+6\Sigma) - 2 + \Omega_m - 6\Omega_\Lambda) + \frac{\varepsilon}{6}Q_0\Sigma(\Omega_m + 18\alpha),$$
(50)

$$\alpha' = \frac{1}{6}(-\Sigma\Omega_m + 2\alpha(9\Sigma(4\Sigma - 1) + \Omega_m - 6\Omega_\Lambda - 1)) + \frac{\varepsilon}{3}Q_0\alpha(\Omega_m + 18\alpha^2), \quad (51)$$

where the algebraic Equation (6) takes the form

$$\Omega_R = 9\Sigma^2 + \Omega_m - 1 \tag{52}$$

in which prime "'" denotes total derivative with respect to the new independent variable $d\tau = \theta dt$. The dynamics for the latter system with $\varepsilon = 0$, has been studied in [23]. However, from all the stationary points we are interested in the asymptotic solutions which satisfy the constraint condition h = 0. Hence from (17) and (11)–(13) with the use of the dimensional variables (48) the constraint equation is

$$18\alpha^2 - \Sigma(1+3\Sigma)\Omega_m + \alpha(2-6\Sigma(1+6\Sigma)+\Omega_m) = 0.$$
(53)

With the use of the constraint Equation (53) the three-dimensional dynamical system (49)–(51) can be reduced by one dimension, that is, every stationary point *P* of the system is defined in the two-dimensional space of variables $P = (\Sigma(P), \alpha(P))$ and describes an exact solution with expansion rate

$$\frac{\dot{\theta}}{\theta^2} = -\frac{1}{6} \Big(2 + 36\Sigma^2 + \Omega_m + \varepsilon Q_0 (\Omega_m + 18\alpha) \Big) = -q(\Sigma, \alpha, \Omega_m).$$
(54)

At every point *P* the expansion rate is found to be $\theta(t) = \frac{1}{q(t-t_0)}$, for $q(\Sigma(P), \alpha(P), \Omega_m(P)) \neq 0$ or $\theta(t) = const.$ for $q(\Sigma(P), \alpha(P), \Omega_m(P)) = 0$.

5.1. Stationary Points of the Szekeres System

We study the dynamics of the classic Szekeres system (50)–(53) without the quantum potential term. The stationary points $P = (\Omega_m(P), \Sigma(P), \alpha(P))$ of the Szekeres system have already been derived in [23]. However, because now we impose the constraint (53), we present the analysis in detail.

From the constraint Equation (53) we see that $\alpha = a(\Omega_m, \Sigma)$ as follows

$$\alpha_{+}(\Omega_{m},\Sigma) = \frac{1}{36} \left(36\Sigma^{2} + 6\Sigma - 2 - \Omega_{m} + \sqrt{72\Sigma(1+3\Sigma)\Omega_{m} + (2 - 6\Sigma(1+6\Sigma) + \Omega_{m})^{2}} \right)$$
(55)

$$\alpha_{-}(\Omega_{m},\Sigma) = \frac{1}{36} \left(36\Sigma^{2} + 6\Sigma - 2 - \Omega_{m} - \sqrt{72\Sigma(1+3\Sigma)\Omega_{m} + (2 - 6\Sigma(1+6\Sigma) + \Omega_{m})^{2}} \right)$$
(56)

5.1.1. Branch α_+

With the use of $\alpha = \alpha_+(\Omega_m, \Sigma)$ we obtain a two-dimensional system which admits the following stationary points $A = (\Omega_m(A), \Sigma(A))$,

$$A_1 = (1,0)$$
, $A_2 = (0,0)$, $A_3 = \left(0, -\frac{1}{3}\right)$, $A_4 = \left(0, \frac{1}{3}\right)$, (57)

$$A_5 = \left(0, \frac{1}{6}\right), \ A_6 = \left(-3, -\frac{1}{3}\right), \ A_7 = \left(-8, -\frac{1}{2}\right).$$
(58)

Point A_1 describes a spatially flat FLRW (-like) universe where the dust fluid dominates, point A_2 describes the asymptotic Milne (-like) universe. The asymptotic solutions at the points A_3 , A_4 describe Bianchi I (-like) spacetimes and specifically Kasner (-like) universes, while the spacetime at point A_5 is that of Kantowski-Sachs (-like) universe. Points A_6 , A_7 are not physically accepted because $\Omega_m(A_6) < 0$ and $\Omega_m(A_7) < 0$.

In order to infer for the stability of the stationary points we determine the eigenvalues of the linearized systems. The eigenvalues $e(A) = (e_1(A), e_2(A))$ are, $e(A_1) = (-\frac{1}{2}, \frac{1}{3})$, $e(A_2) = (-\frac{1}{3}, -\frac{1}{3})$, $e(A_3) = (1, 1)$, $e(A_4) = (\frac{7}{3}, 1)$ and $e(A_5) = (0, \frac{1}{2})$. Therefore we conclude that A_2 is a stable point, A_1 is a saddle point while points A_3 , A_4 and A_5 are sources. Hence, the unique attractor on that branch is the Milne (-like) universe.

5.1.2. Branch α_{-}

On the second branch for which $\alpha = \alpha_{-}(\Omega_{m}, \Sigma)$, the stationary points of the field equations are

$$B_1 = \left(0, -\frac{1}{3}\right), B_2 = \left(0, \frac{1}{6}\right) \text{ and } B_3 = \left(-8, -\frac{1}{2}\right).$$
 (59)

Points B_1 and B_2 have the same physical properties as A_3 and A_5 , respectively, while B_3 is not physically accepted. As far as the stability of the stationary points is concerned, we calculate the eigenvalues of the linearized system around the stationary points from which we get $e(B_1) = (2, 1)$ and $e(B_2) = \left(-\frac{1}{2}, 0\right)$, and infer that B_1 is a source, while for point B_2 in order to infer for the stability we may apply the centre manifold theorem.

The latter is not necessary to be true because we can always reduce the three-dimensional system (49)–(51) into a two-dimensional system for other variables, for the set { Ω_m , α } or { Σ , α } and we can infer the stability properties there. We performed that analysis and found that point B_2 is a saddle point.

Therefore the unique attractor for the Szekeres system with the constraint condition (53) is the Milne universe. In Figure 1 we present the phase-space portrait for the two-dimensional dynamical system (49) and (50) for the two branches α_+ and α_- .



Figure 1. Phase-space portraits for the two-dimensional dynamical system (49) and (50) for $\alpha = \alpha_+(\Omega_m, \Sigma)$ (**left** figure) and $\alpha = \alpha_-(\Omega_m, \Sigma)$ (**right** figure).

5.2. Stationary Points of the Modified Szekeres System

We perform the same analysis for the determination of the stationary points for the modified Szekeres system with the quantum potential term by assuming that $\varepsilon Q_0 \neq 0$.

5.2.1. Branch α_+

For the α_+ branch and for small values of εQ_0 , the stationary points are calculated

$$\bar{A}_1 = (1 - \varepsilon Q_0, 0) , \ \bar{A}_2 = (0, 0) , \ \bar{A}_3 = \left(0, -\frac{1}{3}\right) , \ \bar{A}_4 = \left(0, \frac{1}{3}(1 - \varepsilon Q_0)\right) ,$$
 (60)

$$\bar{A}_5 = \left(0, \frac{1}{6}\right), \ \bar{A}_6 = \left(-3, -\frac{1}{3}\right), \ \bar{A}_7 = \left(-8 + 4\varepsilon Q_0, -\frac{1}{2} + \frac{\varepsilon Q_0}{18}\right).$$
 (61)

Points \bar{A}_6 , \bar{A}_7 are not physically accepted, while \bar{A}_1 cannot be accepted because at this stationary solution the condition $\sqrt{\frac{2}{u}}v \to \infty$ is violated. Therefore, the only stationary solution which is modified is the asymptotic solution of point \bar{A}_4 . The spatial curvature at the point \bar{A}_4 is derived to be $\Omega_R(\bar{A}_4) = -\varepsilon Q_0$, from which we infer that the solution at the point describes a Kantowski-Sachs (-like) universe or a Bianchi III (-like) universe for $\varepsilon Q_0 < 0$. However, because $\varepsilon Q_0 \to 0$, the solution at the point can be seen as the limit of a Kasner (-like) solution.

For the asymptotic solution at the points \bar{A}_4 we calculate $q(\bar{A}_4) = -1 + \frac{4}{3} \epsilon Q_0$. Hence

$$\frac{\dot{\theta}}{\theta^2} = -1 + \frac{4}{3}\varepsilon Q_0 , \qquad (62)$$

that is,

$$\theta(t) = \frac{1}{1 - \frac{4}{3}\varepsilon Q_0} \frac{1}{(t - t_0)} \simeq \left(1 + \frac{4}{3}\varepsilon Q_0\right) \frac{1}{(t - t_0)}$$
(63)

while the anisotropic index σ is

$$\sigma(t) \simeq \frac{1}{3} \left(1 + \frac{\varepsilon Q_0}{3} \right) \frac{1}{(t - t_0)}.$$
 (64)

Therefore, the power law indices of the scale factors follow from the algebraic system $p_1 + 2p_2 = \left(1 + \frac{4}{3}\varepsilon Q_0\right), \ \frac{1}{3}(p_1 - p_2)^2 = \frac{1}{3}\left(1 + \frac{\varepsilon Q_0}{3}\right)$ which gives

$$(p_1, p_2)_A = \left(-\frac{1}{3} + \frac{\varepsilon Q_0}{3}, \frac{2}{3} + \frac{\varepsilon Q_0}{2}\right),$$
 (65)

$$(p_1, p_2)_B = \left(1 + \frac{5\varepsilon Q_0}{9}, \frac{7}{18}\varepsilon Q_0\right).$$
 (66)

It is clear that the asymptotic solution at point \bar{A}_4 is not that of vacuum and that a nonzero energy momentum tensor corresponds to that solution. In the 1 + 3 decomposition, and for the comoving observer $u^{\mu} = \delta_t^{\mu}$, the energy momentum tensor has components

$$T_{\mu\nu} = \mu u_{\mu} u_{\nu} + p h_{\mu\nu} + \pi_{\mu\nu}, \tag{67}$$

where $\mu = T_{\mu\nu}u^{\mu}u^{\nu}$, $p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu}$ and $\pi_{\mu\nu} = T_{\kappa\lambda}\left(h^{\kappa}_{\mu}h^{\lambda}_{\nu} - \frac{1}{3}h^{\kappa\lambda}h_{\mu\nu}\right)$.

Hence, for the power indices $(p_1, p_2)_A$ we calculate

$$\mu_A = \frac{7}{9} \frac{\varepsilon Q_0}{(t - t_0)^2} , \ p_A = -\frac{1}{9} \frac{\varepsilon Q_0}{(t - t_0)^2} , \tag{68}$$

$${}_{A}\pi^{\mu}_{\nu} = diag\left(0, -\frac{26}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}, \frac{10}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}, \frac{10}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}\right).$$
(69)

Furthermore, for the set of the power indices $(p_1, p_2)_B$ we calculate

$$\mu_B = \frac{7}{9} \frac{\varepsilon Q_0}{\left(t - t_0\right)^2} , \ p_B = -\frac{1}{9} \frac{\varepsilon Q_0}{\left(t - t_0\right)^2} , \tag{70}$$

$${}_{B}\pi^{\mu}_{\nu} = diag\left(0, \frac{22}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}, -\frac{14}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}, \frac{14}{27}\frac{\varepsilon Q_{0}}{\left(t-t_{0}\right)^{2}}\right).$$
(71)

Recall that in the later solutions $\dot{t}_0 = 0$ holds, but t_0 is a function of the space variables of the spacetime.

We conclude that a nonzero energy momentum tensor is introduced by quantum corrections in Bohmian mechanics for the Szekeres system. Hence, the Szekeres system admits semi-classical trajectories which do not remain silent in the quantum level.

As far as the stability of the stationary points is concerned, that remains invariant and it is the same with that of the Szekeres system studied above. 5.2.2. Branch α_{-}

For $\alpha = \alpha_{-}(\Omega_m, \Sigma)$ we calculate the stationary points

$$\bar{B}_1 = (1 - Q_0, 0)$$
, $\bar{B}_2 = \left(0, -\frac{1}{3}\right)$, $\bar{B}_3 = \left(0, \frac{1}{6}\right)$, (72)

$$\bar{B}_4 = \left(0, \frac{1}{3}(1 - \varepsilon Q_0)\right), \ \bar{B}_5 = \left(-8 + 4\varepsilon Q_0, -\frac{1}{2} + \frac{\varepsilon Q_0}{18}\right).$$
(73)

The stationary point \bar{B}_5 is not physically accepted, while point \bar{B}_1 violates the condition $\sqrt{\frac{2}{u}}v \to \infty$ and we do not consider it. Points \bar{B}_2 and \bar{B}_3 have the same physical properties and stability with points B_2 , B_3 respectively. \bar{B}_4 is a new point which has the physical properties of point \bar{A}_4 .

Therefore in the branch $\alpha = \alpha_{-}(\Omega_m, \Sigma)$ because of the quantum potential a new stationary point exists. The eigenvalues of the linearized system around the point \bar{A}_4 are found to be $e(\bar{A}_4) = \left(1 - \frac{4}{3}\varepsilon Q_0, \frac{2}{3} - Q_0\varepsilon\right)$ from which we infer that point \bar{A}_4 is a source and the asymptotic solution at the point is always unstable.

6. Conclusions

In this work we studied the quantization process for the Szekeres system and the effects of the quantum corrections in inhomogeneous cosmological models as they are described by the De Broglie–Bohm theory for quantum mechanics. We make use of previous results and we wrote the Szekeres system in its Hamiltonian equivalent. We calculated the conservation laws for the classical system and we derived previous results while we were able to determine new conservation laws which were not found before. One of these conservation laws is the Lewis invariant which is an important adiabatic invariant for the oscillator with many applications in quantum mechanics. The remaining new conservation laws which we derived exist when the Hamiltonian system is conformally invariant, that is, when the conservation law of the "energy" is identical to zero. That means that these new conservation laws, which are constructed by conformal symmetries of the kinetic metric for the Hamiltonian system, exist for a specific set of trajectories for the classical Szekeres system.

These new conservation laws applied to define differential operators are necessary to quantize the Szekeres system. We found new similarity solutions for the wave function of the Szekeres system. From these new wave functions we were able to constructed a nonzero quantum potential by applying the approach of Bohmian mechanics. In order to understand the effects of the quantum potential term in the original Szekeres system we wrote the modified Szekeres system and we studied the dynamics and the asymptotic solutions, for the trajectories with the initial condition the "energy" of the Hamiltonian equivalent system to be zero.

The main result of this analysis is that because of the quantum correction term, the dynamical evolution of the Szekeres system is modified, such that new asymptotic solutions exist which describe approximately Bianchi I spacetimes which modify the Kasner solutions of the Szekeres systems. These new Bianchi I solutions correspond to exact inhomogeneous solutions with a nonzero anisotropic fluid component with nonzero pressure and stress tensor terms. We conclude that quantum corrections can remove the "silent" property of the Szekeres universe, making the Szekeres model more useful in cosmological studies. Therefore, we show that quantum corrections can be seen as a mechanism to introduce negative pressure term in the cosmological fluid which is interesting since it may have applications for the description of the mechanism which starts the inflationary epoch.

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References

- 1. Bojowald, M. Quantum cosmology: A review. Rep. Prog. Phys. 2015, 78, 023901
- 2. Gielen, S.; Sindoni, L. Quantum cosmology from group field theory condensates: A review. Sigma 2016, 12, 82. [CrossRef]
- 3. DeWitt, B.S. Quantum Theory of Gravity. I. The Canonical Theory. Phys. Rev. 1967, 160, 1113. [CrossRef]
- 4. Hartle, J.B.; Hawking, S.W. Wave function of the Universe. Phys. Rev. D 1983, 28, 2960. [CrossRef]
- 5. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I. *Phys. Rev.* **1952**, *85*, 166. [CrossRef]
- 6. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II. *Phys. Rev.* **1952**, *85*, 180. [CrossRef]
- 7. Pinto-Neto, P.; Colistete, R., Jr. Graceful exit from inflation using quantum cosmology. Phys. Lett. A 2001, 290, 219. [CrossRef]
- Terzis, P.A.; Dimakis, N.; Christodoulakis, T. Noether analysis of Scalar-Tensor Cosmology. *Phys. Rev. D* 2014, 90, 123543. [CrossRef]
- 9. Falciano, F.T.; Pinto-Neto, N.; Struyve, W. Wheeler-DeWitt quantization and singularities. *Phys. Rev. D* 2015, *91*, 043524. [CrossRef]
- 10. Krasiński, A. Inhomogeneous Cosmological Models; Cambridge U.P.: Cambridge, UK, 1997.
- 11. Easther, R.; Price, L.C.; Rasero, J. Inflating an inhomogeneous universe. JCAP 2014, 08, 041. [CrossRef]
- 12. Buchert, T.; Obadia, N. Effective inhomogeneous inflation: Curvature inhomogeneities of the Einstein vacuum. *Class. Quantum Grav.* **2011**, *28*, 162002. [CrossRef]
- 13. Clough, K.; Lim, E.A.; DiNunno, B.S.; Fischler, W.; Flauger, R.; Pabah, S. Robustness of inflation to inhomogeneous initial conditions. *JCAP* 2017, *9*, 025. [CrossRef]
- 14. Marsh, M.C.D.; Barrow, J.D.; Ganguly, C. Inhomogeneous Initial Data and Small-Field Inflation. JCAP 2018, 05, 026. [CrossRef]
- 15. Chiba, T. Quintessence, the gravitational constant, and gravity. Phys. Rev. D 1999, 59, 083508. [CrossRef]
- 16. Saulder, C.; Mieske, S.; van Kampen, E.; Zeilinger, W.W. Hubble flow variations as a test for inhomogeneous cosmology. *A&A* **2019**, *622*, A83.
- 17. Clarkson, C.; Regis, M. The cosmic microwave background in an inhomogeneous universe. JCAP 2011, 02, 013. [CrossRef]
- 18. Bolejko, K. The Szekeres Swiss Cheese model and the CMB observations. Gen. Rel. Grav. 2009, 41, 1737. [CrossRef]
- 19. Bolejko, K.; Korzyński, M. Inhomogeneous cosmology and backreaction: Current status and future prospects. *IJMPD* **2017**, *26*, 1730011. [CrossRef]
- 20. Romano, A.E. Corrections to the apparent value of the cosmological constant due to local inhomogeneities. *IJMPD* **2012**, *21*, 1250085. [CrossRef]
- 21. Szekeres, P. A class of inhomogeneous cosmological models. Commun. Math. Phys. 1975, 41, 55. [CrossRef]
- 22. Mustapha, N.; Ellis, G.F.R.; van Elst, H.; Marklund, M. Partially locally rotationally symmetric perfect fluid cosmologies. *Class. Quantum Gravit.* **2000**, *17*, 3135. [CrossRef]
- 23. Bruni, M.; Matarrese, S.; Ornella, P. Dynamics of Silent Universes. Astro. J. 1995, 445, 958. [CrossRef]
- Bolejko, K.; Celerier, M.-N. Szekeres Swiss-cheese model and supernova observations. *Phys. Rev. D* 2010, *82*, 103510. [CrossRef]
 Ishak, M.; Peel, A. Growth of structure in the Szekeres class-II inhomogeneous cosmological models and the matter-dominated era. *Phys. Rev. D* 2012, *85*, 083502. [CrossRef]
- 26. Vrba, D.; Svitek, O. Modelling inhomogeneity in Szekeres spacetime. Gen. Relativ. Grav. 2014, 46, 1808. [CrossRef]
- 27. Bonnor, W.B. Evolution of Inhomogeneous Cosmological Models. MNRAS 1974, 167, 55. [CrossRef]
- 28. Bonnor, W.B. Evolution of Szekeres's Cosmological Models. MNRAS 1976, 175, 85. [CrossRef]
- 29. Bolejko, K.; Stoeger, W.R. Conditions for spontaneous homogenization of the Universe. *Gen. Relat. Gravit.* **2010**, *42*, 2349. [CrossRef]
- 30. Szafron, D.A. Inhomogeneous cosmologies: New exact solutions and their evolution. J. Math. Phys. 1977, 18, 1673. [CrossRef]
- 31. Barrow, J.D.; Stein-Schabes, J. Effect of inflation on anisotropic. cosmologies Phys. Lett. A 1984, 103, 315. [CrossRef]
- 32. Goode, S.W.; Wainwright, J. Characterization of locally rotationally symmetric space-times. *Gen. Relativ. Gravit.* **1986**, *18*, 315. [CrossRef]
- 33. Tomimura, N. A new class of Szekeres inhomogeneous cosmologies with imperfect fluids. Nuovo C. B 1977, 42, 1. [CrossRef]
- 34. Barrow, J.D.; Paliathanasis, A. Szekeres universes with homogeneous scalar fields. EPJC 2018, 78, 767. [CrossRef]
- 35. Barrow, J.D.; Paliathanasis, A. Cyclic Szekeres Universes. EPJC 2019, 79, 379. [CrossRef]
- 36. Paliathanasis, A.; Leon, G.; Barrow, J.D. Einstein-aether theory in Weyl integrable geometry. EPJC 2020, 80, 731. [CrossRef]
- 37. Maartens, R.; Lesame, W.; Ellis, G.F.R. Consistency of dust solutions with divH = 0. Phys. Rev. D 1997, 55, 5219. [CrossRef]
- 38. Paliathanasis, A.; Leach, P.G.L. Symmetries and Singularities of the Szekeres System. Phys. Lett. A 2017, 381, 1277. [CrossRef]
- 39. Gierzkiewicz, A.; Golda, Z.A. On integrability of the Szekeres system. I. J. Nonl. Math. Phys. 2016, 24, 494. [CrossRef]
- 40. Llibre, J.; Valls, C. On the dynamics of the Szekeres system. Phys. Lett. A 2019, 383, 301. [CrossRef]
- Paliathanasis, A.; Zampeli, A.; Christodoulakis, T.; Mustafa, M.T. Quantization of the Szekeres System. *Class. Quantum Grav.* 2018, 35, 125005. [CrossRef]

- 42. Zampelia, A.; Paliathanasis, A. Quantization of inhomogeneous spacetimes with cosmological constant term. *arXiv* 2020, arXiv:2012.10814.
- 43. Christodoulakis, T.; Dimakis, N.; Terzis, P.A.; Doulis, G. Canonical quantization of the BTZ black hole using Noether symmetries. *Phys. Rev. D* 2014, *90*, 024052. [CrossRef]
- 44. Paliathanasis, A. Similarity solutions for the Wheeler–DeWitt equation in f(R)-cosmology. *EPJC* 2019, 79, 1031. [CrossRef]
- 45. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Barrow, J.D. Classical and quantum solutions in Brans-Dicke cosmology with a perfect fluid. *Phys. Rev. D* 2016, *93*, 043528. [CrossRef]
- 46. Lutzky, M. Noether's theorem and the time-dependent harmonic oscillator. Phys. Lett. A 1978, 68, 3. [CrossRef]
- 47. Lewis, H.R., Jr. Class of Exact Invariants for Classical and Quantum Time-Dependent Harmonic Oscillators. J. Math. Phys. 1968, 9, 1976. [CrossRef]
- 48. Kruskal, M. Asymptotic Theory of Hamiltonian and other Systems with all Solutions Nearly Periodic. *J. Math. Phys.* **1962**, *3*, 806. [CrossRef]
- 49. Leach, P.G.L. On a Generalization of the Lewis Invariant for the Time-Dependent Harmonic Oscillator. *Siam J. Appl. Math.* **1978**, 34, 496. [CrossRef]
- 50. Ray, J.R.; Reid, J.L. Ermakov systems, nonlinear superposition, and solutions of nonlinear equations of motion. *Phys. Lett. A* **1979**, 71, 317. [CrossRef]
- 51. Hawkins, R.M.; Lidsey, J.E. The Ermakov-Pinney Equation in Scalar Field Cosmologies. Phys. Rev. D 2002, 66, 023523. [CrossRef]
- 52. Tsamparlis, M.; Paliathanasis, A. Symmetries of differential equations in cosmology. Symmetry 2018, 10, 233. [CrossRef]
- 53. Tsamparlis, M.; Paliathanasis, A.; Basilakos, S.; Capozziello, S. Conformally related metrics and Lagrangians and their physical interpretation in cosmology. *Gen. Rel. Grav.* **2013**, *45*, 2003. [CrossRef]
- 54. Christodoulakis, T.; Dimakis, N.; Terzis, P.A. Lie point and variational symmetries in minisuperspace Einstein gravity. J. Phys. A Math. Theor. 2014, 47, 095202. [CrossRef]
- 55. Terzis, P.A.; Dimakis, N.; Christodoulakis, T.; Paliathanasis, A.; Tsamparlis, M. Variational contact symmetries of constrained Lagrangians. *J. Geom. Phys.* **2016**, *101*, 52. [CrossRef]
- 56. Bialynicki-Birula, I.; Cieplak, M.; Kaminski, J. Theory of Quanta; Oxford University Press: Oxford, UK, 1992.