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Energy–Momentum Pseudotensor and Superpotential for Generally Covariant Theories of Gravity of General Form

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Abstract: The current paper is devoted to the investigation of the general form of the energy–momentum pseudotensor (pEMT) and the corresponding superpotential for the wide class of theories. The only requirement for such a theory is the general covariance of the action without any restrictions on the order of derivatives of the independent variables in it or their transformation laws. As a result of the generalized Noether procedure, we obtain a recurrent chain of the equations, which allows one to express canonical pEMT as a divergence of the superpotential. The explicit expression for this superpotential is also given. We discuss the structure of the obtained expressions and the conditions for the derived pEMT conservation laws to be satisfied independently (fully or partially) by the equations of motion. Deformations of the superpotential form for theories with a change in the independent variables in action are also considered. We apply these results to some interesting particular cases: general relativity and its modifications, particularly mimetic gravity and Regge–Teitelboim embedding gravity.

Keywords: gravitational energy; energy–momentum pseudotensor; energy–momentum superpotential; noether theorem; general covariance; mimetic gravity; Regge–Teitelboim embedding gravity

1. Introduction

One of the traditional issues of General Relativity (GR) is the correct definition of the conserved quantities related to the space–time symmetries, especially the energy. This problem is not special for GR and in general remains for all generally covariant theories. One may point out two different origins of this problem. First of all, the corresponding densities of the integrals of motion—for example, EMT—are not tensors under the diffeomorphisms (for this reason EMT is often called a pseudotensor (pEMT)). Moreover, it is always possible to choose the specific reference frame where, at any given point the densities vanish. As a result, the total energy–momentum density is not well-defined. This observation was done back at the dawn of the GR (for historical review, see [1,2]). Despite the existence of the "covariantization" methods for such densities (see, for example, [3–5]), there is no widely accepted solution for this "non-localizability" problem. Perhaps, the main reason for this is the existence of the newer questions, which have appeared during the historical development of the GR extensions. The correct description of dark matter and dark energy, as well as the inflation theory, at some point, have become central in the context of the topic. The problem of non-localizability has remained a necessary evil, which, however, does not preclude using non-localizable superpotentials for the analysis of the several astrophysical problems [6–8].

Aside from the non-localizability problem, there is another obstacle to giving proper definitions for the energy and other the space–time integrals of motion. Namely, there are infinitely many

conserved pseudotensor currents, which correspond to the same integral of motion. Within the classical set-up, the solution usually comes down to the choice of the number of criteria, which fix the current form [9–12]. This situation is particularly interesting in quantum theory. As discussed, for example, in [13], only certain currents lead to the correct Ward identities. The currents obtained from standard Noether theorem [14] are suitable for such purposes. This fact opens new possibilities for investigations of these quantum identities in the wide class of theories with help of well-known classical results.

In the current paper, we study the generalized Noether procedure to obtain explicit formulae for pEMT and the corresponding superpotential for the maximally general theories of gravity with diffeomorphism-invariant action. In most GR modifications, the form of the independent variables and their transformation laws are often heavily constrained. Usually, the maximal tensor rank of these independent variables is equal to two, as well as the maximal order of derivatives in the action. However, some models violate these restrictions—for example, in [15–17]. To cover most general cases, we impose only one restriction on the action: its gravitational and matter parts must be invariant under the diffeomorphisms separately. There are no additional restrictions on the structure of transformation laws for the independent gravitational variables, or on the maximal (finite) order of their derivatives in the action.

In Section 2 we generalize the Noether procedure for such a large class by using analysis that is similar to that developed in [18]. It is worth noting, that the analysis made in [18] is limited to the most common case of action with no more than first derivatives of the fields with simple transformation law with respect to the gauge symmetry. As a result, we obtain an expression of the pEMT through superpotential and find its explicit form. Despite the lack of the antisymmetricity for the superpotential, we show that the conserving energy–momentum vector can always be expressed through the integral over an infinitely remote surface. The topology of the space–time in this case is assumed trivial. It is worth noting that the general properties of the superpotential in the gauge theories, particularly for the superpotential that corresponds to pEMT, was often discussed in the past (see, for example, [19]). However, in the current paper we use a much simpler analysis, which allows us to find an explicit expression for the superpotential.

The relations obtained in the Section 2 are used in Section 3 to perform the analysis of the superpotential structure. In the Section 4, we briefly discuss how the integrals of motion and the related quantities change in the theories with the smooth change in independent variables in action. The first half of Section 5 is devoted to applying the results obtained in the previous sections for the several well-known particular cases: General Relativity and Palatini formalism. In the second half of this section, we compute new superpotentials for the two modifications of gravity: theories with the disformal change [20] of variables in Einstein–Hilbert action, including mimetic gravity, and lastly the Regge–Teitelboim theory [21].

2. General Form of the Superpotential

Consider the following theory

$$S\left[y_a, \varphi_A\right] = \int dx\, L_{gr}(y_a, .., \partial_{\alpha_1..\alpha_{N_1}} y_a) + \int dx\, L_m(\varphi_A, .., \partial_{\beta_1..\beta_{N_2}} \varphi_A, y_a, .., \partial_{\alpha_1..\alpha_{N_3}} y_a), \qquad (1)$$

where the first term is the gravitational action S_{gr}, and the second one is the matter action S_m. The number of the space–time dimensions does not play any significant role in all following discussions, so it can be considered arbitrary. We will use the notation $\partial_{\alpha_k..\alpha_i}$ for $\partial_{\alpha_k}..\partial_{\alpha_i}$ if $i \geq k$; if $i = k - 1$, we will assume that $\partial_{\alpha_k..\alpha_i} = 1$. For all other cases $\partial_{\alpha_k..\alpha_i} = 0$. Quantities y_a denote the independent variables describing the gravitational field, and φ_A are the matter fields. For the rest of the work we will use

the following decomposition for the transformation law for y_a under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu$:

$$\delta y_a(x) \equiv y'_a(x) - y_a(x) = -\sum_{i=0}^{M} H_{(i)a\rho}{}^{\alpha_1..\alpha_i}\partial_{\alpha_1..\alpha_i}\xi^\rho, \tag{2}$$

where $H_{(i)a\rho}{}^{\alpha_1..\alpha_i}$ are smooth functions of y_a, which are assumed symmetric over indices $\alpha_1..\alpha_i$. Note, that the gravitational variables y_a may not be tensors.

Index a enumerates variables in the set of all independent gravitational variables and, in particular, may be either a space–time index or index of some inner symmetry groups. Similarly, multi-index A for the matter fields φ_A may contain both space–time indices and indices for inner symmetries. For further analysis, the latter is not significant, so we will omit it. For the sake of simplicity we also consider matter fields to be tensors of rank U, so their variations are assumed to be the following:

$$\delta\varphi_A = -\xi^\alpha\partial_\alpha\varphi_A - \sum_{i=1}^{U}\partial_{\mu_i}\xi^\rho\varphi_{(\rho\hat{A})_i}, \tag{3}$$

where

$$\varphi_A \equiv \varphi_{\mu_1..\mu_U}, \qquad \varphi_{(\rho\hat{A})_i} \equiv \varphi_{\mu_1..\mu_{i-1}\rho\mu_{i+1}..\mu_U}. \tag{4}$$

The parameters N_1, N_2, N_3 define the maximal order of the derivatives in action (1). The number M corresponds to the maximal order of the infinitesimal diffeomorphisms parameter ξ^μ in the transformation law of the gravitational variables (2). One may choose φ_A to be the spinor field in order to describe fermions. This choice requires the matter fields φ_A to form a spinor representation of the certain inner gauge group (for 4-dimensional space–time, the proper choice is the Lorentz group $SO(1,3)$). As was already mentioned, the inner symmetries are irrelevant in the current analysis; hence the introduction of spinors does not affect the results much.

We require that the gravitational and the matter parts of the action (1) must be invariant under diffeomorphisms separately. In particular, it means that the total action S is invariant under certain global transformations, which, due to the Noether theorem, lead to the on-shell local conservation laws (continuity equations). In order to study these quantities, we follow the route presented in [18,22,23]. We consider an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu(x)$ and by using the arbitrariness of the vector ξ^μ we will obtain a recurrent chain of identities. As we will see, it allows one to restrict the form of the canonical pEMT and energy–momentum vector.

The general covariance of the action implies that the Lagrangian must be the scalar density under the infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$. This leads to the well-known formula:

$$\delta L = -\partial_\alpha\left(\xi^\mu L\right). \tag{5}$$

The statement is also true for L_{gr} and L_m, but we postpone discussion of this until Section 3. Note that l.h.s. of (5) can be written in terms of the variations of the independent variables δy_a and $\delta\varphi_A$ in the form:

$$\delta L = \sum_{i=0}^{\max(N_1,N_3)}\frac{\partial L}{\partial\partial_{\alpha_1...\alpha_i}y_a}\partial_{\alpha_1..\alpha_i}\delta y_a + \sum_{i=0}^{N_2}\frac{\partial L}{\partial\partial_{\alpha_1...\alpha_i}\varphi_A}\partial_{\alpha_1..\alpha_i}\delta\varphi_A. \tag{6}$$

We can express the terms with $i = 0$ through $\delta S/\delta y_a$, $\delta S/\delta\varphi_A$ and use the product rule:

$$Y^{\alpha_1...\alpha_j}\partial_{\alpha_1..\alpha_j}Q + (-1)^{j+1}(\partial_{\alpha_1..\alpha_j}Y^{\alpha_1..\alpha_j})Q =$$
$$= \partial_\rho\left[\sum_{i=0}^{j-1}(-1)^{i+j+1}(\partial_{\alpha_1..\alpha_{j-i-1}}Y^{\rho\alpha_1..\alpha_{j-1}})(\partial_{\alpha_{j-i}..\alpha_{j-1}}Q)\right], \tag{7}$$

which is valid for the arbitrary symmetric tensor $Y^{\alpha_1...\alpha_j}$, to obtain the following relation:

$$\partial_\rho J^\rho = -\frac{\delta S}{\delta y_a}\delta y_a - \frac{\delta S}{\delta \varphi_A}\delta \varphi_A, \tag{8}$$

where

$$J^\rho \equiv \xi^\rho L + \sum_{j=1}^{\max(N_1,N_3)}\sum_{i=0}^{j-1}(-1)^{i+j+1}(\partial_{\alpha_1..\alpha_{j-i-1}}Z^{a\rho\alpha_1..\alpha_{j-1}})(\partial_{\alpha_{j-i}..\alpha_{j-1}}\delta y_a)+$$

$$+\sum_{j=1}^{N_2}\sum_{i=0}^{j-1}(-1)^{i+j+1}(\partial_{\alpha_1..\alpha_{j-i-1}}N^{A\rho\alpha_1..\alpha_{j-1}})(\partial_{\alpha_{j-i}..\alpha_{j-1}}\delta \varphi_A). \tag{9}$$

Here we define $Z^{a\rho\alpha_1..\alpha_j}$ and $N^{A\rho\alpha_1..\alpha_{j-1}}$ as in the following:

$$Z^{a\alpha_1..\alpha_j} \equiv \frac{\partial L}{\partial \partial_{\alpha_1..\alpha_j}y_a}, \quad N^{A\alpha_1..\alpha_j} \equiv \frac{\partial L}{\partial \partial_{\alpha_1..\alpha_j}\varphi_A}. \tag{10}$$

One can use the transformation laws (2) and (3) in order to write (9) as the linear combination of ξ^α and its derivatives:

$$J^\rho = -\sum_{k=0}^{N_d}K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k}\partial_{\alpha_1..\alpha_k}\xi^\alpha, \tag{11}$$

where $N_d \equiv \max(N_1 + M - 1, N_3 + M - 1, N_2)$. The formulae for the coefficients $K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k}$ are the following (they may be found by rewriting (9) in the form (11)):

$$K_{(0)\alpha}{}^\rho = \sum_{j=1}^{\max(N_1,N_3)}\sum_{i=0}^{j-1}(-1)^i(\partial_{\alpha_{j-i}..\alpha_{j-1}}Z^{a\rho\alpha_1..\alpha_{j-1}})(\partial_{\alpha_1..\alpha_{j-i-1}}H_{(0)a\alpha})+$$

$$+\sum_{j=1}^{N_2}\sum_{i=0}^{j-1}(-1)^i(\partial_{\alpha_{j-i}..\alpha_{j-1}}N^{A\rho\alpha_1..\alpha_{j-1}})(\partial_{\alpha_1..\alpha_{j-i-1}\alpha}\varphi_A) - \delta_\alpha^\rho L, \tag{12}$$

$$K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k} = \theta\left(\max(N_1,N_3)+M-k-\frac{1}{2}\right)\sum_{l=\max(k-\max(N_1,N_3)+1,0)}^{\min(k,M)}\Omega_{(1)}{}^\rho{}_\alpha{}^{\alpha_1..\alpha_k}(k-l,l)+$$

$$+\theta\left(N_2-k+\frac{1}{2}\right)\left(\Omega_{(2)}{}^\rho{}_\alpha{}^{\alpha_1..\alpha_k}(k)+\Omega_{(3)}{}^{\rho\alpha_k}{}_\alpha{}^{\alpha_1..\alpha_{k-1}}(k-1)\right), \quad 1\le k \le N_d, \tag{13}$$

where

$$\Omega_{(1)}{}^\rho{}_\mu{}^{\alpha_1..\alpha_m\beta_1..\beta_l}(m,l) \equiv \left\{\sum_{j=m+1}^{\max(N_1,N_3)}\sum_{i=0}^{j-m-1}(-1)^iC_{j-i-1}^m(\partial_{\alpha_{j-i}..\alpha_{j-1}}Z^{a\rho\alpha_1..\alpha_{j-1}})\times\right.$$

$$\left.\times\partial_{\alpha_{m+1}..\alpha_{j-i-1}}H_{(l)a\mu}{}^{\beta_1..\beta_l}\right\}^{\alpha_1..\alpha_m\beta_1..\beta_l}, \tag{14}$$

$$\Omega_{(2)}{}^\rho{}_\mu{}^{\alpha_1..\alpha_m}(m) \equiv \left\{\sum_{j=m+1}^{N_2}\sum_{i=0}^{j-m-1}(-1)^iC_{j-i-1}^m(\partial_{\alpha_{j-i}..\alpha_{j-1}}N^{A\rho\alpha_1..\alpha_{j-1}})\times\right.$$

$$\left.\times\partial_{\alpha_{m+1}..\alpha_{j-i-1}\mu}\varphi_A\right\}^{\alpha_1..\alpha_m}, \tag{15}$$

$$\Omega_{(3)}{}^{\rho\sigma}{}_{\mu}{}^{\alpha_1..\alpha_m}(m) \equiv \left\{ \sum_{j=m+1}^{N_2} \sum_{i=0}^{j-m-1} (-1)^i C_{j-i-1}^m \times \right.$$
$$\left. \times \left[\sum_{k=1}^{U} (\partial_{\alpha_{j-i}..\alpha_{j-1}} N^{(\sigma A)_k \rho \alpha_1..\alpha_{j-1}}) \partial_{\alpha_{m+1}..\alpha_{j-i-1}} \varphi_{(\mu A)_k} \right] \right\}^{\alpha_1..\alpha_m \sigma}, \quad (16)$$

and $\{..\}^{\alpha_1..\alpha_k}$ denotes symmetrization with respect to the indices $\alpha_1..\alpha_k$ – sum of the expression in the brackets over all permutations of the mentioned indices. Note that $K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k}$ are fully symmetric with respect to the indices $\alpha_1..\alpha_k$ by construction.

By using identity (11), one can write (8) in the more convenient form:

$$\partial_\rho \left(\sum_{k=0}^{N_d} K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k} \partial_{\alpha_1..\alpha_k} \xi^\alpha \right) = \frac{\delta S}{\delta y_a} \delta y_a + \frac{\delta S}{\delta \varphi_A} \delta \varphi_A. \quad (17)$$

The r.h.s of the relations (17) vanishes on-shell, so in this case these relations describe local conservation law, that is satisfied for any choice of $\xi^\alpha(x)$. If one takes $\xi^\alpha = const$, identity (17) gives an expected on-shell equation:

$$\partial_\rho K_{(0)\alpha}{}^\rho = 0. \quad (18)$$

As the reasoning for $\xi^\alpha = const$ is exactly the first Noether theorem for the space–time translation invariance, we will use the proper notation:

$$K_{(0)\alpha}{}^\rho = \mathcal{T}^\rho{}_\alpha, \quad (19)$$

where $\mathcal{T}^\rho{}_\alpha$ is canonical pEMT.

Let us return to the general choice of the $\xi^\alpha(x)$. As it is completely arbitrary, one can independently choose its value and the value of all its derivatives $\partial_{\alpha_1..\alpha_k} \xi^\mu$ at any given point. Hence it is possible to rewrite (17) as the recurrent chain of identities

$$\partial_\rho K_{(0)\alpha}{}^\rho = -\frac{\delta S}{\delta y_a} H_{(0)a\alpha} - \frac{\delta S}{\delta \varphi_A} \partial_\alpha \varphi_A, \quad (20)$$

$$\partial_\rho K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k} + \frac{1}{k!} \left\{ K_{(k-1)\alpha}{}^{\alpha_1..\alpha_k} \right\}^{\alpha_1..\alpha_k} =$$
$$= -\frac{\delta S}{\delta y_a} H_{(k)a\alpha}{}^{\alpha_1..\alpha_k} \theta \left(M - k + \frac{1}{2} \right) - \delta_{1k} \frac{\delta S}{\delta \varphi_A} \sum_{i=1}^{U} \delta_{\mu_i}^{\alpha_1} \varphi_{(\alpha A)_i}, \quad 1 \le k \le N_d, \quad (21)$$

$$\left\{ K_{(N_d)\alpha}{}^{\alpha_1...\alpha_{N_d+1}} \right\}^{\alpha_1...\alpha_{N_d+1}} = 0. \quad (22)$$

The analogous recurrent chain arises in [22] in the context of generalized Belinfante relation. One may refine the recurrent chain (20)–(22) by following the procedure described in [22] on-shell in order to obtain the general form of superpotential for pEMT of theory (1). The resulting superpotential is not nescessarily antisymmetric. Though it prevents one from writing a simple expression of the energy–momentum vector through the surface integral, we will show that even without antisymmetricity, it may be done explicitly.

In the rest of this section, we assume that all equations of motion are satisfied. The equation with $k = 1$ from (21) has the following form (here we used (19)):

$$\mathcal{T}^\rho{}_\alpha = -\partial_\beta K_{(1)\alpha}{}^{\beta\rho}. \quad (23)$$

This equation expresses the canonical pEMT as the full divergence of a certain quantity, so its reasonable to think of $K_{(1)\alpha}{}^{\beta\rho}$ as the superpotential (up to a possible sign convention). However, the definition of the superpotential—i.e., the property

$$\partial_{\beta\rho} K_{(1)\alpha}{}^{\beta\rho} = 0 \tag{24}$$

should be proven first. To see it, one should apply the operator $\partial_{\alpha_1..\alpha_k}$ to (21), and the resulting identity will be the following:

$$\partial_{\rho\alpha_1..\alpha_k} K_{(k)\alpha}{}^{\rho\alpha_1..\alpha_k} = -\partial_{\alpha_1..\alpha_k} K_{(k-1)\alpha}{}^{\alpha_1..\alpha_k}, \tag{25}$$

and by applying $\partial_{\alpha_1..\alpha_{N_d+1}}$ to (22), one can obtain the similar formula:

$$\partial_{\alpha_1..\alpha_{N_d+1}} K_{(N_d)\alpha}{}^{\alpha_1..\alpha_{N_d+1}} = 0. \tag{26}$$

From these equations it is clear that

$$\partial_{\alpha_1..\alpha_{k+1}} K_{(k)\alpha}{}^{\alpha_1..\alpha_{k+1}} = 0, \qquad 0 \le k \le N_d, \tag{27}$$

from which the superpotentiality (24) of $K_{(1)\alpha}{}^{\beta\rho}$ arises. By using (13) with $k = 1$ we finally obtain the general form of the superpotential $\Psi_\alpha{}^{\beta\rho}$:

$$\mathcal{T}^\rho{}_\alpha = \partial_\beta \Psi_\alpha{}^{\beta\rho}, \tag{28}$$

$$\Psi_\alpha{}^{\beta\rho} = -K_{(1)\alpha}{}^{\beta\rho} = -\theta\left(\max(N_1,N_3) + M - \frac{3}{2}\right) \sum_{l=\max(2-\max(N_1,N_3),0)}^{\min(1,M)} \Omega_{(1)}{}^\beta{}_\alpha{}^\rho (1-l,l)$$

$$- \theta\left(N_2 - \frac{1}{2}\right)\left(\Omega_{(2)}{}^\beta{}_\alpha{}^\rho(1) + \Omega_{(3)}{}^{\beta\rho}{}_\alpha(0)\right). \tag{29}$$

For particular cases of (1) with $N_1, N_2, N_3 \le 2$ instead of $\Psi_\alpha{}^{\beta\rho}$ one can [2] use the following superpotential:

$$B^{\beta\rho}{}_\alpha = -K_{(1)\alpha}{}^{\beta\rho} + \partial_\gamma L_\alpha{}^{\beta\gamma\rho}, \qquad B^{\beta\rho}{}_\alpha = -B^{\rho\beta}{}_\alpha, \tag{30}$$

where $L_\alpha{}^{\beta\gamma\rho}$ has the property:

$$\partial_{\beta\gamma} L_\alpha{}^{\beta\gamma\rho} = 0. \tag{31}$$

Then the canonical pEMT can be written in the equivalent form in terms of the described antisymmetric superpotential $B^{\beta\rho}{}_\alpha$:

$$\mathcal{T}^\rho{}_\alpha = \partial_\beta B^{\beta\rho}{}_\alpha. \tag{32}$$

Though the antisymmetricity of the superpotential provides a simple expression for the energy–momentum in terms of the surface integral, it can be difficult to construct $L_\alpha{}^{\beta\gamma\rho}$ explicitly in general. However, as mentioned above, there is another option to achieve the same goal. In order to do it, let us consider (21) with $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$:

$$K_{(k-1)\alpha}{}^{0...0} = -\partial_0 K_{(k)\alpha}{}^{0...0} - \partial_n K_{(k)\alpha}{}^{n0...0}, \tag{33}$$

where index n corresponds to the spatial dimensions. By applying the operator $(-\partial_0)^{i-2}$ to this equation, summing the result from $k = 2$ to N_d and taking into account (22)

$$K_{(N_d)\alpha}{}^{0...0} = 0, \tag{34}$$

it can be shown that

$$K_{(1)\alpha}{}^{00} = -\partial_n \sum_{i=2}^{N_d} (-\partial_0)^{i-2} K_{(i)\alpha}{}^{n0\ldots0}. \tag{35}$$

Equation (35) with assumptions of the trivial space–time topology and the proper field asymptotics on the spatial infinity leads to the expression for the conserving energy–momentum vector based on (23):

$$P_\alpha \equiv \int_{t=const} d\vec{x}\, \mathcal{T}^0{}_\alpha = \lim_{R\to\infty} \int_{S_R} dS_n \left[-K_{(1)\alpha}{}^{n0} - \sum_{i=2}^{N_d} (-\partial_0)^{i-1} K_{(i)\alpha}{}^{n0\ldots0} \right]. \tag{36}$$

Here, S_R is a sphere of radius R in the hyperspace $t = const$.

3. Properties of Gravitational and Matter Contributions into Superpotential

One of the requirements imposed on the action (1) was the separate general covariance of both gravitational and matter parts. It allows one to obtain the recurrent chains similar to the chain (18), (21) and (22) for each of these action contributions. In this section, we show that such chains allow one to lend certain physical sense to the gravitational and matter contributions into the superpotential (23).

Because of the general covariance of the S_m, the lagrangian L_m is obviously the scalar density. It means we can follow the same procedure given by identities (5)–(8), and derive the analogue for (8):

$$\partial_\rho \left[J^\rho \right]_m = -\frac{\delta S_m}{\delta y_a} \delta y_a - \frac{\delta S_m}{\delta \varphi_A} \delta \varphi_A, \tag{37}$$

where J^ρ is defined in (9), and $[..]_m$ denotes the contribution from L_m extracted from the quantity in brackets.

For further analysis of this identity, it is helpful to introduce the additional identities that also follow from the general covariance. Consider the infinitesimal change $x^\mu \to x^\mu + \xi^\mu(x)$ with ξ^μ having a compact support. The general covariance of the total action implies that:

$$\delta S = \int dx \left(\frac{\delta S}{\delta y_a} \delta y_a + \frac{\delta S}{\delta \varphi_A} \delta \varphi_A \right) = 0, \tag{38}$$

where δy_a and $\delta \varphi_A$ are defined in (2) and (3). Integration by parts can remove all the derivatives from ξ^μ because we assumed it has compact support. Then it is not hard to check that the following identities are satisfied:

$$\frac{\delta S}{\delta \varphi_A} \partial_\rho \varphi_A - \sum_{i=1}^{U} \partial_{\mu_i} \left(\frac{\delta S}{\delta \varphi_A} \varphi_{(\rho\hat{A})_i} \right) + \sum_{i=0}^{M} (-1)^i \partial_{\alpha_1..\alpha_i} \left(H_{(i)a\rho}{}^{\alpha_1..\alpha_i} \frac{\delta S}{\delta y_a} \right) = 0. \tag{39}$$

This relation establishes a useful link between different equations of motion. The existence of such identities is a well-known fact for any gauge theory and was first pointed out by Noether in [14]. In the literature (see, for example, [19]), it is also called the second Noether theorem.

By isolating the term with $i = 0$ in the last sum in (39) and by changing S to S_m due to the general covariance of the matter action, it can be shown that:

$$H_{(0)a\rho} \frac{\delta S_m}{\delta y_a} = \sum_{i=1}^{U} \partial_{\mu_i} \left(\frac{\delta S_m}{\delta \varphi_A} \varphi_{(\rho\hat{A})_i} \right) - \frac{\delta S_m}{\delta \varphi_A} \partial_\rho \varphi_A - \sum_{i=1}^{M} (-1)^i \partial_{\alpha_1..\alpha_i} \left(H_{(i)a\rho}{}^{\alpha_1..\alpha_i} \frac{\delta S_m}{\delta y_a} \right). \tag{40}$$

In the particular case of GR (when the gravitational independent variable y_a is reduced to $g_{\mu\nu}$ and hence $M = 1$) the identity (40) takes the familiar form if we assume that the matter equations are satisfied:

$$D_\mu T^{\mu\nu} = 0, \tag{41}$$

where

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} \qquad (42)$$

is the metric (Hilbert) EMT of the matter. The identity (40) arising from the second Noether theorem may be considered independently from the chain Equations (20)–(22). As one shall see below, it can be used to simplify them further.

Now we can use (2) and (3) in (37) and then use (40) and (7) in the result to obtain the following:

$$\partial_\rho \left([J^\rho]_m - \sum_{j=1}^{M} \sum_{i=0}^{j-1} (-1)^{i+j+1} \partial_{\alpha_1..\alpha_{j-i-1}} \left(H_{(j)a\mu}{}^{\rho\alpha_1..\alpha_{j-1}} \frac{\delta S_m}{\delta y_a} \right) \partial_{\alpha_{j-i}..\alpha_{j-1}} \xi^\mu + \right.$$

$$\left. - \sum_{i=1}^{U} \frac{\delta S_m}{\delta \varphi_{(\rho\hat{A})_i}} \varphi_{(\nu\hat{A})_i} \xi^\nu \right) = 0. \quad (43)$$

As with (17), it is clear, that this identity is equivalent to some recurrent chain of equations due to the arbitrariness of ξ^μ. One may expect, that it should look similar to the chain (18), (21) and (22) which arises from (17) in the previous section.

For the further analysis of the chain (43), we will assume that we are working on-shell in the sense of the matter equations of motion $\delta S_m / \delta \varphi_A = 0$. Then the first two equations from the (43) looks like the following:

$$\partial_\rho [\mathcal{T}^\rho{}_\lambda - X^\rho{}_\lambda]_m = 0, \qquad (44)$$

$$[\mathcal{T}^\rho{}_\lambda]_m = [V^\rho{}_\lambda]_m - \partial_\beta \left[K_{(1)\alpha}{}^{\beta\rho} \right]_m, \qquad (45)$$

where we used the notation:

$$X^\rho{}_\mu \equiv \sum_{j=1}^{M} (-1)^j \partial_{\alpha_1..\alpha_{j-1}} \left(H_{(j)a\mu}{}^{\rho\alpha_1..\alpha_{j-1}} \frac{\delta S}{\delta y_a} \right), \qquad (46)$$

$$V^\rho{}_\mu \equiv -H_{(1)a\mu}{}^\rho \frac{\delta S}{\delta y_a}. \qquad (47)$$

Obviously, these quantities vanish on the gravitational equations of motion, so we will use them only in the relations that are satisfied without the appropriate equations of motion.

For the case mentioned above, when the gravitational variable y_a is space–time metric $g_{\mu\nu}$, we have:

$$H_{(1)\mu\nu\alpha}{}^\rho = \left\{ \delta_\mu^\rho g_{\alpha\nu} \right\}_{\mu\nu}, \qquad (48)$$

from which we have

$$\left[X^\rho{}_\mu \right]_m = \left[V^\rho{}_\mu \right]_m = \sqrt{-g}\, T^\rho{}_\mu. \qquad (49)$$

Substituting these formulae in (45), it is not hard to derive the generalization of the well-known Belinfante relation [24] in the curved space–time:

$$[\mathcal{T}^\rho{}_\lambda]_m = \sqrt{-g}\, T^\rho{}_\lambda - \partial_\beta \left[K_{(1)\lambda}{}^{\beta\rho} \right]_m. \qquad (50)$$

One may follow the logic of the derivation of the (44) and (45) and then obtain the analogous relations for the S_{gr}, and hence, the analogous chain of equations. Repeating these first steps, we obtain essentially the same result:

$$\partial_\rho [\mathcal{T}^\rho{}_\lambda - X^\rho{}_\lambda]_{gr} = 0, \qquad (51)$$

$$[\mathcal{T}^\rho{}_\lambda]_{gr} = [V^\rho{}_\lambda]_{gr} - \partial_\beta \left[K_{(1)\alpha}{}^{\beta\rho} \right]_{gr}. \qquad (52)$$

The only difference will be the obvious lack of the matter fields φ_A in the S_{gr}. Therefore, in contrast to (44) and (45), the relations (51) and (52) are satisfied off-shell and hence are true identities.

The relations (45) and (52) give a useful insight into the structure of the general superpotential (29). If we just sum these relations and require the satisfaction of only the matter equations of motion to be able to use generalized Belinfante relation (45), we obtain the expected relation for the total pEMT:

$$\mathcal{T}^{\rho}{}_{\lambda} = -\partial_{\alpha} K_{(1)\lambda}{}^{\alpha\rho} + V^{\rho}{}_{\lambda}, \tag{53}$$

which obviously coincides with (23) on-shell because $V^{\rho}{}_{\lambda} = 0$ on the gravitational equations of motion. In summary, the quantity $K_{(1)\lambda}{}^{\alpha\rho}$ (and hence the superpotential (29)) can be naturally decomposed into the two contributions from the matter and gravitational actions. The first one can be treated (if the matter equations of motion are satisfied) as the generalization of the Belinfante addition (see (50)) to the canonical pEMT. For the theories with no more than first derivatives in action in the flat limit, it can be expressed in terms of the spin tensor (see, for example, [24,25]). The second term is certain superpotential, that depends only on the gravitational independent variables and can be reduced to Møller superpotential [11] in the case of GR.

As already mentioned, $V^{\rho}{}_{\lambda}$ is proportional to the gravitational equations of motion with the coefficient $H_{(1)a\mu}{}^{\rho}$. Suppose, that one field from the set of independent variables y_a does not have first-order derivatives of ξ^{μ} in the transformation law (2) and hence satisfies the condition $H_{(1)a\mu}{}^{\rho} = 0$. Then $V^{\rho}{}_{\lambda}$ does not depend on the equation of motion derived from the variation of the (1) with respect to this field. In the case when $H_{(1)a\mu}{}^{\rho} = 0$ is satisfied for all fields, the relation (53) is drastically simplified (we again assume that the matter equations of motion are satisfied)

$$\mathcal{T}^{\rho}{}_{\lambda} = -\partial_{\alpha} K_{(1)\lambda}{}^{\alpha\rho} \tag{54}$$

without any gravitational equations of motion.

If we additionally cancel the matter contribution in the total action ($S_m = 0$) and impose stronger conditions on the transformation laws:

$$H_{(i)\mu}{}^{\alpha_1..\alpha_i} = 0, \quad i > 0, \tag{55}$$

then the r.h.s of (21) identically vanishes, and hence the superpotentiality condition (24) is automatically satisfied. Thus, the formula (54) can be further strengthened:

$$\partial_{\rho} \mathcal{T}^{\rho}{}_{\lambda} = 0. \tag{56}$$

Like Equation (54), this relation does not need any gravitational equations of motion to be satisfied. This statement may seem strange because the r.h.s. of (20) still depends on the term that is proportional to the equations of motion (note that $H_{(0)}$ is always non-zero). However, it follows from relation (39) that (20) identically equals zero for the considered case.

4. Theories with the Change of Independent Gravitational Variables in Action

In many cases, the consideration of theories of gravity that are modified by the change in independent variables in the GR action may be quite useful. In general, this change may contain derivatives of the new independent variables. Several examples of GR modifications of this kind will be discussed in Section 5. However, there are also other frequently discussed theories—for example, tetrad formulation of GR and theories with non-zero torsion. In general, by the change of variables, we mean the following expression:

$$y_a = f_a(y'_E, .., \partial_{\alpha_1..\alpha_W} y'_E), \tag{57}$$

where y'_E are new gravitational independent variables and f_a is a smooth function. The action of such theory remains unchanged in terms of dependence on the old variables. As discussed in the works [15,26], such modification could lead to richer dynamics in comparison to the original theory. Namely, a new theory has all the solutions from the original one and also has some extra solutions, that may appear useful for the explanations of certain observable effects that are absent in the original theory.

Despite the general form of (57), it constrains the form of the integrals of motion in the modified theory. Indeed, consider the subtraction of (8) for the old theory from the same expression in the modified theory:

$$\partial_\rho \left(J'^\rho - J^\rho \right) = \frac{\delta S}{\delta y_a} \delta y_a - \frac{\delta S}{\delta y'_E} \delta y'_E. \tag{58}$$

Because change (57) is smooth, one can write a polynomial expansion for δy_a in terms of $\delta y'_E$:

$$\delta y_a = \sum_{i=0}^{W} \frac{\partial y_a}{\partial \partial_{\alpha_1..\alpha_i} y'_E} \partial_{\alpha_1..\alpha_i} \delta y'_E, \tag{59}$$

and also derive the expression for $\delta S / \delta y'_a$ in terms of the original equations of motion $\delta S / \delta y_a$:

$$\frac{\delta S}{\delta y'_E} = \sum_{i=0}^{W} (-1)^i \partial_{\alpha_1..\alpha_i} \left(\frac{\delta S}{\delta y_a} \frac{\partial y_a}{\partial \partial_{\alpha_1..\alpha_i} y'_E} \right). \tag{60}$$

By substituting (59) and (60) into (58) and then using (7) for the resulting formula, one can show the following:

$$\partial_\rho \left(J'^\rho - J^\rho \right) = \partial_\rho \sum_{j=1}^{W} \sum_{i=0}^{j-1} (-1)^{i+j+1} \partial_{\alpha_1..\alpha_{j-i-1}} \left(\frac{\delta S}{\delta y_a} \frac{\partial y_a}{\partial \partial_{\rho\alpha_1..\alpha_{j-1}} y'_E} \right) \partial_{\alpha_{j-i}..\alpha_{j-1}} \delta y'_E, \tag{61}$$

Since no equations of motion are required for (61), it is an identity and hence J'^ρ and J^ρ obey the following relation:

$$J'^\rho = J^\rho + \sum_{j=1}^{W} \sum_{i=0}^{j-1} (-1)^{i+j+1} \partial_{\alpha_1..\alpha_{j-i-1}} \left(\frac{\delta S}{\delta y_a} \frac{\partial y_a}{\partial \partial_{\rho\alpha_1..\alpha_{j-1}} y'_E} \right) \partial_{\alpha_{j-i}..\alpha_{j-1}} \delta y'_E + I^\rho, \tag{62}$$

where I^ρ is a divergence-free term:

$$\partial_\rho I^\rho = 0. \tag{63}$$

Note that it follows from (62) that I^ρ necessarily is a local function of the y'_E and diffeomorphism parameter ξ^μ. Unfortunately, the explicit formula for this quantity remains unknown.

5. Examples

In this section, we consider several applications of the general formalism described above. Firstly, we focus on the standard metric description of the GR and then proceed to its modifications. In all examples, we use the standard Einstein–Hilbert action of GR:

$$S_{gr} = -\frac{1}{\varkappa} \int d^4x \sqrt{-g} \, R. \tag{64}$$

For the sake of simplicity will also omit matter contribution by setting $S_m = 0$, so $S = S_{gr}$—nevertheless, all the results below can be easily generalized to the case $S_m \neq 0$.

5.1. General Relativity

The relations (46) and (47) give for (64) the following:

$$X^\rho{}_\mu = \left[V^\rho{}_\mu \right]_{gr} = -\frac{1}{\varkappa} \sqrt{-g}\, G^\rho{}_\mu. \tag{65}$$

By using the following expressions for $Z^{\rho\nu\beta}$ and $Z^{\rho\nu\beta\alpha}$, that are defined in (10):

$$Z^{\rho\nu\beta} = -\frac{\sqrt{-g}}{2\varkappa} \left(g^{\rho\nu} g^{\mu\sigma} \Gamma^\beta_{\mu\sigma} - g^{\mu\sigma} \left\{ g^{\beta\rho} \Gamma^\nu_{\mu\sigma} \right\}^{\rho\nu} + g^{\mu\beta} \left\{ g^{\alpha\rho} \Gamma^\nu_{\mu\alpha} \right\}^{\rho\nu} - g^{\mu\rho} g^{\nu\alpha} \Gamma^\beta_{\mu\alpha} \right), \tag{66}$$

$$Z^{\rho\nu\beta\alpha} = -\frac{\sqrt{-g}}{2\varkappa} \left(\frac{1}{2} \left\{ g^{\alpha\rho} g^{\beta\nu} \right\}^{\alpha\beta} - g^{\rho\nu} g^{\beta\alpha} \right), \tag{67}$$

the following expression for the superpotential (29) can be established:

$$\Psi_{\mathrm{GR}}{}^{\beta\rho}{}_\alpha = \frac{\sqrt{-g}}{2\varkappa} \left(2\Gamma^\beta_{\alpha\gamma} g^{\rho\gamma} - \Gamma^\mu_{\mu\alpha} g^{\rho\beta} - \delta^\beta_\alpha \Gamma^\rho_{\mu\gamma} g^{\mu\gamma} \right). \tag{68}$$

One can substitute this expression in (52) to obtain an off-shell identity for the pEMT:

$$\mathcal{T}^\rho{}_\lambda = -\frac{1}{\varkappa} \sqrt{-g}\, G^\rho{}_\lambda + \frac{1}{2\varkappa} \partial_\beta \left[\sqrt{-g} \left(2\Gamma^\beta_{\lambda\gamma} g^{\rho\gamma} - g^{\rho\beta} \Gamma^\mu_{\mu\lambda} - \delta^\beta_\lambda \Gamma^\rho_{\mu\gamma} g^{\mu\gamma} \right) \right]. \tag{69}$$

One can also construct $L_\alpha{}^{\beta\gamma\rho}$, described by the formulae (30) and (31). In that case expression (68) may be replaced by the antisymmetric quantity $\Psi_{GR}{}^{\beta\rho}{}_\alpha + \partial_\gamma L_\alpha{}^{\beta\gamma\rho}$. This superpotential is well-known for GR and first was obtained by Møller [11]:

$$\Psi_{\mathrm{M}}{}^{\beta\rho}{}_\alpha \equiv \frac{1}{\varkappa} \sqrt{-g} \left(g^{\gamma\rho} \Gamma^\beta_{\gamma\mu} - g^{\gamma\beta} \Gamma^\rho_{\gamma\mu} \right). \tag{70}$$

With that in mind, (69) may be rewritten in the following form:

$$\mathcal{T}^\rho{}_\lambda = -\frac{1}{\varkappa} \sqrt{-g}\, G^\rho{}_\mu + \partial_\beta \left(\Psi_{\mathrm{M}}{}^{\beta\rho}{}_\alpha \right). \tag{71}$$

This identity reproduces the expected standard form of pEMT for the Einstein field on-shell.

It is interesting that the main property (24) of the superpotential $\Psi_{\mathrm{GR}}{}^{\beta\rho}{}_\alpha$ is satisfied identically. For Møller expression, this statement trivially follows from its antisymmetry and for the $\Psi_{\mathrm{GR}}{}^{\beta\rho}{}_\alpha$ it can be seen from the property (31) of the $L_\alpha{}^{\beta\gamma\rho}$.

5.2. Palatini Formalism

By Palatini (or Hilbert–Palatini) formalism one usually means the description of gravity with the symmetric connection and space–time metric as the independent variables. The action in such approach is taken in the standard Einstein–Hilbert form (64). Despite the name of the approach, it was probably proposed for the first time by Einstein in [27] (for the detailed discussion of the topic see [28]). By itself, Palatini formalism does not provide the real modification of GR; however, it is a useful example of the theory with independent non-tensor variables.

Quantities $H_{(k)\lambda}{}^\alpha{}_{\mu\nu}{}^{\alpha_1\ldots\alpha_k}$ for the connection $\Gamma^\alpha_{\mu\nu}$ are the following (recall that symmetrization brackets introduced after (16) is not conventional):

$$H_{(0)\lambda}{}^\alpha{}_{\mu\nu} = \partial_\lambda \Gamma^\alpha_{\mu\nu}, \quad H_{(1)\lambda}{}^\alpha{}_{\mu\nu}{}^\rho = \left\{ \delta^\rho_\mu \Gamma^\alpha_{\lambda\nu} \right\}_{\mu\nu} - \delta^\alpha_\lambda \Gamma^\rho_{\mu\nu}, \quad H_{(2)\lambda}{}^\alpha{}_{\mu\nu}{}^{\rho\nu} = \frac{1}{2} \delta^\alpha_\lambda \left\{ \delta^\rho_\mu \delta^\nu_\nu \right\}_{\mu\nu}. \tag{72}$$

By using (29) and (72) it can be shown that $\Psi^{\beta\rho}{}_\alpha$ for the Palatini formalism coincides with the $\Psi_{GR}{}^{\beta\rho}{}_\alpha$ defined in (68). Thus, the pEMT (52) for the considered theory has the form:

$$\mathcal{T}^\rho{}_\lambda = -\frac{1}{\varkappa}\sqrt{-g}\,G^\rho{}_\lambda +$$

$$+\frac{1}{2\varkappa}\sqrt{-g}\left[g^{\mu\nu}C^\alpha{}_{\alpha\gamma} - g^{\sigma\nu}C^\mu{}_{\sigma\alpha}\delta^\alpha_\gamma\right]^{\alpha\nu}\left(\left\{\delta^\rho_\mu\Gamma^\gamma_{\nu\lambda}\right\}_{\mu\nu} - \delta^\gamma_\lambda\Gamma^\rho_{\mu\nu}\right) + \partial_\beta\Psi_{GR}{}^{\beta\rho}{}_\lambda, \tag{73}$$

where $C^\mu{}_{\sigma\alpha} \equiv \Gamma^\mu_{\sigma\alpha} - \overset{o}{\Gamma}{}^\mu_{\sigma\alpha}$ denotes the difference between total connection $\Gamma^\mu_{\sigma\alpha}$ and the Levi-Civita connection $\overset{o}{\Gamma}{}^\mu_{\sigma\alpha}$, and $[..]_{\mu\nu}$ denotes antisymmetric part (without normalization) of the quantity in the brackets with respect to the given indices.

It is worth noting that, in contrast with GR, for Palatini formalism, one cannot state that the superpotential property (24) of $\Psi^{\beta\rho}{}_\alpha$ is satisfied identically. Indeed, if one considers (21) with $k = 2$ and takes into account that the equations of motion arising from the variation principle with respect to $\Gamma^\mu_{\alpha\nu}$ have the form $C^\mu{}_{\sigma\alpha} = 0$. It can be easily checked that the r.h.s of (21) with $k = 2$ should be proportional to the $C^\mu{}_{\sigma\alpha}$:

$$\partial_\rho K_{(2)\lambda}{}^{\rho\varphi\sigma} + \frac{1}{2}\left\{K_{(1)\lambda}{}^{\varphi\sigma}\right\}^{\varphi\sigma} = -\frac{1}{2\varkappa}\sqrt{-g}\left[g^{\varphi\sigma}C^\alpha{}_{\alpha\lambda} - \frac{1}{2}\{g^{\gamma\varphi}C^\sigma{}_{\gamma\alpha}\delta^\alpha_\lambda\}^{\varphi\sigma}\right]^{\alpha\nu}. \tag{74}$$

It immediately implies that (24) is satisfied only when the equations of motion corresponding to $\Gamma^\rho_{\mu\nu}$ are satisfied, namely when the connection is reduced to the Christoffel symbols. Obviously enough, when all the equations of motion are satisfied, the expression for pEMT in Palatini formalism also coincides with that one for GR.

5.3. Disformal Transformations and Mimetic Gravity

For modification of GR with the action (64) one can use well-known change of the independent variables that is called disformal transformation [20]:

$$g_{\mu\nu} = A(Q,\sigma)\bar{g}_{\mu\nu} + B(Q,\sigma)(\partial_\mu\sigma)(\partial_\nu\sigma), \tag{75}$$

where $\bar{g}_{\mu\nu}$ is an auxiliary metric, σ is a scalar field, $Q \equiv \bar{g}^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)$. From (62) it can be shown that, for the changes in (75), the following relation holds:

$$J'^\rho = J^\rho - \frac{1}{\varkappa}\sqrt{-g}\,G^{\mu\nu}\left((A'_\sigma\bar{g}_{\mu\nu} + B'_\sigma\partial_\mu\sigma\partial_\nu\sigma)\bar{g}^{\rho\alpha}\partial_\alpha\sigma + \delta^\rho_\mu B\partial_\nu\sigma\right)\xi^\alpha\partial_\alpha\sigma + I^\rho, \tag{76}$$

and one should expect the superpotential of the modified theory to differ from the original one (68) only by the the divergence-free term arising from I^ρ. This statement follows from the observation, that second term in the r.h.s depends on ξ^α without any derivatives and hence does not contribute to the $K_{(1)\alpha}{}^{\beta\rho}$. It also can be easily verified by the direct calculation of the (29), which gives the following off-shell expression for the pEMT:

$$\mathcal{T}^\rho{}_\lambda = V^\rho{}_\lambda + \partial_\beta\left(\Psi_{GR}{}^{\beta\rho}{}_\mu + \Psi_{add}{}^{\beta\rho}{}_\mu\right), \tag{77}$$

where

$$V^\rho{}_\lambda = \frac{1}{\varkappa}\sqrt{-g}\left(AG^\rho{}_\lambda - G^{\alpha\beta}\left(A'_Q \bar{g}_{\alpha\beta} + B'_Q \partial_\alpha \sigma \partial_\beta \sigma\right)\partial^\rho \sigma \partial_\lambda \sigma\right),\tag{78}$$

$$\Psi_{\text{add}}{}^{\beta\rho}{}_\mu = -\left[Z^{\varphi\sigma\alpha}\frac{\partial\partial_\alpha g_{\varphi\sigma}}{\partial\partial_\beta \bar{g}_{\rho\nu}}\bar{g}_{\nu\mu} - Z^\rho{}_\mu{}^\beta + Z^{\varphi\sigma\varepsilon\omega}\frac{\partial\partial_{\varepsilon\omega} g_{\varphi\sigma}}{\partial\partial_\beta \bar{g}_{\rho\nu}}\bar{g}_{\mu\nu} + 2Z^{\varphi\sigma\varepsilon\omega}\frac{\partial\partial_{\varepsilon\omega} g_{\varphi\sigma}}{\partial\partial_{\beta\alpha}\bar{g}_{\rho\nu}}\partial_\alpha \bar{g}_{\nu\mu} -\right.$$
$$\left.- 2Z^{\rho\nu\beta\alpha}\partial_\alpha g_{\mu\nu} - \frac{4}{3}\partial_\alpha\left(Z^{\beta\nu\alpha\rho}g_{\mu\nu} + Z^{\varphi\nu\varepsilon\omega}\frac{\partial\partial_{\varepsilon\omega} g_{\varphi\nu}}{\partial\partial_{\beta\alpha}\bar{g}_{\gamma\rho}}\bar{g}_{\gamma\mu}\right)\right]^{\rho\beta},\tag{79}$$

and $Z^{\varphi\sigma\alpha}$, $Z^{\varphi\sigma\varepsilon\omega}$ are defined in (66) and (67). Here we used (66), (67) and (52). We can again replace $\Psi_{\text{GR}}{}^{\beta\rho}{}_\mu$ with the Møller superpotential (70) (see Section 5.1), which leads to the simplified expression for pEMT through the antisymmetric superpotential:

$$\mathcal{T}^\rho{}_\lambda = V^\rho{}_\lambda + \partial_\beta\left(\Psi_{\text{M}}{}^{\beta\rho}{}_\mu + \Psi_{\text{add}}{}^{\beta\rho}{}_\mu\right).\tag{80}$$

As it was noted at the end of the Section 3, if $H_{(1)a\mu}{}^\rho = 0$ for any independent variable, the quantity $V^\rho{}_\lambda$ does not depend on the equations of motion that arise from the variational principle for this variable. Metric theories with the disformal change provide an example exactly of this kind: in this case, σ is the mentioned field. As a result, it can be seen from the (78) for $V^\rho{}_\lambda$ that the equations of motion, which arise from the variation with respect to $\bar{g}_{\mu\nu}$ are sufficient for the continuity equations for the $\mathcal{T}^\rho{}_\lambda$ to be satisfied.

One of the most popular choices of the disformal change is so-called mimetic gravity proposed in the paper [15]. In that case the change takes the simple form:

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}(\partial_\alpha \sigma)(\partial_\beta \sigma),\tag{81}$$

which corresponds to the choice $A(Q,\sigma) = Q$, $B(Q,\sigma) = 0$ in (75). It is crucial that this particular case is somewhat singular because change (81) is not invertible in contrast with the regular case of (75) [29]. For mimetic gravity, additional contribution to the superpotential is simplified to the expression:

$$\Psi_{\text{add}}{}^{\beta\rho}{}_\mu = -\left[Z^\varphi{}_\varphi{}^\rho \partial^\beta \sigma \partial_\mu \sigma + 2Z^{\varphi\sigma\varepsilon\rho}\partial_\varepsilon\left(g_{\varphi\sigma}\partial^\beta \sigma \partial_\mu \sigma\right) - \frac{4}{3}\partial_\chi\left(Z^\varphi{}_\varphi{}^{\rho\chi}\partial^\beta \sigma \partial_\mu \sigma\right)\right]^{\rho\beta}.\tag{82}$$

Substituting this formula in (80) gives the expression for canonical pEMT in mimetic gravity.

Apart from the form of the mimetic gravity induced solely by change (81) in action (64), several generalizations with the potential for σ are often considered (see, for example, [30]). In cases when this potential does not contain derivatives, the formula (82) still holds because (14) (and hence the superpotential (29)) does not depend on $Z_{(\sigma)} \equiv \partial L/\partial\sigma$ (see (10)).

Change (81) extends the dynamic of GR in such way that the modified theory is equivalent to the GR with the additional "mimetic" matter. Since the new matter originates from the gravity only, this behaviour allows one to approach certain problems of the modern cosmology from a new angle. For instance, one may identify this mimetic matter with cold dark matter, which interacts only through gravity and thus may be described as pure gravitational effect (see [31] and its references for the current state of the topic). The additional extension of dynamics can be achieved in two ways. The first possibility is to just change the action—for example, by adding a potential to the scalar field. Another variant is to modify the change (81) itself (see, for example, the change proposed in [32] for which dark matter has a non-potential flow).

It should be noted that theory with change (81) in (64) can be equivalently reformulated as GR with the additional action term by adding the Lagrange multiplier [33]. It can be checked that the superpotential for this reformulated theory coincides with (68), and hence superpotential and pEMT again can be reduced to the Møller superpotential (70) and (71), respectively.

5.4. Regge–Teitelboim Embedding Gravity

Proposed by Regge and Teitelboim [21], embedding gravity (or "embedding theory") is based on the idea of considering four-dimensional GR manifold as an embedded surface $\mathcal{M}$ in a flat space–time of higher dimension d. Action is chosen as usual (64) with the induced metric on $\mathcal{M}$ playing role of the change of variables in action:

$$g_{\mu\nu} = (\partial_\mu y^a)(\partial_\nu y_a). \tag{83}$$

Here, y^a is an embedding function, indices $a, b, ..$ in this subsection denote the tensor representations of the symmetry group $SO(1, d-1)$ of the ambient space flat metric. It is essential that (83) requires the isometricity of the proper embedding. The minimal dimension of the ambient space for the embedding of the smooth four-dimensional Lorentzian manifold, as it was pointed out in [34], is $d = 10$. After the paper [21] the framework of embedding gravity was thoroughly studied in [35]. Since then these ideas were often used in the various investigations, including researches on the quantum gravity (see, for example, [36–44]). In the full analogy to the change (81) the embedding change (83) extends the dynamics of the original GR. Indeed, just like the mimetic gravity, the modified theory appears to be equivalent to GR with an extra matter. Interestingly enough, the number of degrees of freedom corresponding to the extra matter in the resulting theory seems to be big enough to describe dark matter without additional complications [45].

Due to the scalar origin of the independent variables (the embedding function y^a) of the embedding gravity the formula (52) reduces (see the end of Section 3) to the form (54) even without any equations of motion. The direct calculation based on (29) allows one to write (54) in the form:

$$\mathcal{T}^\rho{}_\lambda = \partial_\beta \left(\Psi_{\text{GR}}{}^{\beta\rho}{}_\lambda + \Psi_{\text{add}}{}^{\beta\rho}{}_\lambda \right), \tag{84}$$

where

$$\Psi_{\text{add}}{}^{\beta\rho}{}_\mu = \left[Z^\rho{}_\mu{}^\beta + 2Z^{\rho\nu\beta\alpha}\partial_\alpha g_{\mu\nu} + \frac{4}{3}\partial_\alpha \left(Z^{\beta\nu\rho\alpha} g_{\mu\nu} \right) \right]^{\beta\rho}, \tag{85}$$

and quantities $Z^{\rho\mu\beta}$, $Z^{\rho\nu\beta\alpha}$ are defined in (66), (67). By replacing $\Psi_{\text{GR}}{}^{\beta\rho}{}_\mu$ with the Møller superpotential (70) as above, the canonical pEMT finally takes the antisymmetric form:

$$\mathcal{T}^\rho{}_\lambda = \partial_\beta \left(\Psi_{\text{M}}{}^{\beta\rho}{}_\lambda + \Psi_{\text{add}}{}^{\beta\rho}{}_\lambda \right). \tag{86}$$

Thus, the local conservation law (56) for pEMT (84) is satisfied without any equations of motion, and the pEMT itself can be identically rewritten through the superpotential. Interestingly enough, these properties also hold in the case $S_m \neq 0$ with the satisfied equations of motion of matter. In a way, the embedding gravity can be treated as the "extremal case" of the theory whose metric components are isolated by a change like (81). Mimetic change isolates the conformal mode of the metric into the kinetic term of the scalar field, which makes the corresponding to that scalar field equation of motion unnecessary for the pEMT continuity equation. In contrast, for the embedding gravity, all degrees of freedom find themselves in that position.

An another interesting aspect of embedding gravity is worth mentioning, namely the geometrical interpretation of the second Noether theorem (identity (39)) in this case. At the beginning of Section 3, we noted that the second Noether theorem (39) establishes the relation between the equations of motion arising from the general covariance of the action. For theory with the disformal change (75), this relation comes down to the fact that the equations of motion corresponding to the σ are automatically satisfied for any solution of the equation of motion corresponding to $\bar{g}_{\mu\nu}$. It makes the absence of requirement $\delta S/\delta\sigma = 0$ in the continuity equation for the pEMT (77) (by taking (78) into an account) quite natural. For the embedding theory, identity (39) has the following form:

$$\frac{\delta S}{\delta y^a} H_{(0)}{}^a{}_\mu = 0, \qquad H_{(0)}{}^a{}_\mu = \partial_\mu y^a. \tag{87}$$

It is obvious that the set $\{\partial_\mu y^a, \mu = 0\ldots3\}$ forms a basis in the tangent space of the surface $\mathcal{M}$. So, formula (87) geometrically means that the tangent (with respect to $\mathcal{M}$) part of the equations of motion for the embedding gravity hold identically, which is a well-known fact for the Regge–Teitelboim theory equations (see, for example, [40]). Relation (87) also makes (20) consistent with local conservation law (56) for pEMT (86).

6. Concluding Remarks

Despite the strong connection with the Noether procedure described in formulae (5)–(22), the choice of $J^\rho(\xi)$ precisely in the form (9) for construction of the recurrent chain (20)–(22) is not unique. Indeed, for theories discussed in the Section 4 with the change in the independent variables (57) in action one may start from the identities (8) for the original theory instead of the new ones and rewrite them in form:

$$\partial_\rho J^\rho - \left(\frac{\delta S}{\delta y_a} \delta y_a - \frac{\delta S}{\delta y'_E} \delta y'_E \right) = \frac{\delta S}{\delta y'_E} \delta y'_E \tag{88}$$

(for simplicity we set $S_m = 0$; however, the general case actually does not bring anything new). The second term can be further brought to form of the full divergence by using the same logic which was used in simplifying of the r.h.s. of relation (58). The resulting relation will look like (8) for the modified theory, and the final expression for the new definition of J^ρ will be equal to the r.h.s of (62) with $I^\rho = 0$, which may differ from the direct calculation based on the formulae from Section 2. Another example of the alternative procedure of deriving the densities for conserved charges is provided by the calculation of the superpotential for Regge–Teitelboim embedding gravity in [46]. In that paper, certain identities for the original theory (namely, GR) are used, and in the end, the obtained superpotential $\Psi^{\beta\rho}{}_\mu$ coincided with the Moller one, and hence it differs from the expression in Section 5.4 by addition (85). Nevertheless, it can be shown that J^ρ with the definition from Section 2 in the case of the first-order derivative theory is closely connected with the Ward identities arising in the quantization of the gauge theories, particularly for the generally covariant ones [13]. We expect this property to hold in general. In this regard, it is interesting how obtained formula (62) is reflected in the general form of Ward identities for BMS symmetry (see, for example, [47,48]) for theories with change (57). This topic goes beyond the discussion in the current paper and is a subject of further studies.

Another application of the results obtained is to use formulae (62), (85) and (80) for the analysis of cosmological perturbations in some GR modifications. As it is shown in the papers [3,49], the conservation laws may be used for derivation of the integral constrains, introduced originally in [50] for FLRW metrics. The existence of such constraints significantly reduces the Sachs–Wolfe effect [51] on the mean value of angular fluctuations of the cosmic background radiation. At the moment, this question is scarcely been explored for theories with change (57), and needs to be developed further.

Throughout the paper, formula (17) was used only for the analysis of the canonical pEMT, which corresponds to the conserved energy–momentum vector (36). This identity can also be used to calculate the pseudotensor $U^{\mu\alpha\beta}$ which defines the density of the total angular momentum tensor $M^{\alpha\beta}$ and the superpotential for it. To do this, one should again consider a diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ and write down the infinitesimal diffeomorphism parameter in the special non-covariant form:

$$\xi^\mu(x) = \omega_{\alpha\beta}(x)\hat{M}^{\mu\alpha\beta}(x), \qquad \hat{M}^{\mu\alpha\beta}(x) \equiv (x^\alpha \eta^{\mu\beta} - x^\beta \eta^{\mu\alpha}), \tag{89}$$

where $\eta^{\mu\beta}$ denotes metric of the Minkowski space, and $\omega_{\alpha\beta}(x)$ denoted the arbitrary function with the antisymmetry property $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. If one substitutes this expression for ξ^μ in (17) and then use the arbitrariness of $\omega_{\alpha\beta}$ and its derivatives at one point, the new chain can be derived, similar to the one defined in (20)–(22). This chain leads to the equations for the quantities associated with angular momentum tensor, analogous to (23) and (36) with the recalculated quantities $K_{(1)a\mu}{}^{\alpha_1..\alpha_k}$. Though $U^{\mu\alpha\beta}$ is not a tensor with respect to the diffeomorphisms, it can be easily verified that the

corresponding angular momentum $M^{\alpha\beta}$ is still a tensor with respect to the Lorentz group. The reason for this is obviously one's ability to treat $M^{\alpha\beta}$ as a surface term as for energy–momentum vector (36).

References

1. Baryshev, V.Y. Energy-Momentum of the Gravitational Field: Crucial Point for Gravitation Physics and Cosmology. In Proceedings of the International Conference Problems of Practical Cosmology, St.-Petersburg, Russia, 23–27 June 2008; Volume 1, pp. 276–286.
2. Petrov, A.; Kopeikin, S.; Lompay, R.; Tekin, B. *Metric Theories of Gravity*; De Gruyter: Berlin, Germany, 2017.
3. Katz, J.; Bicak, J.; Lynden-Bell, D. Relativistic conservation laws and integral constraints for large cosmological perturbations. *Phys. Rev. D* **1997**, *55*, 5759. [CrossRef]
4. Babak, S.V.; Grishchuk, L.P. Energy-momentum tensor for the gravitational field. *Phys. Rev. D* **1999**, *61*, 024038. [CrossRef]
5. Paston, S.A. Gravity as a field theory in flat space-time. *Theor. Math. Phys.* **2011**, *169*, 1611–1619. [CrossRef]
6. Thorne, K.; Hartle, J. Laws of motion and precession for black holes and other bodies. *Phys. Rev. D* **1985**, *31*, 1815–1837. [CrossRef]
7. Favata, M. Energy localization invariance of tidal work in general relativity. *Phys. Rev. D* **2001**, *63*, 064013. [CrossRef]
8. Purdue, P. Gauge invariance of general relativistic tidal heating. *Phys. Rev. D* **1999**, *60*, 104054. [CrossRef]
9. Goldberg, J.N. Conservation Laws in General Relativity. *Phys. Rev.* **1958**, *111*, 315–320. [CrossRef]
10. Bergmann, P.G. Non-Linear Field Theories. *Phys. Rev.* **1949**, *75*, 680–685. [CrossRef]
11. Moller, C. On the localization of the energy of a physical system in the general theory of relativity. *Ann. Phys.* **1958**, *4*, 347–371. [CrossRef]
12. Komar, A. Covariant Conservation Laws in General Relativity. *Phys. Rev.* **1959**, *113*, 934–936. [CrossRef]
13. Avery, S.; Schwab, B. Noether's second theorem and Ward identities for gauge symmetries. *J. High Energy Phys.* **2016**, *2016*, 31. [CrossRef]
14. Noether, E. Invariant variation problems. *Gott. Nachr.* **1918**, *1918*, 235–257. [CrossRef]
15. Chamseddine, A.H.; Mukhanov, V. Mimetic dark matter. *J. High Energy Phys.* **2013**, *2013*, 135. [CrossRef]
16. Biswas, T.; Gerwick, E.; Koivisto, T.; Mazumdar, A. Towards Singularity- and Ghost-Free Theories of Gravity. *Phys. Rev. Lett.* **2012**, *108*, 031101. [CrossRef]
17. Frederico Arroja, F.; Bartolo, N.; Karmakar, P.; Matarrese, S. The two faces of mimetic Horndeski gravity: Disformal transformations and Lagrange multiplier. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 51. [CrossRef]
18. Julia, B.; Silva, S. Currents and Superpotentials in classical gauge invariant theories I. Local results with applications to Perfect Fluids and General Relativity. *Class. Quantum Gravity* **1998**, *15*, 2173–2215. [CrossRef]
19. Barnich, G.; Brandt, F. Covariant theory of asymptotic symmetries, conservation laws and central charges. *Nucl. Phys. B* **2002**, *633*, 3–82. [CrossRef]
20. Bekenstein, J.D. Relation between physical and gravitational geometry. *Phys. Rev. D* **1993**, *48*, 3641–3647. [CrossRef]
21. Regge, T.; Teitelboim, C. General relativity à la string: A progress report. *arXiv* **2016**, arXiv:1612.05256.
22. Ilin, R.V.; Paston, S.A. Exact relation between canonical and metric energy-momentum tensors for higher derivative tensor field theories. *Eur. Phys. J. Plus* **2019**, *134*, 21. [CrossRef]
23. Gotay, M.J.; Marsden, J.E. Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula. In *Mathematical Aspects of Classical Field Theory*; Gotay, M.J., Marsden, J.E., Moncrief, V., Eds.; American Mathematical Society: Seattle, WA, USA, 1991; pp. 367–392.
24. Belinfante, F.J. On the spin angular momentum of mesons. *Physica* **1939**, *6*, 887–898. [CrossRef]

25. Belinfante, F.J. On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields. *Physica* **1940**, *7*, 449–474. [CrossRef]

26. Sheykin, A.A.; Solovyev, D.P.; Sukhanov, V.V.; Paston, S.A. Modifications of gravity via differential transformations of field variables. *Symmetry* **2020**, *12*, 240. [CrossRef]

27. Einstein, A. Einheitliche Feldtheorie von Gravitation und Elektrizitat. *Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl.* **1925**, 414–419.

28. Ferraris, M.; Francaviglia, M.; Reina, C. Variational formulation of general relativity from 1915 to 1925 "Palatini's method" discovered by Einstein in 1925. *Gen. Relativ. Gravit.* **1982**, *14*, 243–254. [CrossRef]

29. Barvinsky, A.O. Dark matter as a ghost free conformal extension of Einstein theory. *J. Cosmol. Astropart. Phys.* **2014**, *2014*, 014. [CrossRef]

30. Chamseddine, A.H.; Mukhanov, V.; Vikman, A. Cosmology with Mimetic Matter. *J. Cosmol. Astropart. Phys.* **2014**, *06*, 017. [CrossRef]

31. Sebastiani, L.; Vagnozzi, S.; Myrzakulov, R. Mimetic Gravity: A Review of Recent Developments and Applications to Cosmology and Astrophysics. *Adv. High Energy Phys.* **2017**, *2017*, 3156915. [CrossRef]

32. Paston, S.A. Forms of action for perfect fluid in general relativity and mimetic gravity. *Phys. Rev. D* **2017**, *96*, 084059. [CrossRef]

33. Golovnev, A. On the recently proposed mimetic Dark Matter. *Phys. Lett. B* **2014**, *728*, 39–40. [CrossRef]

34. Friedman, A. Local isometric embedding of Riemannian manifolds with indefinite metric. *J. Math. Mech.* **1961**, *10*, 625.

35. Deser, S.; Pirani, F.A.E.; Robinson, D.C. New embedding model of general relativity. *Phys. Rev. D* **1976**, *14*, 3301–3303. [CrossRef]

36. Pavsic, M. On The Quantization Of Gravity By Embedding Space-Time In A Higher Dimensional Space. *Class. Quantum Gravity* **1985**, *2*, 869. [CrossRef]

37. Tapia, V. Gravitation a la string. *Class. Quantum Gravity* **1989**, *6*, L49. [CrossRef]

38. Estabrook, F.B.; Robinson, R.S.; Wahlquist, H.R. Constraint-free theories of gravitation. *Class. Quantum Gravity* **1999**, *16*, 911–918. [CrossRef]

39. Karasik, D.; Davidson, A. Geodetic Brane Gravity. *Phys. Rev. D* **2003**, *67*, 064012. [CrossRef]

40. Paston, S.A.; Franke, V.A. Canonical formulation of the embedded theory of gravity equivalent to Einstein's general relativity. *Theor. Math. Phys.* **2007**, *153*, 1582–1596. [CrossRef]

41. Cordero, R.; Molgado, A.; Rojas, E. Ostrogradski approach for the Regge-Teitelboim type cosmology. *Phys. Rev. D* **2009**, *79*, 024024. [CrossRef]

42. Faddeev, L.D. New dynamical variables in Einstein's theory of gravity. *Theor. Math. Phys.* **2011**, *166*, 279–290. [CrossRef]

43. Paston, S.A.; Sheykin, A.A. From the Embedding Theory to General Relativity in a result of inflation. *Int. J. Mod. Phys. D* **2012**, *21*, 1250043. [CrossRef]

44. Sheykin, A.A.; Paston, S.A. The approach to gravity as a theory of embedded surface. *AIP Conf. Proc.* **2014**, *1606*, 400.

45. Paston, S.A.; Sheykin, A.A. Embedding theory as new geometrical mimetic gravity. *Eur. Phys. J. C* **2018**, *78*, 989. [CrossRef]

46. Grad, D.A.; Ilin, R.V.; Paston, S.A.; Sheykin, A. Gravitational energy in the framework of embedding and splitting theories. *Int. J. Mod. Phys. D* **2018**, *27*, 1750188. [CrossRef]

47. Bondi, H.; Van der Burg, M.G.J.; Metzner, A.W.K. Gravitational waves in general relativity, VII. Waves from axi-symmetric isolated system. *Proc. R. Soc. Lond. A* **1962**, *269*, 21–52.

48. Hawking, S.; Perry, M.; Strominger, A. Soft Hair on Black Holes. *Phys. Rev. Lett.* **2016**, *116*, 231301. [CrossRef] [PubMed]

49. Petrov, A.; Katz, J. Conserved currents, superpotentials and cosmological perturbations. *Proc. R. Soc. Lond. A* **2002**, *458*, 319–337. [CrossRef]

50. Traschen, J. Constraints on stress-energy perturbations in general relativity. *Phys. Rev. D* **1985**, *31*, 283–289. [CrossRef] [PubMed]

51. Traschen, J.; Eardley, D.M. Large-scale anisotropy of the cosmic background radiation in Friedmann universes. *Phys. Rev. D* **1986**, *34*, 1665–1679. [CrossRef]