

How Extra Symmetries Affect Solutions in General Relativity

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Abstract: To get exact solutions to Einstein’s field equations in general relativity, one has to impose some symmetry requirements. Otherwise, the equations are too difficult to solve. However, sometimes, the imposition of too much extra symmetry can cause the problem to become somewhat trivial. As a typical example to illustrate this, the effects of conharmonic flatness are studied and applied to Friedmann–Lemaître–Robertson–Walker spacetime. Hence, we need to impose some symmetry to make the problem tractable, but not too much so as to make it too simple.

Keywords: general relativity; symmetry; conharmonic flatness; FLRW models

1. Introduction

In 1915, Einstein formulated his theory of general relativity, whose field equations with cosmological term can be written in suitable units as:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}, \quad (1)$$

where R_{ab} is the Ricci tensor, R the Ricci scalar, g_{ab} the metric tensor, Λ the cosmological parameter, and T_{ab} the energy–momentum tensor. The field Equations (1) consist of a set of ten partial differential equations, which need to be solved. Despite great progress, it is worth noting that a clear definition of an exact solution does not exist [1]. What we usually understand by a solution in general relativity is the metric given in terms of elementary functions, such as polynomial, trigonometric, hyperbolic, etc. The idea is to try and find a complete general solution, or, alternatively, as many exact solutions as possible. Then, the physical interpretation of these solutions is presented. It must be borne in mind that, in general relativity, it is not only a local solution to the differential equations that is sought, but also a global analysis and topological methods are also required.

Any metric is a solution to the field Equation (1) if no restriction is placed on the energy–momentum tensor T_{ab} , but in all likelihood, the resulting solution will be unphysical. Thus, what is the alternative? There are various techniques that can be employed. A symmetry can be imposed on the metric, the algebraic structure of the Riemann tensor R_{abcd} or the Weyl tensor C_{abcd} can be restricted, and one can impose initial/boundary conditions, groups of motions, null tetrad methods, spinor, or generating techniques. From a historical point of view, using symmetry considerations, several important solutions have been found. These are static spherical symmetric (e.g., Schwarzschild [2]), stationary axi-symmetric (e.g., Kerr [3]), axisymmetric static (e.g., Weyl solution [4,5]), non-static spherically symmetric (e.g., FLRW [6–9]), and plane symmetric

(e.g., gravitational waves [10]). There do exist some solutions in general relativity which do not have any symmetry [11–16], but these solutions do not appear to be physically applicable as they stand. Some symmetry has to be imposed for physical applications.

In this paper, we consider the symmetry of conformal transformations, and, specifically, a subgroup of this class which imposes an extra symmetry, viz., conharmonic flatness. This symmetry has attracted the attention of several researchers recently. Ma and Pei [17] investigated generalized Lorentzian space-forms, in particular the necessary and sufficient conditions for them to be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric, and their relationships amongst one another. Shaikh et al. [18] have studied the curvature properties of the Vaidya metric in several manifolds, including conharmonically flat ones. The scalar curvature of a projectively flat and conharmonically flat eta-Einstein nearly Kenmotsu manifold have been obtained by Tekin and Atkan [19]. Tripathi and Rastogi [20] investigated an Einstein semisymmetric conharmonically flat $N(k)$ -contact metric manifold. Baishya and Eyasmin [21] proved that a conharmonically flat generalized weakly Ricci-symmetric (CS)(4)-spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ .

The metric connections with torsion of a conformally flat cotangent bundle were studied by Bilen and Gezer [22]. Singh and Kishor [23] provided the conditions to obtain solitons on conharmonically flat Lorentzian paraSasakian manifolds. The conharmonic curvature tensor with respect to the generalized Tanaka–Webster connection was studied by Prakasha and Hadimani [24]. Caliskan [25] investigated quasi-conharmonically flat, xi-conharmonically flat, and phi-conharmonically flat Sasakian Finsler structures on tangent bundles. Using the Schoutenvan Kampen connection, Yildiz [26] studied conharmonically flat three-dimensional f-Kenmotsu manifolds. De et al. [27] showed that, in a conharmonically flat spacetime with cyclic parallel Ricci tensor, the energy–momentum tensor is cyclic parallel and conversely. The flatness conditions of the conharmonic curvature tensor on normal complex contact metric manifolds were studied by Vanli and Unal [28]. Yildirim [29] proved that certain types of complex $\kappa\mu$ spaces cannot be conharmonically flat.

The full implications of this additional symmetry of conharmonic flatness on exact solutions in cosmology do not seem to have been fully appreciated by several authors.

2. Conharmonic Curvature Tensor

A conformal transformation between two manifolds is well-known:

$$\bar{g}_{ab} = e^{2\sigma} g_{ab}, \tag{2}$$

where σ is a scalar function of the coordinates x^c . There have been theories constructed from this transformation, e.g., the scale covariant theory [30] and Weyl gravity [4,5], or which obey the transformation in some way, e.g., Brans–Dicke theory [31]. What is the motivation to study the conharmonic curvature tensor, and conharmonically flat manifolds? A harmonic function $A(x^a)$ is one with zero Laplacian. In tensor notation, this can be written as

$$g^{ab} A_{;ab} = 0, \tag{3}$$

where the semicolon denotes the covariant derivative with respect to the metric g_{ab} . A harmonic function is not invariant in general under a conformal transformation. Ishii [32] was searching for the conditions under which a harmonic function remains invariant? To determine the condition upon σ , let us assume that, under the conformal transformation Equation (2), the function A transforms as

$$\bar{A} = e^{-\sigma} A. \tag{4}$$

We now search for the condition that the function \bar{A} is a harmonic function with respect to the metric \bar{g}_{ab} , i.e., we require

$$\bar{g}^{ab} \bar{A}_{\star ab} = 0, \tag{5}$$

where, by \star , we mean the covariant derivative with respect to the metric \bar{g}_{ab} .

Now, the Christoffel symbols Γ_{ab}^c transform as [30]:

$$\bar{\Gamma}_{ab}^c = \Gamma_{ab}^c + \delta_a^c \sigma_{,b} + \delta_b^c \sigma_{,a} - g_{ab} \sigma_{,c}, \tag{6}$$

where the comma refers to the ordinary derivative. From Equations (2) and (5), we obtain

$$\bar{g}^{ab} \bar{A}_{\star ab} = e^{-3\sigma} [A_{;a}{}^a - \sigma_{,a} \sigma^{,a} A - \sigma_{;a}{}^a A]. \tag{7}$$

Finally, from Equations (3) and (7), we obtain our required condition on σ as

$$\sigma_{;a}{}^a + \sigma_{,a} \sigma^{,a} = 0. \tag{8}$$

The subgroup of the transformation Equation (2) satisfying the condition Equation (8) constitutes a conharmonic transformation. Such transformations preserve the harmonicity property of smooth functions. This answered the question of when a harmonic function is invariant under a conformal transformation

The conharmonic curvature tensor is defined as:

$$L_{bcd}^a = R_{bcd}^a - \frac{1}{2} (g_{bc} R_d^a - g_{bd} R_c^a + \delta_d^a R_{bc} - \delta_c^a R_{bd}). \tag{9}$$

This is invariant under a conharmonic transformation, and has the same symmetry properties as the Riemann tensor. A spacetime in which L_{bcd}^a vanishes everywhere is called conharmonically flat, i.e.,

$$L_{bcd}^a = 0 \iff \text{conharmflat}. \tag{10}$$

The contraction of Equation (9):

$$L_{ab} \equiv L_{acb}^c = -\frac{1}{2} R g_{ab} \tag{11}$$

is also invariant under a conharmonic transformation. There have been several mathematical results which have been obtained for such manifolds, e.g., a conharmonically flat manifold is an Einstein space ($R_{ab} \propto g_{ab}$), and also a space of constant curvature [33].

3. Application to Cosmology

For conharmonically flat spacetime, we have Equation (9):

$$L_{bcd}^a = R_{bcd}^a - \frac{1}{2} (g_{bc} R_d^a - g_{bd} R_c^a + \delta_d^a R_{bc} - \delta_c^a R_{bd}) = 0. \tag{12}$$

Contracting this, we get

$$R g_{ab} = -4 R_{ab}, \tag{13}$$

Substituting this into the field Equations (1), we get

$$3R_{ab} + \Lambda g_{ab} = T_{ab}, \tag{14}$$

or

$$-\frac{3}{4} R g_{ab} + \Lambda g_{ab} = T_{ab}. \tag{15}$$

Several authors have found solutions to the modified field Equations (14) [34–38]. However, there is a serious problem with these solutions. They do not satisfy the “constraint” condition Equation (13).

We illustrate this by means of an example studied by Kumar and Srivastava [39]. For the Robertson–Walker metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{(1+kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{16}$$

the field Equations (14) yield

$$9\frac{\ddot{a}}{a} = -\rho + \Lambda, \tag{17}$$

$$3\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right) = p - \Lambda. \tag{18}$$

In the two equations above, ρ and p refer to the density and pressure, respectively, of a perfect fluid. These are determined from the diagonal components of the corresponding energy–momentum tensor T_{ab} in a comoving coordinate system. By assuming that $H = \text{const} \equiv B$, one of the solutions they found was $a = Ae^{Bt}$. Let us check if Equation (13) is satisfied. Let us examine the (00) component ($Rg_{00} = -4R_{00}$) of Equation (13). The left side is:

$$LHS = Rg_{00} = -12B^2 - \frac{6k}{A^2e^{2Bt}}, \tag{19}$$

whereas the right side is

$$RHS = -4R_{00} = 12B^2. \tag{20}$$

We see that the constraint Equation (13) is not satisfied UNLESS $k = 0, B = 0$, i.e., a static solution. This is also consistent with the other components of Equation (13).

The other solution found by these authors by taking $H \propto a^{-n}$ was a power law solution, $a = (nat + \beta)^{1/n}$. Again, it may be shown from Equation (13) that the only possible solution is the Milne universe ($a \propto t$).

4. Additional Symmetry Requirement

For conharmonic flatness, Equation (13) must be satisfied, i.e., $Rg_{ab} = -4R_{ab}$. However, if we contract this equation, we find that we end up with the “vacuum” condition $R_{ab} = 0$. This is a very strong imposition. Hence, the additional symmetry requirement of conharmonic flatness reduces the space of solutions to some “vacuum” solutions in general relativity. For FLRW spacetimes, we can only get a static ($a = \text{const}$) or Milne-type solution ($a \propto t$). Even if we include a cosmological term, we do not change the above two solutions. This is in contrast to general relativity where, if we include a cosmological term, we have the deSitter exponential solution ($\rho = p = 0, \Lambda = \text{constant}$). A curious feature of conharmonic flatness is that with $\Lambda \neq 0$, we can get solutions with nonzero density ρ and pressure p .

5. Conclusions

Since Einstein’s equations are difficult to solve, we usually have to impose some symmetry requirements. In this work, we have imposed the requirement of conharmonic flatness, and studied its effect in cosmology. It is found that such solutions are reduced to vacuum solutions of general relativistic cosmology. The imposition of too much symmetry can thus make the problem almost trivial. Thus, we have to strike a balance between the complexity of the problem and the symmetry imposed.

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