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Lorentz Group Projector Technique for Decomposing Reducible Representations and Applications to High Spins

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Abstract: The momentum-independent Casimir operators of the homogeneous spin-Lorentz group are employed in the construction of covariant projector operators, which can decompose anyone of its reducible finite-dimensional representation spaces into irreducible components. One of the benefits from such operators is that any one of the finite-dimensional carrier spaces of the Lorentz group representations can be equipped with Lorentz vector indices because any such space can be embedded in a Lorentz tensor of a properly-designed rank and then be unambiguously found by a projector. In particular, all the carrier spaces of the single-spin-valued Lorentz group representations, which so far have been described as $2(2j + 1)$ column vectors, can now be described in terms of Lorentz tensors for bosons or Lorentz tensors with the Dirac spinor component, for fermions. This approach facilitates the construct of covariant interactions of high spins with external fields in so far as they can be obtained by simple contractions of the relevant $SO(1,3)$ indices. Examples of Lorentz group projector operators for spins varying from $1/2$ – 2 and belonging to distinct product spaces are explicitly worked out. The decomposition of multiple-spin-valued product spaces into irreducible sectors suggests that not only the highest spin, but all the spins contained in an irreducible carrier space could correspond to physical degrees of freedom.

Keywords: homogenous Lorentz group; high spins; covariant projectors; decomposition of tensor products

1. Introduction

Particles of high-spins $j \geq 1$, be they massive or mass-less, play a significant role in field theories. In the physics of hadrons, such fields appear as real resonances whose spins can vary from $1/2$ – $17/2$ for baryons and from 0 – 6 for mesons [1]. At hadron colliders, they can emerge as intermediate resonances in a variety of processes, while in gravity, deformations of the metric tensor caused by its coupling to high-spin bosons are of interest [2]. In addition, high spins are fundamental to the physics of rotating black holes [3], not to forget gravitational interactions between high-spin fermions [4]. The traditional methods in the description of high-spin fields were developed in the period between 1939 and 1964 (see [5,6] for recent reviews and [7] for a standard textbook) and are based on the use of carrier (representation) spaces of finite-dimensional representations of the homogeneous Lorentz group. They are associated with the names of Fierz and Pauli (FP) [8], Rarita and Schwinger (RS) [9], Laporte and Uhlenbeck (LU) [10], Cap and Donnert (CD) [11–13], Bargmann and Wigner (BW) [14], as well as to those of Joos [15] and Weinberg [16] (JW). Each one of them has some advantages over the others, which however are as a rule gained at the cost of some specific problems. The key point in

all the methods employed for high-spin description concerns the choice for the representation of the Lorentz group that embeds the spin of interest. In the literature, there are two qualitatively distinct kinds of finite-dimensional representations of the Lorentz group in use, those containing multiple spins as parts of reducible tensor products and those containing one sole spin and represented by irreducible column vectors. The tensor basis characteristic for the former case represents a significant advantage over the latter as it enables construction of covariant interactions with external fields by simple contractions of the $SO(1,3)$ indices. A disadvantage is the necessity to impose on such carrier space conditions, termed as “auxiliary”, and supposed to remove the unwanted spin content, an expectation that is met for free fields, but becomes problematic at the level of interactions. In the current work, we present the technique of the covariant spin-Lorentz group projectors, which allows equipping by Lorentz indices any finite-dimensional irreducible representation space through embedding in properly-designed tensor products. Our special interest concerns representation spaces containing one sole spin, in which case the problematic auxiliary conditions can be avoided, though it is applicable to any type of representations. Our study is basically addressed to practitioners interested in speedy calculations of cross-sections involving high spins, which can be executed by the aid of such symbolic software as FeynCalc and based on Lorentz index contractions.

The paper is organized as follows. In the next section, we bring a glossary of the methods for the high-spin description, which we classify according to the type of representation spaces used, and briefly discuss their virtues and problems. In Section 3, we turn our attention to the spin-Lorentz group projectors, the main subject of this article. Sections 4 and 5 contain examples of explicitly worked out covariant projectors on spins varying from $1/2$ –2. The text closes with a brief summary.

2. Glossary of Methods for High Spin Description: Virtues and Problems

2.1. Multiple-Spin Valued Representations with Lorentz Tensors as Carrier Spaces

2.1.1. Methods Based on Auxiliary Conditions

- The method by Fierz and Pauli: So far, the most popular representations for spin- j boson description are those of multiple spins, whose carrier spaces are Lorentz tensors of rank- j for spin j bosons (Fierz–Pauli (FP) [8]),

$$\begin{aligned} \text{Sym } \Phi_{\mu_1 \dots \mu_j} &\simeq \left(\frac{j}{2}, \frac{j}{2} \right), \\ J^\pi &= 0^+, 1^-, \dots, j^{(-1)^{2j}}. \end{aligned} \quad (1)$$

Here, J^π stands for the spin-values contained in the Lorentz tensor, while $\pi = \pm$ denotes the parity. The wave equations in the Fierz-Pauli approach to high-spin- j bosons read,

$$\begin{aligned} (\partial^2 + m^2) \text{Sym } \Phi_{\mu_1 \dots \mu_i \dots \mu_j} &= 0, \\ \text{tr } \text{Sym } \Phi_{\mu_1 \dots \mu_i \dots \mu_j} &= 0, \\ \partial^{\mu_i} \text{Sym } \Phi_{\mu_1 \dots \mu_i \dots \mu_j} &= 0. \end{aligned} \quad (2)$$

An example for an FP carrier space is given by the four-vector, $(1/2, 1/2)$, which contains the two spins 0^+ and 1^- :

$$\begin{aligned} A_\mu &\simeq \left(\frac{1}{2}, \frac{1}{2} \right), \\ J^\pi &\in 0^+, 1^-. \end{aligned} \quad (3)$$

- The method by Rarita and Schwinger: Fermions of spin- j are interpreted according to the work by Rarita and Schwinger [9] as the highest spin in a Lorentz tensor, $A_{\mu_1 \dots \mu_{j-1/2}}$, of rank- $(j - 1/2)$ with Dirac spinor, ψ , components as,

$$\begin{aligned} \text{Sym } \psi_{\mu_1 \dots \mu_{j-\frac{1}{2}}} &= A_{\mu_1 \dots \mu_{j-\frac{1}{2}}} \otimes \psi \\ &\simeq \left(\frac{j-\frac{1}{2}}{2}, \frac{j-\frac{1}{2}}{2} \right) \otimes \left[\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \right], \\ J^\pi &= \frac{1}{2}^+, \frac{1}{2}^-, \dots, (j-1)^+, (j-1)^-, j^{(-1)^{(2j)}}, \end{aligned} \quad (4)$$

where $\psi \simeq (1/2, 0) \oplus (0, 1/2)$ is the standard notation for a Dirac spinor. The wave equations of this approach are cast as,

$$\begin{aligned} (i\partial - m) \text{Sym } \psi_{\mu_1 \dots \mu_i \dots \mu_{j-\frac{1}{2}}} &= 0, \\ \gamma^{\mu_i} \text{Sym } \psi_{\mu_1 \dots \mu_i \dots \mu_{j-\frac{1}{2}}} &= 0, \\ \partial^{\mu_i} \text{Sym } \psi_{\mu_1 \dots \mu_i \dots \mu_{j-\frac{1}{2}}} &= 0. \end{aligned} \quad (5)$$

The role of the auxiliary conditions is to remove all the lower spin degrees of freedom and guarantee only those corresponding to the maximal spin. An example for an RS field is the four-vector spinor, ψ_μ , the direct product of the four vector $A_\mu \simeq (1/2, 1/2)$ in (3) with the Dirac spinor, $\psi \simeq (1/2, 0) \oplus (0, 1/2)$, which is used for the description of spin-3/2,

$$\begin{aligned} \psi_\mu &= A_\mu \otimes \psi \\ &\simeq \left(\frac{1}{2}, \frac{1}{2} \right) \otimes \left[\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \right], \\ J^\pi &\in \frac{1}{2}^+; \frac{1}{2}^-, \frac{3}{2}^-. \end{aligned} \quad (6)$$

The two spin- $(1/2)^\pm$ sectors of opposite parities are supposed to be removed by the auxiliary conditions.

The advantages of the above two methods are (i) the labeling of the representations by Lorentz (four-vector) indices, which facilitates the construction of interactions with external fields by index contractions, and (ii) the linear character of the differential equations for fermions.

However, the value of the aforementioned advantages is significantly diminished by the problems that plague the above two approaches and that are caused by the fact that neat auxiliary conditions can be formulated only for the case of free particles. For particles interacting with external fields, the auxiliary conditions no longer serve their purpose, and inconsistencies such as acausal propagation of the wave fronts of the solutions in the background of electromagnetic fields can occur, the so-called Velo–Zwanziger problem [17]. Problems of this kind represent furthermore serious obstacles toward the formulation of consistent quantum field theories for high spins [18] and even question the existence of such particles as fundamental objects beyond the experimentally-verified constituent level.

2.1.2. Methods Based on Poincaré Group Spin and Mass Projectors and without Auxiliary Conditions

The auxiliary conditions in the use of multiple-spin valued representations can be avoided by admitting for high-order differential equations. Such a case can emerge for example from projector operators constructed from the two Casimir invariants of the in-homogeneous Lorentz group, the Poincaré group, which are the squared linear momentum, p^2 , and the square, $\mathcal{W}^2$, of the

Pauli–Lubanski pseudovector, $\mathcal{W}_\mu$, a method considered by Aurilia and Umezawa (AU) [19]. The $\mathcal{W}_\mu$ and $\mathcal{W}^2$ operators are defined as:

$$\mathcal{W}_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\tau}M^{\nu\rho}p^\tau, \quad (7)$$

$$\mathcal{W}^2 = -\frac{1}{2}S_{\mu\nu}S^{\mu\nu}p^2 + S_{\mu\nu}S^{\mu\lambda}p^\nu p^\lambda, \quad (8)$$

where $S_{\mu\nu}$ and $M_{\mu\nu}$ are in turn the generators in a given representation (not necessarily reducible) of the homogeneous and inhomogeneous Lorentz groups, discussed in more detail in the opening of Section 3 below, while p_μ are the generators of translations in the external spacetime.

- The method by Aurilia and Umezawa: The action of the p^2 and $\mathcal{W}^2$ invariants on carrier spaces of Lorentz group representations, here generically denoted by $\psi^{(j_1,j_2)\oplus(j_2,j_1)}$, is as follows,

$$p^2\psi^{(j_1,j_2)\oplus(j_2,j_1)} = m^2\psi^{(j_1,j_2)\oplus(j_2,j_1)}, \quad (9)$$

$$\mathcal{W}^2\psi^{(j_1,j_2)\oplus(j_2,j_1)} = -p^2J(J+1)\psi^{(j_1,j_2)\oplus(j_2,j_1)}, \quad J = |j_1 - j_2|, \dots, (j_1 + j_2). \quad (10)$$

The two Casimir invariants can now be employed in the construction of projector operators on spin- J and mass m , here denoted by $\mathcal{P}^{(m,J)}$, exploiting the following properties,

$$\mathcal{P}^{(m,J)}\psi^{(j_1,j_2)\oplus(j_2,j_1)} = \psi^{J\in(j_1,j_2)\oplus(j_2,j_1)}, \quad (11)$$

$$\mathcal{P}^{(m,J)}\psi^{\tilde{J}\in(j_1,j_2)\oplus(j_2,j_1)} = 0, \quad \text{for } \tilde{J} \neq J. \quad (12)$$

The AU method is best illustrated on the example of the particularly simple case of a representation containing two spins, J and $(J-1)$, as is for example the four-vector spinor in (6). The most general two-spin representation is $\left(\frac{1}{2}, j\right) \oplus \left(j, \frac{1}{2}\right)$ with $J = j \pm 1/2$, and has been considered by Hurley in [20] as a $4(2j+1)$ component column vector satisfying a Dirac-type linear differential equation.

Notice that this carrier space, containing spin- $1/2^+$ and spin- $3/2^-$, is reducible according to:

$$\begin{aligned} \psi_\mu &\simeq \left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] \\ &\longrightarrow \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] \oplus \left[\left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right)\right] \\ &\simeq \psi_\mu^{(1/2,0)\oplus(0,1/2)} \oplus \psi_\mu^{(1,1/2)\oplus(1/2,1)}. \end{aligned} \quad (13)$$

In this case, the two operators, $\mathcal{P}^{(m,J)}(p)$, and $\mathcal{P}^{(m,(J-1))}(p)$, in turn defined as,

$$\mathcal{P}^{(m,J)}(p) = -\frac{1}{2J} \left(\frac{\mathcal{W}^2}{m^2} + J(J-1) \frac{p^2}{m^2} \right), \quad (14)$$

$$\mathcal{P}^{(m,(J-1))}(p) = \frac{1}{2J} \left(\frac{\mathcal{W}^2}{m^2} + J(J+1) \frac{p^2}{m^2} \right), \quad (15)$$

act as covariant spin and mass projectors on the irreducible sectors of ψ_μ as follows:

$$\mathcal{P}^{(m,3/2)}(p)\psi_\mu^{(1,1/2)\oplus(1/2,1)} = -m^2\frac{3}{2}\left(\frac{3}{2}+1\right)\psi_\mu^{3/2^-\in(1,1/2)\oplus(1/2,1)}, \quad (16)$$

$$\mathcal{P}^{(m,1/2)}(p)\psi_\mu^{(1,1/2)\oplus(1/2,1)} = -m^2\frac{1}{2}\left(\frac{1}{2}+1\right)\psi_\mu^{1/2^-\in(1,1/2)\oplus(1/2,1)}, \quad (17)$$

$$\mathcal{P}^{(m,1/2)}(p)\psi_\mu^{(1/2,0)\oplus(0,1/2)} = -m^2\frac{1}{2}\left(\frac{1}{2}+1\right)\psi_\mu^{1/2^+\in(1/2,0)\oplus(0,1/2)}, \quad (18)$$

$$\mathcal{P}^{(m,1/2)}(p)\psi_\mu^{3/2^-\in(1,1/2)\oplus(1/2,1)} = 0, \quad (19)$$

$$\mathcal{P}^{(m,3/2)}(p)\psi_\mu^{1/2^-\in(1,1/2)\oplus(1/2,1)} = 0, \quad (20)$$

$$\mathcal{P}^{(m,3/2)}(p)\psi_\mu^{1/2^+\in(1/2,0)\oplus(0,1/2)} = 0. \quad (21)$$

The principle advantage of the covariant spin and mass projector method over the method by Rarita and Schwinger lies in the absence of auxiliary conditions.

The disadvantages are (i) the increase of the order in the momenta of the wave equations like p^{2n} , with n being the number of the distinct redundant spins needed to be removed and (ii) the undetermined parities of the solutions to the projector operators for fermions. The second disadvantage is due to the fact that, while the spin-projector operators are of the order p^{2n} in the momenta, the covariant parity operators are of the order p^{2J} . There are though cases in which the equality $p^{2n} = p^{2J}$ can be reached. Such is the case of spin- 1^- as part of the four-vector, $(1/2, 1/2)$, where only one spin, namely, spin- 0^+ , has to be removed, a reason for which Proca's equation defined by the covariant projector on negative parity coincides with the equation following from the spin- 1^- projector [21]. Furthermore, for the case of spin- 2^+ as part of $(1, 1)$, a space from which the two spins 0^+ and 1^- need to be removed will be of the fourth order in the derivatives, the same as the order prescribed by the relevant parity projector. For fractional spins, in view of the fact that $2n$ is always even, while $2J$ is always odd, $p^{2n} \neq p^{2J}$, and one faces the loss of parity. Nonetheless, in [22,23], the $\mathcal{P}^{(m,J)}(p)$ -based wave equations for spin-1 and spin-3/2, both of second order in the momenta, have been shown to provide a very convenient point of departure toward a more elaborate scheme, specifically well suited for the realistic description of the couplings of such particles to an external electromagnetic field.

- The method by Napsuciale, Kirchbach, and Rodriguez for spin-3/2:

In [24], the Poincaré group projector method was introduced by the authors Napsuciale, Kirchbach, and Rodriguez (NKR), anew and independently of [19], as an alternative to the Rarita–Schwinger framework. There, the projector operator $\mathcal{P}^{(m,3/2)}(p)$ was explicitly constructed and critically analyzed. It has been observed that the only anti-symmetric term contained in it that couples to the anti-commutator of two derivatives, $[\partial^\mu, \partial^\nu]$, is $(-i)M_{\mu\nu}/3$, where $M_{\mu\nu}$ are the generators of the homogeneous Lorentz group in the four-vector spinor representation. Upon gauging, this term prescribes the gyromagnetic factor $g_{\frac{3}{2}}$ of the spin-3/2 particle to take the nonphysical value of $g_{\frac{3}{2}} = 1/3$. Such a case occurs because the squared Pauli–Lubanski operator does not contain the full set of anti-symmetric terms (vanishing at the non-interacting level) and allowed by the Lorentz symmetry to participate in the wave equation. Such terms acquire importance only upon gauging because of:

$$[\partial^\mu, \partial^\nu] \rightarrow [D^\mu, D^\nu] = ieF^{\mu\nu}, \quad D^\mu = \partial^\mu - ieA^\mu, \quad (22)$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor. Stated differently, the space-time symmetries alone are insufficient to predict all the possible couplings to external fields required by the dynamics. In order to remove this ambiguity, the strategy pursued in [24] has been to include at the free particle level all the possible anti-symmetric terms compatible with Lorentz

invariance, weight them by free parameters, and fix the latter by physical arguments. In so doing, the following minimally-gauged wave equation for spin-3/2 transforming within ψ_μ has been obtained:

$$\left((\pi^2 - m^2)g_{\alpha\beta} - ig_{\frac{3}{2}} \left(\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2} g_{\alpha\beta} - eF_{\alpha\beta} \right) + \frac{1}{3}(\gamma_\alpha\pi\!\!\!/ - 4\pi_\alpha)\pi_\beta + \frac{1}{3}(\pi_\alpha\pi\!\!\!/ - \gamma_\alpha\pi^2)\gamma_\beta \right) \psi^\beta = 0, \quad (23)$$

where $\pi^\mu = p^\mu + eA^\mu$ and $g_{\frac{3}{2}}$ is a free parameter. In requiring this equation to satisfy the Current–Hilbert criterion for having solutions whose wave fronts are causally propagating, the $g_{\frac{3}{2}}$ parameter could be fixed to $g_{\frac{3}{2}} = 2$, a value that in addition ensures the unitarity of the forward Compton scattering cross-sections [23]. In this fashion, the compatibility of the Poincaré group projectors with minimal gauging has been demonstrated, and the major Velo–Zwanziger problem of the Rarita–Schwinger framework could be avoided in the NKR method.

The natural way out of the problems of representations of multiple spins is turning one’s attention to single-spin-valued representations.

2.2. Single-Spin-Valued Representations with Column Vectors as Carrier Spaces

The first time that a carrier space of a single-spin-valued bosonic representation has been considered refers to spin-one and is due to work by Laporte and Uhlenbeck [10], who provided an explicit construct of the $(1,0) \oplus (0,1)$ Lorentz group representation space as a totally symmetric tensor of rank-two with $SL(2, C)$ spinor indices, also termed as Weyl spinor indices, and applied it to the description of the electromagnetic field in Maxwell’s theory. Later, Cap and Donnert generalized the method of Laporte and Uhlenbeck to the description of any spin j by means of totally symmetric rank- $2j$ multispinors. Unfortunately, these works remained largely unknown to the community and will be presented in due place below.

The Method of Joos and Weinberg

So far, the method of single spins of the widest spread is due to Joos [15] and Weinberg [16], and the bases of the irreducible representations are described in terms of column-vectors of $2(2j + 1)$ components, the carries spaces of the $(j,0) \oplus (0,j)$ representations of the Lorentz group,

$$\Psi_B^{(j)} \simeq (j,0) \oplus (0,j) = \begin{pmatrix} \psi_1^{(j)} \\ \vdots \\ \psi_{2j+1}^{(j)} \\ \psi_{(2j+1)+1}^{(j)} \\ \vdots \\ \psi_{2(2j+1)}^{(j)} \end{pmatrix}, \quad B \in [1, 2(2j+1)]. \quad (24)$$

The first advantage of the JW method lies in the absence of auxiliary conditions, and the second is that the solutions of (24) are of well-defined parity, this being because the equations have been obtained from the covariant parity projectors on the representation spaces under consideration.

Yet, the idea of switching from multiple- to single-spin-valued representations has not been quite prolific, indeed partly because of the loss of Lorentz indices in the labeling of the latter spaces, which rules out the comfortable construction of interaction vertices through Lorentz index contractions, habitual for the FP and RS methods.

Moreover, differently from the RS approach, the $2(2j + 1)$ -component wave function, $\Psi_B^{(j)}$, satisfies one sole differential equation, which is however of the high-order ∂^{2j} according to:

$$\left(i^{2j} \left[\gamma_{\mu_1 \mu_2 \dots \mu_{2j}} \right]_{AB} \partial^{\mu_1} \partial^{\mu_2} \dots \partial^{\mu_{2j}} - m^{2j} \delta_{AB} \right) \Psi_B^{(j)}(x) = 0. \quad (25)$$

Here, $\left[\gamma_{\mu_1 \mu_2 \dots \mu_{2j}} \right]_{AB}$ are the elements of the generalized Dirac Hermitian matrices of dimensionality $[2(2j + 1)] \times [2(2j + 1)]$, which transform as Lorentz tensors of rank- $2j$. Differential equations of orders higher than two can present various pathologies, among them the so-called Ostrogradsky instability, and allow unwanted nonphysical solutions to exist [25], all problems that motivate searches for different schemes. The Joos–Weinberg column vectors, the only irreducible representation spaces of the homogeneous Lorentz group, are parity invariant and can only be employed in the description of a pair of particles with opposite spatial parities. While for fermions, the opposite parities are associated with particles and anti-particles, in the boson sector, parity doubling is not the rule, despite some isolated examples known for constituent mesons. So far, only the single-spin-one representation space, $(1, 0) \oplus (0, 1)$, enjoys lasting popularity as the field strength tensor for spin-one gauge bosons.

2.3. Single-Spin-Valued Representations with Multispinors as Carrier Spaces

2.3.1. The Method of Laporte and Uhlenbeck for Spin-One

A different approach to single-spin-valued representation spaces has been pioneered by Laporte and Uhlenbeck [10], and it refers to the carrier space of the $(1, 0) \oplus (0, 1)$ Lorentz group representation. There, the $(1, 0)$ part is described as,

$$\begin{aligned} (1, 0) : \quad f_{11} &= \xi_1 \xi_1, & f_{22} &= \xi_2 \xi_2, \\ f_{12} &= f_{21} = \frac{1}{2} (\xi_1 \xi_2 + \xi_2 \xi_1), \end{aligned} \quad (26)$$

while the $(0, 1)$ part is associated with:

$$\begin{aligned} (0, 1) : \quad f^{\dot{1}\dot{1}} &= \eta_{\dot{1}} \eta_{\dot{1}}, & f^{\dot{2}\dot{2}} &= \eta_{\dot{2}} \eta_{\dot{2}}, \\ f^{\dot{1}\dot{2}} &= f^{\dot{2}\dot{1}} = \frac{1}{2} (\eta_{\dot{1}} \eta_{\dot{2}} + \eta_{\dot{2}} \eta_{\dot{1}}). \end{aligned} \quad (27)$$

Here, a two-dimensional vector, ξ , whose complex components are ξ^α with $\alpha = 1, 2$, termed as the Weyl spinor, has been introduced as one of the two nonequivalent fundamental representations of the $SL(2, C)$ group, the universal covering of the Lorentz group. The carrier space of the other fundamental $SL(2, C)$ representation is defined by the so-called co-spinor, here denoted by $\dot{\eta}$, whose components are $\eta_{\dot{\beta}}$, with $\dot{\beta} = \dot{1}, \dot{2}$. The two spinors under discussion correspond in turn to the carrier spaces of the left- and $(1/2, 0)$ and right-handed $(0, 1/2)$ representations,

$$\begin{aligned} \left(0, \frac{1}{2} \right) : \quad \xi &= \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \\ \left(\frac{1}{2}, 0 \right) : \quad \dot{\eta} &= \begin{pmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{pmatrix}. \end{aligned} \quad (28)$$

They are related by charge conjugation according to [26,27],

$$\begin{pmatrix} \eta_1^\bullet \\ \eta_2^\bullet \end{pmatrix} = C \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}^*, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (29)$$

where C is the metric tensor in spinor space, while “ $*$ ” denotes complex conjugation. The C matrix (equal to the two-dimensional Levi-Civita tensor $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$) serves to raise and lower indices in spinor/co-spinor space according to $\zeta_\alpha = \epsilon_{\alpha\beta} \zeta^\beta$, $\zeta^\alpha = \epsilon^{\alpha\beta} \zeta_\beta$, amounting to:

$$\zeta_1 = \epsilon_{12} \zeta^2 = \zeta^2, \quad \zeta_2 = \epsilon_{21} \zeta^1 = -\zeta^1, \quad (30)$$

$$\eta_1^\bullet = \eta^2, \quad \eta_2^\bullet = -\eta^1. \quad (31)$$

Correspondingly, a four-derivative with spinor indices is introduced on the basis of the four-derivative ∂^μ as:

$$i\partial^\mu \longrightarrow i\partial^{\alpha\beta} = i \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} = i\partial^0 \sigma_0 + i \sum_i \partial_i \sigma^i = p^0 \sigma_0 + \vec{p} \cdot \vec{\sigma},$$

$$i\partial_\mu = i \frac{\partial}{\partial x^\mu} \Rightarrow p_\mu, \quad (32)$$

$$i\partial_\mu \longrightarrow i\partial_{\alpha\beta} = i \begin{pmatrix} \partial^0 + \partial^3 & \partial^1 + i\partial^2 \\ \partial^1 - i\partial^2 & \partial^0 - \partial^3 \end{pmatrix} = i\partial^0 [\sigma_0]^T + \sum_i i\partial^i [\sigma^i]^T = p^0 \sigma_0 - \vec{p} \cdot \vec{\sigma}^T,$$

$$i\partial^\mu = i \frac{\partial}{\partial x_\mu} \Rightarrow p^\mu, \quad (33)$$

where the upper script T stands for “transpose”, and we use metric signature $(+, -, -, -)$.

Spinors and co-spinors then satisfy the following kinematic equations,

$$i\partial^{\alpha\beta} \eta_\beta^\bullet = m \zeta^\alpha, \quad (34)$$

$$i\partial_{\alpha\beta} \zeta^\alpha = m \eta_\beta^\bullet, \quad (35)$$

where m is a constant mass. According to the LU approach, the Maxwell equations are encoded by the following equations,

$$i\partial^{\alpha\beta} f_{\beta\gamma}^\bullet = \mu_0 J^\alpha_{\gamma}, \quad J^{\alpha\gamma} = \begin{pmatrix} J^0 - J^3 & -J^1 + iJ^2 \\ -J^1 - iJ^2 & J^0 + J^3 \end{pmatrix}, \quad (36)$$

where J^α_{γ} defines the four-current in spinor space and transforms according to the carrier space of the $(1/2, 1/2)$ representation. Notice that the latter can be written as $(j - 1/2, 1/2)$ for $j = 1/2$, an observation to become relevant in what follows. Similarly, for the $(0, 1)$ part,

$$i\partial^{\alpha\beta} f_{\alpha\delta}^\bullet = \mu_0 J^\beta_{\delta}, \quad (37)$$

holds valid.

2.3.2. The Method of Cap and Donnert for Charged Particles of Any Spin

The Laporte-Uhlenbeck method to single-spin-one has been generalized to any single-spin- j by Cap and Donnert. The carrier spaces of the single-spin-valued $(j, 0) \oplus (0, j)$ representations have been

described in [11–13] in terms of four types of spinor-tensors. The first two are the totally symmetric rank- $2j$ tensors,

$$\begin{aligned}(j, 0) : \quad & f_{v_1 \dots v_{2j}} = \mathbf{Sym} \, \xi_{v_1} \xi_{v_2} \dots \xi_{v_{2j}}, \\ (0, j) : \quad & f^{\dot{\mu}_1 \dots \dot{\mu}_{2j}} = \mathbf{Sym} \, \eta^{\dot{\mu}_1} \eta^{\dot{\mu}_2} \dots \eta^{\dot{\mu}_{2j}},\end{aligned}\tag{38}$$

and the third and the fourth spinor tensors are auxiliary and assumed as:

$$\begin{aligned}\left(j - \frac{1}{2}, \frac{1}{2}\right) : \quad & J^{\dot{\mu}_1}{}_{v_2 \dots v_{2j}}, \\ \left(\frac{1}{2}, j - \frac{1}{2}\right) : \quad & J^{\dot{\mu}_2 \dots \dot{\mu}_{2j}}{}_{v_1},\end{aligned}\tag{39}$$

here in contemporary notations. Then, the wave equations are extensions of the linear Dirac spinor equation and are given by:

$$\partial^{\dot{\mu}_1 v_1} J^{\dot{\mu}_2 \dots \dot{\mu}_{2j}}{}_{v_1} = i \frac{mc}{\hbar} f^{\dot{\mu}_1 \dots \dot{\mu}_{2j}},\tag{40}$$

$$\partial_{\dot{\mu}_1 v_1} f^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{2j}} = i \frac{mc}{\hbar} J^{\dot{\mu}_2 \dots \dot{\mu}_{2j}}{}_{v_1},\tag{41}$$

and:

$$\partial_{\dot{\mu}_1 v_1} J^{\dot{\mu}_1}{}_{v_2 \dots v_{2j}} = i \frac{mc}{\hbar} f_{v_1 v_2 \dots v_{2j}},\tag{42}$$

$$\partial^{\dot{\mu}_1 v_1} f_{v_1 v_2 \dots v_{2j}} = i \frac{mc}{\hbar} J^{\dot{\mu}_1}{}_{v_2 \dots v_{2j}},\tag{43}$$

correspondingly, where m is the mass parameter. The suggestion is to eliminate the generalized currents and end up with the following two second order differential equations,

$$\left(\square - \frac{m^2 c^2}{\hbar^2}\right) f_{v_1 v_2 \dots v_{2j}} = 0,\tag{44}$$

$$\left(\square - \frac{m^2 c^2}{\hbar^2}\right) f^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{2j}} = 0,\tag{45}$$

where $\square$ is the D'Alembertian. The virtue of this framework is that it does not suffer at the classical level the pathology of the Fierz–Pauli and Rarita–Schwinger methods, because upon gauging, it is transformed to:

$$\begin{aligned}\left(D_\mu D^\mu - \frac{m^2 c^2}{\hbar^2}\right) f_{v_1 v_2 \dots v_{2j}} &= \frac{e}{\hbar c} (\vec{S} \cdot \vec{F}) f_{v_1 v_2 \dots v_{2j}}, \\ \left(D_\nu D^\nu - \frac{m^2 c^2}{\hbar^2}\right) f^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{2j}} &= \frac{e}{\hbar c} (\vec{S} \cdot \vec{F}) f^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{2j}}, \\ D_\mu &= \partial_\mu + \frac{ie}{\hbar c} A_\mu, \quad \vec{F} = \vec{H} - i\vec{E}.\end{aligned}\tag{46}$$

Here, A_μ stands for the electromagnetic gauge field, D_μ is the covariant derivative, while $\vec{H}$ and $\vec{E}$ denote in turn the magnetic and electric field strengths. The components of the spin vector $\vec{S}$ are $(2j+1) \times (2j+1)$ matrices, which represent the generators of spin in $(j, 0)$ and $(0, j)$. Equations of this type are currently known as “generalized Feynman–Gell-Mann” equations in reference to a popular work by Feynman and Gell-Mann [28], in which the square of the gauged Dirac equation was shown to be shaped after (46). Second order equations of this type have well-behaved solutions whose wave

fronts satisfy the criterion of causal propagation (see [21] for a recent reference). Moreover, the method is quite promising to avoid inconsistencies also at the quantum level.

Indeed, a Lagrangian for the $f_{\nu_1 \dots \nu_{2j}}$ tensor can be considered along the lines of [29,30] and given by:

$$\begin{aligned} \mathcal{L} = & a \partial_{\mu_1} f_{\nu_1 \nu_2 \dots \nu_{2j}}^{\dagger} \bar{\sigma}^{\mu_1 \nu_1 \nu_1} \bar{\sigma}^{\mu_2 \nu_2 \nu_2} \dots \bar{\sigma}^{\mu_{2j} \nu_{2j} \nu_{2j}} \partial_{\mu_2} \dots \partial_{\mu_{2j}} f_{\nu_1 \nu_2 \dots \nu_{2j}} + b, \partial^{\mu} f^{\nu_1 \nu_2 \dots \nu_{2j}} \partial_{\mu} f_{\nu_1 \nu_2 \dots \nu_{2j}} \\ & + \text{terms with less or equal number of derivatives} + m^{2j} f^{\nu_1 \nu_2 \dots \nu_{2j}} f_{\nu_1 \nu_2 \dots \nu_{2j}} \\ & + \text{Hermitian conjugate terms,} \end{aligned} \quad (47)$$

where $\bar{\sigma}^{\mu \alpha \alpha} = (\sigma_0, -\vec{\sigma})$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, σ_x, σ_y and σ_z are the Pauli matrices, and a and b are constant parameters. In [30], the Laporte–Uhlenbeck case of a tensor-spinor of second rank was worked out along the lines of (47) by means of the canonical constraint Hamiltonian quantization procedure, and there, it could be shown that for $a + b = 1$, the Hamiltonian is free from negative energy solutions and presents itself diagonal in the particle creation and annihilation operators. In comparison to the $(j, 0) \oplus (0, j) \sim \Phi_B$ column-vector field in (25), the Lagrangian in (47) has the advantage that the corresponding equation contains more terms and thus provides more possibilities to avoid instabilities of the Hamiltonian through favorable cancellations in the calculations of Dirac brackets, despite its high order in the derivatives, an observation reported in [30].

2.4. Single-Spin-Valued Representations with Dirac Spinor-Tensors as Carrier Spaces: The Bargmann–Wigner Framework

A further option for spin- j description is given by the totally-symmetric product, of $n = 2j$ copies of a Dirac spinor, $\psi \simeq (1/2, 0) \oplus (0, 1/2)$, where $(1/2, 0)$ and $(0, 1/2)$ are the right- and left-handed Weyl–Van der Waerden two-component spinors corresponding to the two nonequivalent fundamental $SL(2, C)$ representation spaces, also known as “spinor” (ζ) and “co-spinor” ($\dot{\eta}$),

$$\text{Sym } \Psi_{b_1 \dots b_n}^{(n)} = \text{Sym } \psi_{b_1} \dots \psi_{b_n} \simeq \text{Sym} \left[\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \right]_1 \otimes \dots \otimes \left[\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \right]_n, \quad (48)$$

where $b_i = 1, 2, 3, 4$, a scheme known as the Bargmann–Wigner (BW) method [14]. The BW rank- n spinor [14] satisfies also a high-order differential equation, which reads,

$$(\gamma_{\mu} p^{\mu} - m)^{a_1 b_1} \dots (\gamma^{\mu} p^{\mu} - m)^{a_i b_i} \dots (\gamma_{\mu} p^{\mu} - m)^{a_n b_n} \text{Sym } \Psi_{b_1 \dots b_n} = 0. \quad (49)$$

This method is very similar to the Joos–Weinberg approach and shares with the latter almost all the advantages and disadvantages. There is though a distinction, which will be pointed out in the concluding section.

2.5. Single-Spin-Valued Representations with Lorentz Tensors as Carrier Spaces

A method for a single-spin-valued description that is distinct from all the previous ones was suggested by Acosta, Guzmán, and Kirchbach (AGK) [22], where it was shown how carrier spaces of $(j, 0) \oplus (0, j)$ representations can be equipped by $SO(1, 3)$ vector indices. In this way, single-spin-valued representations could be upgraded by one of the main advantages of the multiple-spin-valued representations.

The Method of Acosta, Guzmán, and Kirchbach

In one of the possibilities, a bosonic $(j, 0) \oplus (0, j)$ carrier space can be embedded by direct products of j copies of the totally anti-symmetric Lorentz tensor of second rank, $B_{[\mu\nu]}$, according to,

$$(j, 0) \oplus (0, j) : \Psi_{[\mu_1\nu_1] \dots [\mu_j\nu_j]}^{(j,0) \oplus (0,j)} \in B_{[\mu_1\nu_1]} \otimes \dots \otimes B_{[\mu_j\nu_j]},$$

$$B_{[\mu_1\nu_1]} \simeq (1, 0) \oplus (0, 1), \quad (50)$$

where the brackets indicate index anti-symmetrization. Another option is to embed it into direct products of totally-symmetric Lorentz tensors of second rank, $G_{\{\mu\nu\}}$, as:

$$(j, 0) \oplus (0, j) : \Psi_{\{\mu_1\nu_1\} \dots \{\mu_j\nu_j\}}^{(j,0) \oplus (0,j)} \in G_{\{\mu_1\nu_1\}} \otimes \dots \otimes G_{\{\mu_j\nu_j\}},$$

$$G_{\{\mu_i\nu_i\}} \simeq (1, 1), \quad (51)$$

where the curly brackets indicate index symmetrization. The direct product of the tensors, be it in (50) or (51) with a Dirac spinor, can then be employed in the description of fermionic $(j \pm 1/2, 0) \oplus (0, j \pm 1/2)$ fields. Take as an illustrative example the simplest case of embedding $(3/2, 0) \oplus (0, 3/2)$ in $B_{[\mu\nu]} \otimes \psi$. Direct products of such types are reducible, and specifically, the latter one decomposes into irreducible sectors according to:

$$\psi_{[\mu\nu]}^{(3/2,0) \oplus (0,3/2)} \in B_{[\mu\nu]} \otimes \psi$$

$$= \psi_{[\mu\nu]}^{(3/2,0) \oplus (0,3/2)} \oplus \psi_{[\mu\nu]}^{(1/2,1) \oplus (1,1/2)} \oplus \psi_{[\mu\nu]}^{(1/2,0) \oplus (0,1/2)}$$

$$\simeq \left[\left(\frac{3}{2}, 0 \right) \oplus \left(0, \frac{3}{2} \right) \right] \oplus \left[\left(\frac{1}{2}, 1 \right) \oplus \left(1, \frac{1}{2} \right) \right] \oplus \left[\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \right]. \quad (52)$$

The carrier space under discussion contains the pure spin-3/2, and if there were to exist a possibility to reduce this space without loosing the Lorentz index labeling, it would become possible to describe $(3/2, 0) \oplus (0, 3/2)$ by means of $\psi_{[\mu\nu]}^{(3/2,0) \oplus (0,3/2)}$. Such a possibility does indeed exists and is provided by a momentum-independent projector, here denoted by $\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)}$. The procedure for the construction of such projector operators is the subject of Section 3 and will be presented in more detail there. Wave equations can then be designed in the simplest way, be it in employing the spin and mass projector, $\mathcal{P}^{(m,3/2)}$, or by just setting $\psi_{[\mu,\nu]}^{(3/2,0) \oplus (0,3/2)}$ on its mass shell according to,

$$\square \left[\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)} \right]_{[\mu,\nu]}^{\quad [\eta,\rho]} \quad {}^b_a \left[\psi_{[\eta,\rho]}^{(3/2,0) \oplus (0,3/2)} \right]_b = m^2 \left[\psi_{[\mu,\nu]}^{(3/2,0) \oplus (0,3/2)} \right]_a, \quad (53)$$

where a is the Dirac spinor index. The advantage of the method under discussion consists of (i) adapting one of the advantages of the multiple-spin valued representation spaces, namely their labeling by $SO(1,3)$ vector indices, to those of single spins, and (ii) lowering the order in the momenta of the Joos–Weinberg equations from p^{2j} to the universal p^2 . In this scheme, otherwise cumbersome procedures such as vertex constructions and calculations of scattering amplitudes are significantly simplified.

It needs to be noted that the glossary presented above is highly incomplete, indeed both as regards the variety of methods and the multitude of problems. In particular, the important work by Bhabha [31] on multicomponent linear equations has remained out of the scope of the present work. Moreover, the problems of Lagrangian description and quantization of field theories including high spins, among them seminal works by Singh and Hagen [32], could not be duly attended. The path taken here by us was chosen on the one side for the sake of keeping the presentation of the manuscript more focused on its prime goal concerning the promotion of the representation reduction algorithm and, on the other, for the purpose of avoiding repetitions of material already contained in such exhaustive

reviews on the subject, as is the text in [33]. Specifically in the last reference, the interested reader may find many illuminating discussions on the various aspects of high spin field theories and possibly encounter some inspiring suggestions regarding further developments.

3. Casimir Invariants of the Homogeneous Spin Lorentz Group and Representation Reduction through Covariant Projector Operators

The Lorentz group, which transforms the internal spin degrees of freedom, termed as the spin-Lorentz group and denoted by, $\mathcal{L}$, is a subgroup of the full Lorentz group, which acts as well on the spin as on the external space-time degrees of freedom. The respective $\mathcal{L}$ generators, denoted by $S_{\mu\nu}$, are quadratic $d \times d$ constant matrices, where d is an integer that determines the finite-dimensionality of the internal representation space and encodes the spin value. For the particular case of a single spin, dimensionality and spin are related as $d = 2(2j + 1)$, while for representations of multiple spins, relations of the type $d = \sum_i (2j_i + 1)$, or, $d = 2 \sum_i (2j_i + 1)$, can hold valid. The algebra of the spin-Lorentz group, termed as the homogeneous spin-Lorentz group (HSLG), reads [34],

$$\mathcal{L}: [S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\mu\rho}S_{\nu\sigma} - g_{\nu\rho}S_{\mu\sigma} + g_{\mu\sigma}S_{\rho\nu} - g_{\nu\sigma}S_{\rho\mu}). \quad (54)$$

It has two Casimir invariants, here denoted by F and G , respectively, and defined as,

$$F_{AB} = \frac{1}{4} [S^{\mu\nu}]_{AD} [S_{\mu\nu}]_{DB}, \quad G_{AB} = \frac{1}{8} \epsilon_{\mu\nu}^{\alpha\beta} [S^{\mu\nu}]_{AC} [S_{\alpha\beta}]_{CB}, \quad A, B, C, D, \dots = 1, \dots, d. \quad (55)$$

The $S_{\mu\nu}$ matrices, in being coordinate independent, commute with the generators of external translations, P_μ , and therefore also with the generators of external boosts and rotations, $L_{\mu\nu}$. The generators of the complete homogeneous Lorentz group operating in the internal and external spacetimes are then defined as:

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \quad (56)$$

As a consequence of (54), the HSLG Casimir invariants being independent of the external coordinates commute with $L_{\mu\nu}$ and p_μ without being Casimir invariants of the Poincaré group. This subtle virtue qualifies them as very useful identification tools of finite-dimensional representations by providing their labelings according to $(j_1, j_2) \oplus (j_2, j_1)$, where j_1 and j_2 are termed as right- and left-handed spins [34]. As an important detail, one has to take care that the Casimir operators in (55) are always diagonal in the decomposition into irreducible sectors of the direct product spaces of the types in (50)–(52).

With that in mind, the F operator in (55) unambiguously identifies through its eigenvalues within any direct product space a given irreducible finite-dimensional representation space, here generically denoted by, $\psi^{(j_1, j_2) \oplus (j_2, j_1)} = \phi_R^{(j_1, j_2)} \oplus \phi_L^{(j_2, j_1)}$, where $\phi_R^{(j_1, j_2)}$ and $\phi_L^{(j_2, j_1)}$ are in turn its right- and left-handed chiral components, according to,

$$\begin{aligned} F \psi^{(j_1, j_2) \oplus (j_2, j_1)} &= c_{(j_1, j_2)} \psi^{(j_1, j_2) \oplus (j_2, j_1)}, \\ c_{(j_1, j_2)} &= \frac{1}{2} (K(K+2) + M^2), \\ K = j_1 + j_2, \quad M &= |j_1 - j_2|. \end{aligned} \quad (57)$$

Afterward, by the aid of the G invariant, the chiral parts can be identified as,

$$G \phi_R^{(j_1, j_2)} = r_{(j_1, j_2)} \phi_R^{(j_1, j_2)}, \quad G \phi_L^{(j_2, j_1)} = r_{(j_2, j_1)} \phi_L^{(j_2, j_1)}, \quad (58)$$

$$r_{(j_1, j_2)} = -r_{(j_2, j_1)} = i(K+1)M. \quad (59)$$

In [22], the idea was developed to use the Casimir invariant F in the construction of momentum independent (static) projectors on reflection symmetric irreducible sectors of the Lorentz tensor-spinor in (52) and to explore the consequences.

Such a projector, here denoted by $\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)}$, which identifies the irreducible $(3/2,0) \oplus (0,3/2)$ representation space in (52), has been constructed from F in (57) as:

$$\begin{aligned} \mathcal{P}_F^{(3/2,0) \oplus (0,3/2)} &= \left(\frac{F - c_{(1,1/2)}}{c_{(3/2,0)} - c_{(1,1/2)}} \right) \left(\frac{F - c_{(1/2,0)}}{c_{(3/2,0)} - c_{(1/2,0)}} \right), \\ c_{(1/2,0)} &= \frac{3}{4}, \quad c_{(3/2,0)} = \frac{15}{4}, \quad c_{(1,1/2)} = \frac{11}{4}. \end{aligned} \quad (60)$$

Equation (60) reveals how the operator $\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)}$ nullifies any irreducible representation space, which is different from $(3/2,0) \oplus (0,3/2)$. At the same time, for $(3/2,0) \oplus (0,3/2)$, it acts as the identity operator, which means that $\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)}$ is a projector on this very space. In recalling the notation of the spin-3/2 wave function, $\left[\psi_{[\mu,\nu]}^{(3/2,0) \oplus (0,3/2)} \right]_a$, we find:

$$\left[\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)} \right]_{[\mu,\nu]}^{\left[\eta,\rho \right]_a} \left[\psi_{[\eta,\rho]}^{(3/2,0) \oplus (0,3/2)} \right]_b = \left[\psi_{[\mu,\nu]}^{(3/2,0) \oplus (0,3/2)} \right]_a. \quad (61)$$

The covariant form of this projector (before mentioned in (53)) has been elaborated in [22] and reads,

$$\left[\mathcal{P}_F^{(3/2,0) \oplus (0,3/2)} \right]_{\alpha\beta;\gamma\delta} = \frac{1}{8} (\sigma_{\alpha\beta} \sigma_{\gamma\delta} + \sigma_{\gamma\delta} \sigma_{\alpha\beta}) - \frac{1}{12} \sigma_{\alpha\beta} \sigma_{\gamma\delta}, \quad (62)$$

with $\sigma_{\mu\nu}$ standing for $\sigma_{\mu\nu} = i [\gamma_\mu, \gamma_\nu] / 2$, where γ_μ are the Dirac matrices.

4. Decomposition of the Product Space $T_{\mu\nu} \simeq (1/2, 1/2) \otimes (1/2, 1/2)$ by Spin-Lorentz Group Projectors: A Template Example

The direct product space of two four-vectors, $T_{\mu\nu} = A_\mu \otimes A_\nu \simeq (1/2, 1/2) \otimes (1/2, 1/2)$, is well studied, and its reduction is textbook knowledge. Namely, it decomposes into the following three irreducible parity-invariant representation spaces:

$$(1/2, 1/2) \otimes (1/2, 1/2) = (0, 0) \oplus [(1, 0) \oplus (0, 1)] \oplus (1, 1). \quad (63)$$

According to Chapter 33 in [29], the covariant form of the latter equation reads,

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} T_\alpha^\alpha + [T^A]_{\mu\nu} + [T^S]_{\mu\nu}, \quad (64)$$

where the symmetric $[T^S]_{\mu\nu}$ and antisymmetric $[T^A]_{\mu\nu}$ are given by,

$$[T^A]_{\mu\nu} = \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu}), \quad (65)$$

$$[T^S]_{\mu\nu} = \frac{1}{2} (T_{\mu\nu} + T_{\nu\mu}) - \frac{1}{4} g_{\mu\nu} T_\alpha^\alpha, \quad (66)$$

respectively, while T_α^α stands for the trace. More specifically, $\frac{1}{4} g_{\mu\nu} T_\alpha^\alpha$, $[T^S]_{\mu\nu}$, and $[T^A]_{\mu\nu}$ transform in turn as $(0, 0)$, $(1, 1)$, and $(1, 0) \oplus (0, 1)$. In the following, we shall demonstrate how this result can be reproduced by means of spin-Lorentz group projectors.

4.1. Projectors on Reflection Symmetric Spaces

The spin-Lorentz group projectors on reflection symmetric (parity-invariant) irreducible carrier spaces are built up from the Casimir invariant in (57) using the general expression,

$$\mathcal{P}_F^{(j_1, j_2) \oplus (j_2, j_1)} = \prod_{k, l \neq j_1, j_2} \left(\frac{F - c_{(j_k, j_l)}}{c_{(j_1, j_2)} - c_{(j_k, j_l)}} \right), \quad (67)$$

where $c_{(j_1, j_2)}$ are defined according to (57), which equivalently is rewritten as:

$$F\psi^{(j_1, j_2) \oplus (j_2, j_1)} = c_{(j_1, j_2)}\psi^{(j_1, j_2) \oplus (j_2, j_1)}, \quad c_{(j_1, j_2)} = j_1(j_1 + 1) + j_2(j_2 + 1). \quad (68)$$

Executing the prescription of Equation (67), we constrict below the $[\mathcal{P}_F^{(0,0)}]_{\alpha\beta\gamma\delta}$, $[\mathcal{P}_F^{(1,0) \oplus (0,1)}]_{\alpha\beta\gamma\delta}$, and $[\mathcal{P}_F^{(1,1)}]_{\alpha\beta\gamma\delta}$ projector operators. As a first step toward the construction of the spin-Lorentz projectors, one has to calculate the generators, $[S_{\mu\nu}^T]_{\alpha\beta\gamma\delta}$, in the $T_{\mu\nu}$ product space, which are obtained from the generators, $[S_{\mu\nu}^V]_{\alpha\beta}$, of the four-vector representation space, $(1/2, 1/2)$, as:

$$[S_{\mu\nu}^T]_{\alpha\beta\gamma\delta} = [S_{\mu\nu}^V]_{\alpha\gamma}g_{\beta\delta} + [S_{\mu\nu}^V]_{\beta\delta}g_{\alpha\gamma}. \quad (69)$$

From that, the Casimir invariants F and G emerge as:

$$F_{\alpha\beta\gamma\delta} = 3g_{\alpha\gamma}g_{\beta\delta} + g_{\beta\gamma}g_{\alpha\delta} - g_{\delta\gamma}g_{\alpha\beta}, \quad (70)$$

$$G_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\delta\gamma}, \quad (71)$$

and their respective eigenvalues are:

$$c_{(0,0)} = 0, \quad c_{(1,0)} = 2, \quad c_{(1,1)} = 4, \quad (72)$$

$$r_{(1,0)} = -r_{(0,1)} = 2i. \quad (73)$$

Using (70)–(73), we obtain that the spin-Lorentz projectors for the the second rank tensor space are:

$$\begin{aligned} [\mathcal{P}_F^{(0,0)}]_{\alpha\beta\gamma\delta} &= \left(\frac{F_{\alpha\beta\sigma\rho} - c_{(1,0)}\mathbf{1}_{\alpha\beta\sigma\rho}}{c_{(0,0)} - c_{(1,0)}} \right) \left(\frac{F_{\sigma\rho\gamma\delta} - c_{(1,1)}\mathbf{1}_{\sigma\rho\gamma\delta}}{c_{(0,0)} - c_{(1,1)}} \right) \\ &= \frac{1}{4}g_{\alpha\beta}g_{\delta\gamma}, \end{aligned} \quad (74)$$

$$\begin{aligned} [\mathcal{P}_F^{(1,0) \oplus (0,1)}]_{\alpha\beta\gamma\delta} &= \left(\frac{F_{\alpha\beta\sigma\rho} - c_{(0,0)}\mathbf{1}_{\alpha\beta\sigma\rho}}{c_{(1,0)} - c_{(0,0)}} \right) \left(\frac{F_{\sigma\rho\gamma\delta} - c_{(1,1)}\mathbf{1}_{\sigma\rho\gamma\delta}}{c_{(1,0)} - c_{(1,1)}} \right) \\ &= \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}), \end{aligned} \quad (75)$$

$$\begin{aligned} [\mathcal{P}_F^{(1,1)}]_{\alpha\beta\gamma\delta} &= \left(\frac{F_{\alpha\beta\sigma\rho} - c_{(0,0)}\mathbf{1}_{\alpha\beta\sigma\rho}}{c_{(1,1)} - c_{(0,0)}} \right) \left(\frac{F_{\sigma\rho\gamma\delta} - c_{(1,0)}\mathbf{1}_{\sigma\rho\gamma\delta}}{c_{(1,1)} - c_{(1,0)}} \right) \\ &= \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} + g_{\beta\gamma}g_{\alpha\delta}) - \frac{1}{4}g_{\alpha\beta}g_{\delta\gamma}. \end{aligned} \quad (76)$$

Here, we defined the identity $\mathbf{1}_{\alpha\beta\gamma\delta}$ for the second rank tensor space as:

$$\mathbf{1}_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta}, \quad (77)$$

and in such a way that $\mathbf{1}_{\alpha\beta\gamma\delta}$ has the proper ordering in the indices to carry out the contractions.

Applying the spin-Lorentz projectors $[\mathcal{P}_F^{(0,0)}]_{\alpha\beta\gamma\delta}$, $[\mathcal{P}_F^{(1,0)\oplus(0,1)}]_{\alpha\beta\gamma\delta}$, and $[\mathcal{P}_F^{(1,1)}]_{\alpha\beta\gamma\delta}$ to $T_{\mu\nu}$ amounts to the following decomposition,

$$\begin{aligned} T_{\mu\nu} &= \mathbf{1}_{\mu\nu\alpha\beta} T^{\alpha\beta}, \\ &= \left([\mathcal{P}_F^{(0,0)}]_{\mu\nu\alpha\beta} + [\mathcal{P}_F^{(1,0)\oplus(0,1)}]_{\mu\nu\alpha\beta} + [\mathcal{P}_F^{(1,1)}]_{\mu\nu\alpha\beta} \right) T^{\alpha\beta}, \\ &= \frac{1}{4} g_{\mu\nu} T + [T^A]_{\mu\nu} + [T^S]_{\mu\nu}. \end{aligned} \quad (78)$$

Here, T is the trace of $T_{\mu\nu}$, and $[T^A]_{\mu\nu}$ is an antisymmetric tensor of second rank, while $[T^S]_{\mu\nu}$ is a symmetric traceless second rank tensor. With that, Equations (64)–(66) were reproduced, as expected.

4.2. Chiral Projectors

The parity-invariant irreducible representations are composed by two chiral components, given in more detail in Equations (94) and (95) below, which can be separated by two projectors, denoted by $\mathcal{P}_G^{(j_1,j_2)}$ and $\mathcal{P}_G^{(j_2,j_1)}$, which are based on the G invariant in (59). These projectors are defined in parallel to (67) and are given by,

$$\mathcal{P}_G^{(j_1,j_2)} = \frac{G - r_{(j_2,j_1)}}{r_{(j_1,j_2)} - r_{(j_2,j_1)}}, \quad (79)$$

$$\mathcal{P}_G^{(j_2,j_1)} = \frac{G - r_{(j_1,j_2)}}{r_{(j_2,j_1)} - r_{(j_1,j_2)}}. \quad (80)$$

Through their action on $\psi^{(j_1,j_2)\oplus(j_2,j_1)}$,

$$\begin{aligned} \mathcal{P}_G^{(j_1,j_2)} \psi^{(j_1,j_2)\oplus(j_2,j_1)} &= \phi_R^{(j_1,j_2)}, \\ \mathcal{P}_G^{(j_2,j_1)} \psi^{(j_1,j_2)\oplus(j_2,j_1)} &= \phi_L^{(j_2,j_1)}, \end{aligned} \quad (81)$$

the left- and right-handed chiral components, introduced in (58), are singled out.

Along this line, in the particular case under consideration, the parity-invariant antisymmetric tensor, $(1,0) \oplus (0,1)$, can further be reduced into its right- and left-handed components by the chiral spin-Lorentz projectors, $[\mathcal{P}_G^{(1,0)}]_{\alpha\beta\gamma\delta}$ and $[\mathcal{P}_G^{(0,1)}]_{\alpha\beta\gamma\delta}$, in turn given by:

$$\begin{aligned} [\mathcal{P}_G^{(0,1)}]_{\alpha\beta\gamma\delta} &= \left(\frac{G_{\alpha\beta\gamma\delta} - r_{(1,0)} \mathbf{1}_{\alpha\beta\gamma\delta}}{r_{(0,1)} - r_{(1,0)}} \right), \\ &= \frac{1}{2} g_{\alpha\gamma} g_{\beta\delta} + \frac{1}{4} i \epsilon_{\alpha\beta\gamma\delta}, \end{aligned} \quad (82)$$

$$\begin{aligned} [\mathcal{P}_G^{(1,0)}]_{\alpha\beta\gamma\delta} &= \left(\frac{G_{\alpha\beta\gamma\delta} - r_{(0,1)} \mathbf{1}_{\alpha\beta\gamma\delta}}{r_{(1,0)} - r_{(0,1)}} \right), \\ &= \frac{1}{2} g_{\alpha\gamma} g_{\beta\delta} - \frac{1}{4} i \epsilon_{\alpha\beta\gamma\delta}. \end{aligned} \quad (83)$$

Now, applying $[\mathcal{P}_G^{(1,0)}]_{\alpha\beta\gamma\delta}$ and $[\mathcal{P}_G^{(0,1)}]_{\alpha\beta\gamma\delta}$ to $[T^A]_{\mu\nu}$ amounts to:

$$\begin{aligned} [T^A]_{\mu\nu} &= \mathbf{1}_{\mu\nu\alpha\beta} [T^A]^{\alpha\beta}, \\ &= \left([\mathcal{P}_G^{(1,0)}]_{\mu\nu\alpha\beta} + [\mathcal{P}_G^{(0,1)}]_{\mu\nu\alpha\beta} \right) [T^A]^{\alpha\beta}, \\ &= D_{\mu\nu} + \tilde{D}_{\mu\nu}. \end{aligned} \quad (84)$$

Here, $D_{\mu\nu}$ is a self-dual tensor, which transforms according to $(1, 0)$, and $\tilde{D}_{\mu\nu}$ is an anti-self dual tensor transforming according to $(0, 1)$, which are calculated as:

$$D_{\mu\nu} = \frac{1}{2} [T^A]^{\mu\nu} - \frac{i}{4} \epsilon_{\mu\nu\alpha\beta} [T^A]^{\alpha\beta}, \quad (85)$$

$$\tilde{D}_{\mu\nu} = \frac{1}{2} [T^A]^{\mu\nu} + \frac{i}{4} \epsilon_{\mu\nu\alpha\beta} [T^A]^{\alpha\beta}, \quad (86)$$

and in accord with Chapter 34 in [29]. This template example shows that the spin-Lorentz group projector technique suggested in [22,35] and advocated here provides results equivalent to representation reductions based on index symmetrization and anti-symmetrization. In the subsequent section, we shall decompose the direct product space of a four-vector with a Dirac spinor.

5. Decomposition of the Four-Vector Spinor ψ_μ by Spin-Lorentz Group Projectors

The four-vector spinor space, ψ_μ , decomposes into the two irreducible parity-invariant representation space given in the above Equation (13).

5.1. Projectors on the Reflection Symmetric Carrier Spaces

The aforementioned two sectors can be separated by spin-Lorentz projectors, denoted by $\left[\mathcal{P}_F^{(0,1/2) \oplus (0,1/2)}\right]_{\alpha\beta}$ and $\left[\mathcal{P}_F^{(1,1/2) \oplus (1/2,1)}\right]_{\alpha\beta}$. In the case of our interest, the four-vector-spinor space, the generators are [24,35]:

$$[S_{\mu\nu}]_{\alpha\beta;ab} = [S_{\mu\nu}^V]_{\alpha\beta} \delta_{1b} + g_{\alpha\beta} [S_{\mu\nu}]_{ab}, \quad (87)$$

where $[S_{\mu\nu}^V]_{\alpha\beta}$ are the generators of the four-vector space, $(1/2, 1/2)$, while $[S_{\mu\nu}]_{ab}$ are the generators in the Dirac spinor space, $(1/2, 0) \oplus (0, 1/2)$. The expressions are well known [35] and read,

$$[S_{\mu\nu}^V]_{\alpha\beta} = i (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}), \quad (88)$$

$$S_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu]. \quad (89)$$

Then, the F Casimir in (57) emerges as:

$$F_{\alpha\beta} = \frac{9}{4} g_{\alpha\beta} + \frac{i}{2} \sigma_{\alpha\beta}, \quad (90)$$

and its eigenvalues for the decomposition in (13), which follow from (68), are found as:

$$c_{(1/2,0)} = \frac{3}{4}, \quad c_{(1,1/2)} = \frac{11}{4}. \quad (91)$$

Substituting of (91) and (90) in (67), the spin-Lorentz group projectors under discussion are now calculated as:

$$\left[\mathcal{P}_F^{(0,1/2) \oplus (1/2,0)}\right]_{\alpha\beta} = \frac{1}{4} \gamma_\alpha \gamma_\beta, \quad (92)$$

$$\left[\mathcal{P}_F^{(1,1/2) \oplus (1/2,1)}\right]_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{4} \gamma_\alpha \gamma_\beta, \quad (93)$$

a result due to [35]. When they are contracted with any general four-vector spinor ψ_β , the resulting states transform according to,

$$\left[\mathcal{P}_F^{(0,1/2) \oplus (1/2,0)} \right]_{\alpha\beta} \psi^\beta = \frac{1}{4} \gamma_\alpha \gamma \cdot \psi = \psi_\alpha^{(0,1/2) \oplus (1/2,0)}, \quad (94)$$

$$\left[\mathcal{P}_F^{(1,1/2) \oplus (1/2,1)} \right]_{\alpha\beta} \psi^\beta = \psi_\alpha - \frac{1}{4} \gamma_\alpha \gamma \cdot \psi = \psi_\alpha^{(1,1/2) \oplus (1/2,1)}. \quad (95)$$

In order to make Equation (94) more transparent, more details on ψ need to be given. Besides the spin-3/2[−] spinors, there are two sorts of spin-1/2 momentum space spinors that span ψ_μ . The first is denoted by U_α^{SS} and the second by U_α^{VS} . In [35], they were expressed as:

$$\left[U^{\text{SS}} \right]^\alpha \left(\mathbf{p}, J^\pi = \frac{1}{2}^+, \pm \frac{1}{2} \right) = \frac{p^\alpha}{m} u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right), \quad (96)$$

$$\left[U^{\text{VS}} \right]^\alpha \left(\mathbf{p}, J^\pi = \frac{1}{2}^-, \pm \frac{1}{2} \right) = \frac{1}{\sqrt{3}m} (-p_\alpha + \gamma_\alpha \not{p}) u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right), \quad (97)$$

respectively, where u_+ and u_- are in turn the shorthand for Dirac's u and v spinors. Let us apply on each one of them the $\left[\mathcal{P}_F^{(0,1/2) \oplus (1/2,0)} \right]_{\alpha\beta}$ projectors. In so doing, one arrives at:

$$\begin{aligned} \left[\mathcal{P}_F^{(0,1/2) \oplus (1/2,0)} \right]_{\alpha\beta} \left[U^{\text{SS}} \right]^\alpha \left(\mathbf{p}, J^\pi = \frac{1}{2}^+, \pm \frac{1}{2} \right) &= \frac{1}{4m} \gamma_\alpha \not{p} u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right), \\ \left[\mathcal{P}_F^{(0,1/2) \oplus (1/2,0)} \right]_{\alpha\beta} \left[U^{\text{VS}} \right]^\alpha \left(\mathbf{p}, J^\pi = \frac{1}{2}^+, \pm \frac{1}{2} \right) &= \frac{\sqrt{3}}{4m} \gamma_\alpha \not{p} u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right). \end{aligned} \quad (98)$$

The latter equations show that each one of the two spin-1/2 Rarita–Schwinger spinors U_α^{SS} , and U_α^{VS} contains the spinor:

$$\frac{1}{4m} \gamma_\alpha \not{p} u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right) = \psi_\alpha^{(0,1/2) \oplus (1/2,0)}. \quad (99)$$

This spinor has four independent components, and it is obvious that a contraction by γ^α amounts to $\not{p} u_\pm / m$, the solution of the Dirac equation,

$$\gamma \cdot \psi^{(0,1/2) \oplus (1/2,0)} = \frac{\not{p}}{m} u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right) = u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right). \quad (100)$$

This is the proof that the $\mathcal{P}_F^{(0,1/2)}$ projector correctly identifies the Dirac space in ψ_μ . Moreover, in [35], the spinor in (99) was employed in the calculation of Compton scattering off a spin-1/2 particle, and the precise cross-sections typical for the traditional Dirac spinor were reproduced, thus providing an additional cross check for the correctness of the projector used. In a similar way, all the remaining identities can be verified.

In particular, the expression:

$$\frac{1}{m} \left(p_\alpha - \frac{1}{4} \gamma_\alpha \not{p} \right) u_\pm \left(\mathbf{p}, \pm \frac{1}{2} \right) = \psi_\alpha^{(1,1/2) \oplus (1/2,1)}, \quad (101)$$

could be obtained in [35]. Contraction of the latter equation by p^α leads to:

$$p \cdot \psi^{(1,1/2) \oplus (1/2,1)} = \frac{3}{4} m u_\pm. \quad (102)$$

The $\psi_\alpha^{(1,1/2) \oplus (1/2,1)}$ part of the spin-3/2 Rarita–Schwinger field provides an example of how a representation space of the type $(1/2, j) \oplus (j, 1/2)$ used by Hurley [20] can be equipped by Lorentz indices (one index in this case).

Actually, the genuine Rarita–Schwinger equation could be:

$$\begin{aligned} (\not{p} - m)\psi_\alpha^{(1,1/2) \oplus (1/2,1)} &= 0, \\ p \cdot \psi^{(1,1/2) \oplus (1/2,1)} &= \frac{3}{4}mu_\pm. \end{aligned} \quad (103)$$

Alternatively, the spin-3/2 can be identified also by a second order equation as:

$$\mathcal{P}^{(m,3/2)}\psi_\alpha^{(1,1/2) \oplus (1/2,1)} = -\frac{15}{4}m^2\psi_\alpha^{3/2^- \in (1,1/2) \oplus (1/2,1)}, \quad (104)$$

with $\mathcal{P}^{(m,3/2)}$ in (16). Finally, upon applying to $\psi_\alpha^{(1,1/2) \oplus (1/2,1)}$ the spin-1/2 and mass projector, $\mathcal{P}^{(m,1/2)}$ from (17), a wave equation for spin- $\frac{1}{2}^- \in (1/2, 1) \oplus (1, 1/2)$ was found there as:

$$\mathcal{P}^{(m,1/2)}\psi_\alpha^{(1,1/2) \oplus (1/2,1)} = -\frac{3}{4}m^2\psi_\alpha^{1/2^- \in (1,1/2) \oplus (1/2,1)}. \quad (105)$$

It should be noticed that the Lagrangian description of high spins presents some difficulties already at the classical field theoretic level. Specifically within the Rarita–Schwinger formalism, the problem concerns the incorporation of the auxiliary conditions, which can be done to some extent at the cost of the introduction of a free parameter, which however introduces upon gauging ambiguities through so-called “off-shell parameters”. More details can be found among others in the Introduction to [24]. As a comparison, the second order equation in (104) in its extended version in (23) was derived from a classical Lagrangian in [24]. Furthermore, there, it was shown that for a gyromagnetic factor taking the “natural” value of $g_{3/2} = 2$, the wave fronts of the solutions propagate causally in the background of an electromagnetic field and thus avoid the acausality inconsistency. Admittedly, the quantization of this Lagrangian has not been studied so far. The fact is that the quantization of field theories of high spins coupled to external fields still remains an unsolved problem. We here take for the time being the position that Lagrangians of the type presented in (47) may have chances to provide consistent field theories for at least some of the high spin fields. A further challenge emerging in high-spin theories concerns problems of the stability of quantized high-spin fields placed in external time-dependent potentials [36–38]. For the time being, these issues go beyond the scope of the present work, but should be kept in mind as important topics for future research.

5.2. Projectors on the Chiral Components

The most illustrative example for a chiral projector is provided by the Dirac space. There, one finds:

$$\begin{aligned} G^{(0,1/2)} &= G^{(1/2,0)} = -\frac{3}{4}i\gamma_5, \\ G^{(0,1/2)}\psi^{(0,1/2)} &= -\frac{3}{4}i, \quad G^{(1/2,0)}\psi^{(1/2,0)} = \frac{3}{4}i, \end{aligned} \quad (106)$$

which upon substitution in (79) and (80) amounts to the following projectors,

$$\begin{aligned} \mathcal{P}_G^{(0,1/2)} &= \frac{1 - \gamma_5}{2}, \\ \mathcal{P}_G^{(1/2,0)} &= \frac{1 + \gamma_5}{2}. \end{aligned} \quad (107)$$

These are the very well-known projectors on right- and left-handed Dirac spinors. The example reveals the origin of the γ_5 matrix from the G Casimir invariant of the homogeneous spin-Lorentz group.

In the more complicated case of the four-vector spinor space, the G operator in (55) takes the form:

$$G_{\alpha\beta} = \frac{i}{4}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu} - \frac{3i}{4}\gamma^5 g_{\alpha\beta}, \quad (108)$$

and its eigenvalues are calculated as:

$$r_{(0,1/2)} = \frac{3i}{4}, \quad r_{(1,1/2)} = -\frac{5i}{4}. \quad (109)$$

Substitution in (79) and (80) allows writing down all possible Lorentz projectors of interest as:

$$[\mathcal{P}_G^{(0,1/2)}]_{\alpha\beta} = -\frac{1}{2}\gamma^5 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta} + \frac{1}{6}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}, \quad (110)$$

$$[\mathcal{P}_G^{(1/2,0)}]_{\alpha\beta} = \frac{1}{2}\gamma^5 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta} - \frac{1}{6}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}, \quad (111)$$

$$[\mathcal{P}_G^{(1,1/2)}]_{\alpha\beta} = \frac{3}{10}\gamma^5 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta} - \frac{1}{10}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}, \quad (112)$$

$$[\mathcal{P}_G^{(1/2,1)}]_{\alpha\beta} = -\frac{3}{10}\gamma^5 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}. \quad (113)$$

Finally, combining the projectors $\mathcal{P}_F$ in (92) and (93) with $\mathcal{P}_G$ in (110)–(113), the following identities are obtained,

$$\begin{aligned} [\mathcal{P}_G^{(0,1/2)}]_{\alpha\rho} [\mathcal{P}_F^{(0,1/2)\oplus(1/2,0)}]^{\rho\beta} \psi_\beta &= \frac{1}{8}\gamma^5 \psi_\alpha + \frac{1}{8}\psi_\alpha + \frac{1}{16}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}\psi^\beta - \frac{i}{8}\sigma_{\alpha\beta}\psi^\beta \\ &= [\phi_R^{(0,1/2)}]_\alpha, \end{aligned} \quad (114)$$

$$\begin{aligned} [\mathcal{P}_G^{(1/2,0)}]_{\alpha\rho} [\mathcal{P}_F^{(0,1/2)\oplus(1/2,0)}]^{\rho\beta} \psi_\beta &= -\frac{1}{8}\gamma^5 \psi_\alpha + \frac{1}{8}\psi_\alpha - \frac{1}{16}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}\psi^\beta - \frac{i}{8}\sigma_{\alpha\beta}\psi^\beta \\ &= [\phi_L^{(1/2,0)}]_\alpha, \end{aligned} \quad (115)$$

$$\begin{aligned} [\mathcal{P}_G^{(1,1/2)}]_{\alpha\rho} [\mathcal{P}_F^{(1,1/2)\oplus(1/2,1)}]^{\rho\beta} \psi_\beta &= \frac{3}{8}\gamma^5 \psi_\alpha + \frac{3}{8}\psi_\alpha - \frac{1}{16}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}\psi^\beta + \frac{i}{8}\sigma_{\alpha\beta}\psi^\beta \\ &= [\phi_R^{(1,1/2)}]_\alpha, \end{aligned} \quad (116)$$

$$\begin{aligned} [\mathcal{P}_G^{(1/2,1)}]_{\alpha\rho} [\mathcal{P}_F^{(1,1/2)\oplus(1/2,1)}]^{\rho\beta} \psi_\beta &= -\frac{3}{8}\gamma^5 \psi_\alpha + \frac{3}{8}\psi_\alpha + \frac{1}{16}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu}\psi^\beta + \frac{i}{8}\sigma_{\alpha\beta}\psi^\beta \\ &= [\phi_L^{(1/2,1)}]_\alpha. \end{aligned} \quad (117)$$

6. Conclusions

In the present, work we reviewed the technique suggested in [22,35] for decomposing products of Lorentz tensors into irreducible representation spaces that is based on covariant projectors built up in a transparent way from the Casimir invariants of the homogeneous spin-Lorentz group. Decomposition of product spaces by group projectors is a method fundamental to the construction of basis states in atomic and molecular physics [39], in which for the case of compact groups, use can be made of Weyl's character formula. The technique under discussion in the present work, illustrated in Sections 4 and 5.2 by several new examples of tensor decompositions by projectors, allowed us to equip any finite-dimensional representation space of the Lorentz group by $SO(1,3)$ indices, an example being $(3/2,0) \oplus (0,3/2)$ as part of the direct product of a totally antisymmetric tensor, $B_{[\mu,\nu]}$ of second rank with the Dirac spinor, ψ . The projector that finds this sector in $B_{[\mu\nu]} \otimes \psi$, earlier obtained in [22],

was reported in Equation (62). As a recent application, the presentation of $(3/2, 0) \oplus (0, 3/2)$ as a totally-antisymmetric Lorentz tensor of second rank with Dirac spinor components was employed in [40] as the field-strength tensor of the gravitino, described by a massless Rarita–Schwinger field and in complete parallel to the description in the quantum electrodynamics of a massless photon by means of a field strength tensor transforming as $(1, 0) \oplus (0, 1)$. Moreover, calculations of Compton scattering off such a spin-3/2 revealed differences to spin-3/2 embedded by the four-vector-spinor ψ_μ , and the conclusion could be drawn that particles of the same spin transforming in distinct carrier spaces possess distinct physical characteristics, among them the electromagnetic multipole moments. This conclusion could furthermore be strengthened through the neat separation by the spin-Lorentz group projectors in (92) and (93) of the two spins $1/2^+$ and $1/2^-$ residing in ψ_μ , a finding that hints at the physical nature of these two states.

In this case, the spin- $1/2^+$ and spin- $1/2^-$ particles were shown in [35] to be characterized by the two different gyromagnetic factors, $g_{\frac{1}{2}^+} = 2$ and $g_{\frac{1}{2}^-} = -2/3$, respectively. Finally, in [41], it could be observed that the Bargmann–Wigner approach to pure spin- j was not equivalent to the Joos–Weinberg method. The reason is that although both methods describe spin- j particles in terms of the correct number of $2(2j + 1)$ independent degrees of freedom, in the BW, scheme irreducible representation spaces of the two types $(j, 0) \oplus (0, j)$ and $(j - 1/2, 1/2) \oplus (1/2, j - 1/2)$ get mixed up. In view of the fact that particles of equal spins transforming in distinct representation spaces are different species, the BW method needs to be upgraded through the application on the totally symmetric Dirac spinor-tensors in (49) of a covariant projector of the type in (67), with $j_2 = 0$, this for the sake of avoiding unphysical solutions. In contrast, no such projectors need to be applied to the Weyl spinors in the approach of Cap and Donnert, in which $(j - 1/2, 1/2) \oplus (1/2, j - 1/2)$ played an auxiliary role and were excluded by the second order differential wave equations. This makes, in our opinion, the framework by Cap and Donnert, which is based on irreducible tensors labeled by $SL(2, C)$ indices, the most neat and promising method for the description of single-spin-valued representation spaces. Our covariant spin-Lorentz group projectors allowed us to design in the factor group of $SL(2, C)$, i.e., in $SL(2, C)/Z_2 \simeq SO(1, 3)$, the counterpart to the approach by Cap and Donnert. We believe that the method presented here provides a reliable point of departure for perturbative studies of Lorentz symmetry violating effects in theories with high-spin fields.

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