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Quaternion Electromagnetism and the Relation with Two-Spinor Formalism

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Abstract: By using complex quaternion, which is the system of quaternion representation extended to complex numbers, we show that the laws of electromagnetism can be expressed much more simply and concisely. We also derive the quaternion representation of rotations and boosts from the spinor representation of Lorentz group. It is suggested that the imaginary “ i ” should be attached to the spatial coordinates, and observe that the complex conjugate of quaternion representation is exactly equal to parity inversion of all physical quantities in the quaternion. We also show that using quaternion is directly linked to the two-spinor formalism. Finally, we discuss meanings of quaternion, octonion and sedenion in physics as n -fold rotation.

Keywords: quaternion; electromagnetism; representation theory; Cayley-Dickson algebra; special relativity; twistor theory

1. Introduction

There are several papers claiming that the quaternion or the octonion can be used to describe the laws of classical electromagnetism in a simpler way [1–6]. However, they are mainly limited to describing Maxwell equations. Furthermore, the meaning of quaternion and the reasons electromagnetic laws can be concisely described by them have not been well discussed up to now. Here, we list more diverse quaternion representations of the relations in electromagnetism than previously known and we introduce a new simpler notation to express quaternions. The proposed notation makes the quaternion representation of electromagnetic relations look similar to the differential-form representation of them. Moreover, the classical electromagnetic mass density and the complex Lagrangian can be newly defined and used to represent electromagnetic relations as quaternions.

It has been already well known that the quaternion can describe the Lorentz transformations of four vectors [7]. We here rederive the quaternion representation of the Lorentz boost and the rotation, by using isomorphism between the basis of quaternion and the set of sigma matrices. Hence, we find that not only four vector quantities but also electromagnetic fields can be transformed simply in the quaternion representation. Starting from the 4×4 matrix representation of quaternion, we define a new complex electromagnetic field tensor. By using it, a complex energy–momentum stress tensor of electromagnetic fields and a complex Lagrangian can be nicely expressed. Interestingly, the eigenvalues of the complex energy–momentum stress tensor are the classical electromagnetic mass density up to sign. To define complex tensors, we introduce a new spacetime index called “tilde-spacetime index”. Imaginary number i is usually linked to time so that it can be regarded as imaginary time, but we insist that it is more natural for i to be linked to space. In our representation, we also find that the complex conjugate of a quaternion is equal to the quaternion consisting of the physical quantities with parity inversion.

The two-spinor formalism is known to be a spinor approach, which is useful to deal with the general relativity [8,9]. In the formalism, all world-tensors can be changed to even-indexed spinors and there we derive spinor descriptions of electromagnetism [10]. We here prove that the quaternion representations including Maxwell's equations are equivalent to the spinor representations of electromagnetism. We also explain how spinors in two-spinor formalism are generally linked to the quaternion. Finally, we explore the meaning of quaternion and more extended algebras such as octonion as n -fold rotation.

2. Complex Quaternion

Let us denote quaternions by characters with a lower dot such as $\dot{q}$. Quaternions are generally represented in the form

$$\dot{q} = s + v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \quad (1)$$

where s, v_1, v_2, v_3 are real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the units of quaternions which satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \quad (2)$$

Equation (1) consists of two parts, namely a “scalar” part s and a “quaternion vector” part $v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$. If we denote the quaternion vector part by $\vec{v}$, Equation (1) is written as

$$\dot{q} = s + \vec{v}. \quad (3)$$

All quaternion vectors, denoted by an over-arrow symbol $\vec{}$, can be interpreted as coordinate vectors in $\mathbb{R}^3$. We do not distinguish between vectors and quaternion vectors in this paper.

If $\dot{q}_1 = a + \vec{A}$ and $\dot{q}_2 = b + \vec{B}$ are two quaternions, the multiplication of the quaternions can be described as

$$\dot{q}_1 \dot{q}_2 = (a + \vec{A})(b + \vec{B}) = ab - \vec{A} \cdot \vec{B} + a\vec{B} + b\vec{A} + \vec{A} \times \vec{B}, \quad (4)$$

by applying Equation (2), where $\vec{A} \cdot \vec{B}$ is the dot product and $\vec{A} \times \vec{B}$ is the cross product. The dot product and the cross product, which are operations for three-dimensional vectors are used in quaternion vectors.

The components of quaternions can be extended to complex numbers. We call such a quaternion “complex quaternion”. The general form of complex quaternion is

$$\dot{Q} = a + ib + \vec{c} + i\vec{d}. \quad (5)$$

where a, b and components of $\vec{c}, \vec{d}$ are real numbers, and i is a complex number $\sqrt{-1}$, which differs from the quaternion unit $\mathbf{i}$.

We denote the operation of complex conjugation by a bar $\bar{}$, and the complex conjugate of $\dot{Q}$ is

$$\bar{\dot{Q}} = a - ib + \vec{c} - i\vec{d}. \quad (6)$$

For a quaternion vector $\vec{q} = q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$, the exponential of $\vec{q}$ is defined by

$$\exp(\vec{q}) = e^{\vec{q}} \equiv 1 + \dot{q} + \frac{1}{2!} \dot{q}^2 + \frac{1}{3!} \dot{q}^3 \dots = \cos |\vec{q}| + i \frac{\vec{q}}{q} \sin |\vec{q}|, \quad (7)$$

since $\dot{q}^2 = -|\vec{q}|^2$ [11].

3. Laws of Electromagnetism in the Complex Quaternion Representation

3.1. Electromagnetic Quantities

We use the unit system which satisfies $\epsilon_0 = \mu_0 = c = 1$ where ϵ_0 is vacuum permittivity, μ_0 is vacuum permeability and c is speed of light. The sign conventions for the Minkowski metric is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

In the classical electromagnetism, the density of electromagnetic field momentum $\vec{p}$ and the density of electromagnetic field energy u are defined by

$$\vec{p} \equiv \vec{E} \times \vec{B}, \quad u \equiv \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2), \quad (8)$$

where $\vec{E}$ is an electric field and $\vec{B}$ is a magnetic field [12,13]. In our unit system, the electromagnetic momentum $\vec{p} \equiv \epsilon_0 \vec{E} \times \vec{B}$ (in SI units) is the same as the Poynting vector $\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}$ (in SI units).

We define a complex Lagrangian $\mathcal{L}$ and an electromagnetic mass density m by

$$\mathcal{L} \equiv \frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) + i \vec{E} \cdot \vec{B}, \quad m \equiv \sqrt{u^2 - p^2} = \sqrt{\frac{1}{4}(|\vec{E}|^2 - |\vec{B}|^2)^2 + (\vec{E} \cdot \vec{B})^2}. \quad (9)$$

The electromagnetic mass density m is defined from the energy-momentum relation $m^2 = u^2 - |\vec{p}|^2$ where $(u, \vec{p})$ is four-momentum of a particle of mass m . The meaning of m should be investigated more in detail; however, it is not discussed here. Comparing $\mathcal{L}$ and m , we can see that

$$m = \sqrt{\mathcal{L}\bar{\mathcal{L}}}. \quad (10)$$

3.2. Complex Quaternion Representations of Electromagnetic Relations

Let us define a few physical quantities in the form of complex quaternion,

$$\begin{aligned} \dot{u} &\equiv \gamma + i \gamma \vec{v}, & \dot{A} &\equiv V + i \vec{A}, \\ \dot{F} &\equiv i \vec{E} - \vec{B}, & \dot{J} &\equiv \rho + i \vec{J}, \\ \dot{p} &\equiv u + i \vec{p}, & \dot{f} &\equiv \vec{J} \cdot \vec{E} + i(\rho \vec{E} + \vec{J} \times \vec{B}), \end{aligned} \quad (11)$$

where γ is $1/\sqrt{1-v^2}$ for the velocity v , V is the electric potential, $\vec{A}$ is the vector potential, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density, and $\vec{J}$ is the electric current density. $\dot{F}$ is just a quaternion vector and the terms in $\dot{p}$ are defined in Equation (8). $\dot{J}$ is equal to $\rho_0 \dot{u}$ where ρ_0 is the proper charge density, which is the density in the rest system of the charge. The scalar part of $\dot{f}$ is the rate of work done by electric field on the charge and the vector part is the Lorentz force.

We define a quaternion differential operator by

$$\dot{d} \equiv \frac{\partial}{\partial t} - i \nabla, \quad (12)$$

where t is the time and $\nabla = \partial_x \mathbf{i} + \partial_y \mathbf{j} + \partial_z \mathbf{k}$ is the vector differential operator in the three-dimensional Cartesian coordinate system.

The relations in electromagnetism can be described in the complex quaternion form simply as follows:

$$\begin{aligned}
 (1) \quad \mathcal{D}\bar{\mathcal{D}} &= \square^2 & (\text{d'Alembert Operator}) \\
 (2) \quad \mathcal{A}' &= \mathcal{A} + \mathcal{D}\lambda & (\text{Gauge Transformation}) \\
 (3) \quad \mathcal{D}\bar{\mathcal{A}} &= \mathcal{F} & (\text{Field Strength from Gauge Field}) \\
 (4) \quad \mathcal{D}\bar{\mathcal{F}} &= \mathcal{J} (= \mathcal{D}\bar{\mathcal{D}}\mathcal{A}) & (\text{Electromagnetic Current, Maxwell Equations}) \\
 (5) \quad \mathcal{D}\bar{\mathcal{J}} &= \mathcal{D}\bar{\mathcal{D}}\mathcal{F} = \square^2\mathcal{F} & (\text{Electromagnetic Wave Equation with Source}) \\
 (6) \quad \mathcal{F}\mathcal{J} &= \mathcal{f} + \mathcal{l} & (\text{Lorentz Force}) \\
 (7) \quad \mathcal{F}(\mathcal{D}\bar{\mathcal{F}}) &= (\mathcal{F}\mathcal{D})\bar{\mathcal{F}} (= \mathcal{F}\mathcal{J}) & (\text{Formula with Quaternion Differential Operator}) \\
 (8) \quad \frac{1}{2}\mathcal{F}\bar{\mathcal{F}} &= \mathcal{p} & (\text{Electromagnetic Energy-Momentum}) \\
 (9) \quad \mathcal{D}\bar{\mathcal{p}} &= \frac{1}{2}[(\mathcal{D}\bar{\mathcal{F}})\mathcal{F} + \mathcal{F}(\mathcal{D}\bar{\mathcal{F}})] + i(\mathcal{F} \cdot \nabla)\mathcal{F} & (\text{Conservation of Electromagnetic Energy-Momentum}) \\
 (10) \quad \frac{1}{2}\mathcal{F}\bar{\mathcal{F}} &= \mathcal{L} & (\text{Euclidean Lagrangian of Electromagnetic Fields}) \\
 (11) \quad \mathcal{p}\bar{\mathcal{p}} &= \mathcal{L}\bar{\mathcal{L}} = m^2 & (\text{Electromagnetic Mass Density})
 \end{aligned} \tag{13}$$

where $\mathcal{l} = i\vec{J} \cdot \vec{B} + (-\rho\vec{B} - \vec{E} \times \vec{J})$.

We can check all quaternion relations by expanding multiplications of quaternions using Equation (4). Some expansions are proven in Appendix A. Relations (1), (3) and (4) are already well known in quaternion forms, but the others are not well mentioned thus far. Each quaternion equation in Equation (13) contains several relations, which are known in classical electromagnetism.

Let us discuss in more detail each relation in Equation (13).

- (1) $\mathcal{D}\bar{\mathcal{D}} = \square^2$ is the d'Alembert operator.
- (2) $\mathcal{A}' = \mathcal{A} + \mathcal{D}\lambda$ describes the gauge transformation of gauge fields.

$$V' = V + \frac{\partial \lambda}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla \lambda. \tag{14}$$

- (3) $\mathcal{D}\bar{\mathcal{A}} = \mathcal{F}$ contains three relations. One is Lorentz gauge condition and the others are the relations between fields strength and gauge fields, as shown in Equation (A1),

$$\frac{\partial V}{\partial t} + \nabla \cdot \vec{A} = 0 \tag{15}$$

$$\vec{E} = -\nabla V + \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}. \tag{16}$$

- (4) $\mathcal{D}\bar{\mathcal{F}} = \mathcal{J} (= \mathcal{D}\bar{\mathcal{D}}\mathcal{A})$ contains all four Maxwell's equations in Equation (A2),

$$\nabla \cdot \vec{E} = \rho, \quad \nabla \times \vec{B} = \mathcal{J} + \frac{\partial \vec{E}}{\partial t}, \tag{17}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{18}$$

It can be the wave equations of gauge fields with sources in the Lorentz gauge,

$$\square^2 \vec{V} = \rho, \quad \square^2 \vec{A} = \vec{J}. \tag{19}$$

- (5) $\mathcal{D}\bar{\mathcal{J}} = \mathcal{D}\bar{\mathcal{D}}\mathcal{F} = \square^2\mathcal{F}$ contains the charge conservation relation and the wave equations of $\vec{E}$ and $\vec{B}$ fields in Equation (A3),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0, \tag{20}$$

$$\square^2 \vec{E} = -\nabla \rho + \frac{\partial \mathcal{J}}{\partial t}, \quad \square^2 \vec{B} = \nabla \times \vec{J}. \tag{21}$$

Those can be derived from taking $\mathbb{d}$ operation on the both side of Relation (4) in Equation (13).

- (6) $\mathbb{F}\mathbb{J}$ includes the Lorentz force term $\rho\vec{E} + \vec{J} \times \vec{B}$ and the work done by electromagnetic fields term $\vec{J} \cdot \vec{E}$. However, the meaning of $\mathbb{I} = i\vec{J} \cdot \vec{B} + (-\rho\vec{B} - \vec{E} \times \vec{J})$ is not yet well known.
- (7) We have found that $(\mathbb{F}\mathbb{d})\bar{\mathbb{F}}$ is equal to $\mathbb{F}(\mathbb{d}\bar{\mathbb{F}})(= \mathbb{F}\mathbb{J})$ where $(\mathbb{F}\mathbb{d})$ is the quaternion differential operator. The proof of this is given in Appendix A.2.
- (8) $\frac{1}{2}\mathbb{F}\bar{\mathbb{F}} = \mathbb{p}$ is the quaternion representation of electromagnetic energy and momentum. It can be easily verified, by expanding the left side, that

$$u = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2), \quad \vec{p} = \vec{E} \times \vec{B}. \quad (22)$$

- (9) It can be guessed that $\mathbb{d}\bar{\mathbb{p}} \sim \mathbb{f}$ from the analogy with the force-momentum relation $\frac{Dp_\lambda}{D\tau} = f_\lambda = qu^\mu F_{\mu\lambda}$, where f_λ is the four-force, D is the covariant derivative, τ is the proper time, q is the electric charge, U^μ is the four-velocity, and $F_{\mu\lambda}$ is the electromagnetic tensor, which is the relation of the four-force acting to a charged particle situated in electromagnetic fields. $\mathbb{d}\bar{\mathbb{p}}$ is expanded as

$$\mathbb{d}\bar{\mathbb{p}} = (\partial_t - i\nabla)(u - i\mathbb{p}) = (\partial_t u + \nabla \cdot \mathbb{p}) - i(\partial_t \mathbb{p} + \nabla u) - (\nabla \times \mathbb{p}). \quad (23)$$

Substituting Equation (22) into Equation (23), we get

$$\partial_t u + \nabla \cdot \mathbb{p} = -\vec{J} \cdot \vec{E} \quad (24)$$

$$-(\partial_t \mathbb{p} + \nabla u) = (\rho\vec{E} + \vec{J} \times \vec{B}) - (\nabla \cdot \vec{E})\vec{E}_i - (\vec{E} \cdot \nabla)\vec{E}_i - (\nabla \cdot \vec{B})\vec{B}_i - (\vec{B} \cdot \nabla)\vec{B}_i \quad (25)$$

$$\nabla \times \mathbb{p} = (-\rho\vec{B} + \vec{J} \times \vec{E}) - \vec{E} \times \partial_t \vec{E} - \vec{B} \times \partial_t \vec{B} + \vec{E}\nabla\vec{B} - \vec{B}\nabla\vec{E}, \quad (26)$$

where $(\vec{A}\nabla\vec{B})_i \equiv A_j(\nabla_j B_i)$ for vector fields $\vec{A}$ and $\vec{B}$.

Equation (24) is the work-energy relation in electromagnetism. Equation (25) can be rearranged as

$$\mathbb{f} = (\overleftrightarrow{\nabla} \cdot \mathbb{T}) - \frac{\partial \mathbb{p}}{\partial t}, \quad (27)$$

where

$$(\mathbb{T})_{ij} = (E_i E_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2) + (B_i B_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2) \quad (28)$$

is the Maxwell stress tensor and

$$(\overleftrightarrow{\nabla} \cdot \mathbb{T})_i \equiv (\nabla \cdot \vec{E})E_i + (\vec{E} \cdot \nabla)E_i + (\nabla \cdot \vec{B})B_i + (\vec{B} \cdot \nabla)B_i - \frac{1}{2}\nabla_i(|\vec{E}|^2 + |\vec{B}|^2). \quad (29)$$

Equation (26) is not a well-known relation. The proof of the expansion is given in Appendix B. By looking at Equations (24)–(26), we can observe that it is difficult to find a simple quaternion formula such as $\mathbb{d}\bar{\mathbb{p}} = \mathbb{f} + \mathbb{I}$. The exact formula of $\mathbb{d}\bar{\mathbb{p}}$ is obtained as

$$\mathbb{d}\bar{\mathbb{p}} = \frac{1}{2} [(\mathbb{d}\bar{\mathbb{F}})\mathbb{F} + \bar{\mathbb{F}}(\mathbb{d}\mathbb{F})] + i(\bar{\mathbb{F}} \cdot \nabla)\mathbb{F}. \quad (30)$$

The proof is given in Appendix C.

- (10) $\frac{1}{2}\mathbb{F}\bar{\mathbb{F}} = \mathcal{L}$ is the relation between the complex Lagrangian and electromagnetic fields. The complex Lagrangian $\mathcal{L}$ is defined as $\frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) + i\vec{E} \cdot \vec{B}$. This is, in fact, the Euclidean Lagrangian including topological term [14,15]. The real part $\frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2)$ is the Lagrangian of electromagnetic fields $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = A_{[\mu}\partial_{\nu]}$ for $U(1)$ gauge field A_μ . The variation of this

part gives the first two Maxwell's equations in Equation (17). The complex part $\vec{E} \cdot \vec{B}$ is $\frac{1}{4} F_{\mu\nu} {}^* F^{\mu\nu}$, which is the topological term of gauge fields where ${}^* F^{\mu\nu}$ is Hodge dual of $F^{\mu\nu}$. Its variation gives the other two Maxwell's equations in Equation (18).

(11) $\vec{p} \vec{p} = \mathcal{L} \mathcal{L} = m^2$ is a Lorentz invariant and a gauge invariant quantity.

4. Lorentz Transformation in the Complex Quaternion Representation

For a quaternion basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the algebra of $\{1, i, j, k\}$ is isomorphic to the algebra of sigma matrices $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$, where σ^0 is 2×2 identity matrix and $\sigma^1, \sigma^2, \sigma^3$ are Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (31)$$

It means that complex quaternions that have the form $q = q_0 + i\vec{q}$ are isomorphic to $q_\mu \sigma^\mu$ where $q_\mu = (q_0, \vec{q}) = (q_0, q_1, q_2, q_3)$ and $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$.

We can get the quaternion representation of Lorentz transformation by using isomorphism given above and the spinor representation of the Lorentz group. Let us denote by $S[\Lambda]$ the spinor representation of the Lorentz group, which acts on Dirac spinor $\psi(x)$. Then, Dirac spinor transforms as $\psi(x) \rightarrow S[\Lambda]\psi(\Lambda^{-1}x)$ under a Lorentz transformation $x \rightarrow x' = \Lambda x$.

In the chiral representation of the Clifford algebra, the spinor representation of rotations $S[\Lambda_{rot}]$ and boosts $S[\Lambda_{boost}]$ are

$$S[\Lambda_{rot}] = \begin{pmatrix} e^{+i\vec{\phi} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\phi} \cdot \vec{\sigma}/2} \end{pmatrix}, \quad S[\Lambda_{boost}] = \begin{pmatrix} e^{+\vec{\eta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\eta} \cdot \vec{\sigma}/2} \end{pmatrix}, \quad (32)$$

where $\vec{\phi} = \phi \hat{\phi}$, $\vec{\eta} = \hat{v} \tanh^{-1} |\vec{v}|$, ϕ is the rotation angle, $\hat{\phi}$ is the unit vector of rotation axis, $\vec{v}$ is the boost velocity, and $\hat{v}$ is the unit vector of boost velocity.

Since it is known [16] that

$$S[\Lambda]^{-1} \gamma^\mu S[\Lambda] = \Lambda^\mu_\nu \gamma^\nu, \quad (33)$$

the following relation also holds:

$$S[\Lambda]^{-1} V_\mu \gamma^\mu S[\Lambda] = V_\mu \Lambda^\mu_\nu \gamma^\nu \quad (34)$$

for any four-vector V^μ .

The components of Equation (34) are

$$\begin{pmatrix} 0 & e^{-i\vec{\phi} \cdot \vec{\sigma}/2} V_\mu \sigma^\mu e^{+i\vec{\phi} \cdot \vec{\sigma}/2} \\ e^{-i\vec{\phi} \cdot \vec{\sigma}/2} V_\mu \bar{\sigma}^\mu e^{+i\vec{\phi} \cdot \vec{\sigma}/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_\mu \Lambda^\mu_\nu \sigma^\nu \\ V_\mu \Lambda^\mu_\nu \bar{\sigma}^\nu & 0 \end{pmatrix}, \quad (35)$$

$$\begin{pmatrix} 0 & e^{-\vec{\eta} \cdot \vec{\sigma}/2} V_\mu \sigma^\mu e^{-\vec{\eta} \cdot \vec{\sigma}/2} \\ e^{+\vec{\eta} \cdot \vec{\sigma}/2} V_\mu \bar{\sigma}^\mu e^{+\vec{\eta} \cdot \vec{\sigma}/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_\mu \Lambda^\mu_\nu \sigma^\nu \\ V_\mu \Lambda^\mu_\nu \bar{\sigma}^\nu & 0 \end{pmatrix}, \quad (36)$$

where $\bar{\sigma}^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$. This represents the quaternion Lorentz transformation for the form $q_0 + i\vec{q}$, since $q = q_0 + i\vec{q} \sim q_\mu \sigma^\mu$.

Let us define Lorentz transformation factor $\zeta(\phi, \eta)$ by

$$\zeta(\phi, \eta) \equiv e^{+\frac{1}{2}\vec{\phi}} e^{-\frac{1}{2}i\vec{\eta}} = \left(\cos \frac{\phi}{2} + \hat{\phi} \sin \frac{\phi}{2}\right) \left(\cosh \frac{\eta}{2} - i \hat{\eta} \sinh \frac{\eta}{2}\right), \quad (37)$$

where $\cosh \eta = \gamma$, $\sinh \eta = \gamma v$. Since $(\cosh \frac{\eta}{2} + i \hat{\eta} \sinh \frac{\eta}{2}) = \gamma + i \gamma \vec{v}$ is the quaternion velocity $\mathfrak{u}(\vec{v})$ of a boosted frame with a boost velocity $\vec{v}$, Equation (37) can be rewritten as

$$\zeta(\phi, \eta) = \mathfrak{R}(\vec{\phi}) \mathfrak{u}(\vec{v}), \quad (38)$$

where $\mathfrak{R}(\vec{\phi}) \equiv (\cos \frac{\phi}{2} + \hat{\phi} \sin \frac{\phi}{2})$. The inverse of $\zeta(\phi, \eta)$ and its complex conjugate are defined as

$$\zeta^{-1}(\phi, \eta) = e^{+\frac{1}{2}i\vec{\eta}} e^{-\frac{1}{2}\vec{\phi}}, \quad \overline{\zeta^{-1}}(\phi, \eta) = e^{-\frac{1}{2}i\vec{\eta}} e^{-\frac{1}{2}\vec{\phi}}. \quad (39)$$

From Equations (35) and (36), the Lorentz transformations of a quaternion that has the form $\mathfrak{V} = V^0 + i\vec{V}$ is written as

$$\mathfrak{V}' = \zeta \mathfrak{V} \overline{\zeta^{-1}}. \quad (40)$$

Therefore, the Lorentz transformations of a quaternion gauge field $\mathfrak{A}$ and a quaternion strength field $\mathfrak{F}$ are

$$\mathfrak{A}' = \zeta \mathfrak{A} \overline{\zeta^{-1}}, \quad (41)$$

$$\mathfrak{F}' = \mathfrak{d}' \mathfrak{A}' = \zeta \mathfrak{d} \overline{\zeta^{-1}} \zeta \mathfrak{A} \overline{\zeta^{-1}} = \zeta \mathfrak{F} \overline{\zeta^{-1}}. \quad (42)$$

As an example, if we boost a frame with a speed v along x axis, then

$$\begin{aligned} \zeta &= \mathfrak{u}(\vec{v}) = \cosh \frac{\eta}{2} + i \hat{\eta} \sinh \frac{\eta}{2} = \gamma + i \gamma v \mathbf{i} \\ \mathfrak{A}' &= \zeta \mathfrak{A} \overline{\zeta^{-1}} = (\gamma - i \gamma \vec{v}) \mathfrak{A} (\gamma + i \gamma \vec{v}) \\ &= \gamma(V - A_1 v) + i(\gamma(A_1 - Vv) \mathbf{i} + A_2 \mathbf{i} + A_3 \mathbf{k}), \end{aligned} \quad (43)$$

$$\begin{aligned} \mathfrak{F}' &= \zeta \mathfrak{F} \overline{\zeta^{-1}} = (\gamma - i \gamma \vec{v}) \mathfrak{F} (\gamma + i \gamma \vec{v}) \\ &= i(E_1 \mathbf{i} + \gamma(E_2 - B_3 v) \mathbf{j} + \gamma(E_3 + B_2 v) \mathbf{k}) - (B_1 \mathbf{i} + \gamma(B_2 + E_3 v) \mathbf{j} + \gamma(B_3 - E_2 v) \mathbf{k}), \end{aligned} \quad (44)$$

which is a very efficient representation in computing rotations and boosts.

5. The Role of Complex Number “ i ” in Complex Quaternions

5.1. Complex Space and Real Time

In this section, we explain that it is more natural to attach imaginary number i to the spatial coordinates rather than to the time coordinate. The infinitesimal version of the Lorentz transformation in one dimension is

$$dt' = \gamma(dt - vdx), \quad dx' = \gamma(dx - vdt). \quad (45)$$

This can be manipulated to

$$\begin{aligned} dt' &= \gamma(dt - vdx) \\ &= \frac{1}{\sqrt{1 + (\frac{i dx'_s}{dt'_s})^2}} (dx - vdt) = \frac{1}{\sqrt{(dt'_s)^2 + (i dx'_s)^2}} (dtdt'_s + (i dx)(i dx'_s)), \end{aligned} \quad (46)$$

$$\begin{aligned} i dx' &= i \gamma(dx - vdt) \\ &= \frac{1}{\sqrt{1 + (\frac{i dx'_s}{dt'_s})^2}} (i dx - i vdt) = \frac{1}{\sqrt{(dt'_s)^2 + (i dx'_s)^2}} ((i dx)dt'_s - dt(i dx'_s)), \end{aligned} \quad (47)$$

where $v = dx'_s/dt'_s$ is a boost velocity, dx'_s is an infinitesimal displacement of the moving frame and dt'_s is an infinitesimal time it takes for the frame to move along the displacement.

If we put imaginary number “ i ” to the spatial coordinate as Equations (46) and (47), the Lorentz transformation can be seen as a kind of rotation,

$$\begin{aligned} dt &= r \cos \alpha, & i dx &= r \sin \alpha, \\ dt_s &= r \cos \beta, & i dx'_s &= r \sin \beta, \\ \rightarrow dt' &= r \cos(\alpha - \beta), & i dx' &= r \sin(\alpha - \beta), \end{aligned} \quad (48)$$

for pure imaginary angles α, β and $r = \sqrt{(dt_s')^2 + (i dx_s')^2}$.

In contrast, if we put i to the time coordinate rather than to the spatial coordinate, then

$$\begin{aligned} i dt &= r \cos \alpha, & dx &= r \sin \alpha, \\ i dt_s &= r \cos \beta, & dx'_s &= r \sin \beta, \\ \rightarrow i dt' &\neq r \cos(\alpha - \beta), & dx' &\neq r \sin(\alpha - \beta), \end{aligned} \quad (49)$$

which means that Equations (46) and (47) cannot be regarded as a kind of rotation.

5.2. Parity Inversion and Conjugate of i

All physical quantities that are located in the real part of quaternions, such as ρ , V , $\vec{B}$, $\frac{1}{2}(E^2 - B^2)$, etc., do not change signs under parity inversion; and all physical quantities that are located in the imaginary part of quaternions, such as $\vec{A}$, $\vec{E}$, $\vec{J}$, $\vec{E} \cdot \vec{B}$, etc., change signs under parity inversion. This means that the operation of complex conjugation on a quaternion corresponds to the parity inversion of the physical quantities in the quaternion representation. The reason is related to tilde-spacetime indices, which are defined in Sections 6.1 and 7.2.

All quantities in the imaginary part may be regarded as “imaginary quantities”, not just as “real quantities placed in the imaginary part”, i.e. imaginary space, imaginary momentum, imaginary electric field, etc. It is the same as replacing length units, such as “meter”, with imaginary length unit such as “ i meter”.

6. Complex Electromagnetic Tensor Related to Quaternion and Electromagnetic Laws

6.1. Electromagnetic Tensor with Tilde-Spacetime Index

For a vector $\vec{b} = (b_1, b_2, b_3)$, let us define “vector matrix of $\vec{b}$ ” as

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (50)$$

and the vector matrix by # notation as

$$\mathbf{b}^\# = -\epsilon_{ijk} b^k = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}, \quad (51)$$

where ϵ_{ijk} are the Levi-Civita symbols.

Then, the electromagnetic tensor $F^{\mu\nu}$ can be represented as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{E}^t \\ \mathbf{E} & \mathbf{B}^\# \end{pmatrix}, \quad (52)$$

where $\mathbf{E}, \mathbf{B}$ are vector matrix of $\vec{E}, \vec{B}$ and superscript $\mathbf{E}^t$ means the transpose of a matrix $\mathbf{E}$. The dual tensor can be represented as

$$G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -\mathbf{B}^t \\ \mathbf{B} & -\mathbf{E}^\# \end{pmatrix}, \quad (53)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the rank-4 Levi-Civita symbol with the sign convention $\epsilon^{0123} = +1$.

Now, we define tensor indices with tilde such as " $\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}$..", called "tilde-spacetime indices". $O^{\tilde{\mu}}$ and $O_{\tilde{\mu}}$ for any $O^\mu = (O^0, O^1, O^2, O^3)$ and $O_\mu = (O_0, O_1, O_2, O_3)$ are defined as

$$O^{\tilde{\mu}} = (O^0, iO^1, iO^2, iO^3), \quad O_{\tilde{\mu}} = (O_0, -iO_1, -iO_2, -iO_3). \quad (54)$$

Then, the components of $O^{\tilde{\mu}}$ and $O_{\tilde{\mu}}$ become equal, since $O_0 = O^0, O_1 = -O^1, O_2 = -O^2, O_3 = -O^3$ in Minkowski metric. As an example, $\tilde{\sigma}^{\tilde{\mu}}$ is $(\sigma^0, -i\sigma^1, -i\sigma^2, -i\sigma^3)$. Since this is isomorphic to the quaternion basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$, we can rewrite $\tilde{\sigma}^{\tilde{\mu}}$ as a quaternion basis $\hat{q}^{\tilde{\mu}}$ so that $A_\mu \tilde{\sigma}^\mu = A_{\tilde{\mu}} \hat{q}^{\tilde{\mu}}$.

Generally speaking, the way to convert a quantity with multiple spacetime indices to the quantity with multiple tilde-spacetime indices is multiplying with or dividing by imaginary number i when each spacetime index has the value 1, 2 or 3. As an example, the Minkowski metric with tilde indices is $g_{\tilde{\mu}\tilde{\nu}} = (1, 1, 1, 1)$, since $-(-i)(-i) = 1$.

Applying this rule to electromagnetic tensors, we get

$$F^{\tilde{\mu}\tilde{\nu}} = \partial^{\tilde{\mu}} A^{\tilde{\nu}} - \partial^{\tilde{\nu}} A^{\tilde{\mu}} = \begin{pmatrix} 0 & -iE_1 & -iE_2 & -iE_3 \\ iE_1 & 0 & -B_3 & B_2 \\ iE_2 & B_3 & 0 & -B_1 \\ iE_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\mathbf{E}^t \\ i\mathbf{E} & -\mathbf{B}^\# \end{pmatrix} \quad (55)$$

$$G^{\tilde{\mu}\tilde{\nu}} = \begin{pmatrix} 0 & -i\mathbf{B}^t \\ i\mathbf{B} & \mathbf{E}^\# \end{pmatrix}. \quad (56)$$

6.2. The 4×4 Representation of Complex Quaternions

The basis elements of quaternion, $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, can be represented as 4×4 matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (57)$$

A quaternion such as $q = a + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ can be represented in the tensor representation

$$\begin{aligned}
 T(\mathbf{q}) &= a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + b_1 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} a & -b_1 & -b_2 & -b_3 \\ b_1 & a & -b_3 & b_2 \\ b_2 & b_3 & a & -b_1 \\ b_3 & -b_2 & b_1 & a \end{pmatrix}, \quad (58)
 \end{aligned}$$

where T means the tensor representation. When $a = 0$, $T(\mathbf{q})$ has a simple form $\begin{pmatrix} 0 & -\mathbf{b}^t \\ \mathbf{b} & \mathbf{b}^\# \end{pmatrix}$.

For a quaternion field strength $\mathbf{F} = \vec{E}i - \vec{B} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the tensor form of $\mathbf{F}$ is

$$T(\mathbf{F}) = F_1 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + F_2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + F_3 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{F}^t \\ \mathbf{F} & \mathbf{F}^\# \end{pmatrix}, \quad (59)$$

where $\mathbf{F} \equiv i\mathbf{E} - \mathbf{B}$ which is a vector matrix of the vector $\vec{F} = i\vec{E} - \vec{B}$. This is eventually identical to $F^{\mu\nu} + iG^{\mu\nu}$ [17–19].

6.3. Complex Electromagnetic Tensor and Electromagnetic Laws

Let us define $\mathfrak{F}$ and its conjugate $\mathfrak{F}^*$ as

$$\mathfrak{F} = F^{\mu\nu} + iG^{\mu\nu}, \quad (60)$$

$$\mathfrak{F}^* = F^{\mu\nu} - iG^{\mu\nu}. \quad (61)$$

A few complex tensors can also be defined as follows,

$$\begin{aligned}
 \mathcal{D} &= \left(\frac{\partial}{\partial t}, -i\nabla^t \right), & \mathfrak{J} &= (\rho, i\mathbf{J}^t), \\
 \mathfrak{f} &= \left(\vec{J} \cdot \vec{E}, i(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})^t \right), & \mathcal{T} &= \begin{pmatrix} u & i\mathbf{p}^t \\ i\mathbf{p} & \mathbb{T} \end{pmatrix}, \quad (62)
 \end{aligned}$$

where $\mathbf{J}$ is the vector matrix of $\vec{J}$, $(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})$ is the vector matrix of $\rho\vec{E} + \vec{J} \times \vec{B}$, ∇ is the vector matrix of ∇ , and $\mathbf{p}$ is the vector matrix of $\vec{p}$.

Then, the following tensor relations hold:

$$\mathfrak{F}\mathfrak{F}^* = \mathfrak{F}^*\mathfrak{F}, \quad (63)$$

$$\mathcal{T} = \frac{1}{2}\mathfrak{F}\mathfrak{F}^*, \quad (64)$$

$$\mathcal{D}\mathfrak{F} = \mathfrak{J} \quad (= \mathcal{D}\mathfrak{F}^*), \quad (65)$$

$$\mathcal{D}\mathcal{T} = -\mathfrak{f} = \frac{1}{2}\mathfrak{J}(\mathfrak{F} + \mathfrak{F}^*), \quad (66)$$

$$\frac{1}{2}\mathfrak{F}\mathfrak{F} = \frac{1}{2}\mathfrak{F}^2 = \mathfrak{L}I, \quad (67)$$

$$\mathcal{T}\mathcal{T} = \frac{1}{4}\mathfrak{F}^2\mathfrak{F}^2 = \mathfrak{m}^2I, \quad (68)$$

$$\text{Eigenvalues}(\mathcal{T}) = \pm\mathfrak{m}, \quad (69)$$

where $I = (1, 1, 1, 1)$ is unit matrix. $\mathfrak{L}$ and $\mathfrak{m}$ are the complex Lagrangian and the electromagnetic mass density (Equation (9)). All relations can be easily verified by simple calculations. Actually, the components of $\mathcal{D}$, $\mathfrak{J}$ and $\mathcal{T}$ are equal to the components of $\partial_{\tilde{\mu}}$, $J^{\tilde{\mu}}$ and $T_{EM}^{\tilde{\mu}\tilde{\nu}}$, where ∂_{μ} is the four-gradient, J^{μ} is the electric current density and $T_{EM}^{\mu\nu}$ is the electromagnetic stress–energy tensor defined as

$$T_{EM}^{\mu\nu} = \begin{pmatrix} \mathfrak{u} & \mathfrak{p}^i \\ \mathfrak{p} & -\mathbb{T} \end{pmatrix}. \quad (70)$$

Those listed relations of complex tensors can be verified by using several known tensor relations in electromagnetism and tilde-spacetime indices, instead of the direct calculation. For example, Equation (65), which represents Maxwell’s equations, can be easily verified from $\partial_{\mu}F^{\mu\nu} = 0$ and $\partial_{\mu}G^{\mu\nu} = J^{\nu}$.

The complex electromagnetic stress–energy tensor $\mathcal{T}$ contains the information about electromagnetic energy density $\mathfrak{u}$, momentum density $\vec{\mathfrak{p}}$ and stress $\mathbb{T}$, as shown by Equations (8) and (28). It is interesting that $\mathcal{T}$ is linked to the electromagnetic mass density, as shown in Equations (68) and (69). Especially, Equation (69) cannot be simply derived from known relations of electromagnetism.

By differentiating both sides of the relation in Equation (64), we get

$$\begin{aligned} \mathcal{D}\mathcal{T} &= \mathcal{D}\left(\frac{1}{2}\mathfrak{F}^*\mathfrak{F}\right) = \frac{1}{2}((\mathcal{D}\mathfrak{F}^*)\mathfrak{F} + (\mathfrak{F}^{*T}\mathcal{D}^T)^T\mathfrak{F}) \\ &= \mathcal{D}\left(\frac{1}{2}\mathfrak{F}\mathfrak{F}^*\right) = \frac{1}{2}((\mathcal{D}\mathfrak{F})\mathfrak{F}^* + (\mathfrak{F}^T\mathcal{D}^T)^T\mathfrak{F}^*), \end{aligned} \quad (71)$$

since $\partial_a(A^{ab}B_{bc}) = (\partial_a A^{ab})B_{bc} = (\partial_a A^{ab})B_{bc} + A^{ab}(\partial_a B_{bc})$. Substituting Equation (65) into Equation (71) and comparing it with Equation (66), we further get the following relations:

$$(\mathfrak{F}^T\mathcal{D}^T)^T\mathfrak{F}^* = -(\mathfrak{F}\mathcal{D}^T)^T\mathfrak{F}^* = \mathfrak{J}\mathfrak{F}, \quad (72)$$

$$(\mathfrak{F}^{*T}\mathcal{D}^T)^T\mathfrak{F} = -(\mathfrak{F}^*\mathcal{D}^T)^T\mathfrak{F} = \mathfrak{J}\mathfrak{F}^*. \quad (73)$$

7. Relations between Quaternions and Two-Spinor Formalism

7.1. The Correspondence of Two-Spinor Representations and Quaternion Representations in Electromagnetism

Let us start with some basic contents of two-spinor formalism [8–10]. Mathematically, any null-like spacetime four-vector X^{μ} can be described as a composition of two spinors,

$$X^{\mu} = 1/\sqrt{2} \begin{pmatrix} \xi & \eta \end{pmatrix} \sigma^{\mu} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} = \frac{1}{\sqrt{2}}\psi^A\sigma_{AA'}^{\mu}\bar{\psi}^{A'}, \quad (74)$$

where σ^{μ} are sigma matrices $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$, the components of ψ^A are $\psi^1 = \xi, \psi^2 = \eta$ for proper complex numbers ξ and η , and $(\psi^A)^{\dagger} = \bar{\psi}^{A'}$. It can be rewritten as

$$\frac{1}{\sqrt{2}}X_{\mu}\sigma_{BB'}^{\mu} = \psi_B\bar{\psi}_{B'} \quad (75)$$

by using the relation $\bar{\sigma}^{\mu C'C} = \varepsilon^{C'B'}\varepsilon^{CB}\sigma_{BB'}^{\mu}$ and $\sigma_{AA'}^{\mu}\bar{\sigma}_{\mu}^{BB'} = 2\delta_A^B\delta_{A'}^{B'}$, where $\varepsilon^{AB}, \varepsilon^{A'B'}, \varepsilon_{AB}$, and $\varepsilon_{A'B'}$ are the ε -spinors whose components are $\varepsilon^{12} = \varepsilon_{12} = +1, \varepsilon^{21} = \varepsilon_{21} = -1$ as follows in [10].

We now define a spinor $X_{AA'}$ as

$$X_{AA'} \equiv \frac{1}{\sqrt{2}}X_{\mu}\sigma_{AA'}^{\mu}, \quad (76)$$

which is equivalent to X^μ . The factor, which connects a four-vector to a corresponding spinor, is called “Infeld–van der Waerden symbol” [20], such as $\frac{1}{\sqrt{2}}\sigma_{AA'}^a$ in Equation (76). It can be generally written as $g_{AA'}^a$. We can extend this notation not only to a null-like four-vector but also to any tensors by multiplying more than one Infeld–van der Waerden symbols: any tensor such as $T_{abc..}$ with spacetime indices $a, b, c..$ can be written as a spinor $T_{AA'BB'..}$ with spinor indices $A, A', B, B'..$, by multiplying $T_{abc..}$ with $g_{AA'}^a, g_{BB'}^b, ..$, such as $T_{AA'BB'..} = T_{ab..}g_{AA'}^ag_{BB'}^b..$. This can be simply written as

$$T_{AA'BB'..} = T_{ab..} \quad (77)$$

Any antisymmetric tensor $H_{ab} = H_{AA'BB'}$ can be divided into two parts

$$H_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\psi_{A'B'}, \quad (78)$$

where $\phi_{AB} = \frac{1}{2}H_{ABC'}C'$ and $\psi_{A'B'} = \frac{1}{2}H_C{}^C{}_{A'B'}$ (unprimed spinor indices and primed spinor indices can be rearranged back and forth). If H_{ab} is real, then $\psi_{A'B'} = \bar{\phi}_{A'B'}$ and

$$H_{ab} = H_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\bar{\phi}_{A'B'}. \quad (79)$$

Since an electromagnetic field tensor F_{ab} (Equation (52)) is antisymmetric, it can be written as

$$F_{AA'BB'} = \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\bar{\varphi}_{A'B'} \quad (80)$$

with an appropriate field φ_{AB} . There we find closely related electromagnetic relations [10]:

$$\nabla_{AA'}\Phi^{A'B} = \varphi_A^B, \quad (81)$$

$$\nabla^{A'B}\varphi_B^A = 2\pi J^{AA'}, \quad (82)$$

where $\nabla_{AA'} = \partial_a$ (in Minkowski spacetime) is the four-gradient, $\Phi_{AA'} = \Phi_a$ is the electromagnetic potential and $J_{AA'} = J_a$ is the charge-current vector. The former is the relation of electromagnetic potentials and strength fields, and the latter is equivalent to the two Maxwell’s equations.

Now, we prove that

$$\varphi_A^B = \frac{1}{2}[(-\vec{E} + i\vec{B}) \cdot \vec{\sigma}]_A^B = \frac{1}{2}[(-i\vec{E} - \vec{B}) \cdot \vec{\sigma}/i]_A^B, \quad (83)$$

$$\bar{\varphi}_{B'}^{A'} = \frac{1}{2}[(-\vec{E} - i\vec{B}) \cdot \vec{\sigma}]_{B'}^{A'} = \frac{1}{2}[(-i\vec{E} + \vec{B}) \cdot \vec{\sigma}/i]_{B'}^{A'}. \quad (84)$$

Since $\nabla_{AA'} = \frac{1}{\sqrt{2}}\sigma_{AA'}^a\partial_a$ corresponds to $\bar{\mathbf{d}} = \frac{\partial}{\partial t} + i\nabla$, $\Phi^{B'B} = \frac{1}{\sqrt{2}}\sigma_b^{B'B}\Phi^b$ corresponds to $\mathbf{A} = V + i\vec{A}$, φ_A^B corresponds to $\bar{\mathbf{F}}$, and $J^{AA'} = \frac{1}{\sqrt{2}}\sigma_a^{AA'}J^a$ corresponds to $\mathbf{J} = \rho + i\vec{J}$, Equations (81) and (82) are exactly corresponding to quaternion relations in Equation (13) as follows:

$$\nabla_{AA'}\Phi^{A'B} = \varphi_A^B \leftrightarrow \bar{\mathbf{d}}\bar{\mathbf{A}} = \bar{\mathbf{F}}, \quad (85)$$

$$\nabla^{A'B}\varphi_B^A = 2\pi J^{AA'} \leftrightarrow \bar{\mathbf{d}}\bar{\mathbf{F}} = \bar{\mathbf{J}}. \quad (86)$$

Our proof starts from manipulating $F_{AA'BB'}$ as

$$F_{AA'BB'} = \frac{1}{2}F_{\mu\nu}\sigma_{AA'}^\mu\sigma_{BB'}^\nu = \frac{1}{2}F_{\mu\nu}\sigma_{AA'}^\mu\bar{\sigma}^\nu{}^{C'C}\varepsilon_{C'B'}\varepsilon_{CB}. \quad (87)$$

Then,

$$\begin{aligned}\varphi_{AB} &= \frac{1}{2}F_{AA'B}{}^{A'} = \frac{1}{2}F_{AA'BB'}\varepsilon^{A'B'} \\ &= \frac{1}{4}F_{\mu\nu}\sigma_{AA'}^\mu\bar{\sigma}^{\nu C'C}\varepsilon_{C'B'}\varepsilon_{CB}\varepsilon^{A'B'} = \frac{1}{4}F_{\mu\nu}\sigma_{AA'}^\mu\bar{\sigma}^{\nu A'C}\varepsilon_{CB},\end{aligned}\quad (88)$$

$$\begin{aligned}\bar{\varphi}_{A'B'} &= \frac{1}{2}F_{AA'B'}{}^A = \frac{1}{2}F_{AA'BB'}\varepsilon^{AB} \\ &= \frac{1}{4}F_{\mu\nu}\bar{\sigma}^{\mu C'C}\sigma_{BB'}^\nu\varepsilon_{A'C'}\varepsilon_{AC}\varepsilon^{AB} = \frac{1}{4}F_{\mu\nu}\varepsilon_{A'C'}\bar{\sigma}^{\mu C'B}\sigma_{BB'}^\nu.\end{aligned}\quad (89)$$

Since

$$\sigma_{AA'}^\mu\bar{\sigma}^{\nu A'C} = \begin{pmatrix} \sigma^0\sigma^0 & -\sigma^0\sigma^1 & -\sigma^0\sigma^2 & -\sigma^0\sigma^3 \\ \sigma^1\sigma^0 & -\sigma^1\sigma^1 & -\sigma^1\sigma^2 & -\sigma^1\sigma^3 \\ \sigma^2\sigma^0 & -\sigma^2\sigma^1 & -\sigma^2\sigma^2 & -\sigma^2\sigma^3 \\ \sigma^3\sigma^0 & -\sigma^3\sigma^1 & -\sigma^3\sigma^2 & -\sigma^3\sigma^3 \end{pmatrix}_{A'}^C = \begin{pmatrix} \sigma^0 & -\sigma^1 & -\sigma^2 & -\sigma^3 \\ \sigma^1 & -\sigma^0 & -i\sigma^3 & i\sigma^2 \\ \sigma^2 & i\sigma^3 & -\sigma^0 & -i\sigma^1 \\ \sigma^3 & -i\sigma^2 & i\sigma^1 & -\sigma^0 \end{pmatrix}_{A'}^C, \quad (90)$$

$\varphi_A{}^D = \varepsilon^{DB}\varphi_{AB}$ becomes

$$\begin{aligned}\varphi_A{}^D &= \frac{1}{4}F_{\mu\nu}\sigma_{AA'}^\mu\bar{\sigma}^{\nu A'D} \\ &= \frac{1}{4}\text{Tr} \left[\begin{pmatrix} 0 & -F_{10} & -F_{20} & -F_{30} \\ F_{10} & 0 & F_{12} & F_{13} \\ F_{20} & -F_{12} & 0 & F_{23} \\ F_{30} & -F_{13} & -F_{23} & 0 \end{pmatrix} \begin{pmatrix} \sigma^0 & -\sigma^1 & -\sigma^2 & -\sigma^3 \\ \sigma^1 & -\sigma^0 & -i\sigma^3 & i\sigma^2 \\ \sigma^2 & i\sigma^3 & -\sigma^0 & -i\sigma^1 \\ \sigma^3 & -i\sigma^2 & i\sigma^1 & -\sigma^0 \end{pmatrix}^T \right]_A{}^D \\ &= \frac{1}{2}(F_{i0}\sigma^i - \frac{1}{2}i\epsilon^{ij}{}_k F_{ij}\sigma^k)_A{}^D,\end{aligned}\quad (91)$$

where i, j, k are the three-dimensional vector indices, which have the value 1, 2 or 3, and $\epsilon^{ij}{}_k$ is $\epsilon_{pqk}\delta_p^i\delta_q^j$ for the Levi-Civita symbol ϵ_{ijk} . Einstein summation convention is understood for three-dimensional vector indices i, j and k . Similar to Equations (90) and (92),

$$\bar{\sigma}^{\mu C'B}\sigma_{BB'}^\nu = \begin{pmatrix} \sigma^0\sigma^0 & \sigma^0\sigma^1 & \sigma^0\sigma^2 & \sigma^0\sigma^3 \\ -\sigma^1\sigma^0 & -\sigma^1\sigma^1 & -\sigma^1\sigma^2 & -\sigma^1\sigma^3 \\ -\sigma^2\sigma^0 & -\sigma^2\sigma^1 & -\sigma^2\sigma^2 & -\sigma^2\sigma^3 \\ -\sigma^3\sigma^0 & -\sigma^3\sigma^1 & -\sigma^3\sigma^2 & -\sigma^3\sigma^3 \end{pmatrix}_{B'}^{C'} = \begin{pmatrix} \sigma^0 & \sigma^1 & \sigma^2 & \sigma^3 \\ -\sigma^1 & -\sigma^0 & -i\sigma^3 & i\sigma^2 \\ -\sigma^2 & i\sigma^3 & -\sigma^0 & -i\sigma^1 \\ -\sigma^3 & -i\sigma^2 & i\sigma^1 & -\sigma^0 \end{pmatrix}_{B'}^{C'}, \quad (92)$$

$$\bar{\varphi}^{D'}_{B'} = \varepsilon^{D'A'}\bar{\varphi}_{A'B'} = -\frac{1}{4}F_{\mu\nu}\bar{\sigma}^{\mu D'B}\sigma_{BB'}^\nu = \frac{1}{2}(F_{i0}\sigma^i + \frac{1}{2}i\epsilon^{ij}{}_k F_{ij}\sigma^k)^{D'}_{B'}.\quad (93)$$

Finally, for an electromagnetic tensor $F_{AA'BB'}$, Equations (83) and (84) hold. Equations (92) and (93) also show the link between Equation (64) and the spinor form of the electromagnetic energy–stress tensor $T_{ab} = \frac{1}{2}\varphi_{AB}\bar{\varphi}_{A'B'}$.

7.2. General Relations of Quaternion and Two-Spinor Formalism and the Equivalence between Quaternion Basis and Minkowski Tetrads

Generally speaking, all spinors with spinor indices in two-spinor formalism are directly linked to quaternion. Since $\sigma^{\tilde{a}} = (\sigma^0, i\sigma^1, i\sigma^2, i\sigma^3)$ is isomorphic to quaternion basis $(1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$, $g_{AA'}^{\tilde{a}} = \frac{1}{\sqrt{2}}\sigma_{AA'}^{\tilde{a}}$ are also isomorphic to $\frac{1}{\sqrt{2}}(1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$. For any spinors with two spinor indices in the form $X_{AA'}$, it can be rewritten as $X_{AA'} = X_a g_{AA'}^a = X_{\tilde{a}} g_{AA'}^{\tilde{a}}$. It means that we can think of all spinors of the form $X_{AA'}$ to be obtained by multiplying the four-vector with $g_{AA'}^{\tilde{a}}$.

Any spinor ψ^A can be represented with spin basis o^A, ι^A such as

$$\psi^A = a o^A + b \iota^A \quad (94)$$

where o^A, ι^A is normalized so that $o_A \iota^A = 1$. It is well known that Minkowski tetrads (t^a, x^a, y^a, z^a) , which is a basis of four-vectors, can be constructed from spin basis $o^A, o^{A'}, \iota^A, \iota^{A'}$ [21],

$$g_0^a \equiv t^a = \frac{1}{\sqrt{2}}(o^A o^{A'} + \iota^A \iota^{A'}), \quad (95)$$

$$g_1^a \equiv x^a = \frac{1}{\sqrt{2}}(o^A \iota^{A'} + \iota^A o^{A'}), \quad (96)$$

$$g_2^a \equiv y^a = -\frac{i}{\sqrt{2}}(o^A \iota^{A'} - \iota^A o^{A'}), \quad (97)$$

$$g_3^a \equiv z^a = \frac{1}{\sqrt{2}}(o^A o^{A'} - \iota^A \iota^{A'}). \quad (98)$$

Therefore,

$$K^a = K^{\mathbf{a}} g_{\mathbf{a}}^a = K^0 t^a + K^1 x^a + K^2 y^a + K^3 z^a, \quad (99)$$

where a bold index, which represents a “component”, is distinguished from a normal index. Any spacetime tensor can be divided into components and basis such as $V^a = V^{\mathbf{a}} \delta_{\mathbf{a}}^a$. The component matrix of Minkowski tetrads with respect to the spin basis is

$$g_{\mathbf{A}\mathbf{A}'}^a = (t^a, x^a, y^a, z^a) = \frac{1}{\sqrt{2}}(\sigma^0, \sigma^1, \sigma^2, \sigma^3) = \frac{1}{\sqrt{2}}\sigma_{\mathbf{A}\mathbf{A}'}^a. \quad (100)$$

We can replace g^a by the tilde-tetrads $g^{\tilde{a}}$. The component matrix of tilde-tetrads $g^{\tilde{a}}$ with respect to the spin basis is

$$g_{\mathbf{A}\mathbf{A}'}^{\tilde{a}} = (t^a, ix^a, iy^a, iz^a) = \frac{1}{\sqrt{2}}(\sigma^0, i\sigma^1, i\sigma^2, i\sigma^3), \quad (101)$$

which is isomorphic to $\frac{1}{\sqrt{2}}(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$. From this isomorphism, we can set $g_{\mathbf{A}\mathbf{A}'}^{\tilde{a}} (= g_{\mathbf{a}}^{\tilde{a}}) = \frac{1}{\sqrt{2}}(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$, which is equivalent to

$$\hat{1} \equiv g_0^{\tilde{a}} = t^a, \quad \mathbf{i} \equiv -g_1^{\tilde{a}} = -ix^a, \quad \mathbf{j} \equiv -g_2^{\tilde{a}} = -iy^a, \quad \mathbf{k} \equiv -g_3^{\tilde{a}} = -iz^a. \quad (102)$$

Then, any four-vector with tilde-spacetime index can be written as

$$K_a = K_{AA'} = K_{\tilde{a}} g_{AA'}^{\tilde{a}} = K_0 \hat{1} + (iK_1)\mathbf{i} + (iK_2)\mathbf{j} + (iK_3)\mathbf{k}, \quad (103)$$

$$K^{\tilde{a}} = K^{\mathbf{a}} g_{\mathbf{a}}^{\tilde{a}} = K_0 \hat{1} - K_1 \mathbf{i} - K_2 \mathbf{j} - K_3 \mathbf{k}, \quad (104)$$

where $K^0 = K_0$, $K^1 = -K_1$, $K^2 = -K_2$, $K^3 = -K_3$.

8. Discussion on Meaning of the Quaternion and The Extended Algebra

8.1. The Role of Sigma Matrices and Quaternion Basis as Operators

Let us multiply one of sigma matrices with a tilde-spacetime index by $g^{\tilde{a}}$ as an operator: Multiplying $(\sigma^{\tilde{1}})^{A'}_{B'} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ by $g_{AA'}^{\tilde{a}}$, we then get

$$g_{AA'}^{\tilde{a}}(\sigma^{\tilde{1}})^{A'}_{B'} = \frac{1}{\sqrt{2}}\sigma_{AA'}^{\tilde{a}}(\sigma^{\tilde{1}})^{A'}_{B'} = \frac{1}{\sqrt{2}}(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})(-\mathbf{i}) = \frac{1}{\sqrt{2}}(-\mathbf{i}, -1, -\mathbf{k}, \mathbf{j}). \quad (105)$$

This is the operation of changing the spin basis as

$$o^{A'} \rightarrow i\iota^{A'}, \quad \iota^{A'} \rightarrow io^{A'}. \quad (106)$$

Since $\sigma^{\tilde{\mu}} = (\sigma^0, \sigma^1 i, \sigma^2 i, \sigma^3 i)$ is isomorphic to $(1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$, $g_{AA'}^{\tilde{a}}(\sigma^{\tilde{1}})^{A'}_{B'}$ can be written as $-g^{\tilde{a}}\mathbf{i}$. Multiplying $(\sigma^{\tilde{3}})^{A'}_{B'} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ by $g_{AA'}^{\tilde{a}}$, we can see

$$g_{AA'}^{\tilde{a}}(\sigma^{\tilde{3}})^{A'}_{B'} = \frac{1}{\sqrt{2}}\sigma_{AA'}^{\tilde{a}}(\sigma^{\tilde{3}})^{A'}_{B'} = \frac{1}{\sqrt{2}}(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})(-\mathbf{k}) = \frac{1}{\sqrt{2}}(-\mathbf{k}, -\mathbf{j}, \mathbf{i}, -\hat{1}). \quad (107)$$

This corresponds to changing the spin basis as

$$o^{A'} \rightarrow io^{A'}, \quad \iota^{A'} \rightarrow -i\iota^{A'}, \quad (108)$$

and can be written as $-g^{\tilde{a}}\mathbf{k}$. Multiplying $(\sigma^{\tilde{2}})^{A'}_{B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by $g_{AA'}^{\tilde{a}}$ gives

$$g_{AA'}^{\tilde{a}}(\sigma^{\tilde{2}})^{A'}_{B'} = \frac{1}{\sqrt{2}}\sigma_{AA'}^{\tilde{a}}(\sigma^{\tilde{2}})^{A'}_{B'} = \frac{1}{\sqrt{2}}(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})(-\mathbf{j}) = \frac{1}{\sqrt{2}}(-\mathbf{j}, \mathbf{k}, -\hat{1}, -\mathbf{i}). \quad (109)$$

This corresponds to changing the spin basis as

$$o^{A'} \rightarrow \iota^{A'}, \quad \iota^{A'} \rightarrow -o^{A'} \quad (110)$$

and can be written as $-g^{\tilde{a}}\mathbf{j}$.

Since the component of $(\sigma^{\tilde{2}})^{A'}_{B'}$ is equal to $\varepsilon^{A'B'}$ and

$$\sigma^{\tilde{\mu}}_{A B'} = \sigma^{\tilde{\mu}}_{AA'}\varepsilon^{A'B'} = (\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})\varepsilon = (-\mathbf{j}, \mathbf{k}, -\hat{1}, -\mathbf{i}), \quad (111)$$

$$\sigma^{\tilde{\mu}B}_{A'} = \varepsilon^{BA}\sigma^{\tilde{\mu}}_{AA'} = \varepsilon(\hat{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}) = (-\mathbf{j}, -\mathbf{k}, -\hat{1}, \mathbf{i}), \quad (112)$$

we can interpret that raising or lowering indices means changing spacetime basis.

In summary, the quaternion basis roles as a basis of spacetime itself as well as works as an operator of changing spacetime and spin bases. Similar to the fact that quantities in classical physics act as operators in quantum mechanics, they allow us to think that spacetime might be formed from fundamental operators. The operation on each element of quaternion basis is graphically shown in Figure 1. In the figure, the three types of arrows indicate the operations of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. As shown in the lower right box, if the arrow corresponds to the operation $\mathfrak{D}$ from a to b , then $a\mathfrak{D} = b$ and $b\mathfrak{D} = -a$. As an example, the solid line indicates the operation of $\mathbf{j}$, then $\mathbf{i}(\mathbf{j}) = \mathbf{k}$ and $\mathbf{k}(\mathbf{j}) = -\mathbf{i}$.

Figure 1. Two figures showing the result of using the quaternion basis as an operator set: only the array of elements is different.

8.2. General Discussion on Extended Complex Algebra, and Appropriate Meaning

Quaternion algebra $\mathbb{H}$ is isomorphic to $\mathbb{C} \times \mathbb{C}$ with non-commutative multiplication rule, and the elements of $\mathbb{H}$ can be represented with the secondary complex number j [22]. The set of elements of the form $q = a + bi + (c + di)j = z_1 + z_2j$ where $i^2 = j^2 = -1, ij = -ji$ is isomorphic to the set of quaternions $q = a + bi + cj + dk$. In a similar way, we can construct a larger algebraic system of quaternions, which is called “Octonion” $\mathbb{O}$ by introducing tertiary complex number l , such as $o = q_1 + q_2l$. “Sedenion” $\mathbb{S}$, which is an even larger algebraic system than octonion, can also be derived by performing analogous procedure. This procedure is called Cayley–Dickson construction.

It is still questionable how octonions and sedenions can be used in physics. Since octonions have the similar structure of complex quaternions, they can be used to describe electromagnetism. Furthermore, it is known that a specific octonion is useful to describe SU(3) group, which is the symmetry group of strong interaction [23]. Sedenion is an algebra which have 16 basis elements. We suggest that its basis can be written in the form $q_\mu \otimes q_\nu$, where q_μ is a quaternion basis $(1, i, j, k)$. We also speculate that this may be related to SU(4) group, which has 15 generators, or even to the theory of gravity. Since electromagnetic strength field tensor $F_{ab} = F_{AA'BB'} = \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\bar{\varphi}_{A'B'}$ can be expressed in quaternion representation, Weyl tensor $C_{abcd} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}$ may be expressed by using sedenion. The representation of the basis and possible uses of each algebraic system are listed in Table 1.

Table 1. The representation of the basis and possible uses of each algebraic system. $s_{\mu\nu AB}, s_{\mu\mu' \nu\nu'}$ are sign operators, which are +1 or −1: whether $\mu, \nu..$ is 0 or not and whether A and B is 0 or 1 determines the sign.

Algebraic System	Basis	Products of Basis	Used
$\mathbb{C}$	$1, i$	$i^2 = -1$	Phase rotation
$\mathbb{H} = (\mathbb{C} \times \mathbb{C}, *)$	$1, i, j, ij (= k)$	$i^2 = j^2 = -1, ij = -ji$ ($i \equiv i, j \equiv j, k \equiv ij$)	Vector rotation and Lorentz boost, electromagnetic laws
$\mathbb{O} = (\mathbb{C} \times \mathbb{C} \times \mathbb{C}, *)$	$1, i, j, k$ $1l, il, jl, kl$	$l^2 = -1,$ $e_i = q^\mu l^A = q_A^\mu$ ($l^0 = 1, l^A = l$) ($q^\mu \equiv (1, i, j, k), A = 0, 1, \mu = 0, 1, 2, 3$) $e_i * e_j (= q_A^\mu * q_B^\nu) = s_{\mu\nu AB} q^\mu q^\nu l^{A+B}$	Rotation of gluon, color charges (SU(3)), electromagnetic laws with magnetic monopole
$\mathbb{S} = (\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}, *)$	$1, i, j, k$ $1i', ii', ji', ki'$ $1j', ij', jj', kj'$ $1k', ik', jk', kk'$	$e_i = q^\mu \otimes q^{\mu'} = q^{\mu\mu'}$ ($q^\mu \equiv (1, i, j, k), q^{\mu'} \equiv (1, i', j', k')$) $e_i * e_j (= q^{\mu\mu'} * q^{\nu\nu'}) = s_{\mu\mu' \nu\nu'} q^{\mu\mu'} q^{\nu\nu'}$	Gravity?, SU(4)?

We can think of physical meaning of the algebras made through Cayley–Dickson construction. Multiplying complex numbers by a field implies a change in scale and phase of the field. In this point of view, the spatial rotation can be interpreted as a kind of two-fold rotation because quaternions can describe three-dimensional spatial rotation and they consist of two independent imaginary units i and j . Moreover, it might be that the space itself is constructed from a kind of two-fold rotation. Similarly, since the basis of octonion can be represented with three complex numbers (one quaternion and one complex number), the rotation between gluon color charges can be considered as a three-fold rotation. Likewise, if sedenion has useful relation with the gravity, the metric of spacetime can be deemed as a four-fold rotation.

9. Conclusions

We have seen that quaternions can describe electromagnetism very concisely and beautifully. They can also represent Lorentz boost and spatial rotation in a simpler way. The complex conjugation of complex quaternion corresponds to parity inversion of the physical quantities belonging to the quaternion. We can also take a hint from the 4×4 matrix representation of quaternion and apply it to define the complex tensor, which in turn provides a new representation of electromagnetism. We have verified that the quaternion representation is directly linked to spinor representation in two-spinor formalism, and then investigated meaning of quaternions; not only as a basis but also as an operator.

The use of quaternion could be extended not only for actual calculations, but also to obtain deep insights and new interpretations of physics. Any null-like vectors can be described by two-spinors, and, furthermore, Minkowski tetrads can also be constructed within the two-spinor formalism. This formalism has the implication that the spacetime may come from two-spinor fields. The beautiful conciseness of quaternion representation of electromagnetism and the link between quaternion and two-spinor formalism may imply that spinors are the fundamental ingredients of all fields and the spacetime also consists of two-spinor fields. Conversely, if those conjectures are true, then it is natural to explain why the algebras formed by the Cayley–Dickson procedure, such as quaternion, is useful in the description of nature.

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Appendix A. Expansions of a few Quaternion Products in Equation (13)

Appendix A.1. Expansions of Products in (3), (4) and (5) of Equation (13)

Using the multiplication expression shown in Equation (4), the left sides of the relations (3), (4) and (5) in Equation (13) are expanded as follows.

$$d\vec{A} = \left(\frac{\partial}{\partial t} - i\nabla\right)(V - i\vec{A}) = \left(\frac{\partial V}{\partial t} + \nabla \cdot \vec{A}\right) + i(-\nabla V - \frac{\partial \vec{A}}{\partial t}) - (\nabla \times \vec{A}) \quad (\text{A1})$$

$$\begin{aligned} d\vec{E} &= \left(\frac{\partial}{\partial t} - i\nabla\right)(-i\vec{E} - \vec{B}) \\ &= \nabla \cdot \vec{E} - i\nabla \cdot \vec{B} - (\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}) + i\left(-\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B}\right) \end{aligned} \quad (\text{A2})$$

$$d\vec{J} = \left(\frac{\partial}{\partial t} - i\nabla\right)(\rho - i\vec{J}) = \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J}\right) + i(-\nabla \rho + \frac{\partial \vec{J}}{\partial t}) - (\nabla \times \vec{J}) \quad (\text{A3})$$

Appendix A.2. The proof of (7) in Equation (13)

Here we show that $(\mathbb{F}\mathbb{d})\vec{\mathbb{F}}$ is equal to $\mathbb{F}(\mathbb{d}\vec{\mathbb{F}})$ where $(\mathbb{F}\mathbb{d})$ is the quaternion differential operator.

$$\begin{aligned}
 (\mathbb{F}\mathbb{d})\vec{\mathbb{F}} &= \left[(i\vec{E} - \vec{B})(\partial_t - i\nabla) \right] (-\vec{E}i - \vec{B}) \\
 &= \left[-\vec{E}\partial_t \cdot \vec{E} - (\vec{B} \times \nabla) \cdot \vec{E} - \vec{B}\partial_t \cdot \vec{B} + (\vec{E} \times \nabla) \cdot \vec{B} \right] \\
 &\quad + \left[-i\vec{B}\partial_t \cdot \vec{E} + i(\vec{E} \times \nabla) \cdot \vec{E} + i\vec{E}\partial_t \cdot \vec{B} + i(\vec{B} \times \nabla) \cdot \vec{B} \right] \\
 &\quad + \left[\vec{E}\partial_t \times \vec{E} - (\vec{B} \cdot \nabla)\vec{E} + (\vec{B} \times \nabla) \times \vec{E} + \vec{B}\partial_t \times \vec{B} + (\vec{E} \cdot \nabla)\vec{B} - (\vec{E} \times \nabla) \times \vec{B} \right] \\
 &\quad + \left[i\vec{B}\partial_t \times \vec{E} + i(\vec{E} \cdot \nabla)\vec{E} - i(\vec{E} \times \nabla) \times \vec{E} - i\vec{E}\partial_t \times \vec{B} + i(\vec{B} \cdot \nabla)\vec{B} - i(\vec{B} \times \nabla) \times \vec{B} \right] \\
 &= \vec{J} \cdot \vec{E} + i\vec{B} \cdot \vec{J} + \rho\vec{B} + \vec{J} \times \vec{E} + i(\rho\vec{E} + \vec{J} \times \vec{B}) \\
 &= \mathbb{F}\vec{J},
 \end{aligned} \tag{A4}$$

where we have used the following relations:

$$(\vec{B} \times \nabla) \cdot \vec{E} = \epsilon_{ijk} B_i \nabla_j \vec{E}_k = \vec{B} \cdot (\nabla \times \vec{E}), \tag{A5}$$

$$\begin{aligned}
 [(\vec{B} \times \nabla) \times \vec{E}]_i &= \epsilon_{ipq} (\epsilon_{pjk} B_j \nabla_k) E_q = (\delta_{qj} \delta_{ik} - \delta_{qk} \delta_{ij}) (B_j \nabla_k \vec{E}_q) \\
 &= B_j \nabla_i E_j - B_i (\nabla \cdot \vec{E}),
 \end{aligned} \tag{A6}$$

$$\begin{aligned}
 [\vec{B} \times (\nabla \times \vec{E})]_i &= \epsilon_{iqp} B_q (\epsilon_{pjk} \nabla_j) E_k = (\delta_{ij} \delta_{qk} - \delta_{ik} \delta_{qj}) B_q \nabla_j E_k \\
 &= B_j \nabla_i E_j - (\vec{B} \cdot \nabla) E_i,
 \end{aligned} \tag{A7}$$

$$(\vec{B} \times \nabla) \times \vec{E} = \vec{B} \times (\nabla \times \vec{E}) + (\vec{B} \cdot \nabla) \vec{E} - \vec{B} (\nabla \cdot \vec{E}). \tag{A8}$$

Appendix B. The Proof of Equation (26)

$$\nabla \times \mathbf{p} = \nabla \times (\vec{E} \times \vec{B}) = (\vec{E} \cdot \nabla) \vec{B} - (\vec{B} \cdot \nabla) \vec{E} + \vec{E} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{E}). \tag{A9}$$

$$\begin{aligned}
 \nabla(\vec{E} \cdot \vec{B}) &= \vec{E} \nabla \vec{B} + \vec{B} \nabla \vec{E} \\
 &= \vec{E} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{E}) + (\vec{E} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{E} \quad (\text{from (A7)}) \\
 &= \vec{E} \times (\partial_t \vec{E}) + \vec{E} \times \vec{J} + \vec{B} \nabla \vec{E} + (\vec{E} \cdot \nabla) \vec{B} \\
 &= -\vec{B} \times (\partial_t \vec{B}) + \vec{E} \nabla \vec{B} + (\vec{B} \cdot \nabla) \vec{E}
 \end{aligned} \tag{A10}$$

where $(\vec{A} \nabla \vec{B})_i \equiv A_j (\nabla_i B_j)$ for vector fields $\vec{A}$ and $\vec{B}$. Substituting this into Equation (A9), we get Equation (26).

Appendix C. The Proof of Equation (30)

As we mentioned on Equation (4), $\mathbb{q} = q_0 + i\vec{q}$ is isomorphic to $q_\mu \sigma^\mu$ where $q_\mu = (q_0, \vec{q}) = (q_0, q_1, q_2, q_3)$, $\sigma^\mu = \{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$ and $g_{\mu\nu} = (1, -1, -1, -1)$.

Let us introduce some quantities that are isomorphic to some quaternions

$$\partial_\mu \vec{\sigma}^\mu \sim \mathbb{d} = \partial_t - i\nabla \tag{A11}$$

$$M_\mu \vec{\sigma}^\mu \sim \vec{\mathbb{F}} = i\vec{E} - \vec{B} \tag{A12}$$

$$N_\mu \sigma^\mu \sim \vec{\mathbb{F}} = -i\vec{E} - \vec{B}, \tag{A13}$$

where $\partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$, $M_\mu = (0, -\vec{E} - i\vec{B})$, $N_\mu = (0, -\vec{E} + i\vec{B})$.

Then,

$$\begin{aligned}
 \mathfrak{d}(\bar{\mathbb{F}}\mathbb{F}) &\sim \partial_\rho \bar{\sigma}^\rho (N_\mu \sigma^\mu M_\nu \bar{\sigma}^\nu) = (\partial_\rho N_\mu M_\nu) \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\nu \\
 &= (\partial_\rho N_\mu) M_\nu \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\nu + N_\mu (\partial_\rho M_\nu) \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\nu \\
 &= (\partial_\rho N_\mu) M_\nu \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\nu + N_\mu (\partial_\rho M_\nu) (2g^{\mu\rho} - \bar{\sigma}^\mu \sigma^\rho) \bar{\sigma}^\nu \\
 &= (\partial_\rho N_\mu) M_\nu \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\nu + 2(N^\mu \partial_\mu) M_\nu \bar{\sigma}^\nu - N_\mu (\partial_\rho M_\nu) \bar{\sigma}^\mu \sigma^\rho \bar{\sigma}^\nu \\
 &= (\partial_\rho \bar{\sigma}^\rho N_\mu \sigma^\mu) M_\nu \bar{\sigma}^\nu - N_\mu \bar{\sigma}^\mu (\partial_\rho \sigma^\rho M_\nu \bar{\sigma}^\nu) + 2(N^\mu \partial_\mu) M_\nu \bar{\sigma}^\nu \\
 &\sim (\mathfrak{d}\bar{\mathbb{F}})\mathbb{F} + \bar{\mathbb{F}}(\mathfrak{d}\mathbb{F}) + 2i(\bar{\mathbb{F}} \cdot \nabla)\mathbb{F}.
 \end{aligned} \tag{A14}$$

We have used the relations $(\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu)^{A'}_{B'} = 2g^{\mu\nu} \delta^{A'}_{B'}$ [24] for spinor indices A, B, A' and B' .

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