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Exact Solutions in Poincaré Gauge Gravity Theory

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Abstract: In the framework of the gauge theory based on the Poincaré symmetry group, the gravitational field is described in terms of the coframe and the local Lorentz connection. Considered as gauge field potentials, they give rise to the corresponding field strength which are naturally identified with the torsion and the curvature on the Riemann–Cartan spacetime. We study the class of quadratic Poincaré gauge gravity models with the most general Yang–Mills type Lagrangian which contains all possible parity-even and parity-odd invariants built from the torsion and the curvature. Exact vacuum solutions of the gravitational field equations are constructed as a certain deformation of de Sitter geometry. They are black holes with nontrivial torsion.

Keywords: gauge gravity theory; Poincaré group; coframe; Lorentz connection; odd parity

1. Introduction

The gauge-theoretic understanding of the fundamental physical interactions is one of the solid cornerstones of the modern science. In simple terms, the gauge principle relates physical forces to the underlying symmetry groups. The corresponding Yang–Mills type formalism was developed for the *internal symmetries*, which form the foundation for the theories of electromagnetic, weak and strong interactions, and consistently generalized to the *spacetime symmetries* [1,2] which give rise to the gravitational interaction. In the current research, much attention is paid to the gauge-theoretic models based on the Poincaré group, see [3,4] for an introduction. The monograph [2] provides an extensive list of references.

It is now well established that Einstein's general relativity (GR) theory provides a valid description of the gravitational phenomena on macroscopic scales. Compared to GR, the gauge gravity theory is expected to improve our understanding of the gravitational physics at microscopic scales (and, likewise, at an early stage of the cosmological evolution of the universe), giving rise to GR in a certain macroscopic limit. The simplest version of the Poincaré gauge gravity, known as the Einstein–Cartan theory, is a viable gravitational model which is consistent with all experimental tests, and it only deviates from GR at extremely high matter densities $\frac{2m^2c^4}{\pi G\hbar^2}$ (with m mass of a fermion), where it predicts an avoidance of the cosmological singularity [4]. As one knows, the quantized GR is non-renormalizable, and taking into account the success of the Yang–Mills gauge approach for the strong and electro-weak interactions, one hopes that a development of the gauge-theoretic framework for gravity may help in constructing a consistent quantum theory of the gravitational field.

We focus here on the class of Poincaré gauge gravity models based on the general quadratic Lagrangians of the Yang–Mills type. Both the parity-even and parity-odd terms are included, extending the previous studies which were mainly confined to the parity symmetric theories. Construction of exact solutions of the field equations is important for checking the validity of the theory, and its consistency with GR and experiments. Here, we report on the black hole solutions with dynamical torsion.

Our basic notation and conventions are consistent with [5,6]. In particular, Greek indices $\alpha, \beta, \cdots = 0, \ldots, 3$, denote the anholonomic components (for example, of a coframe ϑ^α), while the Latin indices

$i, j, \cdots = 0, \ldots, 3$, label the holonomic components (e.g., dx^i). The anholonomic vector frame basis e_α is dual to the coframe basis in the sense that $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$, where $\rfloor$ denotes the interior product. The volume 4-form is denoted by η, and the η-basis in the space of exterior forms is constructed with the help of the interior products as $\eta_{\alpha_1 \ldots \alpha_p} := e_{\alpha_p} \rfloor \ldots e_{\alpha_1} \rfloor \eta$, $p = 1, \ldots, 4$. They are related to the ϑ-basis via the Hodge dual operator *, for example, $\eta_{\alpha\beta} = {}^* (\vartheta_\alpha \wedge \vartheta_\beta)$. The Minkowski metric is $g_{\alpha\beta} = \operatorname{diag}(c^2, -1, -1, -1)$. We do not use special unit systems and, accordingly, do not put the fundamental physical constants (such as the light velocity c, Planck's constant $\hbar$, and Newton's gravitational constant G) equal to one, thereby keeping for all objects their natural dimensions. All the objects related to the parity-odd sector (coupling constants, irreducible pieces of the curvature, etc.) are marked by an overline, to distinguish them from the corresponding parity-even objects.

2. Formal Structure of Poincaré Gauge Gravity

The 10-parameter Poincaré symmetry group $G = T_4 \rtimes SO(1,3)$ is a semidirect product of the 4-parameter group of translations and the 6-parameter local Lorentz group, and the Yang–Mills–Utiyama–Kibble formalism can be consistently developed on a spacetime manifold [1,2]. The corresponding gravitational field potentials ("translational" and "rotational", respectively) are then naturally identified with the 1-forms of the coframe and the local Lorentz connection:

$$\vartheta^\alpha = e_i^\alpha dx^i, \tag{1}$$

$$\Gamma^{\alpha\beta} = -\Gamma^{\beta\gamma} = \Gamma_i{}^{\alpha\beta} dx^i. \tag{2}$$

The "translational" and "rotational" field strength 2-forms

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \tag{3}$$

$$R^{\alpha\beta} = d\Gamma^{\alpha\beta} + \Gamma_\gamma{}^\beta \wedge \Gamma^{\alpha\gamma}, \tag{4}$$

have standard geometrical interpretation as the torsion and the curvature of the Riemann–Cartan spacetime. As usual, the covariant differential is denoted D.

2.1. Gravitational Field Equations

The gravitational Lagrangian 4-form is quite generally an arbitrary invariant function of the geometrical variables:

$$V = V(\vartheta^\alpha, T^\alpha, R^{\alpha\beta}). \tag{5}$$

Its variation, with respect to the gravitational (translational and Lorentz) potentials, yields the field equations

$$\frac{\delta V}{\delta \vartheta^\alpha} = -DH_\alpha + E_\alpha = 0, \tag{6}$$

$$\frac{\delta V}{\delta \Gamma^{\alpha\beta}} = -DH_{\alpha\beta} + E_{\alpha\beta} = 0. \tag{7}$$

Here, the Poincaré *gauge field momenta* 2-forms are introduced by

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}}, \tag{8}$$

and the 3-forms of the *canonical* energy-momentum and spin for the gravitational gauge fields are constructed as

$$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma, \tag{9}$$

$$E_{\alpha\beta} := \frac{\partial V}{\partial \Gamma^{\alpha\beta}} = -\vartheta_{[\alpha} \wedge H_{\beta]}. \tag{10}$$

The field Equations (6) and (7) are written here for the *vacuum case*. In the presence of matter, the right-hand sides of Equations (6) and (7) contain the canonical energy-momentum and the canonical spin currents of the physical sources, respectively.

2.2. Quadratic Poincaré Gravity Models

The torsion 2-form can be decomposed into the 3 irreducible parts, whereas the curvature 2-form has 6 irreducible pieces. Their definition is presented in Appendix A.

The general quadratic model is described by the Lagrangian 4-form that contains all possible linear and quadratic invariants of the torsion and the curvature:

$$
\begin{aligned}
V &= \frac{1}{2\kappa c}\left\{\left(a_0\eta_{\alpha\beta} + \bar{a}_0\vartheta_\alpha \wedge \vartheta_\beta\right) \wedge R^{\alpha\beta} - 2\lambda_0\eta - T^\alpha \wedge \sum_{I=1}^{3}\left[a_I{}^*\left({}^{(I)}T_\alpha\right) + \bar{a}_I{}^{(I)}T_\alpha\right]\right\} \\
&\quad - \frac{1}{2\rho}R^{\alpha\beta} \wedge \sum_{I=1}^{6}\left[b_I{}^*\left({}^{(I)}R_{\alpha\beta}\right) + \bar{b}_I{}^{(I)}R_{\alpha\beta}\right].
\end{aligned}
\tag{11}
$$

This Lagrangian has a clear structure: The first line encompasses the terms *linear* in the curvature and the *torsion quadratic* terms (all of which have the same dimension of an area $[\ell^2]$), whereas the second line contains the *curvature quadratic* invariants. For completeness, the cosmological constant is included (with the dimension of an inverse area, $[\lambda_0] = [\ell^{-2}]$). Furthermore, each line is composed of parity-even pieces and parity-odd parts (with the coupling constants marked by an overline). A special case $a_0 = 0$ and $\bar{a}_0 = 0$ describes the purely quadratic model without the Hilbert–Einstein linear term in the Lagrangian.

A general Poincaré gauge model contains a set of the coupling constants which determine the structure of the quadratic part of the Lagrangian: $\rho, a_1, a_2, a_3, b_1, \cdots, b_6$ and $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}_1, \cdots, \bar{b}_6$. It is important to note that not all of the latter constants are independent—we take $\bar{a}_2 = \bar{a}_3$, $\bar{b}_2 = \bar{b}_4$ and $\bar{b}_3 = \bar{b}_6$ because some of the terms in Lagrangian (11) are the same in view of Equations (A12)–(A17). As we already mentioned, the overbar denotes the constants responsible for the parity-odd sector. In recent times, there is a growing interest to such interactions [7–13]. Quite generally, there are no compelling theoretical arguments or experimental evidence which could rule out the violation of parity in gravity, and in 1964 Leitner and Okubo [14] looked into a possibility of extending the gravitational Lagrangian by parity odd terms. Later such extensions were widely studied in the context of the classical and quantum gravity theory [15,16], in particular in Ashtekar's approach and loop quantum gravity [17,18]. Moreover, the inclusion of parity-nonconserving terms is important for the discussion of such fundamental physical issues as the baryon asymmetry of the universe, where the parity-odd terms can be induced by the quantum vacuum structure [19–21].

All coupling constants a_I, $\bar{a}_I$, b_I, and $\bar{b}_I$ are dimensionless, whereas the dimension $[\frac{1}{\rho}] = [\hbar]$. Keeping in mind the importance of the macroscopic limit to GR, we have $\kappa = 8\pi G/c^4$ as Einstein's gravitational constant. The microscopic gravitational phenomena are naturally characterized by the parameter

$$\ell_\rho^2 = \frac{\kappa c}{\rho}. \tag{12}$$

Since $[\frac{1}{\rho}] = [\hbar]$, this new coupling parameter has the dimension of an area, $[\ell_\rho^2] = [\ell^2]$. Below we will see that ℓ_ρ^2 parameter describes the contribution of the curvature square terms in the Lagrangian (11) to the gravitational field dynamics in Equations (17) and (18).

For the Lagrangian (11), from Equations (8)–(10) we derive the gauge gravitational field momenta

$$H_\alpha = \frac{1}{\kappa c}\, h_\alpha\,, \qquad H_{\alpha\beta} = -\frac{1}{2\kappa c}\left(a_0\,\eta_{\alpha\beta} + \bar{a}_0 \vartheta_\alpha \wedge \vartheta_\beta\right) + \frac{1}{\rho}\, h_{\alpha\beta}, \tag{13}$$

and the canonical energy-momentum and spin currents of the gravitational field

$$E_\alpha = \frac{1}{2\kappa c}\left(a_0\,\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + 2\bar{a}_0\, R_{\alpha\beta} \wedge \vartheta^\beta - 2\lambda_0 \eta_\alpha + q_\alpha^{(T)}\right) + \frac{1}{\rho}\, q_\alpha^{(R)}, \qquad E_{\alpha\beta} = \frac{1}{\kappa c}\, h_{[\alpha} \wedge \vartheta_{\beta]}. \tag{14}$$

For convenience, we introduced here the 2-forms that are linear functions of the torsion and the curvature, respectively, by

$$h_\alpha = \sum_{I=1}^{3}\left[a_I\,{}^*\big({}^{(I)}T_\alpha\big) + \bar{a}_I\,{}^{(I)}T_\alpha\right], \qquad h_{\alpha\beta} = \sum_{I=1}^{6}\left[b_I\,{}^*\big({}^{(I)}R_{\alpha\beta}\big) + \bar{b}_I\,{}^{(I)}R_{\alpha\beta}\right], \tag{15}$$

and the 3-forms which are quadratic in the torsion and in the curvature, respectively:

$$q_\alpha^{(T)} = \frac{1}{2}\left[(e_\alpha \rfloor T^\beta) \wedge h_\beta - T^\beta \wedge e_\alpha \rfloor h_\beta\right], \qquad q_\alpha^{(R)} = \frac{1}{2}\left[(e_\alpha \rfloor R^{\beta\gamma}) \wedge h_{\beta\gamma} - R^{\beta\gamma} \wedge e_\alpha \rfloor h_{\beta\gamma}\right]. \tag{16}$$

By construction, $[h_\alpha] = [\ell]$ has the dimension of a length, whereas the 2-form $[h_{\alpha\beta}] = 1$ is obviously dimensionless. Similarly, we find for Equations (16) the dimension of length $[q_\alpha^{(T)}] = [\ell]$, and the dimension of the inverse length, $[q_\alpha^{(R)}] = [1/\ell]$, respectively.

The resulting *vacuum* Poincaré gravity field Equations (6) and (7) then read:

$$\frac{a_0}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + \bar{a}_0 R_{\alpha\beta} \wedge \vartheta^\beta - \lambda_0 \eta_\alpha + q_\alpha^{(T)} + \ell_\rho^2\, q_\alpha^{(R)} - Dh_\alpha \;=\; 0, \tag{17}$$

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma} \wedge T^\gamma + \bar{a}_0\, T_{[\alpha} \wedge \vartheta_{\beta]} + h_{[\alpha} \wedge \vartheta_{\beta]} - \ell_\rho^2\, Dh_{\alpha\beta} \;=\; 0. \tag{18}$$

3. Prelude: De Sitter Spacetime

As a preliminary step, we discuss the de Sitter spacetime in an unusual disguise. This manifold has many faces, and here we consider one of them which is not quite well known. Using a spherical local coordinate system (t, r, θ, φ), it is given by the coframe

$$\widehat{\vartheta}^\alpha = \begin{cases} \widehat{\vartheta}^{\hat{0}} = & \frac{\Delta + mr}{\Delta}\sqrt{\frac{\Delta}{\Sigma}}\left[c dt - j_0 \sin^2\theta\, d\varphi\right] + \frac{mr}{\Delta}\sqrt{\frac{\Sigma}{\Delta}}\, dr, \\[6pt] \widehat{\vartheta}^{\hat{1}} = & \frac{mr}{\Delta}\sqrt{\frac{\Delta}{\Sigma}}\left[c dt - j_0 \sin^2\theta\, d\varphi\right] + \frac{\Delta - mr}{\Delta}\sqrt{\frac{\Sigma}{\Delta}}\, dr, \\[6pt] \widehat{\vartheta}^{\hat{2}} = & \sqrt{\frac{\Sigma}{f}}\, d\theta, \\[6pt] \widehat{\vartheta}^{\hat{3}} = & \sqrt{\frac{f}{\Sigma}}\sin\theta\left[-j_0\, c dt + \left(r^2 + j_0^2\right) d\varphi\right]. \end{cases} \tag{19}$$

Here, the rotation parameter is denoted by j_0 (in order to distinguish it from the torsion coupling constants we avoid a more common notation a), and the functions and constants are defined by

$$\Delta \;:=\; (r^2 + j_0^2)(1 - \lambda\, r^2) - 2mr, \tag{20}$$

$$\Sigma \;:=\; r^2 + j_0^2 \cos^2\theta, \tag{21}$$

$$f \;:=\; 1 + \lambda\, j_0^2 \cos^2\theta, \tag{22}$$

$$m \;:=\; \frac{GM}{c^2}, \tag{23}$$

and $0 < t < \infty, 0 < r < \infty, 0 < \theta < \pi$ and $0 < \varphi < 2\pi$.

The corresponding line element $ds^2 = g_{\alpha\beta}\widehat{\vartheta}^\alpha \otimes \widehat{\vartheta}^\beta = \widehat{g}_{ij}dx^i dx^j$, with the spacetime metric $\widehat{g}_{ij} = \widehat{e}_i^\alpha \widehat{e}_j^\beta g_{\alpha\beta}$, reads:

$$
\begin{aligned}
ds^2 &= \left[f - \lambda(r^2 + j_0^2)\right] c^2 dt^2 - \frac{4mr}{\Delta} c\,dt\,dr + 2\lambda\,(r^2 + j_0^2)\,j_0 \sin^2\theta\,c\,dt\,d\varphi \\
&\quad + \frac{4mr}{\Delta} j_0 \sin^2\theta\,dr\,d\varphi - \frac{\Delta - 2mr}{\Delta^2}\Sigma\,dr^2 - \frac{\Sigma}{f}d\theta^2 - (1 + \lambda j_0^2)(r^2 + j_0^2)\sin^2\theta\,d\varphi^2.
\end{aligned}
\tag{24}
$$

When $m = 0$ and $j_0 = 0$, this line element reduces to

$$
ds^2 = (1 - \lambda r^2)\,c^2 dt^2 - \frac{dr^2}{1 - \lambda r^2} - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2),
\tag{25}
$$

which represents the static spherically symmetric form of the de Sitter spacetime. Quite remarkably, however, also in the general case with $m \neq 0$ and $j_0 \neq 0$, despite a complicated form of the coframe (19) and the metric (24), the components of which appear to be highly nontrivial functions of the spacetime coordinates and parameters m, j_0, λ, the corresponding Riemannian connection satisfies

$$
\begin{aligned}
\widehat{T}^\alpha &= d\widehat{\vartheta}^\alpha + \widehat{\Gamma}_\beta{}^\alpha \wedge \widehat{\vartheta}^\beta = 0, \tag{26}\\
\widehat{R}^{\alpha\beta} &= d\widehat{\Gamma}^{\alpha\beta} + \widehat{\Gamma}_\gamma{}^\beta \wedge \widehat{\Gamma}^{\alpha\gamma} = \lambda\,\widehat{\vartheta}^\alpha \wedge \widehat{\vartheta}^\beta. \tag{27}
\end{aligned}
$$

The components of the connection $\widehat{\Gamma}^{\alpha\beta}$ are explicitly given in Appendix B. By making use of Equations (A19)–(A21), it is straightforward (although the corresponding computation is somewhat lengthy) to verify Equation (27).

In other words, even for nonvanishing m and j_0, the coframe (19) and the metric (24) describe the torsionless de Sitter spacetime of the constant curvature λ. Note that depending on the sign of λ, one sometimes speaks of de Sitter and anti-de Sitter geometries. Here, we do not use this refined language and—irrespective of the sign of the constant curvature—call all these spaces de Sitter.

4. Interlude: From de Sitter to Kerr–de Sitter

As a next step, we introduce the 1-form

$$
k := \sqrt{\frac{\Sigma}{\Delta}}\left(\widehat{\vartheta}^{\hat{0}} - \widehat{\vartheta}^{\hat{1}}\right) = c\,dt - \frac{\Sigma}{\Delta}dr - j_0 \sin^2\theta\,d\varphi,
\tag{28}
$$

and define the covector components by $k_\alpha = \widehat{e}_\alpha \rfloor k$. One can straightforwardly check that this construction yields a null geodetic congruence:

$$
k \wedge {}^{\widehat{*}}k = 0, \qquad k \wedge {}^{\widehat{*}}\widehat{D}k_\alpha = 0.
\tag{29}
$$

Hereafter the hat marks the objects and operators in the de Sitter space ($\widehat{\vartheta}^\alpha, \widehat{\Gamma}^{\alpha\beta}$). Namely, $\widehat{D}$ is the covariant differential corresponding to the connection $\widehat{\Gamma}^{\alpha\beta}$, and ${}^{\widehat{*}}$ is the Hodge duality operator corresponding to the coframe $\widehat{\vartheta}^\alpha$.

Now we use the null 1-form (28) to define a new coframe

$$
\vartheta^\alpha = \widehat{\vartheta}^\alpha - Uk^\alpha k,
\tag{30}
$$

which can be considered as a kind of perturbation of the de Sitter coframe (19). The components of the modified coframe $\vartheta^\alpha = e_i^\alpha dx^i$ are now $e_i^\alpha = \widehat{e}_i^\alpha - Uk^\alpha k_i$, and the corresponding spacetime metric has the Kerr–Schild form

$$
g_{ij} = e_i^\alpha e_j^\beta g_{\alpha\beta} = \widehat{g}_{ij} - 2Uk_i k_j.
\tag{31}
$$

Here $U = U(r, \theta)$ is a function of the radial coordinate r and the angle θ. In order to preserve the stationarity and the axial symmetry, we assume that all the geometric variables do not depend on t and φ. If we choose

$$U = \frac{mr}{\Sigma}, \tag{32}$$

the coframe (30) reads explicitly

$$\vartheta^\alpha = \begin{cases} \vartheta^{\hat{0}} = \sqrt{\frac{\Delta}{\Sigma}} \left[cdt - j_0 \sin^2 \theta \, d\varphi \right], \\ \vartheta^{\hat{1}} = \sqrt{\frac{\Sigma}{\Delta}} \, dr, \\ \vartheta^{\hat{2}} = \sqrt{\frac{\Sigma}{f}} \, d\theta, \\ \vartheta^{\hat{3}} = \sqrt{\frac{f}{\Sigma}} \sin \theta \left[-j_0 \, cdt + (r^2 + j_0^2) \, d\varphi \right]. \end{cases} \tag{33}$$

We immediately recognize the coframe of the *Kerr–de Sitter geometry* [22].

5. Postlude: Ansatz for Poincaré Gauge Theory

In Poincaré gauge theory, the ansatz for the translational potential (coframe 1-form) should be supplemented by the ansatz for the local Lorentz connection 1-form. After all the preparations, we are now in a position to formulate the *Baekler ansatz* for the Poincaré gauge fields:

$$\vartheta^\alpha = \widehat{\vartheta}^\alpha - Uk^\alpha k, \tag{34}$$
$$\Gamma^{\alpha\beta} = \widehat{\Gamma}^{\alpha\beta}. \tag{35}$$

Earlier, a similar technique was successfully used for the construction of the exact plane wave solutions in the Poincaré gauge theory [23,24].

6. Solving Gravitational Field Equations

It is important to notice that the null 1-form (28) preserves its structure with respect to the new coframe (30):

$$k = \sqrt{\frac{\Sigma}{\Delta}} \left(\vartheta^{\hat{0}} - \vartheta^{\hat{1}} \right), \tag{36}$$

and consequently the covector components $k_\alpha = e_\alpha \rfloor k = \sqrt{\frac{\Sigma}{\Delta}}(1, -1, 0, 0)$ have the same values. Moreover, it is still a null geodetic congruence,

$$k \wedge {}^* k = 0, \qquad k \wedge {}^* Dk_\alpha = 0. \tag{37}$$

It is straightforward to derive the torsion and the Riemann–Cartan curvature for the Baekler ansatz (34) and (35). Combining Equations (27), (34), and (35) we find

$$T^\alpha = D\vartheta^\alpha = \widehat{D}\vartheta^\alpha = -\widehat{D}(Uk^\alpha k), \tag{38}$$
$$R^{\alpha\beta} = \widehat{R}^{\alpha\beta} = \lambda \, \widehat{\vartheta}^\alpha \wedge \widehat{\vartheta}^\beta = \lambda \, \vartheta^\alpha \wedge \vartheta^\beta + 2\lambda Uk \wedge k^{[\alpha} \vartheta^{\beta]}. \tag{39}$$

One can check the following properties of the Poincaré gauge field strengths:

$$k_\alpha T^\alpha = 0, \qquad k_\alpha (R^{\alpha\beta} - \lambda \vartheta^\alpha \wedge \vartheta^\beta) = 0, \tag{40}$$
$$\vartheta_\alpha \wedge T^\alpha = 0, \qquad \vartheta_\alpha \wedge R^{\alpha\beta} = 0. \tag{41}$$

Next, we need to find the irreducible parts. Using the identities (40) and (41), we can verify that most of the irreducible parts of the curvature vanish, except for the 4th and 6th:

$$^{(1)}R^{\alpha\beta} = {}^{(2)}R^{\alpha\beta} = {}^{(3)}R^{\alpha\beta} = {}^{(5)}R^{\alpha\beta} = 0, \tag{42}$$

$$^{(4)}R^{\alpha\beta} = 2\lambda U k \wedge k^{[\alpha}\vartheta^{\beta]}, \qquad {}^{(6)}R^{\alpha\beta} = \lambda\,\vartheta^\alpha \wedge \vartheta^\beta. \tag{43}$$

As compared to the curvature (39), the structure of the torsion (38) is more nontrivial. Making use of the components of the connection (Equations (A19)–(A21)) and Equation (32), one can evaluate the covariant derivative in (38) to get

$$T^\alpha = -\,v_5\,k \wedge \vartheta^\alpha + v_4{}^*(k \wedge \vartheta^\alpha) + k^\alpha w, \tag{44}$$

where we introduce the 2-form

$$w = (v_1 + v_5)\,\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} + 3v_4\,\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} - k \wedge (v_2\vartheta^{\hat{2}} + v_3\vartheta^{\hat{3}}), \tag{45}$$

and denoted the functions

$$v_1 = -\frac{m(r^2 - j_0^2\cos^2\theta)}{\Sigma^2}, \quad v_2 = \sqrt{\frac{f}{\Sigma}}\,\frac{mr j_0^2 \sin\theta\cos\theta}{\Sigma^2}, \tag{46}$$

$$v_3 = \sqrt{\frac{f}{\Sigma}}\,\frac{mr^2 j_0 \sin\theta}{\Sigma^2}, \quad v_4 = \frac{mr j_0 \cos\theta}{\Sigma^2}, \quad v_5 = -\frac{mr^2}{\Sigma^2}. \tag{47}$$

Explicitly, the components of the torsion 2-form (44) read:

$$T^\alpha = \begin{cases} T^{\hat{0}} = & k^{\hat{0}}\,[v_1\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} + 2v_4\,\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} - k \wedge (v_2\vartheta^{\hat{2}} + v_3\vartheta^{\hat{3}})], \\ T^{\hat{1}} = & k^{\hat{1}}\,[v_1\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} + 2v_4\,\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} - k \wedge (v_2\vartheta^{\hat{2}} + v_3\vartheta^{\hat{3}})], \\ T^{\hat{2}} = & -k \wedge (v_5\vartheta^{\hat{2}} + v_4\vartheta^{\hat{3}}), \\ T^{\hat{3}} = & -k \wedge (-v_4\vartheta^{\hat{2}} + v_5\vartheta^{\hat{3}}). \end{cases} \tag{48}$$

One can verify that the torsion trace is proportional to the 1-form k:

$$T = e_\alpha \rfloor T^\alpha = (2v_5 - v_1)\,k = -\frac{m}{\Sigma}\,k. \tag{49}$$

As a result, we find the irreducible parts of the torsion: $^{(3)}T^\alpha = 0$ and

$$^{(1)}T^\alpha = -\frac{(v_1 + v_5)}{3}\,k \wedge \vartheta^\alpha + v_4{}^*(k \wedge \vartheta^\alpha) + k^\alpha w, \tag{50}$$

$$^{(2)}T^\alpha = \frac{(v_1 - 2v_5)}{3}\,k \wedge \vartheta^\alpha. \tag{51}$$

The Riemann–Cartan geometry of Baekler's configuration (34) and (35) is globally *regular* in the sense that all the torsion invariants vanish, whereas curvature invariants are constant. In particular,

$$T^\alpha \wedge {}^*T_\alpha = 0, \qquad R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} = 12\lambda^2\eta. \tag{52}$$

To solve the coupled system of the Poincaré gauge field Equations (17) and (18), we have to evaluate $q_\alpha^{(T)}$, $q_\alpha^{(R)}$, h_α, $h_{\alpha\beta}$, and the covariant derivative Dh_α, $Dh_{\alpha\beta}$. We begin by noticing that the structure of the 2-form $h_{\alpha\beta}$ realizes the generalized double-duality ansatz [3], namely:

$$h_{\alpha\beta} = \lambda_1\frac{1}{2}\eta_{\alpha\beta\mu\nu}R^{\mu\nu} + \lambda_2\,R_{\alpha\beta} + \lambda_3\,\eta_{\alpha\beta} + \lambda_4\,\vartheta_\alpha \wedge \vartheta_\beta. \tag{53}$$

In view of Equations (42) and (43), we have explicitly

$$\lambda_1 = -b_4, \quad \lambda_2 = \bar{b}_4, \quad \lambda_3 = (b_4 + b_6)\lambda, \quad \lambda_4 = -(\bar{b}_4 - \bar{b}_6)\lambda. \tag{54}$$

Making use of Equation (53), we find

$$q_\alpha^{(R)} = -\lambda_3 \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} - 2\lambda_4 R_{\alpha\beta} \wedge \vartheta^\beta + 6\lambda\lambda_3 \eta_\alpha, \tag{55}$$

and recast the field equations (17) and (18) into

$$\frac{a_{\text{eff}}}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + \bar{a}_{\text{eff}} R_{\alpha\beta} \wedge \vartheta^\beta - \lambda_{\text{eff}} \eta_\alpha + q_\alpha^{(T)} - Dh_\alpha = 0, \tag{56}$$

$$\frac{a_{\text{eff}}}{2} \eta_{\alpha\beta\gamma} \wedge T^\gamma + \bar{a}_{\text{eff}} T_{[\alpha} \wedge \vartheta_{\beta]} + h_{[\alpha} \wedge \vartheta_{\beta]} = 0, \tag{57}$$

where we introduce the effective constants

$$a_{\text{eff}} = a_0 - 2\ell_\rho^2 \lambda_3, \qquad \bar{a}_{\text{eff}} = \bar{a}_0 - 2\ell_\rho^2 \lambda_4, \qquad \lambda_{\text{eff}} = \lambda_0 - 6\ell_\rho^2 \lambda_3 \lambda. \tag{58}$$

In order to simplify the first field equation (56), we use the explicit form of the curvature (39) to find for the first term

$$\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = \frac{\lambda}{2}\eta_{\alpha\beta\gamma} \wedge \vartheta^\beta \wedge \vartheta^\gamma + \lambda U \eta_{\alpha\beta\gamma} \wedge k k^\beta \wedge \vartheta^\gamma = 3\lambda\,\eta_\alpha + 2\lambda U k_\alpha k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}, \tag{59}$$

whereas the second term vanishes $R_{\alpha\beta} \wedge \vartheta^\beta = 0$ in view of (41).

A direct computation yields

$$q_\alpha^{(T)} = \frac{2m}{3\Sigma} \left[(2a_1 + a_2)v_5 - (\bar{a}_1 - \bar{a}_2)v_4\right] k_\alpha k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}, \tag{60}$$

whereas after a long algebra we find for the components of the derivative Dh_α:

$$
\begin{aligned}
-Dh_{\hat{0}} = {} & -\frac{2(\bar{a}_1 - \bar{a}_2)v_4}{3r}\,\widehat{\vartheta}^{\hat{0}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} + \frac{2a_1 + a_2}{3r}\,k \wedge \vartheta^{\hat{0}} \wedge \left(v_3\vartheta^{\hat{2}} + v_2\vartheta^{\hat{3}}\right) \\
& + \frac{v_5}{3r}\sqrt{\frac{\Delta}{\Sigma}}\,k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} \left\{ -6a_1\lambda\frac{\Sigma^2}{\Delta} + (2a_1 + a_2)\left[1 - \frac{\Sigma}{\Delta}(1 - \lambda j_0^2 - 2\lambda r^2)\right] \right\},
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
-Dh_{\hat{1}} = {} & \frac{2(\bar{a}_1 - \bar{a}_2)v_4}{3r}\,\widehat{\vartheta}^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} - \frac{2a_1 + a_2}{3r}\,k \wedge \vartheta^{\hat{1}} \wedge \left(v_3\vartheta^{\hat{2}} + v_2\vartheta^{\hat{3}}\right) \\
& + \frac{v_5}{3r}\sqrt{\frac{\Delta}{\Sigma}}\,k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} \left\{ 6a_1\lambda\frac{\Sigma^2}{\Delta} + (2a_1 + a_2)\left[1 + \frac{\Sigma}{\Delta}(1 - \lambda j_0^2 - 2\lambda r^2)\right] \right\},
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
-Dh_{\hat{2}} = {} & \frac{1}{3r}\left[(2a_1 + a_2)v_4 + 2(\bar{a}_1 - \bar{a}_2)v_5\right] \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \\
& + \frac{2a_1 + a_2}{3r}\left[v_5\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{3}} + v_2\,k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}\right],
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
-Dh_{\hat{3}} = {} & \frac{1}{3r}\left[(2a_1 + a_2)v_4 + 2(\bar{a}_1 - \bar{a}_2)v_5\right] \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{3}} \\
& + \frac{2a_1 + a_2}{3r}\left[-v_5\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} + v_3\,k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}\right].
\end{aligned}
\tag{64}
$$

As we can see, Equations (60)–(64) are significantly simplified when the coupling constants satisfy $2a_1 + a_2 = 0$ and $\bar{a}_1 = \bar{a}_2$. Namely, we then get $q_\alpha^{(T)} = 0$ and, by making use of Equations (46) and (32), we find

$$-Dh_\alpha = -\frac{6a_1\lambda\,v_5\Sigma}{3r}\,k_\alpha k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} = 2a_1\lambda\,U\,k_\alpha k \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}. \tag{65}$$

Now let us turn to the second field Equation (57) which, in view of the definition (15), imposes an algebraic constraint on the spacetime torsion. Indeed, by making use of the properties of irreducible parts of the torsion, we recast Equation (57) into an equivalent form

$$- (a_{\text{eff}} + a_1)\,{}^{*}\left({}^{(1)}T^{\alpha}\right) + (2a_{\text{eff}} - a_2)\,{}^{*}\left({}^{(2)}T^{\alpha}\right) - (\bar{a}_{\text{eff}} + \bar{a}_1)\,{}^{(1)}T^{\alpha} - (\bar{a}_{\text{eff}} + \bar{a}_2)\,{}^{(2)}T^{\alpha} = 0. \qquad (66)$$

This admits a nontrivial field configuration when $a_{\text{eff}} + a_1 = 0$ and $\bar{a}_{\text{eff}} + \bar{a}_1 = 0$. Combining the definitions (58) with Equation (54), we then finally obtain the constraints on the coupling constants

$$a_0 + a_1 - 2\ell_{\rho}^2 \lambda \, (b_4 + b_6) = 0, \qquad \bar{a}_0 + \bar{a}_1 + 2\ell_{\rho}^2 \lambda \, (\bar{b}_4 - \bar{b}_6) = 0. \qquad (67)$$

Substituting Equation (65) and Equation (59) into Equation (56), we discover that the 1st field equation reduces to a simple relation $3a_{\text{eff}}\lambda = \lambda_{\text{eff}}$. Recalling the definitions of the effective coupling constants (58), the latter is equivalent to

$$3a_0\lambda = \lambda_0. \qquad (68)$$

7. Discussion and Conclusions

The algebraic constraints on the torsion is a well known feature of the double-duality technique [3] which leads to restrictions on the coupling constants. Nevertheless, the class of quadratic theories still remains very wide, and it includes many physically viable models. Moreover, one of the outstanding problems in the Poincaré gauge gravity is the search for the physically meaningful conditions that improve the structure of general Lagrangians so as to pass the "consistency check" with GR. The latter means a possibility of a smooth recovery of GR results in a certain macroscopic limit. In particular, the existence of the black hole solutions can be considered a manifestation of such a consistency with GR for the class of models which allow for the exact solutions obtained above.

In this sense, it is worthwhile to mention one of the most interesting Poincaré gravity models, which was proposed by von der Heyde [25] and attracted much attention in the early stages of development of the gauge approach in gravitational theory [26]. A peculiar feature of the *von der Heyde* Lagrangian is that it does not contain the Hilbert term linear in the curvature, therefore it is a purely quadratic model both in the torsion and the curvature. Explicitly, this Lagrangian reads

$$V^{\text{vdH}} = \frac{1}{2\kappa c}\left\{ (\vartheta^{\alpha} \wedge T^{\beta}) \wedge {}^{*}(\vartheta_{\beta} \wedge T_{\alpha}) + \frac{1}{4\lambda}R^{\alpha\beta} \wedge {}^{*}R_{\alpha\beta} \right\}. \qquad (69)$$

The parity-odd terms are absent, whereas the parity-even sector is described by the set of the coupling constants as $a_0 = 0$, $\lambda_0 = 0$, $b_1 = b_2 = b_3 = b_4 = b_6 = b$, and

$$a_1 = -1, \qquad a_2 = 2, \qquad a_3 = 0, \qquad \frac{b}{\rho} = -\frac{1}{4\kappa c\lambda}. \qquad (70)$$

Another feasible gauge gravity model arises in a de Sitter gauge approach when the Poincaré symmetry is extended to a 10-parameter de Sitter group [27,28]. The corresponding model is described by the Lagrangian

$$V^{\text{dS}} = \frac{1}{2\kappa c}\left\{ R_{\alpha\beta} \wedge \eta^{\alpha\beta} - 6\lambda\eta - \frac{1}{4\lambda}R^{\alpha\beta} \wedge {}^{*}R_{\alpha\beta} \right\}. \qquad (71)$$

The coupling constants set then reduces to $a_0 = 1$, $\lambda_0 = 3\lambda$, $b_1 = b_2 = b_3 = b_4 = b_6 = b$, and

$$a_1 = a_2 = a_3 = 0, \qquad \frac{b}{\rho} = \frac{1}{4\kappa c\lambda}. \qquad (72)$$

As compared to the von der Heyde model (69), the Lagrangian (71) does include the explicit Hilbert term.

The two gravity theories (69) and (71) are explicit examples of the models which satisfy the consistency principle with GR, they both admit the black hole solutions described above. However, the results obtained here are important in view of their widest possible applicability. The quadratic Yang–Mills type Lagrangian (11) describes the class of the most general Poincaré gauge gravity models with the dynamical torsion. The existence of the black hole solutions (with Schwarzschild, Kerr, or Kerr–de Sitter metric) was earlier reported for the case of the parity symmetric theories [29–36]. Now we demonstrated this for the general case with both parity-even and parity-odd sectors taken into account.

In addition, we have clarified the underlying geometrical structure of these exact solutions. Namely, in the framework of the first-order formalism with the Poincaré gauge potentials $(\vartheta^\alpha, \Gamma^{\alpha\beta})$ as the fundamental field variables (as compared to the second-order formalism used in [29–36] with the metric and torsion as the basic variables), the black hole solution is constructed with the help of the beautiful ansatz (34) and (35). Thereby, we have essentially developed a generalization to the post-Riemannian geometries of the Kerr–Schild technique which was successfully used in the Riemannian case [37].

Quite interestingly, in the literature there are similar solutions reported [38,39] with the black hole metric configurations of the same type, however, with different torsion configurations. They are not described by the ansatz (34) and (35), a possibility of constructing an appropriate generalization of this ansatz will be discussed elsewhere. More recently, spherically symmetric solutions were obtained [40,41] for the special case of the parity-even class of quadratic Poincaré gravity models. These configurations do not satisfy the double-duality ansatz, with the dynamical axial trace torsion field playing the role of a Maxwell–Coulomb field which gives rise to an effective Reissner–Nordström-type line element. Such solutions explicitly demonstrate that the generalized Birkhoff's theorem is not valid for the whole class of quadratic Poincaré gauge gravity models, see the relevant discussion in [4,7].

The final remark is as follows. There is common belief in the validity of the statement "black holes do not have hair" which means that there are no non-metric field configurations that satisfy the vacuum field equations and are regular across a black hole horizon. This seems to be generally true for all matter fields with an exception of the electromagnetic field. Our solutions provide a kind of counter-example to the no-hair conjecture in the sense that the corresponding dynamical torsion field is regular across the black hole horizon. The crucial role is played by the properties (52) which show a global regularity of the Riemann–Cartan geometry for our solutions. In other words, we have demonstrated that a black hole may have a nontrivial "geometrical hair" (in the form of the spacetime torsion) which does not affect the structure of a usual Kerr–de Sitter black hole.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Irreducible Decompositions

Appendix A.1. Torsion

The torsion 2-form can be decomposed into the three irreducible pieces, $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, where the torsion trace, the axial torsion, and the purely tensor torsion are defined by

$$^{(2)}T^\alpha \quad = \quad \frac{1}{3}\vartheta^\alpha \wedge (e_\nu \rfloor T^\nu), \tag{A1}$$

$$^{(3)}T^\alpha \quad = \quad \frac{1}{3}e^\alpha \rfloor (T^\nu \wedge \vartheta_\nu), \tag{A2}$$

$$^{(1)}T^\alpha \quad = \quad T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{A3}$$

Appendix A.2. Curvature

The Riemann–Cartan curvature 2-form is decomposed $R^{\alpha\beta} = \sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta}$ into the 6 irreducible parts

$$^{(2)}R^{\alpha\beta} \quad = \quad - {}^*(\vartheta^{[\alpha} \wedge \overline{\Psi}^{\beta]}), \tag{A4}$$

$$^{(3)}R^{\alpha\beta} \quad = \quad -\frac{1}{12}\,{}^*(\overline{X}\,\vartheta^\alpha \wedge \vartheta^\beta), \tag{A5}$$

$$^{(4)}R^{\alpha\beta} \quad = \quad -\vartheta^{[\alpha} \wedge \Psi^{\beta]}, \tag{A6}$$

$$^{(5)}R^{\alpha\beta} \quad = \quad -\frac{1}{2}\vartheta^{[\alpha} \wedge e^{\beta]} \rfloor (\vartheta^\gamma \wedge X_\gamma), \tag{A7}$$

$$^{(6)}R^{\alpha\beta} \quad = \quad -\frac{1}{12}\,X\,\vartheta^\alpha \wedge \vartheta^\beta, \tag{A8}$$

$$^{(1)}R^{\alpha\beta} \quad = \quad R^{\alpha\beta} - \sum_{I=2}^{6} {}^{(I)}R^{\alpha\beta}, \tag{A9}$$

where

$$X^\alpha := e_\beta \rfloor R^{\alpha\beta}, \qquad X := e_\alpha \rfloor X^\alpha, \qquad \overline{X}^\alpha := {}^*(R^{\beta\alpha} \wedge \vartheta_\beta), \qquad \overline{X} := e_\alpha \rfloor \overline{X}^\alpha, \tag{A10}$$

$$\Psi_\alpha := X_\alpha - \tfrac{1}{4}\,\vartheta_\alpha\,X - \tfrac{1}{2}\,e_\alpha \rfloor (\vartheta^\beta \wedge X_\beta), \qquad \overline{\Psi}_\alpha := \overline{X}_\alpha - \tfrac{1}{4}\,\vartheta_\alpha\,\overline{X} - \tfrac{1}{2}\,e_\alpha \rfloor (\vartheta^\beta \wedge \overline{X}_\beta). \tag{A11}$$

Appendix A.3. Elementary Properties

Directly from the definitions (A1)–(A3) and (A4)–(A9), one can prove the relations

$$T^\alpha \wedge {}^{(1)}T_\alpha = {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha, \tag{A12}$$

$$T^\alpha \wedge {}^{(2)}T_\alpha = T^\alpha \wedge {}^{(3)}T_\alpha = {}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha, \tag{A13}$$

$$R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta} = {}^{(1)}R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta}, \tag{A14}$$

$$R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta} = {}^{(5)}R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta}, \tag{A15}$$

$$R^{\alpha\beta} \wedge {}^{(2)}R_{\alpha\beta} = R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta} = {}^{(2)}R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta}, \tag{A16}$$

$$R^{\alpha\beta} \wedge {}^{(3)}R_{\alpha\beta} = R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta} = {}^{(3)}R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta}. \tag{A17}$$

Appendix B. De Sitter Geometry

The Riemannian connection is uniquely determined by the torsion-free condition (26):

$$\widehat{\Gamma}_{\alpha\beta} = \frac{1}{2}\left(\widehat{e}_\alpha \rfloor d\widehat{\vartheta}_\beta - \widehat{e}_\beta \rfloor d\widehat{\vartheta}_\alpha - \widehat{\vartheta}^\gamma\,\widehat{e}_\alpha \rfloor \widehat{e}_\beta \rfloor d\widehat{\vartheta}_\gamma\right). \tag{A18}$$

Explicitly, we find for the components of the local Lorentz connection:

$$\widehat{\Gamma}_{\hat{0}\hat{1}} = -(\beta_1 + \beta_2)\widehat{\vartheta}^{\hat{0}} + \beta_2\widehat{\vartheta}^{\hat{1}} + \alpha_1\widehat{\vartheta}^{\hat{3}}, \qquad\qquad \widehat{\Gamma}_{\hat{2}\hat{3}} = -\alpha_2\widehat{\vartheta}^{\hat{0}} + \alpha_4\widehat{\vartheta}^{\hat{1}} - \beta_5\widehat{\vartheta}^{\hat{3}}, \tag{A19}$$

$$\widehat{\Gamma}_{\hat{0}\hat{2}} = \alpha_3\widehat{\vartheta}^{\hat{0}} + \beta_3\widehat{\vartheta}^{\hat{2}} + \alpha_2\widehat{\vartheta}^{\hat{3}}, \qquad\qquad \widehat{\Gamma}_{\hat{3}\hat{1}} = \alpha_1\widehat{\vartheta}^{\hat{0}} - \alpha_4\widehat{\vartheta}^{\hat{2}} + \beta_4\widehat{\vartheta}^{\hat{3}}, \tag{A20}$$

$$\widehat{\Gamma}_{\hat{0}\hat{3}} = \alpha_1\widehat{\vartheta}^{\hat{1}} - \alpha_2\widehat{\vartheta}^{\hat{2}} + \beta_3\widehat{\vartheta}^{\hat{3}}, \qquad\qquad \widehat{\Gamma}_{\hat{1}\hat{2}} = -\alpha_3\widehat{\vartheta}^{\hat{1}} - \beta_4\widehat{\vartheta}^{\hat{2}} - \alpha_4\widehat{\vartheta}^{\hat{3}}, \tag{A21}$$

where we introduced the abbreviations

$$\alpha_1 = \sqrt{\tfrac{f}{\Sigma}}\,\tfrac{j_0 r \sin\theta}{\Sigma}, \qquad \alpha_2 = \tfrac{1}{\sqrt{\Delta\Sigma}}\,\tfrac{j_0 \cos\theta (\Delta + mr)}{\Sigma}, \qquad \alpha_3 = \sqrt{\tfrac{f}{\Sigma}}\,\tfrac{j_0^2 r \sin\theta \cos\theta}{\Sigma}, \tag{A22}$$

$$\alpha_4 = \tfrac{1}{\sqrt{\Delta\Sigma}}\,\tfrac{j_0 mr \cos\theta}{\Sigma}, \quad \beta_1 = \tfrac{1}{2}\sqrt{\tfrac{\Delta}{\Sigma}}\left(\tfrac{\Delta'}{\Delta} - \tfrac{\Sigma'}{\Sigma}\right), \quad \beta_2 = \tfrac{m}{2\sqrt{\Delta\Sigma}}\left(2 - r\tfrac{\Delta'}{\Delta} - r\tfrac{\Sigma'}{\Sigma}\right), \tag{A23}$$

$$\beta_3 = \tfrac{1}{\sqrt{\Delta\Sigma}}\,\tfrac{mr^2}{\Sigma}, \quad \beta_4 = \tfrac{1}{\sqrt{\Delta\Sigma}}\,\tfrac{r(\Delta + mr)}{\Sigma}, \quad \beta_5 = \sqrt{\tfrac{f}{\Sigma}}\left[\cot\theta + j_0^2 \sin\theta \cos\theta \left(\tfrac{1}{\Sigma} - \tfrac{\lambda}{f}\right)\right]. \tag{A24}$$

Here, the prime denotes the derivative with respect to the radial coordinate $' = \partial_r$. We split the coefficients into two groups: $\alpha_1, \dots, \alpha_4$ vanish in the absence of rotation (when $j_0 = 0$), whereas $\beta_1, \dots, \beta_5$ are always nontrivial.

References

1. Blagojević, M.; Hehl, F.W. (Eds.) *Gauge Theories of Gravitation. A Reader with Commentaries*; Imperial College Press: London, UK, 2013.
2. Ponomarev, V.N.; Barvinsky, A.O.; Obukhov, Y.N. *Gauge Approach and Quantization Methods in Gravity Theory*; Nauka: Moscow, Russia, 2017.
3. Obukhov, Y.N. Poincaré gauge gravity: Selected topics. *Int. J. Geom. Methods Mod. Phys.* **2006**, *3*, 95–137. [CrossRef]
4. Obukhov, Y.N. Poincaré gauge gravity: An overview. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15* (Suppl. 1), 1840005. [CrossRef]
5. Hehl, F.W.; McCrea, J.D.; Mielke, E.W.; Ne'eman, Y. Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance. *Phys. Rep.* **1995**, *258*, 1–177. [CrossRef]
6. Hehl, F.W.; Obukhov, Y.N. *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*; Birkhäuser: Boston, MA, USA, 2003.
7. Obukhov, Y.N.; Ponomariev, V.N.; Zhytnikov, V.V. Quadratic Poincaré gauge theory of gravity: A comparison with the general relativity theory. *Gen. Relat. Grav.* **1989**, *21*, 1107–1142. [CrossRef]
8. Ho, J.K.; Nester, J.M. Poincaré gauge theory with coupled even and odd parity spin-0 modes: Cosmological normal modes. *Annalen der Physik* **2012**, *524*, 97–106. [CrossRef]
9. Diakonov, D.; Tumanov, A.G.; Vladimirov, A.A. Low-energy general relativity with torsion: A systematic derivative expansion. *Phys. Rev. D* **2011**, *84*, 124042. [CrossRef]
10. Baekler, P.; Hehl, F.W. Beyond Einstein-Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms. *Class. Quantum Grav.* **2011**, *28*, 215017. [CrossRef]
11. Baekler, P.; Hehl, F.W.; Nester, J.M. Poincaré gauge theory of gravity: Friedman cosmology with even and odd parity modes: Analytic part. *Phys. Rev. D* **2011**, *83*, 024001. [CrossRef]
12. Karananas, G.K. The particle spectrum of parity-violating Poincaré gravitational theory. *Class. Quantum Grav.* **2015**, *32*, 055012. [CrossRef]
13. Blagojević, M.; Cvetković, B. General Poincaré gauge theory: Hamiltonian structure and particle spectrum. *Phys. Rev. D* **2018**, *98*, 024014. [CrossRef]
14. Leitner, J.; Okubo, S. Parity, charge conjugation, and time reversal in the gravitational interaction. *Phys. Rev.* **1964**, *136*, B1542–B1546. [CrossRef]
15. Hari Dass, N.D. Experimental test for some quantum effects in gravitation. *Ann. Phys.* **1977**, *107*, 337–359. [CrossRef]
16. Hojman, R.; Mukku, C.; Sayed, W.A. Parity violation in metric-torsion theories of gravitation. *Phys. Rev. D* **1980**, *22*, 1915–1921. [CrossRef]
17. Holst, S. Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action. *Phys. Rev. D* **1996**, *53*, 5966–5969. [CrossRef]
18. Kiefer, C. *Quantum Gravity*, 3rd ed.; Oxford University Press: Oxford, UK, 2012.
19. Randono, A. Gravity from a fermionic condensate of a gauge theory. *Class. Quantum Grav.* **2010**, *27*, 215019. [CrossRef]
20. Popławski, N. Matter-antimatter asymmetry and dark matter from torsion. *Phys. Rev. D* **2011**, *83*, 084033. [CrossRef]

21.　Bjorken, J.D. Emergent photons and gravitons: The problem of vacuum structure. In *CPT and Lorentz Symmetry, Proceedings of the Fifth Meeting, Bloomington, IN, USA, 28 June–2 July 2010*; Kostelecký, V.A., Ed.; World Scientific: Singapore, 2010; pp. 1–5.

22.　Heinicke, C.; Hehl, F.W. Schwarzschild and Kerr solutions of Einstein's field equation: An Introduction. *Int. J. Mod. Phys. D* **2015**, *24*, 1530006. [CrossRef]

23.　Obukhov, Y.N. Gravitational waves in Poincaré gauge gravity theory. *Phys. Rev. D* **2017**, *95*, 084028. [CrossRef]

24.　Blagojević, M.; Cvetković, B.; Obukhov, Y.N. Generalized plane waves in Poincaré gauge theory of gravity. *Phys. Rev. D* **2017**, *96*, 064031. [CrossRef]

25.　Von der Heyde, P. Is gravitation mediated by the torsion of spacetime? *Z. Naturforsch.* **1976**, *31*, 1725–1726.

26.　Hehl, F.W.; Nitsch, J.; von der Heyde, P. Gravitation and the Poincaré gauge field theory with quadratic Lagrangian. In *General Relativity and Gravitation: One Hundred Years after the Birth of Albert Einstein*; Held, A., Ed.; Plenum Press: New York, NY, USA, 1980; Volume 1, pp. 329–355.

27.　Stelle, K.S.; West P.C. Spontaneously broken de Sitter symmetry and the gravitational holonomy group. *Phys. Rev. D* **1980**, *21*, 1466–1481. [CrossRef]

28.　Tseytlin, A.A. Poincaré and de Sitter gauge theories of gravity with propagating torsion. *Phys. Rev. D* **1982**, *26*, 3327–3341. [CrossRef]

29.　Baekler, P. A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials. *Phys. Lett. B* **1981**, *99*, 329–332. [CrossRef]

30.　Baekler, P.; Gürses, M. Exact solutions of the Poincaré gauge theory from its linearized field equations. *Lett. Math. Phys.* **1987**, *14*, 185–191. [CrossRef]

31.　Baekler, P.; Gürses, M.; Hehl, F.W. A new method to solve the field equations of Poincare gauge theories. *Class. Quantum Grav.* **1988**, *5*, L105–L112. [CrossRef]

32.　Baekler, P.; Gürses, M.; Hehl, F.W.; McCrea, J.D. The exterior gravitational field of a charged spinning source in the Poincaré gauge theory: A Kerr-Newman metric with dynamical torsion. *Phys. Lett. A* **1988**, *128*, 245–250. [CrossRef]

33.　Baekler, P.; Hehl, F.W. A charged Taub-NUT metric with torsion. A new axially symmetric solution of the Poincaré gauge field theory. *Phys. Lett. A* **1984**, *100*, 392–396. [CrossRef]

34.　McCrea, J.D.; Baekler, P.; Gürses, M. A Kerr-like solution of the Poincaré gauge field equations. *Nuovo Cimento B* **1987**, *99*, 171–177. [CrossRef]

35.　McCrea, J.D.; Mielke, E.W.; Hehl, F.W. A remark on the axisymmetric Chen et al. solution of the Poincaré gauge theory. *Phys. Lett. A* **1988**, *127*, 65–69. [CrossRef]

36.　Lee, C.H. A spherically symmetric electro-vacuum solution of the Poincaré gauge field theory of gravitation. *Phys. Lett. B* **1983**, *130*, 257–259. [CrossRef]

37.　Gürses, M.; Şişman, T.C.; Tekin, B. New exact solutions of quadratic curvature gravity. *Phys. Rev. D* **2012**, *86*, 024009. [CrossRef]

38.　Chen, H.H.; Chern, D.-C.; Hsu, R.R.; Yeung, W.B. Rotonless solutions in the Poincaré gauge theory of gravitation. *Phys. Lett. A* **1985**, *111*, 693–696. [CrossRef] [PubMed]

39.　Chen, H.H.; Chern, D.-C.; Hsu, R.R.; Yeung, W.B. An axisymmetric solution to the Poincaré gauge theory of gravitation with Kerr metric and a nonvanishing torsion. *Chin. J. Phys.* **1986**, *24*, 115–121.

40.　Cembranos, J.A.R.; Valcarcel, J.G. New torsion black hole solutions in Poincaré gauge theory. *JCAP* **2017**. [CrossRef]

41.　Cembranos, J.A.R.; Valcarcel, J.G. Extended Reissner-Nordström solutions sourced by dynamical torsion. *Phys. Lett. B* **2018**, *779*, 143–150. [CrossRef]