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Gravitational Qubits

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Abstract: We report on the behavior of two-level quantum systems, or qubits, in the background of rotating and non-rotating metrics and provide a method to derive the related spin currents and motions. The calculations are performed in the external field approximation.

Keywords: spin; rotation; quantum gravity; relativity

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1. Introduction

Spin effects in gravity straddle the boundary between quantum and classical physics. The difference between quantum and classical behavior becomes particularly transparent with spin, which is relevant to low energy approaches to quantum gravity.

Quantum beats are an unequivocal indication that the system considered obeys the laws of quantum mechanics. Quantum systems with a two-dimensional Hilbert space are also called qubits. This definition is borrowed from quantum computing where two-level quantum systems play a predominant role. Qubits provide examples of systems that are genuinely quantum mechanical and, at the same time, simple, can be studied within the confines of first quantization, and are ideal in the study of relativistic gravity close to, or at the quantum level. There are gravitational qubits in the universe. Some of them are disussed below.

In what follows, use is made of the external field approximation [1,2] that treats gravity as a classical theory when it interacts with quantum particles. This approximation can be applied successfully to all those problems involving gravitational sources of weak to intermediate strength for which the full-fledged use of general relativity is not required [3–11], it is encountered in the solution of relativistic wave equations, and takes different forms according to the statistics obeyed by the particles [2,6,12–14]. The approximation can also be applied to theories in which acceleration has an upper limit [15–24] and that allow for the resolution of astrophysical and cosmological singularities in quantum gravity [25,26]. It is of interest to those theories of asymptotically safe gravity that can be expressed as Einstein gravity coupled to a scalar field [27], and can produce results complementary to those of the method of space-time deformation [28].

At the same time, theoretical developments by Mashhoon [19,20,29–31], and by other authors [12,32–36] in the field of spin-gravity coupling require a scheme that involves all components of the metric tensor. Finally, recent experimental observations of important rotation-related classical effects [37–39], of spin-rotation coupling for photons [40] and neutrons [41], the development of a spin rotator for neutron interferometry [42], and the generation of spin currents via spin-rotation coupling [43] indicate the degree of maturity and breadth of scope reached by the field.

In the formalism introduced in references [1,2], the effect of gravity on wave functions is contained in a phase factor. If phase differences develop in processes involving the qubits studied below, measurements become possible. This is the common thread that links the various sections of this work.

If gravitation produces qubits, then these may be observable and yield useful experimental results. Quantum physics and gravity may meet well before the onset of the quantum gravity regime usually associated with Planck's length and there still are interesting problems to investigate at lower scales. For instance, in addition to the important classical effect observed and discussed in references [37–39], there also is a quantum Lense–Thirring effect, that represents the action of the Lense–Thirring metric on a particle wave function. By applying the procedure of [1,2], one finds [36] that the phase difference produced by a gravitational source of mass M, radius R and angular velocity ω is $\Delta\chi_{LT} = \Omega_{LT}\Pi$, where $\Omega_{LT} = 2GM\omega/(5c^2R)$ is the effective Lense–Thirring frequency of a gyroscope and $\Pi = 4m\ell^2/\hbar$ replaces the period of a satellite in the classical calculation. Its observation with neutron interferometers of typical dimension $\ell \sim 3$cm, still seems difficult but would complete nicely what we know at present about rotation in relativity.

For the sake of completeness, some essential points are being repeated. The key player in what follows is the covariant Dirac equation

$$[i\gamma^{\mu}(x)\mathcal{D}_{\mu} - m]\Psi(x) = 0\,, \tag{1}$$

that determines the behavior of spin-1/2 particles in the presence of a gravitational field $g_{\mu\nu}$. In Equation (1), $\mathcal{D}_{\mu} = \nabla_{\mu} + i\Gamma_{\mu}(x)$, ∇_{μ} is the covariant derivative, $\Gamma_{\mu}(x)$ the spin connection and the matrices $\gamma^{\mu}(x)$ satisfy the relations $\{\gamma^{\mu}(x), \gamma^{\nu}(x)\} = 2g^{\mu\nu}$. Both $\Gamma_{\mu}(x)$ and $\gamma^{\mu}(x)$ can be obtained from the usual constant Dirac matrices $\gamma^{\hat{\alpha}}$ by using the vierbein fields $e^{\mu}_{\hat{\alpha}}$ and the relations

$$\gamma^{\mu}(x) = e^{\mu}_{\hat{\alpha}}(x)\gamma^{\hat{\alpha}}\,, \qquad \Gamma_{\mu}(x) = -\frac{1}{4}\sigma^{\hat{\alpha}\hat{\beta}}e^{\nu}_{\hat{\alpha}}e_{\nu\hat{\beta};\,\mu}\,, \tag{2}$$

where $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$. We use units $\hbar = c = 1$ and the notations are as in [3].

Equation (1) can be solved exactly [12,44] to first order in the metric deviation $\gamma_{\mu\nu}(x) = g_{\mu\nu} - \eta_{\mu\nu}$, where the Minkowski metric $\eta_{\mu\nu}$ has signature -2. This is achieved by first transforming Equation (1) into the equation

$$[i\tilde{\gamma}^{\nu}(x)\nabla_{\nu} - m]\tilde{\Psi}(x) = 0\,, \tag{3}$$

where

$$\tilde{\Psi}(x) = S^{-1}\Psi(x)\,, \qquad S(x) = e^{-i\Phi_s(x)}\,, \qquad \Phi_s(x) = \mathcal{P}\int_P^x dz^{\lambda}\Gamma_{\lambda}(z)\,, \qquad \tilde{\gamma}^{\mu}(x) = S^{-1}\gamma^{\mu}(x)S\,. \tag{4}$$

By multiplying Equation (3) on the left by $(-i\tilde{\gamma}^{\nu}(x)\nabla_{\nu} - m)$, we obtain the equation

$$(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + m^2)\tilde{\Psi}(x) = 0\,, \tag{5}$$

whose solution

$$\tilde{\Psi}(x) = e^{-i\hat{\Phi}_G(x)}\Psi_0(x)\,, \tag{6}$$

is exact to first order. The operator $\hat{\Phi}_G(x)$ is defined as

$$\hat{\Phi}_G = -\frac{1}{4}\int_P^x dz^{\lambda}\left[\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)\right]\hat{L}^{\alpha\beta}(z) + \frac{1}{2}\int_P^x dz^{\lambda}\gamma_{\alpha\lambda}\hat{k}^{\alpha}\,, \tag{7}$$

$$[\hat{L}^{\alpha\beta}(z), \Psi_0(x)] = \left((x^{\alpha} - z^{\alpha})\hat{k}^{\beta} - (x^{\beta} - z^{\beta})\hat{k}^{\alpha}\right)\Psi_0(x)\,, \qquad [\hat{k}^{\alpha}, \Psi_0(x)] = i\partial^{\alpha}\Psi_0\,,$$

and $\Psi_0(x)$ satisfies the usual free Dirac equation

$$\left(i\gamma^{\hat{\mu}}\partial_{\mu} - m\right)\Psi_0(x) = 0\,. \tag{8}$$

In Equations (4) and (7), the path integrals are taken along the classical world line of the particle starting from an arbitrary reference point P. Only the path to $\mathcal{O}(\gamma_{\mu\nu})$ needs to be known in the integrations indicated because Equation (4) already is a first order solution. The positive energy solutions of Equation (8) are given by

$$\Psi_0(x) = u(\mathbf{k})e^{-ik_\alpha x^\alpha} = N \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{E+m}\,\phi \end{pmatrix} e^{-ik_\alpha x^\alpha}\,, \tag{9}$$

where $N = \sqrt{\frac{E+m}{2E}}$, $u^+u = 1$, $\bar{u} = u^+\gamma^0$, $u_1^+u_2 = u_2^+u_1 = 0$ and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ represents the Pauli matrices. In addition, ϕ can take the forms ϕ_1 and ϕ_2 where $\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\hat{L}_{\alpha\beta}$ and $\hat{k}^\alpha$ are the angular and linear momentum operators of the particle. It follows from Equations (6) and (4) that the solution of Equation (1) can be written in the form [6]

$$\Psi(x) = e^{-i\Phi_s}\left(-i\tilde{\gamma}^\mu(x)\nabla_\mu - m\right)e^{-i\Phi_G}\,\Psi_0(x) \equiv \hat{T}\Psi_0\,, \tag{10}$$

and also as

$$\Psi(x) = -\frac{1}{2m}\left(-i\gamma^\mu(x)\mathcal{D}_\mu - m\right)e^{-i\Phi_T}\Psi_0(x) \equiv \hat{T}\Psi_0\,, \tag{11}$$

where $\Phi_T = \Phi_s + \Phi_G$ is of first order in $\gamma_{\alpha\beta}(x)$. The factor $-1/2m$ on the r.h.s. of Equation (11) appears because both sides of the equation must agree when the gravitational field vanishes.

On multiplying Equation (1) on the left by $(-i\gamma^\nu(x)\mathcal{D}_\nu - m)$ and using the relations

$$\nabla_\mu\Gamma_\nu(x) - \nabla_\nu\Gamma_\mu(x) + i[\Gamma_\mu(x), \Gamma_\nu(x)] = -\frac{1}{4}\sigma^{\alpha\beta}(x)R_{\alpha\beta\mu\nu}\,, \tag{12}$$

and

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\frac{i}{4}\sigma^{\alpha\beta}(x)R_{\alpha\beta\mu\nu}\,, \tag{13}$$

we obtain the equation

$$\left(g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu - \frac{R}{4} + m^2\right)\Psi(x) = 0\,. \tag{14}$$

In Equations (13) and (14), $R_{\alpha\beta\lambda\mu}(z) = -\frac{1}{2}\left(\gamma_{\alpha\lambda,\beta\mu} + \gamma_{\beta\mu,\alpha\lambda} - \gamma_{\alpha\mu,\beta\lambda} - \gamma_{\beta\lambda,\alpha\mu}\right)$ is the linearized Riemann tensor, R the corresponding Ricci scalar, and $\sigma^{\alpha\beta}(x) = (i/2)[\gamma^\alpha(x), \gamma^\beta(x)]$.

By using Equation (4), we also find

$$(-i\gamma^\nu(x)\mathcal{D}_\nu - m)\,S\,(i\tilde{\gamma}^\mu\nabla_\mu - m)\tilde{\Psi}(x) = S\,(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\tilde{\Psi}(x) = 0\,. \tag{15}$$

Equation (14) implies that the gyro-gravitational ratio of a massive Dirac particle is one, as found in [45–47].

The transformations of coordinates $x_\mu \to x_\mu + \xi_\mu$, with $\xi_\mu(x)$ small of first order, lead to the "gauge" transformations $\gamma_{\mu\nu} \to \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$. It is therefore necessary to show that Φ_T in Equation (11) is gauge invariant. In fact, on applying Stokes theorem to a closed spacetime path C and using Equation (12), we find that Φ_T changes by

$$\Delta\Phi_T = \frac{1}{4}\int_\Sigma d\tau^{\mu\nu}J^{\alpha\beta}R_{\mu\nu\alpha\beta}\,, \tag{16}$$

where Σ is a surface bound by C and $J^{\alpha\beta}$ is the total angular momentum of the particle. Equation (16) shows that Equations (10) and (11) are gauge invariant and confirms that, to first order in the gravitational field, the gyro-gravitational ratio of a Dirac particle is one. Use of Equation (10) or Equation (11) assures the correct treatment of both spin and angular momentum.

The plan of this work is as follows. In Section 2 we discuss qubits represented by particles in accelerators. In Section 3 we derive the gravitational deflection of particles propagating in a gravitational background represented by the Lense–Thirring metric and obtain the contribution due to the rotation of the source. The neutrino helicity transitions are derived in Section 4 and some astrophysical consequences are discussed in Section 5. Spin currents and spin motion are presented in Section 6 and are followed by a summary.

2. Spin-Rotation Coupling in Accelerators

The spin-rotation effect described by Mashhoon is conceptually important, since it extends our knowledge of rotational inertia to the quantum level and violates the principle of equivalence [29,31] that is well-tested at the classical level.

It has, of course, been argued that the principle of equivalence does not hold true in the quantum world. This is the case for phase shifts in particle interferometers [44,48] and wave functions depend on the masses of the particles involved [49]. In addition, the equivalence principle does not apply in the context of the causal interpretation of quantum mechanics as shown by Holland [50]. Several models predicting quantum violations of the equivalence principle have also been discussed in the literature [51], also in connection with neutrino oscillations [52–55]. The Mashhoon term, in particular, yields different potentials for different particles and for different spin states and cannot, therefore, be regarded as universal. It plays, nonetheless, an essential role in precise measurements of the $g-2$ factor of the muon.

The experiment [56,57] involves muons in a storage ring. Muons on equilibrium orbits within a small fraction of the maximum momentum are almost completely polarized with spin vectors pointing in the direction of motion. As the muons decay, those electrons projected forward in the muon rest frame are detected around the ring. Their modulated angular distribution reflects the precession of the muon spin along the cyclotron orbits.

Our calculations use the covariant Dirac equation and are performed in the rotating frame of the muon and do not therefore require a relativistic treatment of inertial spin effects [58]. Then the vierbein formalism yields $\Gamma_i = 0$ and

$$\Gamma_0 = -\frac{1}{2}\, a_i \sigma^{0i} - \frac{1}{2}\, \omega_i \sigma^i \,, \tag{17}$$

where a_i and ω_i are the three-acceleration and three-rotation of the observer and

$$\sigma^{0i} \equiv \frac{i}{2}\,[\gamma^0, \gamma^i] = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

in the chiral representation of the usual Dirac matrices. The second term in Equation (17) represents the Mashhoon effect. The first term drops out. The remaining contributions to the Dirac Hamiltonian, to first order in a_i and ω_i, are [32,44]

$$\begin{aligned} H \;\approx\;& \vec{\alpha}\cdot\vec{p} + m\beta + \frac{1}{2}[(\vec{a}\cdot\vec{x})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{x})] \\ & -\vec{\omega}\cdot\left(\vec{L} + \frac{\vec{\sigma}}{2}\right) . \end{aligned} \tag{18}$$

All quantities in H are time-independent and are referred to a left-handed set of three axes rotating about the x_2-axis in the clockwise direction of motion of the muons. The muon momentum is directed along the x_3-axis which is tangent to the muon orbits. The magnetic field is $B_2 = -B$. Only the Mashhoon term then couples the helicity states of the muon. The remaining terms contribute to the overall energy E of the states, and we indicate by H_0 the corresponding part of the Hamiltonian.

Before decay the muon states can be represented as

$$|\psi(t)>= a(t)|\psi_+> + b(t)|\psi_->, \tag{19}$$

where $|\psi_+>$ and $|\psi_->$ are the right and left helicity states of the Hamiltonian H_0 and satisfy the equation

$$H_0|\psi_{+,-}>= E|\psi_{+,-}> \, .$$

The total effective Hamiltonian is $H_{eff} = H_0 + H'$, where

$$H' = -\frac{1}{2}\,\omega_2\sigma^2 + \mu B\sigma^2 \, . \tag{20}$$

$\mu = \left(1 + \frac{g-2}{2}\right)\mu_0$ represents the total magnetic moment of the muon and μ_0 is the Bohr magneton. We will neglect the presence of electric fields that also affect the muon spin. Their effects can be controlled in suitable ways [57].

The coefficients $a(t)$ and $b(t)$ in Equation (19) evolve in time according to

$$i\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} , \tag{21}$$

where M is the matrix

$$M = \begin{bmatrix} E - i\frac{\Gamma}{2} & i\left(\frac{\omega_2}{2} - \mu B\right) \\ -i\left(\frac{\omega_2}{2} - \mu B\right) & E - i\frac{\Gamma}{2} \end{bmatrix} \tag{22}$$

and Γ represents the width of the muon and is not particularly relevant to what follows. Equations (21) and (22) describe a two-dimensional qubit. The non-diagonal form of M (when $B = 0$) implies that rotation does not couple universally to matter.

M has eigenvalues

$$\begin{aligned} h_1 &= E - i\frac{\Gamma}{2} + \frac{\omega_2}{2} - \mu B \, , \\ h_2 &= E - i\frac{\Gamma}{2} - \frac{\omega_2}{2} + \mu B \, , \end{aligned}$$

and eigenstates

$$\begin{aligned} |\psi_1> &= \frac{1}{\sqrt{2}}\left[i|\psi_+> + |\psi_->\right] , \\ |\psi_2> &= \frac{1}{\sqrt{2}}\left[-i|\psi_+> + |\psi_->\right] . \end{aligned}$$

The muon states that satisfy Equation (21), and the condition $|\psi(0)>= |\psi_->$ at $t = 0$, are

$$\begin{aligned} |\psi(t)> &= \frac{e^{-\Gamma t/2}}{2}e^{-iEt}\left\{i\left[e^{-i\tilde{\omega}t} - e^{i\tilde{\omega}t}\right]|\psi_+> \right. \\ &\left. + \left[e^{-i\tilde{\omega}t} + e^{i\tilde{\omega}t}\right]|\psi_->\right\} , \end{aligned} \tag{23}$$

where

$$\tilde{\omega} \equiv \frac{\omega_2}{2} - \mu B \, .$$

The spin-flip probability is therefore

$$\begin{aligned} P_{\psi_- \to \psi_+} &= |<\psi_+|\psi(t)>|^2 \\ &= \frac{e^{-\Gamma t}}{2}\left[1 - \cos(2\mu B - \omega_2)t\right] . \end{aligned} \tag{24}$$

The Γ-term in Equation (24) accounts for the observed exponential decrease in electron counts due to the loss of muons by radioactive decay [57]. The term in square brackets represents the well known phenomenon of quantum beats which one should expect because muons are quantum systems. It also represents the characteristic behavior of a two-dimensional qubit.

The spin-rotation contribution to $P_{\psi_-\to\psi_+}$ is represented by ω_2 which is the cyclotron angular velocity $\frac{eB}{m}$ [57]. The spin-flip angular frequency is then

$$\begin{aligned}\Omega &= 2\mu B - \omega_2 \\ &= \left(1+\frac{g-2}{2}\right)\frac{eB}{m} - \frac{eB}{m} \\ &= \frac{g-2}{2}\frac{eB}{m} \equiv \frac{1}{\Delta},\end{aligned} \tag{25}$$

which is precisely the observed modulation frequency of the electron counts [57,59] and yields the value Δ of the energy level splitting. This result is independent of the value of the anomalous magnetic moment of the particle. It is therefore the spin-rotation coupling that gives evidence to the $g-2$ term in Ω by exactly cancelling, in $2\mu B$, the much larger contribution μ_0 that one would get if the fermion had no anomalous magnetic moment. The cancellation is made possible by the non-diagonal form of M and is therefore a direct consequence of the violation of the equivalence principle.

It is perhaps surprising that spin-rotation coupling as such has almost gone unnoticed for such a long time. It is, however, significant that its effect is observed in an experiment that has already provided crucial tests of quantum electrodynamics and a test of Einstein's time-dilation formula to better than a 0.1 percent accuracy.

Applications of these ideas to compound spin systems like heavy ions in accelerators can be found in [60–62].

3. Geometrical Optics of Spin-1/2 Particles

In this Section we study the propagation of a spin-1/2 particle in the Lense–Thirring metric [63] represented, in its post-Newtonian form, by

$$\gamma_{00} = 2\varphi\,, \quad \gamma_{ij} = 2\varphi\delta_{ij}\,, \quad \gamma_{0i} = h_i = \frac{2}{r^3}(\mathbf{J}\wedge\mathbf{r})_i\,, \tag{26}$$

where

$$\varphi = -\frac{GM}{r}\,, \quad \mathbf{h} = \frac{4GMR^2\omega}{5r^3}(y,-x,0)\,, \tag{27}$$

and M, R, $! = (0,0,\omega)$ and $\mathbf{J}$ are mass, radius, angular velocity, and angular momentum of the source. The vierbein field to $\mathcal{O}(\gamma_{\mu\nu})$ is

$$e^0_{\hat{i}} = 0\,, \quad e^0_{\hat{0}} = 1-\varphi\,, \quad e^i_{\hat{0}} = h_i\,, \quad e^l_{\hat{k}} = (1+\varphi)\,\delta^l_k\,. \tag{28}$$

The gravitational contribution in Equation (28) can be further isolated by writing $e^\mu_{\hat{\alpha}} \simeq \delta^\mu_{\hat{\alpha}} + h^\mu_{\hat{\alpha}}$. The components of the spin connection can be calculated using Equations (2) and (28) and are

$$\begin{aligned}\Gamma_0 &= -\frac{1}{2}\varphi_{,j}\,\sigma^{\hat{0}\hat{j}} - \frac{1}{8}(h_{i,j}-h_{j,i})\sigma^{\hat{i}\hat{j}} \\ \Gamma_i &= -\frac{1}{8}(h_{i,j}-h_{j,i})\sigma^{\hat{0}\hat{j}} - \frac{1}{2}\varphi_{,j}\,\sigma^{\hat{i}\hat{j}}\,,\end{aligned} \tag{29}$$

and have the explicit form

$$
\begin{aligned}
\Gamma_0 &= -\frac{GM}{2r^3}\left(x\sigma^{\hat{0}\hat{1}} + y\sigma^{\hat{0}\hat{2}} + z\sigma^{\hat{0}\hat{3}}\right) + \frac{GMR^2\omega}{5r^5}\left[(r^2-3z^2)\sigma^{\hat{1}\hat{2}} + 3yz\sigma^{\hat{1}\hat{3}} - 3xz\sigma^{\hat{2}\hat{3}}\right] \\
\Gamma_1 &= \frac{3GMR^2\omega}{5r^5}\left[2xy\sigma^{\hat{0}\hat{1}} + (y^2-x^2)\sigma^{\hat{0}\hat{2}} + yz\sigma^{\hat{0}\hat{3}}\right] + \frac{GM}{2r^3}\left(y\sigma^{\hat{1}\hat{2}} + z\sigma^{\hat{1}\hat{3}}\right) \\
\Gamma_2 &= \frac{3GMR^2\omega}{5r^5}\left[(y^2-x^2)\sigma^{\hat{0}\hat{1}} - 2xy\sigma^{\hat{0}\hat{2}} - xz\sigma^{\hat{0}\hat{3}}\right] + \frac{GM}{2r^3}\left(-x\sigma^{\hat{1}\hat{2}} + z\sigma^{\hat{2}\hat{3}}\right) \\
\Gamma_3 &= \frac{3GMR^2\omega}{5r^5}\left(yz\sigma^{\hat{0}\hat{1}} - xz\sigma^{\hat{0}\hat{2}}\right) + \frac{GM}{2r^3}\left(x\sigma^{\hat{1}\hat{3}} + y\sigma^{\hat{2}\hat{3}}\right).
\end{aligned} \tag{30}
$$

In what follows, use is made of the Dirac representation of the $\gamma^{\hat{\mu}}$, of the first derivative of Φ_G with respect to x^μ

$$
\Phi_{G,\mu} = -\frac{1}{2}\int_P^x dz^\lambda(\gamma_{\mu\lambda,\beta} - \gamma_{\beta\lambda,\mu})k^\beta + \frac{1}{2}\gamma_{\alpha\mu}k^\alpha\,, \tag{31}
$$

and of the second derivative

$$
\Phi_{G,\mu\nu} = k_\alpha\Gamma^\alpha_{\mu\nu}\,, \tag{32}
$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffell symbols of the second type.
For the Lense–Thirring metric and to $\mathcal{O}(\gamma_{\mu\nu})$, these are

$$
\begin{aligned}
\Gamma^0_{00} &= 0\,, \quad \Gamma^0_{0i} = \varphi_{,i}\,, \quad \Gamma^0_{ij} = \frac{1}{2}(h_{i,j} + h_{j,i})\,, \\
\Gamma^i_{00} &= \varphi_{,i}\,, \quad \Gamma^i_{0j} = \frac{1}{2}(h_{j,i} - h_{i,j})\,, \quad \Gamma^i_{jk} = \delta^j_k\varphi_{,i} - \delta^i_j\varphi_{,k} - \delta^i_k\varphi_{,j}\,.
\end{aligned} \tag{33}
$$

In the geometrical optics approximation $|\partial_i\gamma_{\mu\nu}| \ll |k\gamma_{\mu\nu}|$, where k is the momentum of the particle, the geometrical phase Φ_G is sufficient to reproduce the classical angle of deflection, as it should, but also some effects due to the angular velocity of rotation of the source.

The deflection angle δ is defined by

$$
\tan\delta = \frac{\sqrt{-g_{ij}p^i_\perp p^j_\perp}}{p_\parallel} \simeq \frac{|\mathbf{p}_\perp|}{k_\parallel}\,, \tag{34}
$$

where $k_\parallel = p_\parallel$ is the unperturbed momentum and $|\mathbf{p}_\perp| = \sqrt{-\eta_{ij}p^i_\perp p^j_\perp}$, for $p^i_\perp \sim \mathcal{O}(\gamma_{\mu\nu})$.

It follows from Equations (7) and (10) that, once $\Psi_0(x)$ is chosen to be a plane wave solution of the flat spacetime Dirac equation, the geometrical phase of a particle of four-momentum k^μ is given by

$$
v(x) = -k_\alpha x^\alpha - \Phi_G(x)\,, \tag{35}
$$

where $\hat{\Phi}_G\Psi_0 = \Phi_G\Psi_0$ and

$$
\Phi_G(x) = -\frac{1}{4}\int_P^x dz^\lambda\left[\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)\right]\left((x^\alpha - z^\alpha)k^\beta - (x^\beta - z^\beta)k^\alpha\right) + \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}k^\alpha\,. \tag{36}
$$

The components of $\mathbf{p}_\perp$ can be determined from the equation

$$
\begin{aligned}
p_i = \frac{\partial v}{\partial x^i} &= -k_i - \Phi_{G,i} = \\
&= -k_i - \frac{1}{2}\gamma_{\alpha i}(x)k^\alpha + \frac{1}{2}\int_P^x dz^\lambda(\gamma_{i\lambda,\beta}(z) - \gamma_{\beta\lambda,i}(z))k^\beta\,.
\end{aligned} \tag{37}
$$

We consider the two cases of propagation along the z-axis, which is parallel to the angular momentum of the source, and along the x-axis, orthogonal to it. In both instances, the fermions are assumed to be ultrarelativistic, i.e., $dz^0 \simeq dz(1+m^2/2E^2), E \simeq k(1+m^2/2E^2)$.

When motion is along the z-direction $\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (i.e., $\phi_{1,2}$ are eigenstates of σ^3).

We consider fermions starting from $z = -\infty$ with impact parameter $b \geq R$ and propagating along $x = b, y = 0$. We find

$$\begin{aligned} p_1 &= -\frac{1}{2}\left[\int_{-\infty}^{z} dz^0 \gamma_{00,1} k^0 + \int_{-\infty}^{z} dz^3 \gamma_{33,1} k^3\right] \\ &= -2k\left(1+\frac{m^2}{2E^2}\right)\int_{-\infty}^{z} \varphi_{,1} dz\,, \\ p_2 &= -\frac{1}{2}\gamma_{02}k^0 + \frac{1}{2}\int_{-\infty}^{z} dz^0 \gamma_{20,3} k^3 = 0 \end{aligned} \tag{38}$$

and

$$\begin{aligned} (p_\perp)^1 &= g^{1\mu} p_\mu \simeq -p_1 = -\frac{2GMk}{b}\left(1+\frac{m^2}{2E^2}\right)\left(1+\frac{z}{r}\right), \\ (p_\perp)^2 &= g^{2\mu} p_\mu \simeq h_2 E = -\frac{4GMR^2\omega bk}{5r^3}\left(1+\frac{m^2}{2E^2}\right). \end{aligned} \tag{39}$$

We finally obtain

$$\delta = \frac{2GM}{b}\left(1+\frac{m^2}{2E^2}\right)\sqrt{\left(1+\frac{z}{r}\right)^2 + \left(\frac{2R^2b^2\omega}{5r^3}\right)^2}, \tag{40}$$

which is the deflection predicted by general relativity for photons, with corrections due to the fermion mass and to ω. In the limit $z \to \infty$ Equation (40) reduces to

$$\delta = \frac{4GM}{b}\left(1+\frac{m^2}{2E^2}\right). \tag{41}$$

When the fermions propagate along x, the deflection angle is

$$\delta = \frac{2GM}{b}\left(1-\frac{2R^2\omega}{5b}\right)\left(1+\frac{m^2}{2E^2}\right)\left(1+\frac{x}{r}\right). \tag{42}$$

The first term is just that predicted by general relativity.

Contrary to the case of propagation along z, the contribution of ω does not vanish in the limit $x \to \infty$. In fact, in this limit we get

$$\delta = \frac{4GM}{b}\left(1-\frac{2R^2\omega}{5b}\right)\left(1+\frac{m^2}{2E^2}\right). \tag{43}$$

4. Neutrino Helicity Transitions

In what follows, it is convenient to write the left and right neutrino wave functions in the form

$$\Psi_0(x) = \nu_{0L,R} e^{-ik_\alpha x^\alpha} = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \nu_{L,R} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{E+m}\nu_{L,R} \end{pmatrix} e^{-ik_\alpha x^\alpha}, \tag{44}$$

where $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ represents the Pauli matrices. $\nu_{L,R}$ are eigenvectors of $(\boldsymbol{\sigma}\cdot\mathbf{k})$ corresponding to negative and positive helicity and $\bar{\nu}_{0\,L,R}(k) \equiv \nu^{\dagger}_{0\,L,R}(k)\gamma^{\hat{0}}$, $\nu^{\dagger}_{0\,L,R}(k)\nu_{0\,L,R}(k) = 1$. This notation already takes into account the fact that if $\nu_{\pm}$ are the helicity states, then we have $\phi_2 \simeq \nu_-$, $\phi_1 \simeq \nu_+$ for relativistic neutrinos. The propagation is *in vacuo*.

In general, the spin precesses during the motion of the neutrino. This can be expected because of the presence of Φ_s in Φ_T.

We now study the helicity flip of one flavor neutrinos as they propagate in the gravitational field produced by a rotating mass. The neutrino state vector can be written as

$$|\psi(\lambda)\rangle = \alpha(\lambda)|\nu_R\rangle + \beta(\lambda)|\nu_L\rangle\,, \tag{45}$$

where $|\alpha|^2 + |\beta|^2 = 1$ and λ is an affine parameter along the world-line. In order to determine α and β, we can write Equation (10) as

$$|\psi(\lambda)\rangle = \hat{T}(\lambda)|\psi_0(\lambda)\rangle\,, \tag{46}$$

where

$$\hat{T} = -\frac{1}{2m}\left(-i\gamma^{\mu}(x)\mathcal{D}_{\mu} - m\right)e^{-i\Phi_T}\,, \tag{47}$$

and $|\psi_0(\lambda)\rangle$ is a plane wave solution of Equation (9). The latter can be written as

$$|\psi_0(\lambda)\rangle = e^{-ik\cdot x}\left[\alpha(0)|\nu_R\rangle + \beta(0)|\nu_L\rangle\right]\,. \tag{48}$$

$|\psi(\lambda)\rangle$ should also be normalized. However, this is unnecessary, because it is shown below that $\alpha(\lambda)$ is already of $\mathcal{O}(\gamma_{\mu\nu})$ and can only produce higher order terms. From Equations (45), (46), and (48) we obtain

$$\alpha(\lambda) = \langle\nu_R|\psi(\lambda)\rangle = \alpha(0)\langle\nu_R|\hat{T}|\nu_R\rangle + \beta(0)\langle\nu_R|\hat{T}|\nu_L\rangle\,. \tag{49}$$

An equation for β can be derived in an entirely similar way.

If we consider neutrinos which are created in the left-handed state, then $|\alpha(0)|^2 = 0$, $|\beta(0)|^2 = 1$, and we obtain

$$P_{L\to R} = |\alpha(\lambda)|^2 = \left|\langle\nu_R|\hat{T}|\nu_L\rangle\right|^2 = \left|\int_{\lambda_0}^{\lambda}\langle\nu_R|\dot{x}^{\mu}\partial_{\mu}\hat{T}|\nu_L\rangle d\lambda\right|^2\,, \tag{50}$$

where $\dot{x}^{\mu} = k^{\mu}/m$. As remarked in [64], $\dot{x}^{\mu}$ need not be a null vector if we assume that the neutrino moves along an "average" trajectory. We also find, to lowest order,

$$\begin{aligned}
\partial_{\mu}\hat{T} &= \frac{1}{2m}\left(-i2m\Phi_{G,\mu} - i(\gamma^{\hat{\alpha}}k_{\alpha} + m)\Phi_{s,\mu} + \gamma^{\hat{\alpha}}(h^{\beta}_{\hat{\alpha},\mu}k_{\beta} + \Phi_{G,\alpha\mu})\right) \\
\Phi_{s,\lambda} &= \Gamma_{\lambda}\,, \quad \Phi_{G,\alpha\mu} = k_{\beta}\Gamma^{\beta}_{\alpha\mu}\,, \quad \nu_0^{\dagger}(\gamma^{\hat{\alpha}}k_{\alpha} + m) = 2E\nu_0^{\dagger}\gamma^{\hat{0}}\,,
\end{aligned} \tag{51}$$

where $\Gamma^{\beta}_{\alpha\mu}$ are the usual Christoffel symbols, and

$$\langle\nu_R|\dot{x}^{\mu}\partial_{\mu}\hat{T}|\nu_L\rangle = \frac{E}{m}\left[-i\,\frac{k^{\lambda}}{m}\bar{\nu}_R\Gamma_{\lambda}\nu_L + \frac{k^{\lambda}k_{\mu}}{2mE}\,(h^{\mu}_{\hat{\alpha},\lambda} + \Gamma^{\mu}_{\alpha\lambda})\nu_R^{\dagger}\gamma^{\hat{\alpha}}\nu_L\right]. \tag{52}$$

In order to solve the evolution equations for α and β and complete the equations describing this two-dimensional qubit, one also needs the terms $\langle\nu_L|\dot{x}^{\mu}\partial_{\mu}\hat{T}|\nu_L\rangle$ and $\langle\nu_R|\dot{x}^{\mu}\partial_{\mu}\hat{T}|\nu_R\rangle$ of the usual qubit matrix M. In what follows, we compute the probability amplitude Equation (52) for neutrinos propagating along the z and the x directions explicitly.

For propagation along the z-axis, we have $k^0 = E$ and $k^3 \equiv k \simeq E(1 - m^2/2E^2)$ and we choose $y = 0$, $x = b$. We find

$$\begin{aligned} -i\frac{k^\lambda}{m}\bar{\nu}_R\Gamma_\lambda\nu_L &= \frac{k}{m}\varphi_{,1} + i\frac{m}{4E}h_{2,3}\,, \\ \frac{k^\lambda k_\mu}{2mE}\,(h^\mu_{\hat{\alpha},\lambda} + \Gamma^\mu_{\alpha\lambda})\nu_R^\dagger\gamma^{\hat{\alpha}}\nu_L &= -\frac{k}{2m}\left(1+\frac{k^2}{E^2}\right)\frac{GM}{2b}\,. \end{aligned} \tag{53}$$

Summing up, and neglecting terms of $\mathcal{O}((m/E)^2)$, Equation (52) becomes

$$\langle\nu_R|\dot{x}^\mu\partial_\mu\hat{T}|\nu_L\rangle = \frac{1}{2}\varphi_{,1} + \frac{i}{4}h_{2,3}\,. \tag{54}$$

As a consequence

$$\frac{d\alpha}{dz} \simeq \frac{m}{E}\frac{d\alpha}{d\lambda} = \frac{m}{E}\left(\frac{1}{2}\varphi_{,1} + \frac{i}{4}h_{2,3}\right)\,, \tag{55}$$

and the probability amplitude for the $\nu_L \to \nu_R$ transition is of $\mathcal{O}(m/E)$, as expected.

Integrating Equation (55) from $-\infty$ to z, yields

$$\begin{aligned} \alpha &\simeq \frac{m}{E}\left[\frac{1}{2}\int_{-\infty}^{z} dz\varphi_{,1} + \frac{i}{4}h_2(z)\right] \\ &= \frac{m}{E}\frac{GM}{2b}\left[1+\frac{z}{r} - i\frac{2\omega R^2 b^2}{5r^3}\right]\,. \end{aligned} \tag{56}$$

It also follows that

$$P_{L\to R}(-\infty, z) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left[\left(1+\frac{z}{r}\right)^2 + \left(\frac{2\omega b^2 R^2}{5r^3}\right)^2\right]\,. \tag{57}$$

In this qubit, the first term in Equation (57) comes from the mass of the gravitational source. The second from the source's angular momentum and vanishes for $r \to \infty$ because the contribution from $-\infty$ to 0 exactly cancels that from 0 to $+\infty$. In fact, if we consider neutrinos propagating from 0 to $+\infty$, we obtain

$$P_{L\to R}(0, +\infty) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left[1 + \left(\frac{2\omega R^2}{5b}\right)^2\right]\,. \tag{58}$$

According to semiclassical spin precession equations [65], there should be no spin motion because spin and $\vec{\omega}$ are parallel. The probabilities Equations (57) and (58) mark therefore a departure from expected results. They yield however results that are small of second order. Both expressions vanish for $m \to 0$, as it should because helicity is conserved [66]. It is interesting to observe that spin precession also occurs when ω vanishes [67,68]. In the case of Equation (57) the mass contribution is larger when $b < (r/R)\sqrt{\frac{5r}{2\omega}}$, which, close to the source, with $b \sim r \sim R$, becomes $R\omega < 5/2$ and is always satisfied. In the case of Equation (58), the rotational contribution is larger if $b/R < 2\omega R/5$ which restricts the region of dominance to a strip about the z-axis in the equatorial plane, if the source is compact and ω is relatively large.

In proximity of the source where the gravitational field is stronger and $r \sim b \sim R$ the evolution equations for α and β are $\frac{d\alpha}{d\lambda} \approx \tilde{D}\beta$ and $\frac{d\beta}{d\lambda} \approx \tilde{D}^*\alpha$, $\tilde{D}$ is almost constant, and α and β oscillate with frequency

$$\Omega = \sqrt{\tilde{D}\tilde{D}^*} \approx \frac{m}{E}\frac{GM}{2b^2}\left\{1+\left(\frac{6\omega b}{5}\right)^2\right\}^{\frac{1}{2}}\,. \tag{59}$$

The contribution of the source rotation is therefore $(\omega b/c)^2 \sim (v/c)^2$. The helicity oscillations discussed are, in principle, relevant in astrophysics because right-handed neutrinos are considered sterile. This point is discussed in the next section.

When propagation is along x, we put $k^0 = E$, $k^1 \equiv k \simeq E(1 - m^2/2E^2)$. The calculation can be simplified by assuming that the motion is in the equatorial plane with $z = 0$, $y = b$. We then have

$$\begin{aligned} -i\frac{k^\lambda}{m}\bar{\nu}_R\Gamma_\lambda\nu_L &= i\frac{k}{m}\varphi_{,2} + i\frac{E^2+k^2}{4mE}h_{1,2} - i\frac{E^2-k^2}{4mE}h_{2,1}\,, \\ \frac{k^\lambda k_\mu}{2mE}\,(h^\mu_{\hat{\alpha},\,\lambda} + \Gamma^\mu_{\alpha\lambda})\nu_R^\dagger\gamma^{\hat{\alpha}}\nu_L &= -i\frac{k}{2m}\left(1+\frac{k^2}{E^2}\right)\varphi_{,2} - i\frac{k^2}{2mE}h_{1,2}\,. \end{aligned} \tag{60}$$

Summing up, and neglecting terms of $\mathcal{O}(m/E)^2$, Equation (52) becomes

$$\langle \nu_R|\dot{x}^\mu\partial_\mu\hat{T}|\nu_L\rangle = \frac{i}{2}\varphi_{,2} + \frac{i}{4}(h_{1,2} - h_{2,1})\,. \tag{61}$$

The contributions to $\mathcal{O}((E/m)^2)$ again vanish and we get

$$\frac{d\alpha}{dx} \simeq \frac{m}{E}\frac{d\alpha}{d\lambda} = \frac{m}{E}\left[\frac{i}{2}\varphi_{,2} + \frac{i}{4}(h_{1,2} - h_{2,1})\right] \sim \mathcal{O}(m/E)\,. \tag{62}$$

Integrating Equation (62) from $-\infty$ to x, we obtain

$$\alpha \simeq i\frac{m}{E}\frac{GM}{2b}\left(1 - \frac{2\omega R^2}{5b}\right)\left(1 + \frac{x}{r}\right) \tag{63}$$

and

$$P_{L\to R}(-\infty, x) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left(1 - \frac{2\omega R^2}{5b}\right)^2\left(1 + \frac{x}{r}\right)^2\,. \tag{64}$$

Obviously, the contribution of M is the same as for z-axis propagation. However, the two cases differ substantially in the behavior of the term containing ω. In this case, in fact, the term does not vanish for $r \to \infty$. If we consider neutrinos generated at $x = 0$ and propagating to $x = +\infty$, we find

$$P_{L\to R}(0, +\infty) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left(1 - \frac{2\omega R^2}{5b}\right)^2\,. \tag{65}$$

The M term is larger when $\frac{2\omega R^2}{5b} < 1$. At the poles $b \sim R$ and the M term dominates because the condition $\omega R < 5/2$ is always satisfied. The angular momentum contribution prevails in proximity of the equatorial plane. The transition probability vanishes at $b = 2\omega R^2/5$.

An altogether different type of qubit is represented by neutrino flavor oscillations. They have been discussed in the context of the Lense–Thirring metric in [6]. The qubit frequency is in this case proportional to $\Delta m^2 = m_2^2 - m_1^2$, where m_1 and m_2 are the masses of the neutrino mass eigenstates.

5. Neutrino Conversion in Supernovae

The results of the previous section may be applied to the propagation of a beam of neutrinos in vacuo. The presence of a medium is realized by means of a potential V. The neutrinos are massive and may therefore have a magnetic moment μ. In the presence of an external magnetic field $\vec{B}$ and of the Mashhoon term proportional to the angular velocity of the source $\vec{\Omega}$, the evolution equations become [33,69]

$$\begin{aligned} i\frac{d\nu}{dz} &= \frac{m^2}{2E} - \frac{1}{2}\left(P_+\vec{\Omega}\cdot\vec{\sigma}P_- + P_-\vec{\Omega}\cdot\vec{\sigma}P_+\right)\nu \\ &\quad -\mu\left(P_+\vec{B}\cdot\vec{\sigma}P_- + P_-\vec{B}\cdot\vec{\sigma}P_+\right)\nu, \end{aligned} \tag{66}$$

where $\nu = \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix}$ and $P_\pm = (1 \pm \sigma_3)/2$ are the R and L projection operators. Equation (66) leads to neutrino oscillations.

The frequency of oscillation is then $\Omega_\perp/2$ where $\Omega_\perp$ is the component of the angular velocity normal to the neutrino trajectory. In particular, if a beam of neutrinos consists of $N_L(0)$ particles at $z = 0$, the relative numbers of ν_L and ν_R at z will be

$$N_L(z) = N_L(0)\cos^2\left(\frac{\Omega_\perp z}{2}\right), \qquad N_R(z) = N_R(0)\sin^2\left(\frac{\Omega_\perp z}{2}\right), \tag{67}$$

These oscillations are interesting because the ν_R's, if they exist, do not interact. They would therefore provide an energy dissipation mechanism with possible astrophysical implications. The conversion rate is not large for galaxies and white dwarfs. In fact one can obtain from Equation (67) $N_R \sim 10^{-6} N_L(0)$ for galaxies of size L for which $\Omega_\perp L \sim 200\,\mathrm{km/s}$. Similarly, for white dwarfs for which $\Omega_\perp \sim 1.0\,s^{-1}$, one finds $N_R \sim 10^{-4} N_L(0)$. On the other hand, the ν_L's diffuse out of a canonical neutron star in a time 1 to 10 s, during which they travel a maximum distance 3×10^9 cm between collisions. This and the fact that for a millisecond pulsar the conversion rate $\nu_L \to \nu_R$ is ~ 0.5 at distances $L \sim 5 \times 10^6$ cm suggest that the dynamics of the star could be affected by such a cooling mechanism. Indeed the star may even cool too rapidly at higher rotational speeds for a pulsar to form.

The magnetic moment of the neutrino does not appear in the calculations because magnetic spin-flip rates of magnitude comparable to Equation (67) would require magnetic moments in excess of the value $\mu \sim 10^{-19}\mu_B\left(\frac{m_\nu}{1eV}\right)$ predicted by the standard model.

The behavior of neutrinos in a medium is modified by a potential V that vanishes for ν_R's. In the core of a supernova V can be written as [70]

$$V(\nu_e) = 14\mathrm{eV}\frac{\rho}{\rho_c} y(\vec{r}, t), \tag{68}$$

where $y(\vec{r}, t) \equiv 3Y_e(\vec{r}, t) + 4Y_{\nu_e}(\vec{r}, t) - 1$, the Y's represent the lepton fractions present, and $\rho_c = 4 \times 10^{14}\,\mathrm{g/cm^3}$. For supernovae V can be large, of the order of several electron volts, and rotation may be neglected. Only the acceleration term in Equation (67) need be considered and the effective Hamiltonian then has the form [69]

$$H = \begin{vmatrix} V & \frac{\hbar a_\perp}{2c} \\ \frac{\hbar a_\perp}{2c} & \frac{c^4 \delta m^2}{2E} \end{vmatrix}, \tag{69}$$

where $\delta m^2 \equiv m^2_{\nu_L} - m^2_{\nu_R}$ and $a_\perp$ is the component of the acceleration transverse to the neutrino trajectory. If the initial state is pure ν_L and the number of particles in this state is N_0 at $z = 0$, then the corresponding numbers of ν_L and ν_R at z are

$$N_L = N_0\left[\cos^2(\tilde{\Omega} z) + \cos^2(2\theta_a)\sin^2(\tilde{\Omega} z)\right], \qquad N_R = N_0 \sin^2(2\theta_a)\sin^2(\tilde{\Omega} z), \tag{70}$$

where

$$\sin^2(2\theta_a) \equiv \frac{\left(\frac{\hbar a_\perp}{2c}\right)^2}{\left(\frac{\hbar a_\perp}{2c}\right)^2 + 4\left(V - \frac{c^4 \delta m^2}{2E}\right)}; \quad \tilde{\Omega} \equiv \sqrt{\frac{1}{c^2}\left(\frac{a_\perp}{2c}\right)^2 + \frac{1}{4\hbar^2 c^2}\left(V - \frac{c^4 \delta m^2}{2E}\right)^2}. \tag{71}$$

If $\frac{1}{2}\left(V - \frac{c^4 \delta m^2}{2E}\right) > \frac{\hbar a_\perp}{2c}$, spin precession is strongly suppressed and the flux of particles at z consists mainly of ν_L's. The conversion takes place at resonance if $V = \frac{c^4 \delta m^2}{2E}$.

Summarizing, the components of acceleration transverse to the particle path couple to its spin. This and the Mashhoon term applied to massive neutrinos produce $\nu_L \leftrightarrow \nu_R$ oscillations, which may have macroscopic effects if the ν_R's are sterile, as frequently assumed. In fact, $\nu_L \rightarrow \nu_R$ conversion by rotation-spin coupling may help to explain why pulsars of period shorter than a millisecond are relatively rare.

6. Spin Currents

The realization that the flow of spin angular momentum can be separated from that of charge has recently stimulated intense interest in fundamental spin physics [71], particularly in view of its applications [43,72,73].

In this section we study the generation and control of spin currents by rotation and acceleration [3]. In this context the fundamental tool still is the covariant Dirac equation.

We use the first order solutions of Equation (1) that have the form

$$\Psi(x) = \hat{T}(x)\Psi_0(x)\,, \tag{72}$$

where $\Psi_0(x)$ is a solution of Equation (8) and the operator $\hat{T}$ is given by Equation (10) or Equation (11).

When acceleration and rotation are present, $\gamma_{\mu\nu}$ is given by [32,44]

$$\gamma_{00} \approx 2(\mathbf{a}\cdot\mathbf{x}) + (\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{\Omega}^2\mathbf{x}^2 + (\mathbf{\Omega}\cdot\mathbf{x})^2\,, \gamma_{0i} = -(\mathbf{\Omega}\times\mathbf{x})_i\,, \gamma_{ij} = \eta_{ij}\,, \tag{73}$$

where $\mathbf{a}$ and $\mathbf{\Omega}$ represent acceleration and rotation respectively. To first order the tetrad is given by

$$\begin{aligned} e^{\mu}_{\hat{\alpha}} &\approx \delta^{\mu}_{\alpha} + h^{\mu}_{\hat{\alpha}}\,, h^{0}_{\hat{0}} = -\mathbf{a}\cdot\mathbf{x}\,, h^{0}_{\hat{i}} = 0\,, h^{k}_{\hat{i}} = 0\,, h^{i}_{\hat{0}} = -\varepsilon^{ijk}\Omega_j x_k\,, \\ h^{\hat{0}}_{0} &= \mathbf{a}\cdot\mathbf{x}\,, h^{\hat{k}}_{0} = \varepsilon_{ijk}\Omega^i x^j\,, h^{\hat{0}}_{i} = 0, h^{\hat{k}}_{i} = \delta^k_i\,. \end{aligned} \tag{74}$$

from which the spinorial connection can be calculated in the usual way. The result is $\Gamma_i = 0$ and $\Gamma_0 = -\frac{1}{2}a_i\sigma^{\hat{0}\hat{i}} - \frac{1}{2}\mathbf{\Omega}\cdot\mathbf{œ}I$.

For electrons, u_1 corresponds to the choice $\phi = \phi_1$ and u_2 to $\phi = \phi_2$. Substituting into Equation (9), one finds the spinors u_1 and u_2. These are not eigenspinors of the matrix $\Sigma^3 = \sigma^3 I$ and do not, therefore, represent the spin components in the z-direction. They become however eigenspinors of Σ^3 when $k^1 = k^2 = 0$, or when $\mathbf{k} = 0$ (electron rest frame).

The appropriate way to determine whether there is transfer of angular momentum between the external non-inertial field and the electron spin is to use the third rank spin current tensor [74]

$$S^{\rho\mu\nu} = \frac{1}{4im}\left[\left(\nabla^{\rho}\bar{\Psi}\right)\sigma^{\mu\nu}(x)\Psi - \bar{\Psi}\sigma^{\mu\nu}(x)\left(\nabla^{\rho}\Psi\right)\right]\,, \tag{75}$$

that in Minkowski space satisfies the conservation law $S^{\rho\mu\nu}{}_{,\rho} = 0$ when all $\gamma_{\alpha\beta}(x)$ vanish and yields in addition the expected result $S^{\rho\mu\nu} = S^{0\mu\nu}$ in the rest frame of the particle. Writing $\sigma^{\mu\nu}(x) \approx \sigma^{\hat{\mu}\hat{\nu}} + h^{\mu}_{\hat{\tau}}\sigma^{\hat{\tau}\hat{\nu}} + h^{\nu}_{\hat{\tau}}\sigma^{\hat{\mu}\hat{\tau}}$, using the relation $\Phi_{G,\mu\nu} = k_\alpha\Gamma^{\alpha}_{\mu\nu}$ and substituting Equations (72) and (11) into Equation (75), one obtains, to $\mathcal{O}(\gamma_{\alpha\beta})$,

$$\begin{gathered} S^{\rho\mu\nu} = \frac{1}{16im^3}\bar{u}_0\left\{8im^2k^{\rho}\sigma^{\hat{\mu}\hat{\nu}} + 8imk^{\rho}h^{[\mu}_{\hat{\tau}}\sigma^{\hat{\tau}\hat{\nu}]} + \right. \\ 4imk^{\rho}\left(\Phi_{G,\alpha} + k_\sigma h^{\sigma}_{\hat{\alpha}}\right)\left\{\sigma^{\hat{\mu}\hat{\nu}}, \gamma^{\hat{\alpha}}\right\} - 8imk^{\rho}\Phi_G k^{[\mu}\gamma^{\hat{\nu}]} + \\ 4mk^{\rho}k_\alpha\left[\sigma^{\hat{\mu}\hat{\nu}}, \left(\gamma^{\hat{\alpha}}\Phi_S - \gamma^{\hat{0}}\Phi_S^{+}\gamma^{\hat{0}}\gamma^{\hat{\alpha}}\right)\right] + 4m^2k^{\rho}\left[\sigma^{\hat{\mu}\hat{\nu}}, \left(\Phi_S - \gamma^{\hat{0}}\Phi_S^{+}\gamma^{\hat{0}}\right)\right] - \\ 8m^2k^{\rho}h^{0}_{\hat{\alpha}}\left[\gamma^{\hat{0}}, \left[\sigma^{\hat{0}\hat{\alpha}}, \sigma^{\hat{\mu}\hat{\nu}}\right]\right] - 8im^2k_\sigma\left(\Gamma^{\sigma}_{\alpha\beta}\eta^{\beta\rho} + \partial^{\rho}h^{\sigma}_{\hat{\alpha}}\right)\eta^{\alpha[\mu}\gamma^{\hat{\nu}]} + \end{gathered} \tag{76}$$

$$8im^2\partial^\rho\Phi_G\left(4m\sigma^{\hat{\mu}\hat{\nu}}-2ik^{[\mu}\gamma^{\hat{\nu}]}\right)+4im^2\gamma^{\hat{0}}\Gamma^{\rho+}\gamma^{\hat{0}}\left\{\left(\gamma^{\hat{\alpha}}k_\alpha+m\right),\sigma^{\hat{\mu}\hat{\nu}}\right\}\Gamma^\rho\right\}u_0\,.$$

It is therefore possible to separate $S^{\rho\mu\nu}$ in inertial and non-inertial parts. The first term on the r.h.s. of Equation (76) gives the result expected when $\vec{k}=0$ and the external field vanishes. From Equation (76) one finds

$$\partial_\rho S^{\rho\mu\nu}=\frac{1}{16im^3}\bar{u}_0\left\{8imk^\rho\partial_\rho h^{[\mu}_{\hat{\tau}}\sigma^{\hat{\tau}\hat{\nu}]}-8imk^\rho\Phi_{G,\rho}k^{[\mu}\gamma^{\hat{\nu}]}+\right. \tag{77}$$

$$4imk^\rho\left(k_\sigma\Gamma^\sigma_{\alpha\rho}+\partial_\rho h^\sigma_{\hat{\alpha}}k_\sigma\right)\left\{\sigma^{\hat{\mu}\hat{\nu}},\gamma^{\hat{\alpha}}\right\}+4mk^\rho k_\alpha\left[\sigma^{\hat{\mu}\hat{\nu}},\left(\gamma^{\hat{\alpha}}\Gamma_\rho-\gamma^{\hat{0}}\Gamma^+_\rho\gamma^{\hat{0}}\gamma^{\hat{\alpha}}\right)\right]+$$

$$4m^2k^\rho\left[\sigma^{\hat{\mu}\hat{\nu}},\left(\Gamma_\rho-\gamma^{\hat{0}}\Gamma^+_\rho\gamma^{\hat{0}}\right)\right]+8m^2k^\rho\partial_\rho h^0_{\hat{\alpha}}\left[\gamma^{\hat{0}},\left[\sigma^{\hat{0}\hat{\alpha}},\sigma^{\hat{\mu}\hat{\nu}}\right]\right]-$$

$$8im^2k_\sigma\partial^\rho\Gamma^\sigma_{\alpha\rho}\eta^{\alpha[\mu}\gamma^{\hat{\nu}]}+8im^2k_\sigma\Gamma^\sigma_{\rho\tau}\eta^{\tau\rho}\left(4m\sigma^{\hat{\mu}\hat{\nu}}-2ik^{[\mu}\gamma^{\hat{\nu}]}\right)+$$

$$\left.8im^2k^\alpha\Gamma^\rho_{\alpha\rho}\sigma^{\hat{\mu}\hat{\nu}}+8im^2k^\rho\Gamma^\mu_{\alpha\rho}\sigma^{\hat{\alpha}\hat{\nu}}-8im^2k^\rho\Gamma^\nu_{\alpha\rho}\sigma^{\hat{\alpha}\hat{\mu}}\right\}u_0\,,$$

where terms containing $\Gamma_{0,0}=0$ and $\partial_\alpha\partial_\beta h^\mu_{\hat{\nu}}=0$ have been eliminated. It therefore follows that the external field invalidates the simple conservation law $\partial_\rho S^{\rho\mu\nu}=0$ and that there is in this case continual interchange between spin and orbital angular momentum. The result is entirely similar to that found for external electromagnetic fields [74]. Thus, in principle, one can use non-inertial fields to generate spin currents. In the rest frame of the particle and when $\mathbf{\Omega}=(0,0,\Omega)$, one finds

$$\partial_\rho S^{\rho\mu\nu}=\partial_i S^{i12}=\frac{1}{2}\left(\Gamma^\rho_{0\rho}+\Gamma^1_{10}+\Gamma^2_{20}\right)\bar{u}_0\sigma^{\hat{1}\hat{2}}u_0=\frac{E+m}{2E}\frac{\Omega a_2 x}{1+\mathbf{a}\cdot\mathbf{x}}\,, \tag{78}$$

and $\partial_i S^{i13}=\partial_i S^{i23}=0$. In Equation (78) u_0 corresponds to u_1. The direct coupling of the non-inertial field to the particle's spin current violates the law $\partial_\rho S^{\rho\mu\nu}=0$. Conservation is restored if the parameters Ω, or a_2, or both vanish.

Qubits appear in the actual spin motion. In general, transfer of angular momentum between external and non-inertial fields occurs when the operator $\hat{T}$ has some non-diagonal matrix elements. If in fact at time $t=0$ a beam of electrons is entirely of the u_2 variety, at time t the fraction of u_1 is $|\langle u_1|\hat{T}|u_2\rangle|^2$. The last expression becomes, along the electron world line,

$$P_{2\to 1}=\left|\langle u_1|\hat{T}|u_2\rangle\right|^2=\left|\int_{\lambda_0}^{\lambda}\langle u_1|\dot{x}^\mu\partial_\mu\hat{T}|u_2\rangle d\lambda\right|^2, \tag{79}$$

where, as usual, $\dot{x}^\mu=k^\mu/m$ and λ is the affine parameter along the world line. From [6]

$$\partial_\nu\hat{T}=\frac{1}{2m}\left\{h^\mu_{\hat{\alpha};\nu}\gamma^{\hat{\alpha}}k_\mu+\gamma^{\hat{\mu}}\Phi_{G,\mu\nu}-2im\left(\Phi_{G,\nu}+\Gamma_\nu-eA_\nu\right)\right\}\,, \tag{80}$$

one can see that

$$\langle u_1|\frac{k^\nu}{m}\partial_\nu\hat{T}|u_2\rangle=-i\frac{k^0}{m}\langle u_1|\Gamma_0|u_2\rangle=-i\frac{k^0}{m}\langle u_1|\left\{-\frac{1}{2}a_i\sigma^{\hat{0}\hat{i}}-\frac{1}{2}\Omega_i\sigma^i I\right\}|u_2\rangle. \tag{81}$$

A useful way to visualise the spin motion under the action of rotation and acceleration follows from the Mashhoon term $H_M=-\mathbf{\Omega}\cdot\mathbf{s}$, where $\mathbf{s}=\frac{\boldsymbol{\sigma}}{2}$, and from $\frac{1}{2}a_i\sigma^{\hat{0}\hat{i}}=\frac{i}{2}a_i\alpha^{\hat{i}}$. The two interaction terms lead to the first order equation of motion [75]

$$\frac{d\mathbf{s}}{dt}=\mathbf{s}\times(\mathbf{\Omega}+\mathbf{v}\times\mathbf{a})\;. \tag{82}$$

Note that A_μ, introduced by writing $\Phi_T=\Phi_S+\Phi_G+\Phi_{EM}$, where $\Phi_{EM}=e\int_P^x dz^\lambda A_\lambda(z)$, does not contribute to Equation (81) because $\langle u_1|u_2\rangle=0$. Note also that the terms $ie(h^\mu_{\hat{\alpha}}\gamma^{\hat{\alpha}}k_\mu+\gamma^{\hat{\mu}}\Phi_{G,\mu})A_\nu\frac{k^\nu}{m}$ drop out, in the particle rest frame, on account of $\langle u_1|\gamma^{\hat{\mu}}|u_2\rangle=0$. No mixed effects of first order in rotation or acceleration and first order in the electromagnetic field are therefore present in this calculation. This applies to all terms containing the magnetic field $\mathbf{B}$, like the Zeeman term, and electric fields, like the spin-orbit interaction, that are present in the lowest order Dirac Hamiltonian that can be derived from Equation (1) [44]. To $\mathcal{O}(\gamma_{\mu\nu})$, contributions to Equation (81) from the electromagnetic field are present in the actual determination of the electron's path, as stated above.

From Equation (81) one obtains

$$\begin{aligned} \frac{2m}{iE}\langle u_1|\frac{k^\nu}{m}\partial_\nu \hat{T}|u_2\rangle &= -i\frac{k^3}{E}a_1 - \frac{k^3}{E}a_2 + i\frac{k^1-ik^2}{E}a_3 \\ &+ \Omega^3\frac{k^3}{E}\frac{-k^1+ik^2}{E+m} \\ &+ \Omega^1\frac{E+m}{2E}\left(1+\frac{(k^3)^2}{(E+m)^2}-\frac{(k^1-ik^2)^2}{(E+m)^2}\right) \\ &- i\Omega^2\frac{E+m}{2E}\left(1+\frac{(k^3)^2}{(E+m)^2}+\frac{(k^1-ik^2)^2}{(E+m)^2}\right) \equiv A_{12}\,, \end{aligned} \tag{83}$$

where $k^0 \equiv E$. The parameter k_μ corresponds to the electron four-momentum when $\mathbf{\Omega} = 0$ and $\mathbf{a} = 0$.

Some general conclusions can be drawn from Equation (83).

1. If $k^3 = 0$, the particles move in the (x, y)-plane. If, in addition, $\Omega^1 = \Omega^2 = 0$, then $A_{12} \neq 0$ only if $a_3 \neq 0$.
2. If, however, $\mathbf{a}$ is also due to rotation, the conditions $\Omega_1 = \Omega_2 = 0$ imply $a_3 = 0$ and therefore $A_{12} = 0$. This is the relevant case of motion in the (x, y)-plane with rotation along an axis perpendicular to it. One cannot, therefore, have a rotation-induced spin current in this instance.
3. Even for $\mathbf{k} = 0$ one can have $A_{12} \neq 0$ if one of Ω_1 and Ω_2 does not vanish. This is a direct consequence of the spin rotation interaction or Mashhoon term contained in Equation (1) [19,20,44].

A few examples are discussed below.

Consider an electron wave packet moving along the x-axis of a frame rotating about the same axis. Then $\mathbf{a} = 0$ and the remaining parameters are $\mathbf{k} = (k, 0, 0)$ and $\mathbf{\Omega} = (\Omega, 0, 0)$. While the electron propagates along x, u_1 and u_2 propagate in opposite directions along x because of Equations (81) and (82). For a beam the spin current generated by rotation is therefore $I_s = I_\downarrow - I_\uparrow$, with obvious meaning of the symbols. One finds $P_{2\to1} = |\frac{i\Omega(E+m)}{4m}\int_0^t dt\left(1-\frac{k^2}{(E+m)^2}\right)|^2 = (\frac{\Omega t}{2})^2$ which holds for $t \le 2/\Omega$ on account of the requirement $P_{2\to1} \le 1$.

Consider next a wave packet moving in the plane $z = 0$ itself moving along z with velocity k^3/E, while rotating about the z-axis with $\mathbf{\Omega} = (0, 0, \Omega)$. The other parameters are $\mathbf{k} = (k\cos\Omega t, k\sin\Omega t, k^3)$ and $\mathbf{a} = (-\Omega^2 x, -\Omega^2 y, 0)$. From Equation (83) one finds $A_{12} = (\frac{k^3}{E})(-ia^1 - a^2 + \Omega\frac{-k^1+ik^2}{E+m})$ and $P_{2\to1} = \left[\frac{k^3}{E}\left(\Omega R + \frac{k}{E+m}\right)\sin 2\Omega t\right]^2$, where R is the radius of the circle described by the wave packet in the plane $z = 0$. The motion of the center of mass of the wave packet is helical and so is the motion of the spin components which propagate, however, in opposite directions giving rise to a spin current.

In the case of motion occurring in the plane $z = 0$ ($k^3 = 0$) and $\mathbf{\Omega} = (\Omega, 0, 0)$, one also gets $\mathbf{a} = (-\Omega^2 x, 0, 0)$ and

$$A_{12} = \Omega\frac{E+m}{2E}\left\{1-\frac{(k^1-ik^2)^2}{(E+m)^2}\right\}\,, \tag{84}$$

where $k^1 = k\cos\omega t$, $k^2 = k\sin\omega t$, and $\omega = \frac{eB}{m\gamma}$ is the cyclotron frequency of the electrons along the circular path determined by the constant magnetic field B. Substituting Equations (84) in (79) one finds

$$\begin{aligned} P_{2\to 1} &= \left| i\Omega \frac{E+m}{2E} \int_0^t \left\{ 1 - \frac{k^2(\cos 2\omega t - i\sin 2\omega t)}{(E+m)^2} \right\} dt \right|^2 \\ &= \left(\Omega t \frac{E+m}{4E} \right)^2 \left\{ \frac{\sin^4 \omega t}{\omega^2 t^2} + \left(1 - \frac{k^2}{(E+m)^2} \frac{\sin 2\omega t}{2\omega t} \right)^2 \right\}, \end{aligned} \tag{85}$$

which holds for all t for which $P_{2\to 1} \leq 1$. While both spin up and down electrons move on a circle of radius R about the z-axis, they propagate in opposite directions because of the spin-rotation coupling, thus generating a spin current.

Summarizing, rotation and acceleration can be used to generate and control spin currents. This follows from the covariant Dirac equation and its exact solutions to $\mathcal{O}(\gamma_{\mu\nu})$. To this order, external electromagnetic fields can be taken into account through the particle motion. The transition amplitude for the conversion spin-up to spin-down is proportional to A_{12} and is expressed as a function of $\mathbf{\Omega}$, $\mathbf{a}$, and of the electron four-momentum k_μ before the onset of rotation and acceleration. The same expression suggests criteria for the generation of spin currents and the transfer of momentum and angular momentum to them. No energy can, of course, be transferred from the non-inertial fields to the spin currents as long as $\gamma_{\mu\nu}$ remains stationary.

The particular forms of Equation (83) discussed above provide additional examples of gravitational qubits.

7. Summary

Gravitational qubits are the simplest quantum systems that can be employed to study gravity. They exist in the laboratory and astrophysical conditions. Attention has been focussed on spin-1/2 fermions because of their simple eigenstate structure. Spin-flip transitions occur in nature frequently. We have considered here some of those that are characterized by sustained oscillations.

To the laboratory belong particles rotating in accelerators. Their quantum beats refer to spin oscillations mediated by the Mashhoon spin-rotation interaction. They have been observed and play an important role in important measurement of the anomalous magnetic moment of the muon.

Section 3 is entirely devoted to the calculation of the deflection of fermions in a gravitational background described by the Lense–Thirring metric. The interest is limited here to the action of rotation on the particle spin. The procedure can be also applied to bosons. Basically, the part of the deflection that does not depend on rotation is that predicted by general relativity. Rotation of the source yields in general additional corrections which are due to the particle spin and are therefore quantum mechanical. These, for instance, are present in the deflection Equations (40) and (42), but with a noticeable difference: in the case of Equation (42) the contribution of the source angular momentum does not vanish in the limit $x \to \infty$.

The neutrino helicity oscillations in vacuo described in Section 4 are small, but intriguing because ν_L evolve into ν_R which are sterile. This energy dissipation mode could be relevant to compact astrophysical objects. The introduction in Section 5 of a medium in which neutrinos propagate increases the $\nu_L \to \nu_R$ transition probability if the medium potential is attuned to the difference of the mass squared of the neutrino mass eigenstates. The fact that pulsars of period shorter than a millisecond are rare, lends support to the dissipation mechanism implied by the $\nu_L \to \nu_R$ conversion by rotation-spin coupling.

Spin currents and spin motion are introduced in Section 6. It is shown that gravitational fields violate the usual formulation of the spin conservation law. Some particular qubits are discussed together with the associated spin currents. We also consider the case of fermions accelerated to the maximal acceleration limits contemplated by Caianiello [15–17]. The calculations [76] confirm that continual interchange between spin and angular momentum can occur in this instance, but only if the acceleration is time-dependent. This requires a transfer of energy from a very compact star, or a black hole and the particle. Even in the case, uniform acceleration produces no observable effects on the particle spin, in agreement with [77].

Near-neighbor momenta oscillations of a gas of particles in a gravitational field are discussed in [78].

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References

1. Cai, Y.Q.; Papini, G. Particle interferometry in weak gravitational fields. *Class. Quantum Grav.* **1989**, *6*, 407.
2. Papini, G. Spin–gravity coupling and gravity-induced quantum phases. *Gen. Relativ. Grav.* **2008**, *40*, 1117–1144.
3. Papini, G. Spin currents in non-inertial frames. *Phys. Lett. A* **2013**, *377*, 960–963.
4. Papini, G. Classical and quantum aspects of particle propagation in external gravitational fields. *Int. J. Mod. Phys. D* **2017**, *26*, 1750137.
5. Papini, G. Structured objects in quantum gravity. The external field approximation. *Int. J. Mod. Phys. D* **2018**, *27*, 1850104.
6. Lambiase, G.; Papini, G.; Punzi, R.; Scarpetta, G. Neutrino optics and oscillations in gravitational fields. *Phys. Rev. D* **2005**, *71*, 073011
7. Capozziello, S.; Lambiase, G. Inertial effects on Berry's phase of neutrino oscillations. *Eur. Phys. J. C* **2000**, *16*, 155–159.
8. Papini, G. Radiative processes in external gravitational fields. *Phys. Rev. D* **2010**, *82*, 024041.
9. Papini, G. Perspectives on gravity-induced radiative processes in astrophysics. *Galaxies* **2015**, *3*, 72–83.
10. Papini, G. FERMION–ANTIFERMION MIXING IN GRAVITATIONAL FIELDS. *Mod. Phys. Lett. A* **2013**, *28*, 1350071.
11. Papini, G. Covariance and gauge invariance in relativistic theories of gravity. *Mod. Phys. Lett. A* **2014**, *29*, 1450075.
12. Cai, Y.Q.; Papini, G. Neutrino helicity flip from gravity-spin coupling. *Phys. Rev. Lett.* **1991**, *66*, 1259.
13. Papini, G.; Scarpetta, G.; Feoli, A.; Lambiase, G. Optics of spin-1 particles from gravity-induced phases. *Int. J. Mod. Phys. D* **2009**, *18*, 485–499.
14. Papini, G. Spin-2 particles in gravitational fields. *Phys. Rev. D* **2007**, *75*, 044022.
15. Caianiello, E.R. Is there a maximal acceleration? *Lett. Nuovo Cimento* **1981**, *32*, 65–70.
16. Caianiello, E.R. Quantum and other physics as systems theory. *Riv. Nuovo Cimento* **1992**, *15*, 4.
17. Caianiello, E.R. Maximal acceleration as a consequence of Heisenberg's uncertainty relations. *Lett. Nuovo Cimento* **1984**, *41*, 370–372.
18. Brandt, H.E. Maximal proper acceleration and the structure of spacetime. *Found. Phys. Lett.* **1989**, *2*, 39–58.
19. Mashhoon, B. Limitations of spacetime measurements. *Phys. Lett. A* **1990**, *143*, 176–182.
20. Mashhoon, B. The hypothesis of locality in relativistic physics. *Phys. Lett. A* **1990**, *145*, 147–153.
21. Mashhoon, B. *Black Holes: Theory and Observation*; Hehl, F.W., Kiefer, C., Metzler, R., Eds.; Springer: Berlin/Heidelberg, Germany, 1998; pp. 269–284.
22. Toller, M. Geometries of maximal acceleration. *arXiv* **2003**, arXiv:0312016.
23. Feoli, A.; Lambiase, G.; Papini, G.; Scarpetta, G. Schwarzschild field with maximal acceleration corrections. *Phys. Lett. A* **1999**, *263*, 147–153.
24. Punzi, R.; Schuller, F.P.; Wohlfarth, M.N.R. Geometric obstruction of black holes. *Ann. Phys.* **2007**, *322*, 1335–1372.
25. Rovelli, C.; Vidotto, F. Evidence for maximal acceleration and singularity resolution in covariant loop quantum gravity. *Phys. Rev. Lett.* **2013**, *111*, 091303.
26. Bruneton, J.-P. Notes on several phenomenological laws of quantum gravity. *arXiv* **2013**, arXiv:gr-qc/1308.4044.
27. Cai, Y.-F.; Easson, D.A. Asymptotically safe gravity as a scalar-tensor theory and its cosmological implications.*Phys. Rev. D* **2011**, *84*, 103502.

28. Capozziello, S.; Lambiase, G.; Sakellariadou, M.; Stabile, A.; Stabile, A. Constraining models of extended gravity using Gravity Probe B and LARES experiments. *Phys. Rev. D* **2015**, *91*, 044012.
29. Mashhoon, B. Neutron interferometry in a rotating frame of reference. *Phys. Rev. Lett.* **1988**, *61*, 2639.
30. Mashhoon, B. Mashhoon replies. *Phys. Rev. Lett.* **1992**, *68*, 3812.
31. Mashhoon, B. Gravitational couplings of intrinsic spin. *Class. Quantum Gravity* **2000**, *17*, 2399.
32. Hehl, F.W.; Ni, W.-T. Inertial effects of a Dirac particle. *Phys. Rev. D* **1990**, *42*, 2045.
33. Cai, Y.Q.; Lloyd, D.G.; Papini, G. Helicity precession of accelerated fermions from spin-rotation coupling. *Phys. Lett. A* **1993**, *178*, 225–230.
34. Papini, G. Parity and time reversal in the spin-rotation interaction. *Phys. Rev. D* **2002**, *65*, 077901.
35. Papini, G.Quantum systems in weak gravitational fields. In *Advances in the Interplay between Quantum and Gravity Physics*; Bergmann, P.G., de Sabbata, V., Eds.; Kluwer Academic: Dordrecht, The Netherlands, 2002; pp. 317–338.
36. Papini, G. Quantum Physics in Inertial and Gravitational Fields. In *Relativity in Rotating Frames*; Rizzi, G., Ruggiero, M.L., Eds.; Kluwer Academic: Dordrecht, The Netherlands, 2004; pp. 335–359
37. Everitt, C.W.F.; DeBra, D.B.; Parkinson, B.W.; Turneaure, J.P.; Conklin, J.W.; Heifetz, M.I.; Keiser, G.M.; Silbergleit, A.S.; Holmes, T.; Kolodziejczak, J.; et al. Gravity Probe B: Final results of a space experiment to test general relativity. *Phys. Rev. Lett.* **2011**, *106*, 221101.
38. Iorio, L.; Ruggiero, M.L.; Corda, C. Novel considerations about the error budget of the LAGEOS-based tests of frame-dragging with GRACE geopotential models. *Acta Astronaut.* **2013**, *91*, 141–148.
39. Iorio, L.; Lichtenegger, H.I.M.; Ruggiero, M.L.; Corda, C. Phenomenology of the Lense-Thirring effect in the solar system. *Astrophys. Space Sci.* **2011**, *331*, 351–395.
40. Ashby, N. Relativity in the global positioning system. *Liv. Rev. Relativ.* **2003**, *6*, 1.
41. Demirel, B.; Sponar, S.; Hasegawa, Y. Measurement of the spin–rotation coupling in neutron polarimetry. *New J. Phys.* **2015**, *17*, 023065.
42. Danner, A.; Demirel, B.; Sponar, S.; Hasegawa, Y. Development and perfomance of a miniaturised spin rotator suitable for neutron interferometer experiments. *J. Phys. Commun.* **2019**, *3*, 035001.
43. Kobayashi, D.; Yoshikawa, T.; Matsuo, M.; Iguchi, R.; Maekawa, S.; Saitoh, E.; Nozaki, Y. Spin current generation using a surface acoustic wave generated via spin-rotation coupling. *Phys. Rev. Lett.* **2017**, *119*, 077202.
44. Singh, D.; Papini, G. Spin-1/2 particles in non-inertial reference frames: low-and high-energy approximations. *Nuovo Cimento B* **2000**, *115*, 223.
45. de Oliveira, C.G.; Tiomno, J. Representations of Dirac equation in general relativity. *Nuovo Cimento* **1962**, *24*, 672–687.
46. Audretsch, J. Trajectories and spin motion of massive spin-1/2 particles in gravitational fields. *J. Phys. A Math. Gen.* **1981**, *14*, 411.
47. Kannenberg, L. Mean motion of Dirac electrons in a gravitational field. *Ann. Phys.* **1977**, *103*, 64–73.
48. Lämmerzahl, C. On the equivalence principle in quantum theory. *Gen. Rel. Grav.* **1996**, *28*, 1043–1070.
49. Greenberger, D.M. The neutron interferometer as a device for illustrating the strange behavior of quantum systems. *Rev. Mod. Phys.* **1983**, *55*, 875.
50. Holland, P.R. The validity of the principle of equivalence in the WKB limit of quantum mechanics. *Found. Phys. Lett.* **1989**, *2*, 471–485.
51. Peres, A. Test of equivalence principle for particles with spin. *Phys. Rev. D* **1978**, *18*, 2739.
52. Halprin, A.; Leung, C.N. Can the sun shed light on neutrino gravitational interactions?. *Phys. Rev. Lett.* **1991**, *67*, 1833.
53. Butler, M.N.; Nozawa, S.; Malaney, R.A.; Boothroyd, A.I. Gravitationally induced neutrino oscillations. *Phys. Rev. D* **1993**, *47*, 2615.
54. Bozza, V.; Lambiase, G.; Papini, G.; Scarpetta, G. Quantum violations of the equivalence principle in a modified Schwarzschild geometry. Neutrino oscillations. *Phys. Lett. A* **2001**, *279*, 163–168.
55. Adunas, G.Z.; Rodriguez-Milla, E.; Ahluwalia, D.V. Probing quantum violations of the equivalence principle. *Gen. Rel. Grav.* **2001**, *33*, 183–194.
56. Bailey, J. et al. [CERN-Mainz-Daresbury Collaboration]. Final report on the CERN muon storage ring including the anomalous magnetic moment and the electric dipole moment of the muon, and a direct test of relativistic time dilation, *Nucl. Phys. B* **1979**, *150*, 1–75.

57. Farley, F.J.M.; Picasso, E. *Quantum Electrodynamics*. In *Advanced Series in High-Energy Physics*; Kinoshita, T., Ed.; World Scientific: Singapore, 1990; Volume 7, p. 479.
58. Ryder, L. Relativistic treatment of inertial spin effects. *J. Phys. A Math. Gen.* **1998**, *31*, 2465.
59. Farley, F.J.; Picasso, E. The muon (g-2) experiments. *Ann. Rev. Nucl. Part. Sci.* **1979**, *29*, 243–282.
60. Lambiase, G.; Papini, G. Spin-rotation coupling in compound spin objects. *Phys. Lett. A* **2013**, *377*, 1021–1023.
61. Lambiase, G.; Papini, G.; Scarpetta, G. GSI anomaly and spin–rotation coupling. *Phys. Lett. B* **2013**, *718*, 998–1001.
62. Lambiase, G.; Papini, G.; Scarpetta, G. The role of spin–rotation coupling in the non-exponential decay of hydrogen-like heavy ions. *Ann. Phys. (N. Y.)* **2012**, *332*, 143–165.
63. Lense, J.; Thirring, H. Über den Einfluss der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie. *Z. Phys.* **1918**, *19*, 156.
64. Cardall, C.Y.; Fuller, G.M. Neutrino oscillations in curved spacetime: A heuristic treatment. *Phys. Rev. D* **1997**, *55*, 7960.
65. Montague, B.W. Polarized beams in high energy storage rings. *Phys. Rep.* **1984**, *113*, 1–96.
66. Singh, D.; Mobed, N.; Papini, G. Helicity precession of spin-1/2 particles in weak inertial and gravitational fields. *J. Phys. A Math. Gen.* **2004**, *37*, 8329.
67. Aldrovandi, R.; Matsas, G.E.A.; Novaes, S.F.; Spehler, D. Fermion helicity flip in weak gravitational fields. *Phys. Rev. D* **1994**, *50*, 2645.
68. Casini, H.; Montemayor, R. Chirality transitions in gravitational fields. *Phys. Rev. D* **1994**, *50*, 7425.
69. Papini, G. Quantum Phases and Spin-Gravity Coupling. In *General Relativity and Relativistic Astrophysics, Proceedings of the 5th Canadian Conference, Waterloo, ON, Canada, 13–15 May 1993;* Mann, R.B., McLenaghan, R.G., Eds.; World Scientific: Singapore, 1994; p. 107.
70. Kainulainen, K.; Maalampi, J.; Peltoniemi, J.T. Inert neutrinos is supernovae. *Nucl. Phys.* **1991**, *B358*, 435–446.
71. *Spin Electronics*; Ziese, M., Thornton, M.J., Eds.; Springer-Verlag: Berlin, Germany, 2001.
72. Žutić, I.; Fabian, J.; Sarma, S.D. Spintronics: Fundamentals and applications. *Rev. Mod. Phys.* **2004**, *76*, 323.
73. Bauer, G.E.; Bretzel, S.; Brataas, A.; Tserkovnyak, Y. Nanoscale magnetic heat pumps and engines. *Phys. Rev. B* **2010**, *81*, 024427.
74. Grandy, W.T., Jr. *Relativistic Quantum Mechanics of Leptons and Fields*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1990.
75. Jackson, J.D. *Classical Electrodynamics*, 2nd ed.; John Wiley and Sons: New York, NY, USA, 1975.
76. Papini, G. Spin and maximal acceleration. *Galaxies* **2017**, *5*, 103.
77. Bini, D.; Cherubini, C.; Mashhoon, B. Spin, acceleration and gravity. *Class. Quantum Grav.* **2004**, *21*, 3893.
78. Papini, G. Long range order in gravity. *arXiv* **2019**, arXiv:1902.09940v1[gr-qc].