

Free Scalar Fields in Finite Volume Are Holographic

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Abstract: This brief note presents a back-of-the-envelope calculation showing that the number of degrees of freedom of a free scalar field in expanding flat space equals the surface area of the Hubble volume in Planck units. The logic of the calculation is the following. The amount of energy in the Hubble volume scales with its linear size, consequently the volume can only contain a finite number of quantized field modes. Since the momentum of the lowest energy mode scales inversely with the linear size of the volume, the maximal number of such modes in the volume scales with its surface area. It is possible to show that when the number of field modes is saturated the modes are confined to the surface of the volume. Gravity only enters this calculation as a regulator, providing a finite volume that contains the field, the entire calculation is done in flat space. While this toy model is bound to be incomplete, it is potentially interesting because it reproduces the defining aspects of holography, and advocates a regularization of the quantum degrees of freedom based on Friedmann's equation.

Keywords: quantum field theory; finite volume; vacuum energy; holography

1. Introduction

Nearly five decades ago it was noticed that the entropy of black holes and some cosmologies scales with the area of their event horizons [1–4]. Prompted by this, the number of degrees of freedom in gravitating quantum systems was conjectured to be equal to the area of their horizon measured in Planck units [5–7]. After the conjecture was proven for extreme black holes [8], a cosmological version of the conjecture was proposed [9,10], and the emerging holographic principle was put on covariant foundations [11,12]. Since then, various quantum systems were found to be holographic, including non-gravitating objects such as solitons and instantons [13–17].

Using general relativity and Bekenstein's entropy-mass bound it is possible to show that a Friedmann-Lemaître-Robertson-Walker universe is holographic [1,11,12]. This is true even for an 'empty' universe, which only contains vacuum [11]. In this brief note, using quantum field theory, I show that a flat Friedmann-Lemaître-Robertson-Walker universe containing bosonic vacuum does indeed exhibit holographic properties. The main challenge in this task is the calculation of the physical properties (energy and number of degrees of freedom) of vacuum, which is a notoriously difficult problem in quantum field theory due to the need for a non-trivial ultraviolet regulator [18–65].

The curtailing of the vacuum modes of the field is achieved in two steps in this work. The first step is the application of the well known fact that in finite volume the Hamiltonian admits a discrete spectrum. This reduces the maximal degrees of freedom of the system to a countable infinite number [66]. The second step imposes Friedmann's equation to further restrict the number of quantum degrees of freedom within the volume to a finite number. Friedmann's equation and quantization in finite volume then quantitatively

yields that the number of degrees of freedom of a free scalar field scales with the area of the volume containing it.

There are three potentially compelling aspects of this calculation. Its first merit is simplicity and its main reliance on quantum theory. Not many holographic systems exist in which the degrees of freedom can easily be counted. In this example, albeit minimalistic, counting the quantized modes is straightforward and transparent. The calculation is done in stationary flat space, gravity enters the calculation only as a regulator by establishing a finite volume for the system. Other than this, only basic properties of quantum fields in finite volume are used.

The second interesting feature of this calculation is the novel method of regularizing the energy of the vacuum modes. The quantum vacuum contribution is typically bounded by an upper limit of the integral over the momentum of the vacuum modes. In finite volume, for a discrete spectrum, such limit is not applicable. Thus, in this calculation the ultraviolet regulator is Friedmann’s condition itself, limiting the total energy of the system by its linear size in flat space.

Finally, in the gravitational context it is well known that an empty system obeying Friedmann’s equation is holographic [11,12]. The novelty added by this work is the calculation showing how quantum modes fit in the volume. In particular, the result that the number of degrees of freedom is only saturated if the quantum modes reside on the surface.

2. Vacuum Degrees of Freedom in Finite Volume

To carry out the outlined analysis consider a three dimensional, flat, slowly expanding space which obeys Friedmann’s equation, written in natural units as

$$H^2 = \frac{8\pi}{3}\rho. \tag{1}$$

This equation relates the Hubble parameter, $H = \dot{a}/a$, to the total average energy density, ρ , within the space. When the time derivative of the spatial scale parameter is positive, $\dot{a} > 0$, the Hubble radius $R = H^{-1}$ marks an event horizon and, due to the lack of curvature, an apparent horizon around an observer in this space. This horizon encloses a flat spherical region of the space. Customarily, I refer to this spatial region with a volume of $V = 4\pi R^3/3$ as the Hubble volume, or the volume for short. For simplicity I assume that the Hubble radius is not changing with time, or changes slowly enough so that the field in the volume can be considered stationary and described by conventional quantum field theory.

Next, assume that the volume is filled with a generic, real, spin zero, free quantum field $\phi(x)$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{2}$$

In this work I choose to remain agnostic about the time evolution of the field, consequently it suffices to consider its spatial dependence. Thus, it is appropriate to quantize the field using its Fourier expansion in the Schrödinger representation

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_p}} \left(a_p e^{i\mathbf{x}\cdot\mathbf{p}} + a_p^\dagger e^{-i\mathbf{x}\cdot\mathbf{p}} \right). \tag{3}$$

Here, on shell, the energy and momentum of each mode are related by the mass of the field as $\epsilon_p^2 - \mathbf{p}^2 = m^2$.

The above expansion, however, is only valid in static space, that is when $\dot{a} = 0$ implying $V = \infty$, because it contains all possible Fourier modes. In finite volume only certain Fourier modes are allowed and the field admits a discrete expansion over these Matsubara modes [66]

$$\phi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2V\epsilon_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{i\mathbf{p}\cdot\mathbf{x}} \right). \tag{4}$$

In this case, as for any quantum system with countable states, the creation and annihilation operators satisfy the equal time commutation relations

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger} \right] = \delta_{\mathbf{p}\mathbf{q}}, \quad \left[a_{\mathbf{p}}, a_{\mathbf{q}} \right] = \left[a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger} \right] = 0. \tag{5}$$

The energy of a quantum state of the field is given by the expectation value of the Hamiltonian

$$H = \frac{1}{2} \int d^3x \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \right). \tag{6}$$

According to Equation (1) such expectation value must remain finite even for states $|s\rangle$ that maximize its average energy [11]:

$$\langle s|H|s\rangle = \frac{R}{2}. \tag{7}$$

This, and the fact that all quantized modes carry energy, implies that the state $|s\rangle$ can only contain a finite number of modes. (Unless the space is static, as typically assumed in quantum field theory, in which case $R = \infty$ and the number of modes allowed in $|s\rangle$ is infinite.)

Since in finite volume $|s\rangle$ contains a finite number of modes, in the Fock representation it is written as

$$|s\rangle = a_{\mathbf{p}_1}^{\dagger} \dots a_{\mathbf{p}_N}^{\dagger} |0\rangle. \tag{8}$$

In static space the volume is infinite, consequently any modes can be excited from the vacuum $|0\rangle$. In expanding space once the maximal energy is reached within the volume, no more modes can be included in $|s\rangle$, not even vacuum modes. This regulation of the vacuum contribution can be thought of as the finite volume equivalent of the ultraviolet cut-off of the momentum integral in the infinite volume case, although operationally and conceptually it is very different [18,20,21,23,24,43,46].

The regulator used here is imposed on states rather than on operators acting on these states, which allows us to keep the structure of quantum theory in finite space intact. Another important difference is that this regularization acts both in the ultraviolet and the infrared. In the ultraviolet, according to Equation (7), R limits the total energy of the system and its infrared effect will become clear soon. Perhaps most importantly, the regulator is not a fundamental scale of gravity, rather it is the horizon size. Since the ultraviolet regulator arises from Friedmann’s equation itself, the cosmologically observed energy density becomes an input in this calculation.

To explore the consequences of the above regulator I calculate the energy of the state $|s\rangle$

$$\langle s|H|s\rangle = \sum_{i=1}^N \left(n_i + \frac{1}{2} \right) \epsilon_i. \tag{9}$$

Here the eigenvalues of the number density operators, n_i , count the excited (non-vacuum) states with energy ϵ_i , associated with the momenta \mathbf{p}_i included in the state $|s\rangle$. The $\epsilon_i/2$ terms are the familiar vacuum contribution from the modes which are allowed in the volume by Equation (7). In this work I equate the number of degrees of freedom within the volume with the number of modes, N , included in the state $|s\rangle$.

To determine how the number of degrees of freedom within the volume scales with R , I will find the state $|s\rangle$ that simultaneously saturates the energy and contains the maximal number of modes, N_{\max} . In mathematical terms this implies looking for the values of n_i and ϵ_i that allow for the maximal possible number of terms in Equation (9). Since the total energy of the state is fixed, it is easy to see that the state minimizing n_i and ϵ_i maximizes N . The minimal value of n_i is zero for each i , implying that the state $|s\rangle$ is a vacuum state. Assuming that the minimal possible value of ϵ_i is ϵ_{\min} , the energy of this state is

$$\langle s|H|s\rangle = \frac{1}{2}N_{\max}\epsilon_{\min}. \tag{10}$$

Due to the uncertainty principle the size of the volume also acts as an infrared regulator. From dimensional considerations, ϵ_{\min} must scale inversely with the linear size of the volume: $\epsilon_{\min} \sim R^{-1}$. Using this and (7) and (10) we can already conclude that N_{\max} goes as R^2 . But we can do better.

We can determine ϵ_{\min} more precisely using the fact that vacuum states of harmonic modes saturate the uncertainty relation, written in natural units as [67]

$$\Delta x \Delta p = \frac{1}{2}. \tag{11}$$

Within a sphere of maximal radial variance $\Delta x = \langle x^2 \rangle - \langle x \rangle^2 = R$, Equation (11) implies a minimal energy of $\epsilon = (2R)^{-1}$ for a massless field mode [68,69]. This, however, is not the lowest energy that can be attained by a mode in a spherical field configuration. If the field is confined close to, or on, the surface of the volume then ϵ is further reduced. This is because the spatial variance on a spherical surface, $\Delta x = 2\pi R$, is larger than that within the volume resulting

$$\epsilon_{\min} = \frac{1}{4\pi R}. \tag{12}$$

Combining Equations (7), (10) and (12) yields

$$N_{\max} = A, \tag{13}$$

where $A = 4\pi R^2$ is the surface area of the flat spherical volume.

3. Discussion

The result that the maximal number of degrees of freedom equals the surface area surrounding the volume in Planck units parallels gravitational holography [5–7,9–12]. Additionally, for the case of a free scalar field, the maximal number of degrees of freedom can only be achieved if the field modes are confined to the surface. This is an intriguing example of a simple quantum system in which the holographic property follows from basic laws of quantum field theory.

Only a single piece of input is used from general relativity to derive this result: Friedmann’s equation. Friedmann’s equation plays a dual role in this calculation. Its first task is to provide the finite volume setting. Quantized modes in finite volume are then automatically regulated in the infrared: they are ‘frozen out’ if their de Broglie wavelength is longer than the Hubble scale. Friedmann’s equation also limits the total energy of the system, including the vacuum contribution: quantum modes with the shortest wavelengths are excluded, ‘vaporized’, from the system because they would oversaturate the energy. In turn, this leads to an upper limit on the number of degrees of freedom. This argument could be turned around by assuming holography, which would lead to an upper limit on the energy (density) of the quantum system, including the vacuum contribution.

An interesting feature of these results is that they offer a different perspective on the regularization of vacuum degrees of freedom in quantum field theory. In this work the properties of cosmological vacuum (such as its energy density or number of degrees of freedom) are not set by fundamental aspects of gravity, rather they are limited by an accidental scale, the prevailing size of the apparent cosmological horizon. This perspective has advantages compared to the mainstream one. It is minimal: without the need for any exotic unknown physics, it renders the traditional vacuum energy problem of quantum field theory moot. It is self-consistent: after imposing it, quantum field theory leads to a result already known from gravity, namely the holography of an empty Friedmann-type universe. It is simple: it can be formulated extremely transparently.

This toy example is by no means intended to account for the quantum gravity description of holography. Due to its simplicity, however, it provides some potentially valuable insight into the holographic behaviour of quantum systems and may advance our understanding of regularizing the vacuum energy.

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