

On the Real Part of a Conformal Field Theory

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Received: 7 August 2018; Accepted: 17 September 2018; Published: 19 September 2018



Abstract: Every conformal field theory has the symmetry of taking each field to its adjoint. We consider here the quotient (orbifold) conformal field theory obtained by twisting with respect to this symmetry. A general method for computing such quotients is developed using the Coulomb gas representation. Examples of parafermions, $SU(2)$ current algebra and the $N = 2$ minimal models are described explicitly. The partition functions and the dimensions of the disordered fields are given. This result is a tool for finding new theories. For instance, it is of importance in analyzing the conformal field theories of exceptional holonomy manifolds.

Keywords: conformal field theory; string theory

Conformal field theories enjoy several operations which result in different conformal field theories. Examples are toroidal orbifolds [1,2], and coset type models [3,4]. Every conformal field theory (CFT), C , contains the fields A in the Hilbert space along with their conjugate A^\dagger . In general, $A \neq A^\dagger$. Thus we may consider the quotient theory (abstract orbifold),

$$\tilde{C} = C/w, \quad \text{where } w(A) = A^\dagger. \quad (1)$$

From the point of view of CFT, this quotient is quite complicated since every field in the Hilbert space transforms independently, and is not organized by some characters of an extended algebra. We found, however, the following method to compute the partition function. Many, if not all, known rational conformal theories may be described by a system of free bosons moving on a Lorentzian lattice with a background charge. This is called the Coulomb gas method. By describing C as such a system, the quotient by w becomes a \mathbb{Z}_2 orbifold where some of the free bosons flip sign. This on the other hand is straightforward to compute. Thus, we derive the partition function of \tilde{C} in the cases of \mathbb{Z}_k parafermions, $SU(2)_k$ current algebra and k th $N = 2$ minimal model, and leave to further work the consideration of other models.

The results described here for $N = 2$ minimal models are of importance in the study of the conformal field theories of compactification of string theory on exceptional holonomy manifolds [5]. This is since these theories can be considered as the real part of string theories compactified on Calabi-Yau manifolds. Our main interest is the compactification to two dimensions from 10 dimensions on spin(7) holonomy manifolds [5]. It was shown by Figueroa-O'Farrill [6] that the spin(7) algebra can be constructed from a four-dimensional Calabi-Yau algebra (which is the $N = 2$ superconformal algebra) by a twist which is taking the conjugate of the fields in the algebra. In particular, for the minimal models compactification, this is taking the real part of a tensor product of minimal $N = 2$ superconformal models at the central charge $c = 12$.

Thus, it behooves us to study the real part of minimal $N = 2$ superconformal field theory, as this will give us solvable spin(7) compactifications. This is what is described in this note. Moreover, the case of the orbifold of parafermionic theories turns out to be useful to calculate string functions at level 2 [7].

The left right symmetric \mathbb{Z}_k parafermion partition function is given by [8]

$$Z = \frac{|\eta(\tau)|^2}{2} \sum_{l-m=0 \bmod 2} |c_m^l(\tau)|^2, \quad (2)$$

where $l = 0, 1, \dots, k$ and m is any integer modulo $2k$. Our considerations below may easily be generalized for any modular invariant, but for simplicity we consider only the left right symmetric one. Here $\eta(\tau)$ is the Dedekind's eta function,

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2n\pi i \tau}), \quad (3)$$

and $c_m^l(\tau)$ is the so-called string function, which is a generating function for the L_0 of the states in the representation of $SU(2)_k$ current algebra which have isospin $l/2$ and $J_3 = m/2$,

$$c_m^l(\tau) = q^\alpha \text{Tr}_{\mathcal{H}_m^l} q^{L_0}, \quad (4)$$

where

$$\alpha = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} - \frac{k}{8(k+2)}, \quad (5)$$

is a factor which was originally added to ensure modular covariance [9], but is necessary to ensure that ηc_m^l is the parafermion partition function [8]. Kac and Peterson [10] expressed c_m^l as Hecke indefinite modular forms,

$$c_m^l(\tau) = \eta(\tau)^{-3} \sum_{\substack{-|x| < y \leq |x| \\ (x,y) \text{ or } (\frac{1}{2}-x, \frac{1}{2}+y) \in (\frac{l+1}{2(k+2)}, \frac{m}{2k}) + \mathbb{Z}^2}} \text{sign}(x) e^{2\pi i \tau [(k+2)x^2 - ky^2]}. \quad (6)$$

As was noted in ref. [11], Equation (6) can be interpreted as the partition function of two bosons, ϕ_1 and ϕ_2 , (after multiplying by eta function) moving on a Lorentzian lattice of signature $(1, -1)$ whose stress tensor is,

$$T(z) = -\frac{1}{4}(\partial\phi_1)^2 + \frac{1}{4}(\partial\phi_2)^2 + \frac{i}{2\sqrt{k+2}} \partial^2\phi_1, \quad (7)$$

where we included a background charge to get the correct central charge. The parafermions can be written as,

$$\begin{aligned} \psi(z) &= \frac{1}{2} \left(\partial\phi_2 - \sqrt{\frac{k+2}{k}} \partial\phi_1 \right) \exp(i\phi_2/\sqrt{k}), \\ \psi^\dagger(z) &= \frac{1}{2} \left(\partial\phi_2 + \sqrt{\frac{k+2}{k}} \partial\phi_1 \right) \exp(-i\phi_2/\sqrt{k}). \end{aligned} \quad (8)$$

The lattice on which these bosons move is read from Equation (6) and is a rectangular one of dimensions $(\sqrt{k+2}, \sqrt{k})$.

Similarly, for subsequent reference, $SU(2)_k$ and the $N = 2$ minimal models can be constructed using an additional free boson (the connection between parafermions and $SU(2)_k$ was described in ref. [12], whereas the connection with $N = 2$ was described in ref. [13]), with no background charge, ϕ_3 ,

$$T(z) = -\frac{1}{4}(\partial\phi_1)^2 + \frac{1}{4}(\partial\phi_2)^2 - \frac{1}{4}(\partial\phi_3)^2 + \frac{i}{2\sqrt{k+2}} \partial^2\phi_1, \quad (9)$$

where $\phi_{1,2}$ move on the same lattice as before and ϕ_3 moves on a lattice of radius \sqrt{k} for $SU(2)_k$, and of radius $2k(k+2)$ for the $N = 2$ case.

The $SU(2)_k$ currents, J^\pm and J_3 , are given by

$$\begin{aligned} J^+(z) &= \frac{1}{2}\sqrt{k} \left(\partial\phi_2 - \sqrt{\frac{k+2}{k}} \partial\phi_1 \right) \exp \left[\frac{i}{\sqrt{k}}(\phi_2 + \phi_3) \right], \\ J^-(z) &= \frac{1}{2}\sqrt{k} \left(\partial\phi_2 + \sqrt{\frac{k+2}{k}} \partial\phi_1 \right) \exp \left[-\frac{i}{\sqrt{k}}(\phi_2 + \phi_3) \right], \\ J^3(z) &= \sqrt{k} \partial\phi_3. \end{aligned} \quad (10)$$

Similarly, the currents for $N = 2$, G^\pm and J , are obtained by the same expressions with the rescaling of ϕ_3 by a factor of $\sqrt{k+2}$.

Now, it is clear from these expressions that the operation of taking $A \rightarrow A^\dagger$ is equivalent to taking $\phi_2 \rightarrow -\phi_2$ and $\phi_3 \rightarrow -\phi_3$, i.e., it is a bosonic twist. We wish to compute the partition function, of the quotient by this bosonic twist. Since $\phi_{2,3}$ do not have a background charge, this is almost a standard orbifold. Consider the parafermion case. We have four sectors $Z(\delta_1, \delta_2)$, where $\delta_i = 0, 1 \pmod{2}$, describing the path integral on the torus with boundary condition $(-1)^{\delta_1} ((-1)^{\delta_2})$ in the time (space) directions. $Z(0,0)$ is the parafermionic partition function Z described before, Equation (2). $Z(1,0)$ is the partition function

$$Z(1,0) = \text{Tr}_{\mathcal{H}} (-1)^w q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}, \quad (11)$$

where $w(A) = A^\dagger$. $Z(1,0)$ receives contributions only from states that have zero momentum in the ϕ_2 direction, since a state of momentum p , A_p , goes to A_{-p}^\dagger which is different and we can form the pair of states, $A_p \pm A_{-p}^\dagger$, which have the eigenvalues ± 1 of w , thus giving a net contribution zero to Equation (11). For $p = 0$ we have only the contribution of the moments of $\partial\phi_2$ which gives $\eta(\tau)\eta(2\tau)^{-1}$ by standard arguments. Thus, $Z(1,0)$ setting $y = 0$ in Equation (6) and multiplying by the above prefactor, for the contribution from \mathcal{H}_0^l sector (which imply that l is even) we get a contribution,

$$B^l(\tau) = Y(1,0) \sum_{n=-\infty}^{\infty} (-1)^{nk} e^{2(k+2)\pi i \tau (n + \frac{l+1}{2(k+2)})^2} \quad (12)$$

for l even, and $B^l = 0$ for l odd, where for the parafermions the prefactor is

$$Y(1,0) = \eta(2\tau)^{-1}. \quad (13)$$

Note that we changed some of the signs in Equation (11). This is required for modular invariance as we will see below. The full partition function $Z(1,0)$ is given by

$$Z(1,0) = \sum_{\substack{l=0 \\ l \equiv 0 \pmod{2}}}^k B_l B_l^*. \quad (14)$$

For $SU(2)$ and $N = 2$ we can do the same. The partition function of $SU(2)_k$ is given by ref. [14]

$$Z = \sum_{l=0}^k \chi_l(\tau, 0, 0) \chi_l^*(\tau, 0, 0), \quad (15)$$

where $\chi_l(\tau, z, u)$ is the character of the affine $SU(2)_k$ representation with isospin $l/2$,

$$\chi^l = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}, \quad (16)$$

and where the level m classical theta function is defined by,

$$\Theta_{n,m}(\tau, z, u) = e^{2\pi i u} \sum_{j \in n/(2m) + \mathbb{Z}} e^{2\pi i \tau m j^2 + 2\pi i j z}, \quad (17)$$

where n is defined modulo $2m$.

For the $N = 2$ partition function we have [15]

$$Z = \sum_{l,m,s} \left| \chi_m^{l(s)}(\tau, 0, 0) \right|^2, \quad (18)$$

where the sum is over $l = 0, 1, \dots, k$ and m and s modulo $2(k+2)$ and 4 respectively. $s = 0, 2$ in the NS sector, and $s = 1, 3$ in the Ramond sector. The $\chi_m^{l(s)}(\tau, z, u)$ are the characters of the k th $N = 2$ minimal model and are given by,

$$\chi_m^{l(s)}(\tau, z, u) = \sum_{j \bmod k} c_{m+4j-s}^l \Theta_{2m+(4j-s)(k+2), 2k(k+2)}(\tau, 2kz, u). \quad (19)$$

Looking at the expressions for the characters in Equation (19), we see that again this is just the appropriate partition function for the appropriate system of bosons. Thus we get the contribution of the l representation by simply setting the momentum in the $\phi_{2,3}$ directions to zero. We get the same answer as for the parafermions, Equation (12), with only a different prefactor to account for the extra boson,

$$Y(1, 0) = \eta(\tau) \eta(2\tau)^{-1}. \quad (20)$$

It is the same prefactor for $SU(2)$ and $N = 2$.

We can get the partition function for the twisted sectors $Z(0, 1)$ and $Z(1, 1)$ by modular transformations. We have $Z(1, 0)(-1/\tau) = Z(0, 1)(\tau)$ and $Z(1, 1)(\tau) = Z(0, 1)(\tau + 1)$. For consistency it is required that all partition functions would be invariant under $\tau \rightarrow \tau + 2$ and $Z(1, 1)(-1/\tau) = Z(1, 1)(\tau)$. The total partition function is,

$$Z = \frac{1}{2} \sum_{\delta_1, \delta_2=0,1} Z(\delta_1, \delta_2). \quad (21)$$

The partition function for the untwisted sector, namely fields invariant under $A \rightarrow A^\dagger$, is $\frac{1}{2}(Z(0, 0) + Z(1, 0))$. Similarly, the partition function for the twisted sector, disorder fields, is $\frac{1}{2}(Z(0, 1) + Z(1, 1))$.

We can write the sum in the expression for B_l , Equation (12) as a theta function,

$$B_l/Y(1, 0) = \Theta_{l+1, k+2}(\tau, k/2, 0), \quad (22)$$

where we used the definition, Equation (17). The theta functions transform nicely under modular transformations [9],

$$\Theta_{n,m} \left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{z^2}{2\tau} \right) = \frac{1}{\sqrt{2m}} (-i\tau)^{\frac{1}{2}} \sum_{l \bmod 2m} e^{-\pi i l n/m} \Theta_{l,m}(\tau, z, u). \quad (23)$$

Using this, we find the partition function $Z(0, 1)$ by calculating $Z(1, 0)(-1/\tau)$. We find,

$$Z(0, 1) = |Y(0, 1)|^2 \sum_{\substack{\gamma, \tilde{\gamma} \bmod 2(k+2) \\ \gamma - \tilde{\gamma} = 0 \bmod k+2}} e^{\pi i (\gamma - \tilde{\gamma})/(k+2)} \Theta_{\gamma + \frac{p}{2}, k+2}(\tau, 0, 0) \Theta_{\tilde{\gamma} + \frac{p}{2}, k+2}(\tau, 0, 0)^*, \quad (24)$$

where $p = 0$ or $p = 1$, $p = k \bmod 2$.

$$D_\gamma = Y(0,1)(\Theta_{\gamma+\frac{p}{2},k+2} - \Theta_{\gamma+\frac{p}{2}+k+2,k+2})/a \quad (25)$$

are the characters of the twisted sector, with respect to the appropriate extended chiral algebra. a is a numerical factor equal to $\sqrt{2}$ for parafermions and $a = 2$ for $N = 2$ and $SU(2)$. The lowest dimension field in D_γ ('primary fields') are the disorder fields A_γ of dimension,

$$\Delta_\gamma = \frac{(\gamma + p/2)^2}{4(k+2)} + s + c/24, \quad (26)$$

where $s = 0$ for $SU(2)$ and $N = 2$, and $s = -1/48$ for parafermions, c is the central charge and γ is defined modulo $k+2$. Actually, for parafermions, the dimensions of these disorder fields were already calculated by Zamolodchikov [13], and our results agree in this case. The novelty here is that we obtained the characters, as well.

In terms of these, the partition function $Z(0,1)$ is left-right symmetric. The prefactor, $Y(0,1)$, comes from the modular transform of the $Y(1,0)$. We find,

$$\begin{aligned} Y(0,1) &= \frac{\sqrt{2}}{\eta(\tau/2)}, & \text{for parafermions,} \\ Y(0,1) &= \frac{2\eta(\tau)}{\eta(\tau/2)^2}, & \text{for } SU(2) \text{ and } N = 2, \end{aligned} \quad (27)$$

where we used the well know transformation properties of the eta function, e.g., [9].

Author Contributions: Both authors contributed equally to this work.

Funding: This research received no external funding.

Acknowledgments: We thank CERN for the kind hospitality. H.P. would like to thank Weizmann Institute of Science for hospitality.

Conflicts of Interest: The authors declare no conflict of interest.

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