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Higher Spin Superfield Interactions with the Chiral Supermultiplet: Conserved Supercurrents and Cubic Vertices

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Abstract: We investigate cubic interactions between a chiral superfield and higher spin superfields corresponding to irreducible representations of the $4D$, $\mathcal{N}=1$ super-Poincaré algebra. We do this by demanding an invariance under the most general transformation, linear in the chiral superfield. Following Noether's method we construct an infinite tower of higher spin supercurrent multiplets which are quadratic in the chiral superfield and include higher derivatives. The results are that a single, massless, chiral superfield can couple only to the half-integer spin supermultiplets $(s+1, s+1/2)$ and for every value of spin there is an appropriate improvement term that reduces the supercurrent multiplet to a minimal multiplet which matches that of superconformal higher spins. On the other hand a single, massive, chiral superfield can couple only to higher spin supermultiplets of type $(2l+2, 2l+3/2)$ (only odd values of s, $s=2l+1$) and there is no minimal multiplet. Furthermore, for the massless case we discuss the component level higher spin currents and provide explicit expressions for the integer and half-integer spin conserved currents together with a R-symmetry current.

Keywords: supersymmetry; off-shell supermultiplets; higher spin

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1. Introduction

Higher spin theories [1–10] have a considerable history and for a number of years drove the development of many ideas in theoretical physics. However, their role in fundamental interactions is still not clear. On the one hand, all the elementary particles observed in nature so far seem to be concentrated in a region of spin values (s) such that $s \leq 2$. Moreover, this observation appears to be supported by a substantial list of No-Go theorems [11–26] (for reviews look in [27,28]) suggesting that nature stops with spin 2. On the other hand, if we want to understand relativistic field theories and their quantum aspects in full generality, there is no a priori reason to exclude higher spin fields. In recent decades, this point was made undeniable due to the crucial part that massless and massive higher spin particles play in (*i*) the softness of string interactions at high energy scales, (*ii*) the possibilities to describe string effects in the framework of field theory, and (*iii*) investigations of some aspects of the holographic principle[1].

The construction of fully interacting higher spin theories is an extremely exciting topic but also very difficult, mostly due to the road blocks placed by the no-go results and maybe due to current lack of (still unknown) general principles. Also, one cannot exclude that higher spin field theory is an effective theory for an underlying, so far unknown, more fundamental theory. Nevertheless, there are few examples of successful approaches to higher spin theory such as Vasiliev's theory [10,29–31] (for reviews look in [32–34]) and the 3D Cherns-Simon higher spin-gravity formulation [35–37]. Despite their actual successes, these theories still appear very complicated. For example, Vasiliev's theory provides an infinite set of on-shell equations of motion and many conceptual questions about observables, Lagrangian formulation, locality[2] and quantization require continued study. In addition, the Chern-Simons description of interacting higher spins is restricted to 3D and has, in the massless case, no local degrees of freedom. Therefore, many important questions concerning higher spin field theory are still open[3].

In higher spin theories the structure of possible interaction vertices is essentially fixed by higher-spin symmetries. We will consider the construction of the simplest vertices in the supersymmetric higher-spin models. In this case, one can expect that the supersymmetry will impose the additional restrictions on the form of vertices and therefore one can hope to uncover clarifications and simplifications in comparison to non-supersymmetric higher-spin models.

The simplest higher spin interaction is described by the cubic vertex. Therefore, we will begin with the construction of a cubic vertex for supersymmetric field theory. It is well known that supersymmetric field models can be formulated on-shell in terms of component fields or off-shell in terms of appropriate superfields (see the text books [56,57]). Both these ways of constructing supersymmetric field models have their own advantages and disadvantages and complement each other. In this paper we will follow the superfield approach which allows us to keep manifest supersymmetry off-shell.

One kind of cubic interaction vertex for two types of fields can be written in the form jh, where j is a current constructed from fields of type ϕ (matter fields) and h is a field of another type (gauge fields). Because the gauge field h is defined up to gauge transformations, the current j must satisfy some conservation laws, i.e., it is conserved. Higher-spin interactions on the base of conserved current have been constructed and explored by many authors (see e.g., [58–67])[4].

In this work we will present the construction of the conserved $\mathcal{N} = 1$ higher superspin supercurrent and supertrace that generate the cubic interactions between super-Poincaré higher spin supermultiplets which play the role of gauge fields and the chiral supermultiplet which will play the role of matter. The higher spin supercurrent and higher spin supertrace together constitute the higher spin supercurrent multiplet and are the corresponding analogues to the low-spin supercurrent and supertrace of conventional supersymmetric theory (see [56,57]).

The strategy we follow is that of Noether's method, which is a perturbative procedure that allows one to constrain the allowed interactions by imposing invariance order by order in the number of (super)fields. Such a treatment of interactions will be very clear and useful for the cubic order. In our case the corresponding transformation for the matter superfield is the most general transformation, consistent with its chiral nature and up to linear order terms in the superfield and for the higher spin superfields is their gauge transformation.

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1 For example, Fradkin-Vasiliev cubic interaction vertex of massless higher spin fields with gravity requires the AdS background.

2 See e.g., [38,39].

3 At present time, there is an extensive literature on different aspects of higher spin field theory. For example see the recent papers [40–55] and references therein.

4 A BRST approach to the construction of cubic vertex has been developed in [68].

The paper is organized as follows. Section 2 is devoted to discussing Noether's procedure and specific features of $4D$, $\mathcal{N}=1$ super-Poincaré higher spin theories. In Section 3, we find the most general transformation of chiral superfield up to linear order and observe that the parameters of this transformation match the structure of the gauge transformations of specific higher spin supermultiplets. Sections 4–6 devoted to the construction of the higher spin supercurrent multiplet of a free massless chiral and generate the cubic interactions with higher spins. We find that the massless chiral can be coupled only to higher spin supermultiplets of type $(s+1, s+1/2)$. In Section 7, we show that for every value of integer s there are two types of higher spin supercurrent multiplets, the canonical and the minimal and one can go from the canonical to the minimal by an appropriate choice of improvement terms. Furthermore, we demonstrate that the minimal multiplet coincides with the supercurrent multiplet generated by superconformal higher spins. In Section 8, we discuss the on-shell superspace conservation equations for both supercurrent multiplets. For the case of minimal multiplet, we use the conservation equation alone to derive a simpler expression for the higher spin supercurrent. In Section 9, we project to components and find explicit expressions for the spacetime conserved integer spin, half-integer spin and R-symmetry currents. The integer spin current has two contributions, one of the boson - boson type that matches the known expressions for the integer spin currents of a complex scalar and the other is of the fermion - fermion type which agrees with the known expressions of integer spin currents of a spinor. The half-integer spin and R-symmetry currents, as far as we know, appear in the literature for the first time. Section 10, is devoted to the massive chiral superfield. We find that it can couple only to higher spin supermultiplets of type $(2l+2\ ,\ 2l+3)$ and we present new expressions for the higher spin supercurrent multiplet. For the massive chiral there is no minimal multiplet. In Section 11, we summarize and discuss the results.

2. Noether's Method

In general, finding *consistent* interactions is a very difficult problem if there is no guiding principle. For the cases of spin 2 (GR) and spin 1 (YM) there is a very well developed geometrical understanding (*Riemannian Manifolds* and *Principle Bundles* respectively) that plays the role of the guiding principle, but for higher spins we do not have this geometrical input. In some extent, the geometrical interpretation of higher spin fields is still mysterious. Therefore, we have to use alternative methods. The idea is to relax any geometrical prejudice and have only algebraic requirements. In this case the physical guiding principle is that of gauge invariance and *consistent* interactions are the ones that are in agreement with gauge symmetries. Keep in mind that this is a physical requirement in order for the interacting theory to have the same degrees of freedom as the free theory.

Noether's method is a systematic, perturbative, analysis of the invariance requirement. In this approach one expands the action $S[\phi, h]$ and the transformation of fields in a power series of a coupling constant g

$$S[\phi,h] = S_0[\phi] + gS_1[\phi,h] + g^2 S_2[\phi,h] + \dots\ , \tag{1}$$

$$\delta\phi = \delta_0[\xi] + g\delta_1[\phi,\xi] + g^2\delta_2[\phi,\xi] + \dots\ , \tag{2}$$

$$\delta h = \delta_0[\zeta] + g\delta_1[h,\zeta] + g^2\delta_2[h,\zeta] + \dots \tag{3}$$

where $S_i[\phi,h]$ includes the interaction terms of order $i+2$ in the number of fields and δ_i is the part of transformations with terms of order i in the number of fields. Hence, invariance can now be written iteratively up to the order we desire to investigate. For the free theory (g^0) and the cubic interactions (g^1), which is the first step beyond free theory, invariance demands:

$$g^0:\quad \frac{\delta S_0}{\delta\phi}\delta_0\phi + \frac{\delta S_0}{\delta h}\delta_0 h = 0\ , \tag{4a}$$

$$g^1:\quad g\frac{\delta S_0}{\delta\phi}\delta_1\phi + g\frac{\delta S_1}{\delta\phi}\delta_0\phi + g\frac{\delta S_0}{\delta h}\delta_1 h + g\frac{\delta S_1}{\delta h}\delta_0 h = 0\ . \tag{4b}$$

In our case, the role of matter will be played by the chiral supermultiplet, described by a chiral superfield Φ ($\bar{\mathrm{D}}_{\dot{\alpha}}\Phi = 0$). At the free theory level the chiral superfield does not have any gauge transformation, $\delta_0\Phi = 0$.

For the role of gauge fields we consider the massless, higher spin, irreducible representations of the $4D$, $\mathcal{N} = 1$, super-Poincaré algebra. In the pioneer papers [5,6,69], using a component formulation, free $\mathcal{N} = 1$ supersymmetric massless higher spin models in four dimensions have been constructed. A superfield formulation was proposed in [70–72] and further developed in subsequent papers [73,74] and generalized by different authors[5]. The results are[6]:

1. The integer superspin $Y = s$ supermultiplets $(s + 1/2, s)$ are described by a pair of superfields $\Psi_{\dot{\alpha}(s)\dot{\alpha}(s-1)}$ and $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ with the following zero order gauge transformations

$$\delta_0\Psi_{\dot{\alpha}(s)\dot{\alpha}(s-1)} = -\mathrm{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \tfrac{1}{(s-1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1})}\Lambda_{\alpha(s)\dot{\alpha}(s-2)}\,, \tag{5a}$$
$$\delta_0 V_{\alpha(s-1)\dot{\alpha}(s-1)} = \mathrm{D}^{\alpha_s} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{L}_{\alpha(s-1)\dot{\alpha}(s)}\,. \tag{5b}$$

2. The half-integer superspin $Y = s + 1/2$ supermultiplets $(s + 1, s + 1/2)$ have two descriptions. One of them use the pair of superfields $H_{\alpha(s)\dot{\alpha}(s)}$, $\chi_{\alpha(s)\dot{\alpha}(s-1)}$ with the following zero order gauge transformations

$$\delta_0 H_{\alpha(s)\dot{\alpha}(s)} = \tfrac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \tfrac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))}\,, \tag{6a}$$
$$\delta_0\chi_{\alpha(s)\dot{\alpha}(s-1)} = \bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \mathrm{D}^{\alpha_{s+1}}\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \tag{6b}$$

and the other one use the superfields $H_{\alpha(s)\dot{\alpha}(s)}$, $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ with

$$\delta_0 H_{\alpha(s)\dot{\alpha}(s)} = \tfrac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \tfrac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))}\,, \tag{7a}$$
$$\delta_0\chi_{\alpha(s-1)\dot{\alpha}(s-2)} = \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}L_{\alpha(s)\dot{\alpha}(s-1)} + \tfrac{s-1}{s}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}L_{\alpha(s)\dot{\alpha}(s-1)} + \tfrac{1}{(s-2)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-2}}J_{\alpha(s-1)\dot{\alpha}(s-3))}\,. \tag{7b}$$

Consequently, the cubic interactions of the chiral superfield with the higher spin multiplets, according to (4) must satisfy:

$$\frac{\delta S_0}{\delta\Phi}\delta_1\Phi + \frac{\delta S_1}{\delta\mathcal{A}}\delta_0\mathcal{A} = 0 \tag{8}$$

where $\mathcal{A}$ is the set of superfields that participate in the description of higher spin supermultiplets for any value of s. In this language, the collection of *non-trivial* supercurrents that generate the cubic interaction terms correspond to the terms $\frac{\delta S_1}{\delta\mathcal{A}}$. The word *non-trivial* means that (*i*) the chiral superfield may not interact with all possible higher spin supermultiplets (trivially zero supercurrents) and (*ii*) for the ones that it interacts with, we must check that these interactions can not be adsorbed by redefinitions of the chiral superfield.

3. First Order Gauge Transformation for Chiral Superfield

In the previous section, we saw that the higher spin supercurrents of a chiral superfield are controlled by $\delta_1\Phi$. That is the part of the transformation of Φ which is linear in Φ. Examples of transformations of this type are generated by superdiffeomorphisms or the superconformal group and have been used in the past [76,77] in order to find the coupling of the chiral supermultiplet to supergravities.

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5 See also a formulation of supersymmetric gauge theory in the framework of BRST approach [75].
6 This is the "economical" description according to [74].

In this section we present the higher spin version of this transformation. The most general ansatz one can write for such a transformation is[7]:

$$\begin{aligned}\delta_g \Phi = g \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \Big\{ & A_l^{\alpha(k+1)\dot{\alpha}(k)} \, \Box^l \, \mathrm{D}_{\alpha_{k+1}} \bar{\mathrm{D}}_{\dot{\alpha}_k} \mathrm{D}_{\alpha_k} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \\ & + \Gamma_l^{\alpha(k)\dot{\alpha}(k+1)} \, \Box^l \, \bar{\mathrm{D}}_{\dot{\alpha}_{k+1}} \mathrm{D}^2 \bar{\mathrm{D}}_{\dot{\alpha}_k} \mathrm{D}_{\alpha_k} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \\ & + \Delta_l^{\alpha(k)\dot{\alpha}(k)} \, \Box^l \, \bar{\mathrm{D}}_{\dot{\alpha}_k} \mathrm{D}_{\alpha_k} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \\ & + E_l^{\alpha(k)\dot{\alpha}(k)} \, \Box^l \, \mathrm{D}^2 \bar{\mathrm{D}}_{\dot{\alpha}_k} \mathrm{D}_{\alpha_k} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \Big\} \end{aligned} \tag{9}$$

and depends on four infinite families of coefficients $\{A^l_{\alpha(k+1)\dot{\alpha}(k)}, \Gamma^l_{\alpha(k)\dot{\alpha}(k+1)}, \Delta^l_{\alpha(k)\dot{\alpha}(k)}, E^l_{\alpha(k)\dot{\alpha}(k)}\}$ with independently symmetrized dotted and undotted indices. To make this transformation consistent with the chiral nature of Φ we must have ($\bar{\mathrm{D}}_{\dot{\beta}} \delta_g \Phi = 0$):

$$A^l_{\alpha(k+1)\dot{\alpha}(k)} = -\tfrac{k+1}{k+2} \, \bar{\mathrm{D}}^{\dot{\alpha}_{k+1}} \Delta^l_{\alpha(k+1)\dot{\alpha}(k+1)} \,, \tag{10a}$$

$$\Gamma^l_{\alpha(k)\dot{\alpha}(k+1)} = \tfrac{1}{(k+1)!} \, \bar{\mathrm{D}}_{(\dot{\alpha}_{k+1}} \Delta^{l+1}_{\alpha(k)\dot{\alpha}(k))} \,, \tag{10b}$$

$$E^l_{\alpha(k)\dot{\alpha}(k)} = \bar{\mathrm{D}}^2 \Delta^{l+1}_{\alpha(k)\dot{\alpha}(k)} \,, \tag{10c}$$

$$\bar{\mathrm{D}}_{(\dot{\beta}} \Delta^0_{\alpha(k)\dot{\alpha}(k))} = 0 \,, \tag{10d}$$

$$\bar{\mathrm{D}}_{\dot{\beta}} \Delta^0 = 0 \,. \tag{10e}$$

The conclusion is that parameters $A^l_{\alpha(k+1)\dot{\alpha}(k)}$, $\Gamma^l_{\alpha(k)\dot{\alpha}(k+1)}$, $E^l_{\alpha(k)\dot{\alpha}(k)}$ are not independent and furthermore

$$\Delta^0 = \bar{\mathrm{D}}^2 \ell \,, \tag{11a}$$

$$\Delta^0_{\alpha(k)\dot{\alpha}(k)} = \tfrac{1}{k!} \bar{\mathrm{D}}_{(\dot{\alpha}_k} \ell_{\alpha(k)\dot{\alpha}(k-1))} \,, \tag{11b}$$

$$\Delta^l_{\alpha(k)\dot{\alpha}(k)} \text{ is unconstrained for } l \geq 1 \tag{11c}$$

where ℓ, $\ell_{\alpha(k)\dot{\alpha}(k-1)}$ are arbitrary.

From Equation (8) it is evident that the parameters which appear in the transformation of Φ must also appear in the zeroth order gauge transformation of the higher spin superfields. Looking at the gauge parameters that appear in (5)–(7) we find that there is no unconstrained parameter with the structure of $\Delta^{l+1}_{\alpha(k)\dot{\alpha}(k)}$, but Equations (6) and (7) include unconstrained gauge parameters which match the structure of $\ell_{\alpha(k)\dot{\alpha}(k-1)}$. The conclusion is that in order to construct invariant theories where the chiral superfield couples to purely higher spin supermultiplets we have to consider the following transformation of Φ:

$$\begin{aligned}\delta_g \Phi = -g \sum_{k=0}^{\infty} \Big\{ & \bar{\mathrm{D}}^2 \ell^{\alpha(k+1)\dot{\alpha}(k)} \, \mathrm{D}_{\alpha_{k+1}} \bar{\mathrm{D}}_{\dot{\alpha}_k} \mathrm{D}_{\alpha_k} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \\ & - \tfrac{1}{(k+1)!} \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))} \, \bar{\mathrm{D}}_{\dot{\alpha}_{k+1}} \mathrm{D}_{\alpha_{k+1}} \ldots \bar{\mathrm{D}}_{\dot{\alpha}_1} \mathrm{D}_{\alpha_1} \Phi \Big\} \\ + g \bar{\mathrm{D}}^2 \ell \, \Phi \ . \end{aligned} \tag{12}$$

[7] We use the conventions of *Superspace* [56] which include $\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}}$, $\mathrm{D}^\alpha \mathrm{D}_\alpha = 2\mathrm{D}^2$ and $\bar{\mathrm{D}}^{\dot{\alpha}} \bar{\mathrm{D}}_{\dot{\alpha}} = 2\bar{\mathrm{D}}^2$.

The last term of (12) will generate coupling to the vector multiplet, thus in order to consider purely higher spin interactions we should ignore it. However, for the sake of completeness we will not do that.

The second conclusion we can already reach, is that a theory of a single chiral superfield can couple *only* to half-integer superspin $Y = s + 1/2$ supermultiplets. This is a consequence of the constraint (10d) whose solution matches the structure of the transformation of bosonic superfields of half-integer superspin theories but crucially not that of integer superspin.

4. Constructing the Higher Spin Supercurrents I: Varying the Action

Having found the appropriate first order transformation for the chiral superfield, we use it to perform Noether's procedure for the cubic order terms, as described in Section 2 and construct the higher spin supercurrents of the chiral supermultiplet. We consider a free massless chiral superfield, so we start from the free action

$$S_o = \int d^8z\ \Phi\bar{\Phi} \tag{13}$$

and calculate its variation under $\delta_g\Phi$ [8]:

$$\begin{aligned}\delta_g S_o = -g\int\sum_{k=0}^{\infty}\Big\{&\bar{\mathrm{D}}^2\ell^{\alpha(k+1)\dot{\alpha}(k)}\ \mathrm{D}_{\alpha_{k+1}}\bar{\mathrm{D}}_{\dot{\alpha}_k}\mathrm{D}_{\alpha_k}\dots\bar{\mathrm{D}}_{\dot{\alpha}_1}\mathrm{D}_{\alpha_1}\Phi\ \bar{\Phi}\ + c.c.\\ &-\tfrac{1}{(k+1)!}\ \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}}\ell^{\alpha(k+1)\dot{\alpha}(k))}\ \bar{\mathrm{D}}_{\dot{\alpha}_{k+1}}\mathrm{D}_{\alpha_{k+1}}\dots\bar{\mathrm{D}}_{\dot{\alpha}_1}\mathrm{D}_{\alpha_1}\Phi\ \bar{\Phi}\ + c.c.\Big\}\\ +g\int\Big\{\bar{\mathrm{D}}^2\ell + \mathrm{D}^2\bar{\ell}\Big\}&\ \Phi\ \bar{\Phi}\ .\end{aligned} \tag{14}$$

However, in the above expression we can freely add any pair of terms $A_{\alpha(k+1)\dot{\alpha}(k)}$, $B_{\alpha(k+1)\dot{\alpha}(k+1)}$ such that they identically satisfy the equation

$$\bar{\mathrm{D}}^2 A_{\alpha(k+1)\dot{\alpha}(k)} = \bar{\mathrm{D}}^{\dot{\alpha}_{k+1}} B_{\alpha(k+1)\dot{\alpha}(k+1)} \quad . \tag{15}$$

These terms play the role of improvement terms. We can prove that there are at least two pairs of them

1. $A_{\alpha(k+1)\dot{\alpha}(k)} = W_{\alpha(k+1)\dot{\alpha}(k)}$, $B_{\alpha(k+1)\dot{\alpha}(k+1)} = \frac{k+1}{(k+2)(k+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{k+1}}W_{\alpha(k+1)\dot{\alpha}(k))}$,
2. $A_{\alpha(k+1)\dot{\alpha}(k)} = \frac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\bar{U}_{\alpha(k))\dot{\alpha}(k+1)}$, $B_{\alpha(k+1)\dot{\alpha}(k+1)} = \frac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^2\bar{U}_{\alpha(k))\dot{\alpha}(k+1)}$

which will be relevant for our discussion. Hence, we can write for the variation of the S_o action:

$$\begin{aligned}\delta_g S_o = -g\int\sum_{k=0}^{\infty}\Big\{&\bar{\mathrm{D}}^2\ell^{\alpha(k+1)\dot{\alpha}(k)}\ \mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)}\ + c.c.\\ &-\tfrac{1}{(k+1)!}\ \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}}\ell^{\alpha(k+1)\dot{\alpha}(k))}\ \mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}\ + c.c.\Big\}\\ +g\int\Big\{\bar{\mathrm{D}}^2\ell + \mathrm{D}^2\bar{\ell}\Big\}&\ \mathcal{J}\end{aligned} \tag{16}$$

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[8] From this point forward, when the integration is over the entire superspace the measure d^8z will not be explicitly written but it will be implied.

where

$$\mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)} = \tfrac{1}{(k+1)!k!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}_{(\dot{\alpha}_k}\mathrm{D}_{\alpha_k}\dots\bar{\mathrm{D}}_{\dot{\alpha}_1)}\mathrm{D}_{\alpha_1)}\Phi\,\bar{\Phi} + W_{\alpha(k+1)\dot{\alpha}(k)} + \tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\bar{U}_{\alpha(k))\dot{\alpha}(k+1)}\,, \tag{17a}$$

$$\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)} = \tfrac{1}{(k+1)!(k+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{k+1}}\mathrm{D}_{(\alpha_{k+1}}\dots\bar{\mathrm{D}}_{\dot{\alpha}_1)}\mathrm{D}_{\alpha_1)}\Phi\,\bar{\Phi} + \tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^2\bar{U}_{\alpha(k))\dot{\alpha}(k+1)} + \tfrac{k+1}{(k+2)(k+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{k+1}}W_{\alpha(k+1)\dot{\alpha}(k))}\,, \tag{17b}$$

$$\mathcal{J} = \Phi\bar{\Phi}\qquad . \tag{17c}$$

It is important to observe that these objects are not uniquely defined, but there is some freedom. For example $\mathcal{J}$ is defined up to terms $\mathrm{D}^{\alpha}\bar{\mathrm{D}}^2\lambda_{\alpha} + \bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}^2\bar{\lambda}_{\dot{\alpha}}$ for an arbitrary λ_{α}[9], whereas $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ is defined up to terms $\bar{\mathrm{D}}^{\dot{\alpha}_{k+2}}\Xi_{\alpha(k+1)\dot{\alpha}(k+2)}$. Also $\mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)}$ has the freedom

$$\mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)} \sim \mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)} + \bar{\mathrm{D}}_{(\dot{\alpha}_k}P^{(1)}_{\alpha(k+1)\dot{\alpha}(k-1))} + \bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}P^{(2)}_{\alpha(k+1)\dot{\alpha}(k+1)} + \mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^2R^{(1)}_{\alpha(k))\dot{\alpha}(k)} + \mathrm{D}^{\alpha_{k+2}}\bar{\mathrm{D}}^2R^{(2)}_{\alpha(k+2)\dot{\alpha}(k)}\,. \tag{18}$$

Furthermore, Equation (16) points towards a coupling of the chiral with the first formulation (6) of $(s+1, s+1/2)$ supermultiplets, but for that to happen we must have $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ to be real. This is a consequence of the reality of superfield $H_{\alpha(s)\dot{\alpha}(s)}$ and transformation (6a). Thus, in order to couple the theory purely to half-integer superspin multiplet, we must make sure that we can select the improvement terms such that $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$=$\bar{\mathcal{J}}_{\alpha(k+1)\dot{\alpha}(k+1)}$. This will depend on the detailed structure of the real and imaginary part of the term $\frac{1}{(k+1)!(k+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{k+1}}\mathrm{D}_{(\alpha_{k+1}}\dots\bar{\mathrm{D}}_{\dot{\alpha}_1)}\mathrm{D}_{\alpha_1)}\Phi\,\bar{\Phi}$. The investigation of these structures is the purpose of the following section. Due to the chiral nature of Φ, this term can be simply written as $i^{k+1}\partial^{(k+1)}\Phi\,\bar{\Phi}$, where for simpicity we omit the uncontracted indices and complete symmetrization of them with appropriate symmetrization factors is understood. The symbol $\partial^{(k)}$ denotes a string of k spacetime derivatives.

5. The Combinatorics of the Imaginary Part

First of all, we decompose the quantity $i^{k+1}\partial^{(k+1)}\Phi\,\bar{\Phi}$ to a real and an imaginary part

$$i^{k+1}\partial^{(k+1)}\Phi\,\bar{\Phi} = \tfrac{i^{k+1}}{2}\left[\partial^{(k+1)}\Phi\,\bar{\Phi} + (-1)^{k+1}\Phi\,\partial^{(k+1)}\bar{\Phi}\right] + \tfrac{i^{k+1}}{2}\left[\partial^{(k+1)}\Phi\,\bar{\Phi} - (-1)^{k+1}\Phi\,\partial^{(k+1)}\bar{\Phi}\right] \tag{19}$$

and then we focus at the imaginary part with the goal to clarify whether the various improvement terms ($W_{\alpha(k+1)\dot{\alpha}(k)}$, $U_{\alpha(k+1)\dot{\alpha}(k)}$) can modify it in order to make $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ real. Notice the difference between even and odd values of $k+1$

$$\mathcal{I}^{(k+1)} \equiv i\mathrm{Im}[\,i^{k+1}\partial^{(k+1)}\Phi\,\bar{\Phi}\,] = \begin{cases} \frac{i}{2}(-1)^l\left(\partial^{(2l+1)}\Phi\,\bar{\Phi} + \Phi\,\partial^{(2l+1)}\bar{\Phi}\right), \text{ for } k+1 = 2l+1,\ l = 0,1,\dots \\ \frac{1}{2}(-1)^l\left(\partial^{(2l)}\Phi\,\bar{\Phi} - \Phi\,\partial^{(2l)}\bar{\Phi}\right), \text{ for } k+1 = 2l,\ l = 1,2,\dots \end{cases} \tag{20}$$

---

[9] λ_{α} has its own redundancy $\lambda_{\alpha} \sim \lambda_{\alpha} + \bar{\mathrm{D}}^{\dot{\alpha}}\zeta_{\alpha\dot{\alpha}} + i\mathrm{D}_{\alpha}\varrho$ with $\varrho = \bar{\varrho}$.

The type of terms that appear above are a special case to the more general type $\partial^{(m)}\Phi\,\partial^{(n)}\bar{\Phi}$ terms. It is easy to prove that this type of terms satisfy the following recursion relations:

$$\partial^{(m)}\Phi\,\partial^{(n)}\bar{\Phi} = \partial\left(\partial^{(m-1)}\Phi\,\partial^{(n)}\bar{\Phi}\right) - \partial^{(m-1)}\Phi\,\partial^{(n+1)}\bar{\Phi} \ , \tag{21}$$

$$\partial^{(m)}\Phi\,\partial^{(n)}\bar{\Phi} = \partial\left(\partial^{(m)}\Phi\,\partial^{(n-1)}\bar{\Phi}\right) - \partial^{(m+1)}\Phi\,\partial^{(n-1)}\bar{\Phi} \ . \tag{22}$$

Using these recursion formulas, one can prove that

$$\partial^{(2l+1)}\Phi\,\bar{\Phi} + \Phi\,\partial^{(2l+1)}\bar{\Phi} = \sum_{n=0}^{l} c_n\,\partial^{(2n+1)}\left\{\partial^{(l-n)}\Phi\,\partial^{(l-n)}\bar{\Phi}\right\} \ , \tag{23}$$

$$\partial^{(2l)}\Phi\,\bar{\Phi} - \Phi\,\partial^{(2l)}\bar{\Phi} = \sum_{n=0}^{l-1} d_n\,\partial^{(2n+1)}\left\{\partial^{(l-n)}\Phi\,\partial^{(l-n-1)}\bar{\Phi} - \partial^{(l-n-1)}\Phi\,\partial^{(l-n)}\bar{\Phi}\right\} \tag{24}$$

with

$$c_n = (-1)^{l-n}\left[\binom{l+n+1}{l-n} + \binom{l+n}{l-n-1}\right] \ , \ d_n = (-1)^{l-n-1}\binom{l+n}{l-n-1} \ . \tag{25}$$

These identities hold in general, not just for the chiral but for any (super)function Φ. An alternative proof of them can be found by expanding the right hand side using the identity

$$\partial^{(m)}\,(A\,B) = \sum_{i=0}^{m}\binom{m}{i}\,\partial^{(m-i)}A\,\partial^{i}B \tag{26}$$

and matching the coefficients of the various terms with those of the left hand side. Doing that, one will find the following consistency conditions

$$\sum_{i=0}^{l} c_i\binom{2i+1}{l+i-p+1} = \begin{cases} 1 \text{ for } p=0 \\ 0 \text{ for } p=1,2,\ldots,l \end{cases} \ , \tag{27}$$

$$\sum_{i=0}^{l-1} d_i\left[\binom{2i+1}{l+i-p+1} - \binom{2i+1}{l+i-p}\right] = \begin{cases} -1 \text{ for } p=0 \\ 0 \text{ for } p=1,2,\ldots,l-1 \end{cases} \tag{28}$$

which define the coefficients c_n, d_n recursively and have (25) as solutions. Furthermore, due to (27) and (28) the coefficients c_i and d_i also satisfy

$$\sum_{i=0}^{l} c_i\binom{2i}{l-p+i} = (-1)^p \ , \ \sum_{i=0}^{l-1} d_i\left[\binom{2i}{l-p+i} - \binom{2i}{l-p+i-1}\right] = (-1)^{p+1} \ . \tag{29}$$

5.1. Odd Values of $K+1$

With the above in mind, for the general odd case we get:

$$\mathcal{I}^{2l+1} = \sum_{n=0}^{l} \frac{(-1)^l}{2} c_n\,\partial^{(2n)}\left\{\mathrm{D},\bar{\mathrm{D}}\right\}\left[\partial^{(l-n)}\Phi\,\partial^{(l-n)}\bar{\Phi}\right] \ , \ l = 0,1,\ldots \tag{30}$$

where using the supersymmetry algebra we have converted $i\partial$ to the anticommutator of the spinorial covariant derivatives. Notice that with the exception of this part of the expression, everything else is real. So it will be beneficial if we convert the anticommutator of spinorial derivatives to a commutator of spinorial derivatives using the following identity,

$$\left\{\mathrm{D},\bar{\mathrm{D}}\right\} = \left[\mathrm{D},\bar{\mathrm{D}}\right] + 2\bar{\mathrm{D}}\mathrm{D} \ . \tag{31}$$

The part with the commutator will be a real contribution and the left over term has the structure $\bar{\mathrm{D}}\mathrm{D}\,(\ldots)$. According to (17b) these terms can always be removed by an appropriate choice of the improvement term $W_{\alpha(2l+1)\dot{\alpha}(2l)}$, thus the reality of $\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)}$ can always be guaranteed. Specifically we get:

$$\begin{aligned}\mathcal{I}^{2l+1} = (-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)} \left[\partial^{(l-n)}\mathrm{D}\Phi\, \partial^{(l-n)}\bar{\mathrm{D}}\bar{\Phi}\right] \\ - \tfrac{i}{2}(-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)} \left[\partial^{(l-n+1)}\Phi\, \partial^{(l-n)}\bar{\Phi} - \partial^{(l-n)}\Phi\, \partial^{(l-n+1)}\bar{\Phi}\right] \\ + (-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)}\bar{\mathrm{D}}\mathrm{D} \left[\partial^{(l-n)}\Phi\, \partial^{(l-n)}\bar{\Phi}\right] .\end{aligned} \tag{32}$$

The conclusion of this analysis is that the term $i^{2l+1}\partial^{(2l+1)}\Phi\,\bar{\Phi}$ which appears in the expression of $\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)}$ can be written as:

$$i^{2l+1}\partial^{(2l+1)}\Phi\,\bar{\Phi} = \mathrm{X}^{(2l+1)}_{\alpha(2l+1)\dot{\alpha}(2l+1)} + \tfrac{1}{[(2l+1)!]^2}\bar{\mathrm{D}}_{(\dot{\alpha}_{2l+1}}\mathrm{D}_{(\alpha_{2l+1}}Z^{(2l+1)}_{\alpha(2l))\dot{\alpha}(2l))} \tag{33}$$

where

$$\begin{aligned}\mathrm{X}^{(2l+1)}_{\alpha(2l+1)\dot{\alpha}(2l+1)} = \tfrac{i}{2}(-1)^l \left[\partial^{(2l+1)}\Phi\,\bar{\Phi} - \Phi\,\partial^{(2l+1)}\bar{\Phi}\right] \\ - \tfrac{i}{2}(-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)} \left[\partial^{(l-n+1)}\Phi\, \partial^{(l-n)}\bar{\Phi} - \partial^{(l-n)}\Phi\, \partial^{(l-n+1)}\bar{\Phi}\right] \\ + (-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)} \left[\partial^{(l-n)}\mathrm{D}\Phi\, \partial^{(l-n)}\bar{\mathrm{D}}\bar{\Phi}\right] ,\end{aligned} \tag{34}$$

$$Z^{(2l+1)}_{\alpha(2l)\dot{\alpha}(2l)} = (-1)^l \sum_{n=0}^{l} c_n\, \partial^{(2n)} \left[\partial^{(l-n)}\Phi\, \partial^{(l-n)}\bar{\Phi}\right] \tag{35}$$

and both these quantities are real. These expressions can be further simplified using (26) and (29) to

$$\mathrm{X}^{(2l+1)}_{\alpha(2l+1)\dot{\alpha}(2l+1)} = i(-1)^l \sum_{p=1}^{2l} (-1)^p\, \partial^{(p)}\Phi\, \partial^{(2l+1-p)}\bar{\Phi} + (-1)^l \sum_{p=0}^{2l} (-1)^p\, \partial^{(p)}\mathrm{D}\Phi\, \partial^{(2l-p)}\bar{\mathrm{D}}\bar{\Phi} , \tag{36}$$

$$Z^{(2l+1)}_{\alpha(2l)\dot{\alpha}(2l)} = (-1)^l \sum_{p=0}^{2l} (-1)^p\, \partial^{(p)}\Phi\, \partial^{(2l-p)}\bar{\Phi} . \tag{37}$$

5.2. Even Values of $K+1$

The same analysis can be done for the general even case. For that situation we get

$$\begin{aligned}\mathcal{I}^{(2l)} = \tfrac{1}{2}(-1)^{(l-1)} \sum_{n=0}^{l-1} d_n\, \partial^{(2n)} \left[\partial^{(l-n+1)}\Phi\, \partial^{(l-n-1)}\bar{\Phi} - 2\partial^{(l-n)}\Phi\, \partial^{(l-n)}\bar{\Phi} + \partial^{(l-n-1)}\Phi\, \partial^{(l-n+1)}\bar{\Phi}\right] \\ + i(-1)^{(l-1)} \sum_{n=0}^{l-1} d_n\, \partial^{(2n)} \left[\partial^{(l-n)}\mathrm{D}\Phi\, \partial^{(l-n-1)}\bar{\mathrm{D}}\bar{\Phi} - \partial^{(l-n-1)}\mathrm{D}\Phi\, \partial^{(l-n)}\bar{\mathrm{D}}\bar{\Phi}\right] \\ + i(-1)^{(l-1)} \sum_{n=0}^{l-1} d_n\, \partial^{(2n)}\bar{\mathrm{D}}\mathrm{D} \left[\partial^{(l-n)}\Phi\, \partial^{(l-n-1)}\bar{\Phi} - \partial^{(l-n-1)}\Phi\, \partial^{(l-n)}\bar{\Phi}\right] .\end{aligned} \tag{38}$$

Hence, the term $i^{2l}\partial^{(2l)}\Phi\,\bar{\Phi}$ can be expressed in the following way:

$$i^{2l}\partial^{(2l)}\Phi\,\bar{\Phi} = \mathrm{X}^{(2l)}_{\alpha(2l)\dot{\alpha}(2l)} + \tfrac{1}{[(2l)!]^2}\bar{\mathrm{D}}_{(\dot{\alpha}_{2l}}\mathrm{D}_{(\alpha_{2l}}Z^{(2l)}_{\alpha(2l-1))\dot{\alpha}(2l-1))} \tag{39}$$

where

$$
\begin{aligned}
X^{(2l)}_{\alpha(2l)\dot\alpha(2l)} =\ & \tfrac{1}{2}(-1)^{l}\left[\partial^{(2l)}\Phi\,\bar\Phi + \Phi\,\partial^{(2l)}\bar\Phi\right] \\
& + \tfrac{1}{2}(-1)^{(l-1)}\sum_{n=0}^{l-1} d_n\,\partial^{(2n)}\left[\partial^{(l-n+1)}\Phi\,\partial^{(l-n-1)}\bar\Phi - 2\partial^{(l-n)}\Phi\,\partial^{(l-n)}\bar\Phi + \partial^{(l-n-1)}\Phi\,\partial^{(l-n+1)}\bar\Phi\right] \\
& + i(-1)^{(l-1)}\sum_{n=0}^{l-1} d_n\,\partial^{(2n)}\left[\partial^{(l-n)}\mathrm{D}\Phi\,\partial^{(l-n-1)}\bar{\mathrm{D}}\bar\Phi - \partial^{(l-n-1)}\mathrm{D}\Phi\,\partial^{(l-n)}\bar{\mathrm{D}}\bar\Phi\right] \ , \\
Z^{(2l)}_{\alpha(2l-1)\dot\alpha(2l-1)} =\ & i(-1)^{(l-1)}\sum_{n=0}^{l-1} d_n\,\partial^{(2n)}\left[\partial^{(l-n)}\Phi\,\partial^{(l-n-1)}\bar\Phi - \partial^{(l-n-1)}\Phi\,\partial^{(l-n)}\bar\Phi\right] \ .
\end{aligned} \tag{40}
$$

As in the previous case, both $X^{(2l)}_{\alpha(2l)\dot\alpha(2l)}$ and $Z^{(2l)}_{\alpha(2l-1)\dot\alpha(2l-1)}$ are real. Using (26) and (29) we can simplify these expressions further

$$
X^{(2l)}_{\alpha(2l)\dot\alpha(2l)} = (-1)^{(l-1)}\sum_{p=1}^{2l-1}(-1)^p\,\partial^{(p)}\Phi\,\partial^{(2l-p)}\bar\Phi + i(-1)^{l}\sum_{p=0}^{2l-1}(-1)^p\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(2l-1-p)}\bar{\mathrm{D}}\bar\Phi \ , \tag{41}
$$

$$
Z^{(2l)}_{\alpha(2l-1)\dot\alpha(2l-1)} = i(-1)^{l}\sum_{p=0}^{2l-1}(-1)^p\,\partial^{(p)}\Phi\,\partial^{(2l-1-p)}\bar\Phi \ . \tag{42}
$$

6. Constructing the Higher Spin Supercurrents II: Gauge Invariance and Cubic Interactions

The main point of the previous section is to prove that for every value of integer m we can write

$$
i^{(k+1)}\partial^{(k+1)}\Phi\,\bar\Phi = X^{(k+1)}_{\alpha(k+1)\dot\alpha(k+1)} + \tfrac{1}{[(k+1)!]^2}\bar{\mathrm{D}}_{(\dot\alpha_{k+1}}\mathrm{D}_{(\alpha_{k+1}}Z^{(k+1)}_{\alpha(k))\dot\alpha(k))} \tag{43}
$$

where $X^{(k+1)}_{\alpha(k+1)\dot\alpha(k+1)}$ and $Z^{(k+1)}_{\alpha(k)\dot\alpha(k)}$ are:

$$
X^{(k+1)}_{\alpha(k+1)\dot\alpha(k+1)} = (-i)^{k-1}\sum_{p=1}^{k}(-1)^p\,\partial^{(p)}\Phi\,\partial^{(k+1-p)}\bar\Phi + (-i)^{k}\sum_{p=0}^{k}(-1)^p\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\bar{\mathrm{D}}\bar\Phi \ , \tag{44}
$$

$$
Z^{(k+1)}_{\alpha(k)\dot\alpha(k)} = (-i)^{k}\sum_{p=0}^{k}(-1)^p\,\partial^{(p)}\Phi\,\partial^{(k-p)}\bar\Phi \ . \tag{45}
$$

Thus the expression for $\mathcal{J}_{\alpha(k+1)\dot\alpha(k+1)}$ (17b) becomes:

$$
\begin{aligned}
\mathcal{J}_{\alpha(k+1)\dot\alpha(k+1)} =\ & X^{(k+1)}_{\alpha(k+1)\dot\alpha(k+1)} + \tfrac{1}{(k+1)!(k+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{k+1}}\mathrm{D}_{(\alpha_{k+1}}Z^{(k+1)}_{\alpha(k))\dot\alpha(k))} \\
& + \tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^2\bar U_{\alpha(k))\dot\alpha(k+1)} + \tfrac{k+1}{(k+2)(k+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{k+1}}W_{\alpha(k+1)\dot\alpha(k))} \ .
\end{aligned} \tag{46}
$$

This is useful because it makes obvious that we can always make $\mathcal{J}_{\alpha(k+1)\dot\alpha(k+1)}$ real by choosing

$$
W_{\alpha(k+1)\dot\alpha(k)} = -\tfrac{k+2}{k+1}\mathrm{D}^2U_{\alpha(k+1)\dot\alpha(k)} - \tfrac{k+2}{k+1}\tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}Z^{(k+1)}_{\alpha(k))\dot\alpha(k)} \ . \tag{47}
$$

With this choice we get

$$
\mathcal{J}_{\alpha(k+1)\dot\alpha(k+1)} = X^{(k+1)}_{\alpha(k+1)\dot\alpha(k+1)} + \tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\bar{\mathrm{D}}^2\bar U_{\alpha(k))\dot\alpha(k+1)} - \tfrac{1}{(k+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{k+1}}\mathrm{D}^2U_{\alpha(k+1)\dot\alpha(k))} \ , \tag{48}
$$

$$
\mathcal{T}_{\alpha(k+1)\dot\alpha(k)} = \tfrac{1}{(k+1)!}\mathrm{D}_{(\alpha_{k+1}}\mathcal{T}_{\alpha(k))\dot\alpha(k)} \ , \tag{49}
$$

$$
\mathcal{T}_{\alpha(k)\dot\alpha(k)} = i^k\partial^{(k)}\Phi\,\bar\Phi - \tfrac{k+2}{k+1}Z^{(k+1)}_{\alpha(k)\dot\alpha(k)} + \tfrac{k+2}{k+1}\mathrm{D}^{\alpha_{k+1}}U_{\alpha(k+1)\dot\alpha(k)} + \bar{\mathrm{D}}^{\dot\alpha_{k+1}}\bar U_{\alpha(k)\dot\alpha(k+1)} \ . \tag{50}
$$

Due to Equation (49), the variation of the action can be enhanced from (16) to the following, with the addition of the $\lambda_{\alpha(k+2)\dot{\alpha}(k)}$ term:

$$\begin{aligned}\delta_g S_o =& -g\int\sum_{k=0}^{\infty}\Big\{\left[\bar{\mathrm{D}}^2\ell^{\alpha(k+1)\dot{\alpha}(k)}-\mathrm{D}_{\alpha_{k+2}}\lambda^{\alpha(k+2)\dot{\alpha}(k)}\right]\mathrm{D}_{\alpha_{k+1}}\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}+c.c.\\ &-\tfrac{1}{(k+1)!}\bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}}\ell^{\alpha(k+1)\dot{\alpha}(k))}\,\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}+c.c.\Big\}\\ &+g\int\left\{\bar{\mathrm{D}}^2\ell+\mathrm{D}^2\bar{\ell}\right\}\mathcal{J}\end{aligned} \tag{51}$$

In order to complete Noether's procedure and get an invariant theory we have to add to the starting action S_o the following higher spin, cubic interaction terms

$$\begin{aligned}S_{\text{HS-}\Phi\text{ cubic interactions}} =& \,g\int\sum_{k=0}^{\infty}\Big\{\;H^{\alpha(k+1)\dot{\alpha}(k+1)}\;\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}\\ &+\chi^{\alpha(k+1)\dot{\alpha}(k)}\;\mathrm{D}_{\alpha_{k+1}}\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}+c.c.\Big\}\\ &-g\int\;V\,\mathcal{J}\end{aligned} \tag{52}$$

where V is the real scalar superfield that describes the vector supermultiplet and has the gauge transformation $\delta_0 V=\bar{\mathrm{D}}^2\ell+\mathrm{D}^2\bar{\ell}$ and $H_{\alpha(k+1)\dot{\alpha}(k+1)}$, $\chi_{\alpha(k+1)\dot{\alpha}(k)}$ are the superfields that describe the super-Poincaré higher spin $(k+2, k+3/2)$ supermultiplet with the gauge transformations of (6). These cubic interaction terms generate the higher spin supercurrent $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ and the higher spin supertrace $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$.

As expected, the supercurrent $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ and supertrace $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$ include higher derivative terms. This is a corollary of the *Metsaev bounds* [78], where the number of derivatives that appear in a non-trivial cubic vertex is bounded from below by the highest spin involved and from above by the sum of the spins involved. In our case, there is no upper bound on the spins involved, which is consistent with the higher spin algebra structure[10] [79,80] thus making the number of derivatives that appear in (52) unbounded (as in string field theory).

Due to the higher derivative terms and the fixed engineering dimensions of $H_{\alpha(k+1)\dot{\alpha}(k+1)}$, $\chi_{\alpha(k+1)\dot{\alpha}(k)}$ from the free theory of massless, super-Poincaré higher spins [73,74], we need to have an appropriate dimensionful parameter M in order to balance the engineering dimensions of (52)[11], but since this effect can be easily tracked, for the sake of simplicity we will not explicitly include it. However, it is important to remember its presence since it introduces a scale into the theory. Also the parameter M gives the connection between the gauge parameters $\ell_{\alpha(k+1)\dot{\alpha}(k)}, \lambda_{\alpha(k+2)\dot{\alpha}(k)}$ that appear in (51) with the gauge parameters $L_{\alpha(k+1)\dot{\alpha}(k)}, \Lambda_{\alpha(k+2)\dot{\alpha}(k)}$ that appear in (6).

The conclusion of this section is that a single chiral superfield can have cubic interactions with only the half-integer superspin supermultiplets $(s+1, s+1/2)$ through the higher spin supercurrent and supertrace that have been constructed above, but more importantly although there are two possible descriptions of the $(s+1, s+1/2)$ supermultiplet, the chiral superfield has a preference to only one of them. The one that it chooses to interact with, is the one that appears in the higher spin, $\mathcal{N}$=2 theories as presented in [81].

---

[10] The Jacobi identity requires an infinite tower of fields with unbounded spin.

[11] Multiply the terms inside the curly bracket with $\left(\frac{1}{M}\right)^{k+1}$.

7. Minimal Multiplet of Noether Higher Spin Supercurrents

In the previous section, we found explicit expressions for the higher spin supercurrent and supertrace of the chiral superfield. Using the terminology of [77] these define the *canonical* multiplet of Noether higher spin supercurrents $\left\{\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}, \mathcal{T}_{\alpha(k)\dot{\alpha}(k)}\right\}$. In this section we will show that for any value of the non-negative integer parameter k, there is another higher spin supercurrent multiplet, called the *minimal* multiplet $\left\{\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}, \mathcal{T}^{min}_{\alpha(k)\dot{\alpha}(k)}\right\}$ and we arrive at it by an appropriate choice of the improvement terms such that $\mathcal{T}^{min}_{\alpha(k)\dot{\alpha}(k)} = 0$. In order to get some intuition about this process, it will be useful to examine first a simple example.

7.1. Coupling to Supergravity

For the case of $k = 0$ the canonical multiplet of supercurrents we obtain is

$$\begin{aligned} \mathcal{J}_{\alpha\dot{\alpha}} &= \mathrm{D}_\alpha\Phi\,\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\Phi} + \mathrm{D}_\alpha\bar{\mathrm{D}}^2\bar{U}_{\dot{\alpha}} - \bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2U_\alpha\,, \\ \mathcal{T} &= -\Phi\bar{\Phi} + 2\mathrm{D}^\beta U_\beta + \bar{\mathrm{D}}^{\dot{\beta}}\bar{U}_{\dot{\beta}} \end{aligned} \tag{53}$$

and they generate the cubic interactions between the chiral and non-minimal supergravity supermultiplet. To investigate whether U_α has the potential to completely eliminate one of these supercurrents or reduce it to the point of being zero up to redefinitions of Φ, we consider the following ansatz

$$U_\alpha = f_1\,\mathrm{D}_\alpha\Phi\,\bar{\Lambda} + f_2\,\Phi\,\mathrm{D}_\alpha\bar{\Lambda} \tag{54}$$

where Λ is the prepotential of the chiral field (i.e., $\Phi = \bar{\mathrm{D}}^2\Lambda$). It is straight forward to find that:

$$\begin{aligned} \mathcal{J}_{\alpha\dot{\alpha}} = &\,[1+2f_1-2f_2]\;\mathrm{D}_\alpha\Phi\,\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\Phi} - i[f_1-f_2]\,\partial_{\alpha\dot{\alpha}}\Phi\,\bar{\Phi} + i[f_1-f_2]\,\Phi\partial_{\alpha\dot{\alpha}}\bar{\Phi} \\ &+[f_1-f_2]\,\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}_\alpha\left[\mathrm{D}^2\Phi\,\bar{\Lambda}\right] - [f_1-f_2]\,\mathrm{D}_\alpha\bar{\mathrm{D}}_{\dot{\alpha}}\left[\bar{\mathrm{D}}^2\bar{\Phi}\,\Lambda\right]\,, \end{aligned} \tag{55}$$

$$\begin{aligned} \mathcal{T} = &\,[-1+3f_2-3f_1]\;\Phi\bar{\Phi} + 2[f_1-f_2]\;\mathrm{D}^2\Phi\,\bar{\Lambda} + [f_1-f_2]\;\bar{\mathrm{D}}^2\bar{\Phi}\,\Lambda \\ &+2[f_1+f_2]\;\mathrm{D}^2[\Phi\,\bar{\Lambda}] + [f_1+f_2]\bar{\mathrm{D}}^2[\bar{\Phi}\,\Lambda]\;. \end{aligned} \tag{56}$$

It is obvious that there is no choice of coefficients, f_1 and f_2 that can make $\mathcal{T}$ vanish. However, there is a choice that makes $\mathcal{T}$ proportional to the zeroth order (free theory) equation of motion of Φ. This is important because terms of this type can be absorbed by field redefinitions. If we choose $-f_1 = f_2 = 1/6$ we find

$$\begin{aligned} \mathcal{J}_{\alpha\dot{\alpha}} &= \tfrac{1}{3}\left\{\mathrm{D}_\alpha\Phi\,\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\Phi} + i\partial_{\alpha\dot{\alpha}}\Phi\,\bar{\Phi} - i\Phi\partial_{\alpha\dot{\alpha}}\bar{\Phi}\right\} + \tfrac{1}{3}\left[\mathrm{D}_\alpha\bar{\mathrm{D}}_{\dot{\alpha}}(\Lambda\bar{\mathrm{D}}^2\bar{\Phi}) + c.c.\right] \\ \mathcal{T} &= -\tfrac{2}{3}\mathrm{D}^2\Phi\bar{\Lambda} - \tfrac{1}{3}\bar{\mathrm{D}}^2\bar{\Phi}\Lambda \end{aligned} \tag{57}$$

and therefore by redefining Φ in the following manner

$$\Phi \to \Phi + \tfrac{1}{3}g\bar{\mathrm{D}}^2(\Lambda\,\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}^\alpha H_{\alpha\dot{\alpha}}) - \tfrac{1}{3}g\bar{\mathrm{D}}^2(\Lambda\,\mathrm{D}^\alpha\chi_\alpha) - \tfrac{2}{3}g\bar{\mathrm{D}}^2(\Lambda\,\bar{\mathrm{D}}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}) \tag{58}$$

the S_o term will cancel the parts of the supercurrent and supertrace that have a $\mathrm{D}^2\Phi$, $\bar{\mathrm{D}}^2\bar{\Phi}$ dependence. The outcome of this procedure is the minimal multiplet of Noether supercurrent for the case of supergravity $\{\mathcal{J}^{min}_{\alpha\dot{\alpha}},\ \mathcal{T}^{min}\}$, which is in agreement with the well known results in [77,82][12]

$$\mathcal{J}^{min}_{\alpha\dot{\alpha}} = \tfrac{1}{3}\left\{\mathrm{D}_\alpha\Phi\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\Phi} + i\left(\partial_{\alpha\dot{\alpha}}\Phi\right)\bar{\Phi} - i\Phi\left(\partial_{\alpha\dot{\alpha}}\bar{\Phi}\right)\right\}\ , \tag{59a}$$

$$\mathcal{T}^{min} = 0\ . \tag{59b}$$

Furthermore, the cubic interaction of the chiral superfield with supergravity becomes

$$S_{\text{SG-}\Phi\text{ cubic interactions}} = g\int H^{\alpha\dot{\alpha}}\ \mathcal{J}^{min}_{\alpha\dot{\alpha}}\quad . \tag{60}$$

Nevertheless, we must keep in mind that Φ's redefinition (58) will generate order g^2 terms which we ignore because we focus on the cubic interaction terms. However, an interesting observation is that part of these g^2 terms modify our starting action S_o in the following way

$$\int\Phi\bar{\Phi}\rightarrow\int\left\{1-\tfrac{1}{9}g^2\left[\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{D}^\alpha H_{\alpha\dot{\alpha}} - \mathrm{D}^\alpha\chi_\alpha - 2\bar{\mathrm{D}}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}\right]\left[\mathrm{D}^\alpha\bar{\mathrm{D}}^{\dot{\alpha}}H_{\alpha\dot{\alpha}} + 2\mathrm{D}^\alpha\chi_\alpha + \bar{\mathrm{D}}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}\right]\right\}\Phi\bar{\Phi}\ . \tag{61}$$

Of course this is nothing else than the perturbative construction of the volume element as one should expect for a theory that couples to supergravity.

7.2. Coupling to Higher Superspin Supermultiplets

Based on the previous example, we should check whether the minimal multiplet exists for the general case or not. According to (50), $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$ is a linear combination of terms $\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi}$ for various values of the non-negative integer p. Therefore a relevant ansatz for the improvement term is:

$$U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)} = f_1^{(p)}\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\bar{\Lambda} + f_2^{(p)}\partial^{(p)}\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda}\ . \tag{62}$$

Following the instructions of (50) we calculate $\mathrm{D}^{\alpha_{k+1}}U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)}$

$$\begin{aligned}\mathrm{D}^{\alpha_{k+1}}U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)} =&\ f_2^{(p)}\tfrac{k+2}{k+1}\,\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi} + f_1^{(p)}\tfrac{k+2}{k+1}\,\partial^{(p)}\mathrm{D}^2\Phi\,\partial^{(k-p)}\bar{\Lambda}\\ &+ f_2^{(p)}\,\partial^{(p)}\mathrm{D}^{\alpha_{k+1}}\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda} - f_1^{(p)}\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\mathrm{D}^{\alpha_{k+1}}\bar{\Lambda}\ .\end{aligned} \tag{63}$$

To avoid potential confusion, the explicit expression of the term $\partial^{(p)}\mathrm{D}^{\alpha_{k+1}}\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda}$ is

$$\tfrac{1}{(k+1)k!}\partial_{(\alpha_1(\dot{\alpha}_1}\dots\partial_{\alpha_p\dot{\alpha}_p}\mathrm{D}^{\alpha_{k+1}}\Phi\,\partial_{\alpha_{p+1}\dot{\alpha}_{p+1}}\dots\partial_{\alpha_k\dot{\alpha}_k)}\mathrm{D}_{\alpha_{k+1})}\bar{\Lambda}$$

and by expanding the symmetrization of the indices, one can show that

$$\begin{aligned}\partial^{(p)}\mathrm{D}^{\alpha_{k+1}}\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda} =&\\ &-\tfrac{k-p+1}{k+1}\,\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi} + i\tfrac{k-p}{k+1}\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\bar{\Phi}\\ &+ i\tfrac{p}{k+1}\,\partial^{(p-1)}\bar{\mathrm{D}}\mathrm{D}^2\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda} + i\tfrac{k-p}{k+1}\,\partial^{(p)}\mathrm{D}^2\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\mathrm{D}\bar{\Lambda} - \tfrac{1}{k+1}\,\partial^{(p)}\mathrm{D}^2\Phi\,\partial^{(k-p)}\bar{\Lambda}\\ &- i\tfrac{k-p}{k+1}\mathrm{D}^2\left[\partial^{(p)}\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\mathrm{D}\bar{\Lambda}\right] + \tfrac{1}{k+1}\mathrm{D}^2\left[\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Lambda}\right]\ .\end{aligned} \tag{64}$$

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[12] Keep in mind the difference in conventions for the covariant spinorial derivatives.

Similarly for the term $\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\mathrm{D}^{\alpha_{k+1}}\bar{\Lambda}$ we get

$$\begin{aligned}
&\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\mathrm{D}^{\alpha_{k+1}}\bar{\Lambda} = \\
&\qquad \tfrac{p+1}{k+1}\,\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi} + i\tfrac{k-p}{k+1}\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\bar{\Phi} \\
&\qquad +i\tfrac{p}{k+1}\,\partial^{(p-1)}\bar{\mathrm{D}}\mathrm{D}^2\Phi\,\partial^{(k-p)}\mathrm{D}\bar{\Lambda} - i\tfrac{k-p}{k+1}\,\partial^{(p)}\mathrm{D}^2\Phi\,\partial^{(k-p-1)}\mathrm{D}\bar{\mathrm{D}}\bar{\Lambda} + \tfrac{p+1}{k+1}\,\partial^{(p)}\mathrm{D}^2\Phi\,\partial^{(k-p)}\bar{\Lambda} \\
&\qquad -i\tfrac{k-p}{k+1}\mathrm{D}^2\left[\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\bar{\Lambda}\right] - \tfrac{p+1}{k+1}\mathrm{D}^2\left[\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Lambda}\right]\,.
\end{aligned} \tag{65}$$

Putting together all the above, we get

$$\begin{aligned}
\mathrm{D}^{\alpha_{k+1}}U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)} = {}& \tfrac{p+1}{k+1}\left(f_2^{(p)} - f_1^{(p)}\right)\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi} + i\tfrac{k-p}{k+1}\left(f_2^{(p)} - f_1^{(p)}\right)\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p-1)}\bar{\mathrm{D}}\bar{\Phi} \\
&+\mathrm{D}^2\left[\vartheta\right] + \mathcal{O}(\mathrm{D}^2\Phi)
\end{aligned} \tag{66}$$

where $\mathrm{D}^2\left[\vartheta\right]$ is the sum of the terms that have the structure $\mathrm{D}^2\left[\dots\right]$ and $\mathcal{O}(\mathrm{D}^2\Phi)$ is the sum of the terms that depend on the combination $\mathrm{D}^2\Phi$. Therefore the contribution of $U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)}$ to $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$ is

$$\begin{aligned}
&\tfrac{k+2}{k+1}\mathrm{D}^{\alpha_{k+1}}U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)} + \bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\bar{U}^{(p)}_{\alpha(k)\dot{\alpha}(k+1)} = \\
&\tfrac{k+2}{k+1}\left(f_2^{(p)} - f_1^{(p)}\right)\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi} + \tfrac{p+1}{k+1}\left(f_2^{(p)} - f_1^{(p)}\right)^*\partial^{(k-p)}\Phi\,\partial^{(p)}\bar{\Phi} - \tfrac{k-p}{k+1}\left(f_2^{(p)} - f_1^{(p)}\right)^*\partial^{(k-p-1)}\Phi\,\partial^{(p+1)}\bar{\Phi} \\
&+\tfrac{k+2}{k+1}\mathrm{D}^2\left[\vartheta\right] + \bar{\mathrm{D}}^2\left[\bar{\vartheta}\right] + \tfrac{k+2}{k+1}\mathcal{O}(\mathrm{D}^2\Phi) + \bar{\mathcal{O}}(\bar{\mathrm{D}}^2\bar{\Phi}) + \mathrm{D}\zeta
\end{aligned} \tag{67}$$

where we used $\partial^{(m)}\mathrm{D}\Phi\,\partial^{(n)}\bar{\mathrm{D}}\bar{\Phi} = \mathrm{D}\left(\,\partial^{(m)}\Phi\,\partial^{(n)}\bar{\mathrm{D}}\bar{\Phi}\,\right) - i\partial^{(m)}\Phi\,\partial^{(n+1)}\bar{\Phi}$ and $\mathrm{D}\zeta$ is the sum of terms that have the structure $\mathrm{D}(\dots)$. It is important to observe that due to (i) Equation (49), (ii) the freedom in the definition of $\mathcal{T}_{\alpha(k+1)\dot{\alpha}(k)}$ (18) and (iii) the freedom to redefine the chiral superfield in a manner similar to Section 7.1, all the terms in the last line of (67) can be ignored. Furthermore, the terms in the first line contribute to the appropriate terms of $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$. Hence, if we consider

$$U_{\alpha(k+1)\dot{\alpha}(k)} = \sum_{p=0}^{k} U^{(p)}_{\alpha(k+1)\dot{\alpha}(k)} \tag{68}$$

we have enough freedom to completely cancel $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$. To illustrate this let us do this cancellation for $k=1$ and $k=2$ and then for the general case.

1. <u>$k=1$</u>: The canonical supertrace is $i\partial\Phi\,\bar{\Phi} - \frac{3}{2}Z^{(2)} = -\frac{i}{2}\,\partial\Phi\,\bar{\Phi} + i\frac{3}{2}\,\Phi\,\partial\bar{\Phi}$.
   The contribution of $U^{(1)}$ is $\frac{3}{2}f^{(1)}\,\partial\Phi\,\bar{\Phi} + f^{(1)*}\Phi\,\partial\bar{\Phi}$, where $f^{(1)} = f_2^{(1)} - f_1^{(1)}$.
   The contribution of $U^{(0)}$ is $\frac{1}{2}f^{(0)*}\partial\Phi\,\bar{\Phi} + \left[\frac{3}{2}f^{(0)} - \frac{1}{2}f^{(0)*}\right]\Phi\,\partial\bar{\Phi}$, where $f^{(0)} = f_2^{(0)} - f_1^{(0)}$.
   We can cancel the supertrace competely if we select

$$\begin{aligned}
\tfrac{3}{2}f^{(1)} + \tfrac{1}{2}f^{(0)*} &= \tfrac{i}{2} \\
f^{(1)*} + \tfrac{3}{2}f^{(0)} - \tfrac{1}{2}f^{(0)*} &= -\tfrac{3i}{2}
\end{aligned} \quad \Rightarrow\; f^{(1)} = \tfrac{i}{10}\,,\; f^{(0)} = -\tfrac{7i}{10}\,. \tag{69}$$

2. <u>$k=2$</u>: The canonical supertrace is $\frac{1}{3}\partial^2\Phi\,\bar{\Phi} - \frac{4}{3}\partial\Phi\,\partial\bar{\Phi} + \frac{4}{3}\Phi\,\partial^2\bar{\Phi}$.
   The contribution of $U^{(2)}$ is $\frac{4}{3}f^{(2)}\,\partial^2\Phi\,\bar{\Phi} + f^{(2)*}\,\Phi\,\partial^2\bar{\Phi}$, where $f^{(2)} = f_2^{(2)} - f_1^{(2)}$.
   The contribution of $U^{(1)}$ is $\left[\frac{4}{3}f^{(1)} + \frac{2}{3}f^{(1)*}\right]\,\partial\Phi\,\partial\bar{\Phi} - \frac{1}{3}f^{(1)*}\,\Phi\,\partial^2\bar{\Phi}$, where $f^{(1)} = f_2^{(1)} - f_1^{(1)}$.
   The contribution of $U^{(0)}$ is $\frac{1}{3}f^{(0)*}\,\partial^2\Phi\,\bar{\Phi} - \frac{2}{3}f^{(0)*}\,\partial\Phi\,\partial\bar{\Phi} + \frac{4}{3}f^{(0)}\,\Phi\,\partial^2\bar{\Phi}$, where $f^{(0)} = f_2^{(0)} - f_1^{(0)}$.
   If we select

$$\begin{aligned}
\tfrac{4}{3}f^{(2)} + \tfrac{1}{3}f^{(0)*} &= -\tfrac{1}{3} \\
\tfrac{4}{3}f^{(1)} + \tfrac{2}{3}f^{(1)*} - \tfrac{2}{3}f^{(0)*} &= \tfrac{4}{3} \\
f^{(2)*} - \tfrac{1}{3}f^{(1)*} + \tfrac{4}{3}f^{(0)} &= -\tfrac{4}{3}
\end{aligned} \quad \Rightarrow\; f^{(2)} = -\tfrac{1}{35}\,,\; f^{(1)} = \tfrac{13}{35}\,,\; f^{(0)} = -\tfrac{31}{35} \tag{70}$$

then we completely cancel the supertrace.

3. General k: For the general case, using (68) we can show that up to terms that can be ignored due to chiral redefinition and the freedom in the definitions of the supertrace (18) and (49) we get:

$$\tfrac{k+2}{k+1}\mathrm{D}^{\alpha_{k+1}}U_{\alpha(k+1)\dot{\alpha}(k)} + \bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\bar{U}_{\alpha(k)\dot{\alpha}(k+1)} =$$

$$\left\{\tfrac{k+2}{k+1}f^{(k)} + \tfrac{1}{k+1}f^{(0)^*}\right\}\partial^{(k)}\Phi\,\bar{\Phi} + \sum_{p=0}^{k-1}\left\{\tfrac{k+2}{k+1}f^{(p)} + \tfrac{k+1-p}{k+1}f^{(k-p)^*} - \tfrac{p+1}{k+1}f^{(k-1-p)^*}\right\}\partial^{(p)}\Phi\,\partial^{(k-p)}\bar{\Phi}$$

where $f^{(p)} = f_2^{(p)} - f_1^{(p)}$. Then in order to cancel the supertrace, according to (45) and (50) we must have

$$(k+2)f^{(k)} + f^{(0)^*} = (i)^k\,, \tag{71a}$$

$$(k+2)f^{(p)} + (k+1-p)f^{(k-p)^*} - (p+1)f^{(k-1-p)^*} = (-1)^{k+p}\,(i)^k\,(k+2)\,, \quad p = 0,1,\ldots,k-1\,. \tag{71b}$$

This is a system of $k+1$ linear equations for the $k+1$ parameters $f^{(p)}$, $p = 0,1,...,k$. The solution is

$$f^{(p)} = (-1)^{k+p}\,(i)^k\,\frac{\sum_{j=0}^{k-p}\binom{k+j+1}{p+j+1}\binom{k+1-j}{p+1}}{\binom{2k+3}{k+2}}\,,\quad p = 0,1,...,k\,. \tag{72}$$

The result is that for any value of k, we can find an improvement term in order to go to the minimal multiplet of higher spin supercurrents $\{\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}\,,\,\mathcal{T}^{min}_{\alpha(k)\dot{\alpha}(k)}\}$ where

$$\begin{aligned}\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)} = {}& if^{(k)}\,\partial^{(k+1)}\Phi\,\bar{\Phi} - if^{(k)^*}\,\Phi\,\partial^{(k+1)}\bar{\Phi}\\ &+i\sum_{p=1}^{k}\left\{(-1)^{k+p}\,(i)^k + f^{(p-1)} - f^{(k-p)^*}\right\}\partial^{(p)}\Phi\,\partial^{(k+1-p)}\bar{\Phi}\\ &+\sum_{p=0}^{k}\left\{(-1)^{k+p}\,(i)^k - f^{(p)} - f^{(k-p)^*}\right\}\partial^{(p)}\mathrm{D}\Phi\,\partial^{(k-p)}\bar{\mathrm{D}}\bar{\Phi}\,,\end{aligned} \tag{73}$$

$$\mathcal{T}^{min}_{\alpha(k)\dot{\alpha}(k)} = 0\,. \tag{74}$$

For $k = 1$ and $k = 2$ we get

$$\mathcal{J}^{min}_{\alpha\beta\dot{\alpha}\dot{\beta}} = -\tfrac{1}{10}\,\partial^{(2)}\Phi\,\bar{\Phi} - \tfrac{1}{10}\,\Phi\,\partial^{(2)}\bar{\Phi} + \tfrac{2}{5}\,\partial\Phi\,\partial\bar{\Phi} - \tfrac{1}{5}i\,\mathrm{D}\Phi\,\partial\bar{\mathrm{D}}\bar{\Phi} + \tfrac{1}{5}i\,\partial\mathrm{D}\Phi\,\bar{\mathrm{D}}\bar{\Phi}\,, \tag{75}$$

$$\begin{aligned}\mathcal{J}^{min}_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} = {}& -\tfrac{i}{35}\,\partial^{(3)}\Phi\,\bar{\Phi} + \tfrac{i}{35}\,\Phi\,\partial^{(3)}\bar{\Phi} + i\tfrac{9}{35}\,\partial^{(2)}\Phi\,\partial\bar{\Phi} - i\tfrac{9}{35}\,\partial\Phi\,\partial^{(2)}\bar{\Phi}\\ &-\tfrac{3}{35}\,\partial^{(2)}\mathrm{D}\Phi\,\bar{\mathrm{D}}\bar{\Phi} - \tfrac{3}{35}\,\mathrm{D}\Phi\,\partial^{(2)}\bar{\mathrm{D}}\bar{\Phi} + \tfrac{9}{35}\,\partial\mathrm{D}\Phi\,\partial\bar{\mathrm{D}}\bar{\Phi}\,.\end{aligned} \tag{76}$$

These expressions match the results of [41] which give the superconformal higher spin supercurrent. In the minimal supercurrent multiplet, the cubic interactions of the chiral supermultiplet with the higher spin supermultiplets are

$$S_{\text{HS-}\Phi\text{ cubic interactions}} = g\int\sum_{k=0}^{\infty}\,H^{\alpha(k+1)\dot{\alpha}(k+1)}\,\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}\,. \tag{77}$$

8. On-Shell Conservation Equations

Using Noether's method, we have constructed an invariant action up to order g. Hence, for every unconstrained parameter $\ell_{\alpha(k+1)\dot{\alpha}(k)}$ and ℓ we generate a Bianchi identity, which express the invariance of the action. Once we go on-shell and take into account the equation of motion of Φ, the Bianchi identities reduce to the following on-shell conservation equations for the *canonical* multiplet of the higher spin supercurrents.

$$\bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)} = \frac{1}{(k+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{k+1}}\mathcal{T}_{\alpha(k))\dot{\alpha}(k)} \;,\; k = 0,1,2,\dots \;, \tag{78}$$

$$\bar{\mathrm{D}}^2\mathcal{J} = 0 \;. \tag{79}$$

It is straightforward to verify the validity of these on-shell equations using the expressions (48)–(50). For the *minimal* multiplet, the conservation equation takes the much simpler form

$$\bar{\mathrm{D}}^{\dot{\alpha}_{k+1}}\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)} = 0 \;,\; k = 0,1,2,\dots \tag{80}$$

After a bit of work, one can verify that Equation (73) satisfies this conservation equation. However, instead of using (73) we can get a simpler expression for the minimal higher spin supercurrent by using the conservation equation to define the coefficients of the various terms. From the previous section we know that the general ansatz for the minimal, higher spin supercurrent is

$$\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)} = \sum_{p=0}^{s} a_p\,\partial^{(p)}\Phi\,\partial^{(s-p)}\bar{\Phi} + \sum_{p=0}^{s-1} b_p\,\partial^{(p)}\mathrm{D}\Phi\,\partial^{(s-p-1)}\bar{\mathrm{D}}\bar{\Phi}\,. \tag{81}$$

We also know that $\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)}$ must be real, hence

$$a_p = a^*_{s-p} \;,\; p = 0,1,...,s \;, \tag{82}$$

$$b_p = b^*_{s-p-1} \;,\; p = 0,1,...,s-1 \tag{83}$$

and the on-shell conservation ($\bar{\mathrm{D}}^{\dot{\alpha}_s}\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)} = 0$), also gives the constraint

$$i\,a_{p+1}\left[\tfrac{p+1}{s}\right] + b_p\left[\tfrac{s-p}{s}\right] = 0 \;,\; p = 0,1,...,s-1 \;. \tag{84}$$

The constraints (82)–(84) fix a_p and b_p to be (up to a real proportionality constant)

$$a_p = (-1)^p(i)^s\binom{s}{p}^2 \;, \tag{85}$$

$$b_p = (-1)^p(i)^{s+1}\left(\frac{s-p}{p+1}\right)\binom{s}{p}^2 \tag{86}$$

and $\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)}$ is proportional to

$$\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)} \sim (i)^s\sum_{p=0}^{s}(-1)^p\binom{s}{p}^2\left\{\partial^{(p)}\Phi\,\partial^{(s-p)}\bar{\Phi} + i\left(\frac{s-p}{p+1}\right)\partial^{(p)}\mathrm{D}\Phi\,\partial^{(s-p-1)}\bar{\mathrm{D}}\bar{\Phi}\right\}. \tag{87}$$

We can fix the overall constant of proportionality by comparing this expression to (73), thus we get

$$\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)} = \frac{(-i)^s}{\binom{2s+1}{s+1}}\sum_{p=0}^{s}(-1)^p\binom{s}{p}^2\left\{\partial^{(p)}\Phi\,\partial^{(s-p)}\bar{\Phi} + i\left(\frac{s-p}{p+1}\right)\partial^{(p)}\mathrm{D}\Phi\,\partial^{(s-p-1)}\bar{\mathrm{D}}\bar{\Phi}\right\}. \tag{88}$$

It is easy to check that this expression agrees with Equations (59a), (73), (75) and (76) and up to an overall coefficient it also agrees with the results in [41].

9. Component Discussion

In the literature there are various sets of conserved currents that generate the cubic interactions of a complex scalar (two spin 0) and a spinor (one spin 1/2) with higher spins [58–63]. It is important to find how the results of previous sections translate at the component description.

In principle, we can start with Equation (77) and switch to the component formulation by evaluating the θ integrals in order to find the component analogue. However, for the purpose of identifying the higher spin, conserved currents, a conceptual cleaner approach would be to start with the superspace conservation Equation (80) and project it down to the component level, in order to derive the spacetime conservation equation of the currents. The latter is the approach that we will follow and the definition of components we will use is:

$$\begin{aligned}
\Phi^{(0,0)}_{\alpha(n)\dot\alpha(m)} &= \Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~, & \Phi^{(1,0)}_{\beta\alpha(n)\dot\alpha(m)} &= D_\beta\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~,\\
\Phi^{(0,1)}_{\alpha(n)\dot\beta\dot\alpha(m)} &= \bar D_{\dot\beta}\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~, & \Phi^{(1,1)}_{\beta\alpha(n)\dot\beta\dot\alpha(m)} &= -\tfrac{1}{2}\left[D_\beta,\bar D_{\dot\beta}\right]\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~,\\
\Phi^{(2,0)}_{\alpha(n)\dot\alpha(m)} &= -D^2\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~, & \Phi^{(0,2)}_{\alpha(n)\dot\alpha(m)} &= -\bar D^2\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~,\\
\Phi^{(2,1)}_{\alpha(n)\dot\beta\dot\alpha(m)} &= -\tfrac{1}{2}\left\{D^2,\bar D_{\dot\beta}\right\}\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~, & \Phi^{(1,2)}_{\beta\alpha(n)\dot\alpha(m)} &= -\tfrac{1}{2}\left\{\bar D^2,D_\beta\right\}\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~,\\
\Phi^{(2,2)}_{\alpha(n)\dot\alpha(m)} &= \tfrac{1}{2}\{D^2,\bar D^2\}\Phi_{\alpha(n)\dot\alpha(m)}| - \tfrac{1}{4}\Box\Phi_{\alpha(n)\dot\alpha(m)}|_{\theta=0} ~.
\end{aligned} \tag{89}$$

The various components are labeled by the name of the superfield they come from and their position (n,m) in its θ expansion

$$\begin{aligned}
\Phi_{\alpha(n)\dot\alpha(m)} = {}&\Phi_{\alpha(n)\dot\alpha(m)} + \theta^\beta\Phi^{(1,0)}_{\beta\alpha(n)\dot\alpha(m)} + \bar\theta^{\dot\beta}\Phi^{(0,1)}_{\alpha(n)\dot\beta\dot\alpha(m)} + \theta^2\Phi^{(2,0)}_{\alpha(n)\dot\alpha(m)} + \bar\theta^2\Phi^{(0,2)}_{\alpha(n)\dot\alpha(m)}\\
&+\theta^\beta\bar\theta^{\dot\beta}\Phi^{(1,1)}_{\beta\alpha(n)\dot\beta\dot\alpha(m)} + \theta^\beta\bar\theta^2\Phi^{(1,2)}_{\beta\alpha(n)\dot\alpha(m)} + \theta^2\bar\theta^{\dot\beta}\Phi^{(2,1)}_{\alpha(n)\dot\beta\dot\alpha(m)} + \theta^2\bar\theta^2\Phi^{(2,2)}_{\alpha(n)\dot\alpha(m)} ~.
\end{aligned} \tag{90}$$

Furthermore, components with more than one index of the same type can be decomposed into symmetric (S) and anti-symmetric (A) pieces as follows

$$\begin{aligned}
F_{\beta\alpha(n)\dot\alpha(m)} &= F^{(S)}_{\beta\alpha(n)\dot\alpha(m)} + \tfrac{n}{(n+1)!}C_{\beta(\alpha_n}F^{(A)}_{\alpha(n-1))\dot\alpha(m)} ~,\\
F^{(S)}_{\beta\alpha(n)\dot\alpha(m)} &= \tfrac{1}{(n+1)!}F_{(\beta\alpha(n))\dot\alpha(m)} ~,~ F^{(A)}_{\alpha(n-1)\dot\alpha(m)} = C^{\beta\alpha_n}F_{\beta\alpha(n)\dot\alpha(m)} ~.
\end{aligned} \tag{91}$$

Using the above, it is straightforward to project Equation (80) and the results we find for the bosonic components are:

$$\partial^{\alpha_s\dot\alpha_s}\mathcal{J}^{min~(0,0)}_{\alpha(s)\dot\alpha(s)} = 0 ~, \tag{92a}$$

$$\mathcal{J}^{min~(0,2)}_{\alpha(s)\dot\alpha(s)} = 0 ~, \tag{92b}$$

$$\mathcal{J}^{min~(1,1)(S,A)}_{\alpha(s+1)\dot\alpha(s-1)} = -\tfrac{i}{2(s+1)!}\partial_{(\alpha_{s+1}}{}^{\dot\alpha_s}\mathcal{J}^{min~(0,0)}_{\alpha(s))\dot\alpha(s)} ~, \tag{92c}$$

$$\mathcal{J}^{min~(1,1)(A,A)}_{\alpha(s-1)\dot\alpha(s-1)} = 0 ~, \tag{92d}$$

$$\partial^{\alpha_{s+1}\dot\alpha_{s+1}}\mathcal{J}^{min~(1,1)(S,S)}_{\alpha(s+1)\dot\alpha(s+1)} = 0 ~, \tag{92e}$$

$$\mathcal{J}^{min~(2,2)}_{\alpha(k+1)\dot\alpha(k+1)} = -\tfrac{1}{4}\Box\mathcal{J}^{(0,0)}_{\alpha(k+1)\dot\alpha(k+1)} \tag{92f}$$

and for the fermionic components we get:

$$\mathcal{J}^{min~(0,1)(A)}_{\alpha(s)\dot\alpha(s-1)} = 0 ~, \tag{93a}$$

$$\mathcal{J}^{min\ (1,2)(S)}_{\alpha(s+1)\dot{\alpha}(s)} \ = \ \tfrac{i}{2(s+1)!}\partial_{(\alpha_{s+1})}{}^{\dot{\alpha}_{s+1}}\mathcal{J}^{min\ (0,1)(S)}_{\alpha(s)\dot{\alpha}(s+1)} \ , \tag{93b}$$

$$\mathcal{J}^{min\ (1,2)(A)}_{\alpha(s-1)\dot{\alpha}(s)} \ = \ 0 \ , \tag{93c}$$

$$\partial^{\alpha_{s+1}\dot{\alpha}_s}\mathcal{J}^{min\ (1,0)(S)}_{\alpha(s+1)\dot{\alpha}(s)} \ = \ 0 \ . \tag{93d}$$

The lesson is that the component $\mathcal{J}^{min\ (1,1)(S,S)}_{\alpha(s+1)\dot{\alpha}(s+1)}$ is the *minimal* integer spin current and Equation (92e) is its conservation equation. The cubic interactions it generates are of the type

$$\int d^4x \sum_{s=0}^{\infty} h^{\alpha(s+1)\dot{\alpha}(s+1)}\,\mathcal{J}^{min\ (1,1)(S,S)}_{\alpha(s+1)\dot{\alpha}(s+1)} \tag{94}$$

where the field $h_{\alpha(s+1)\dot{\alpha}(s+1)}$ is the symmetric, traceless part of the free, massless, integer spin $j = s+1$ ($h_{\alpha(s+1)\dot{\alpha}(s+1)} \sim \left[\mathrm{D}_{(\alpha_{s+1}}, \bar{\mathrm{D}}_{(\dot{\alpha}_{s+1}}\right] H_{\alpha(s))\dot{\alpha}(s))}|$). From Equation (88) we get

$$\begin{aligned}\mathcal{J}^{min\ (1,1)(S,S)}_{\alpha(s+1)\dot{\alpha}(s+1)} \sim (-i)^s \sum_{p=0}^{s}(-1)^p \binom{s}{p}^2 \Big\{ & i\,\partial^{(p)}\phi\,\partial^{(s+1-p)}\bar{\phi} \ - \ i\left[\tfrac{2s+1-p}{p+1}\right]\,\partial^{(p+1)}\phi\,\partial^{(s-p)}\bar{\phi} \\ & + \left[\tfrac{s+p+2}{p+1}\right]\,\partial^{(p)}\chi\,\partial^{(s-p)}\bar{\chi} \ - \ \left[\tfrac{s-p}{p+1}\right]\,\partial^{(p+1)}\chi\,\partial^{(s-p-1)}\bar{\chi} \Big\} \ .\end{aligned} \tag{95}$$

Observe, that there are two contributions into these integer spin currents. The first one is the boson—boson contribution and includes the two terms of the first line, where $\phi = \Phi|$. This corresponds to the bosonic integer spin current that appears in [59] and also the traceless part of the currents in [58,63]. The second contribution is the fermion—fermion one and includes the two terms of the second line, where $\chi_\alpha = \mathrm{D}_\alpha\Phi|$. This corresponds to the fermionic integer spin current that appears in [59].

Furthermore, Equation (93d) gives the conservation of the half-integer spin current $\mathcal{J}^{min\ (1,0)(S)}_{\alpha(s+1)\dot{\alpha}(s)}$. The cubic interactions we get are:

$$\int d^4x \sum_{s=0}^{\infty} \psi^{\alpha(s+1)\dot{\alpha}(s)}\,\mathcal{J}^{min\ (1,0)(S)}_{\alpha(s+1)\dot{\alpha}(s)} + c.c. \tag{96}$$

where $\psi_{\alpha(s+1)\dot{\alpha}(s)}$ is the symmetric, traceless and γ-traceless part of the free, massless, half-integer spin $j = s+1/2$ ($\psi_{\alpha(s+1)\dot{\alpha}(s)} \sim \left\{\mathrm{D}_{(\alpha_{s+1}}, \bar{\mathrm{D}}^2\right\} H_{\alpha(s))\dot{\alpha}(s)}|$). Again using (88) we get

$$\mathcal{J}^{min\ (1,0)(S)}_{\alpha(s+1)\dot{\alpha}(s)} \sim (-i)^s \sum_{p=0}^{s} \ (-1)^p \binom{s}{p}^2 \binom{s+1}{p+1} \ \partial^{(p)}\chi\,\partial^{(s-p)}\bar{\phi} \ . \tag{97}$$

This is the half-integer spin current and appears for the first time in the literature and it has only one contribution of the fermion—boson type.

Finally, we notice that Equation (92a) is the conservation of another current. This corresponds to the $\mathcal{R}$-symmetry current and it has the form

$$\mathcal{J}^{min\ (0,0)}_{\alpha(s)\dot{\alpha}(s)} \sim (-i)^s \sum_{p=0}^{s} \ (-1)^p \binom{s}{p}^2 \Big\{ \ \partial^{(p)}\phi\,\partial^{(s-p)}\bar{\phi} + i\left(\frac{s-p}{p+1}\right) \ \partial^{(p)}\chi\,\partial^{(s-p-1)}\bar{\chi} \ \Big\} \ . \tag{98}$$

10. Massive Chiral Superfield

10.1. Higher Spin Supercurrent and Supertrace

So far we have discussed the higher spin supercurrent multiplet of a free, massless chiral superfield. In this section, we repeat the analysis for a massive chiral superfield, with a starting action $S_o + S_m$ where S_o is given by (13) and S_m is the mass term:

$$S_m = \tfrac{m}{2} \int d^6z \, \Phi^2 \; + c.c. \,. \tag{99}$$

The variation of this extra term under (12) is

$$\begin{aligned}\delta_g S_m = -gm \sum_{k=0}^{\infty} \int d^6z \Big\{ &\bar{\mathrm{D}}^2 \ell^{\alpha(k+1)\dot{\alpha}(k)} \; i^k \partial^{(k)} \mathrm{D}\Phi \, \Phi \; + c.c. \\ &- \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))} \; i^{k+1} \partial^{(k+1)} \Phi \, \Phi \; + c.c. \Big\} \\ +gm \int d^6z \, \bar{\mathrm{D}}^2 \ell \, \Phi \, \Phi \; + c.c. \qquad &.\end{aligned} \tag{100}$$

It is straight forward to show that:

$$\begin{aligned}\bar{\mathrm{D}}^2 \ell^{\alpha(k+1)\dot{\alpha}(k)} \; i^k \partial^{(k)} \mathrm{D}\Phi \, \Phi \; - \; \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))} \; i^{k+1} \partial^{(k+1)} \Phi \, \Phi = \\ \tfrac{1}{2} \bar{\mathrm{D}}^2 \Big[\bar{\mathrm{D}}^2 \ell^{\alpha(k+1)\dot{\alpha}(k)} \; \Big\{ i^k \partial^{(k)} \mathrm{D}\Lambda \, \Phi \; + \; i^k \partial^{(k)} \mathrm{D}\Phi \, \Lambda \Big\} \\ - \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))} \; \Big\{ i^{k+1} \partial^{(k+1)} \Lambda \, \Phi \; + \; i^{k+1} \partial^{(k+1)} \Phi \, \Lambda \Big\} \Big] \,,\end{aligned} \tag{101}$$

$$\bar{\mathrm{D}}^2 \ell \Phi \, \Phi = \bar{\mathrm{D}}^2 \Big[\bar{\mathrm{D}}^2 \ell \, \Lambda \, \Phi \Big] \tag{102}$$

and by absorbing the overall $\bar{\mathrm{D}}^2$ factor, we can convert the integration over the entire superspace:

$$\begin{aligned}\delta_g S_m = \tfrac{g}{2} m \int \sum_{k=0}^{\infty} \Big\{ &\bar{\mathrm{D}}^2 \ell^{\alpha(k+1)\dot{\alpha}(k)} \; \Big[i^k \partial^{(k)} \mathrm{D}\Lambda \, \Phi \; + \; i^k \partial^{(k)} \mathrm{D}\Phi \, \Lambda \Big] \; + c.c. \\ &- \bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))} \; \Big[i^{k+1} \partial^{(k+1)} \Lambda \, \Phi \; + \; i^{k+1} \partial^{(k+1)} \Phi \, \Lambda \Big] \; + c.c. \Big\} \\ -gm \int \; \bar{\mathrm{D}}^2 \ell \, \Lambda \, \Phi \; + c.c. \qquad &.\end{aligned} \tag{103}$$

From this expression we can extract the contribution of the mass term to Equations (17a) and (17b). However, in order to couple the theory purely to higher spin supermultiplets the coefficient of $\bar{\mathrm{D}}^{(\dot{\alpha}_{k+1}} \ell^{\alpha(k+1)\dot{\alpha}(k))}$ must be written as a real term plus total spinorial or spacetime derivative terms. For the massless theory, we have proven this property via Equation (43) and it holds for any value of k. The story for a massive chiral is different as we will show that only the even values of $k = 2l$ can satisfy such a requirement.

The relevant quantity for the mass term is $i^{k+1} \partial^{(k+1)} \Lambda \, \Phi + i^{k+1} \partial^{(k+1)} \Phi \, \Lambda$. It is easy to show that this combination can be written in the following manner:

$$
\begin{aligned}
& i^{k+1}\partial^{(k+1)}\Lambda\,\Phi \;+\; i^{k+1}\partial^{(k+1)}\Phi\,\Lambda = \\
&= \begin{cases} i\partial\left[\;\sum\limits_{n=0}^{2l}\;(-1)^{l+n}\,\partial^{(n)}\Lambda\,\partial^{(2l-n)}\Phi\;\right]\;,\; for\; k=2l\;,\; l=0,1,2,\ldots \\ \partial\left[\;\sum\limits_{n=0}^{l}\;(-1)^{l+n+1}\,\partial^{(n)}\Lambda\,\partial^{(2l+1-n)}\Phi\;+\;\sum\limits_{n=l+1}^{2l+1}(-1)^{l+n}\,\partial^{(n)}\Lambda\,\partial^{(2l+1-n)}\Phi\;\right]+2\,\partial^{(l+1)}\Lambda\,\partial^{(l+1)}\Phi\;, \\ \qquad for\; k=2l+1\;,\; l=0,1,2,\ldots \end{cases}
\end{aligned}
\tag{104}
$$

therefore, for odd values of k and due to the presence of the term $\partial^{(l+1)}\Lambda\;\partial^{(l+1)}\Phi$, there is no improvement term $W_{\alpha(2l+2)\dot\alpha(2l+1)}$ to eliminate the imaginary part of $\mathcal{J}_{\alpha(2l+2)\dot\alpha(2l+2)}$. Hence, in order to construct an invariant theory of a massive chiral interacting with irreducible higher spin supermultiplets, all terms in $\delta_g\,(S_o+S_m)$ that correspond to an odd value of k must be set to zero. For that reason the parameters ℓ and $\ell_{\alpha(2l+2)\dot\alpha(2l+1)}$ for $l=0,1,2,\ldots$ must vanish and the transformation of Φ we must consider in this massive case is reduced to:

$$
\begin{aligned}
\delta_g\Phi =& -g\sum_{l=0}^{\infty}\Big\{\;\bar{\mathrm{D}}^2\ell^{\alpha(2l+1)\dot\alpha(2l)}\;\mathrm{D}_{\alpha_{2l+1}}\bar{\mathrm{D}}_{\dot\alpha_{2l}}\mathrm{D}_{\alpha_{2l}}\ldots\bar{\mathrm{D}}_{\dot\alpha_1}\mathrm{D}_{\alpha_1}\Phi \\
&\qquad -\tfrac{1}{(2l+1)!}\bar{\mathrm{D}}^{(\dot\alpha_{2l+1}}\ell^{\alpha(2l+1)\dot\alpha(2l))}\;\bar{\mathrm{D}}_{\dot\alpha_{2l+1}}\mathrm{D}_{\alpha_{2l+1}}\ldots\bar{\mathrm{D}}_{\dot\alpha_1}\mathrm{D}_{\alpha_1}\Phi\Big\}\;\;.
\end{aligned}
\tag{105}
$$

Moreover, we can show that for the case of $k=2l$ the quantity $i^{2l}\partial^{(2l)}\mathrm{D}\Lambda\,\Phi+i^{2l}\partial^{(2l)}\mathrm{D}\Phi\,\Lambda$ which appears in (103) as the coefficient of $\bar{\mathrm{D}}^2\ell^{\alpha(2l+1)\dot\alpha(2l)}$ can be expressed in the following way:

$$
i^{2l}\partial^{(2l)}\mathrm{D}\Lambda\,\Phi+i^{2l}\partial^{(2l)}\mathrm{D}\Phi\,\Lambda=\mathrm{D}\left[\;(-1)^{l}\,\Lambda\,\partial^{(2l)}\Phi\;\right]+\partial\left[\;\sum_{n=0}^{2l-1}\;(-1)^{l+n+1}\,\partial^{(n)}\mathrm{D}\Lambda\,\partial^{(2l-1-n)}\Phi\;\right]\;. \tag{106}
$$

With all the above into account, we get that

$$
\begin{aligned}
\mathcal{J}_{\alpha(2l+1)\dot\alpha(2l+1)}=&\;X^{(2l+1)}_{\alpha(2l+1)\dot\alpha(2l+1)}+\tfrac{1}{(2l+1)!^2}\bar{\mathrm{D}}_{(\dot\alpha_{2l+1}}\mathrm{D}_{(\alpha_{2l+1}}Z^{(2l+1)}_{\alpha(2l))\dot\alpha(2l))}-\tfrac{im}{2(2l+1)!^2}\partial_{(\alpha_{2l+1}(\dot\alpha_{2l+1}}Y_{\alpha(2l))\dot\alpha(2l))} \\
&+\tfrac{1}{(2l+1)!}\mathrm{D}_{(\alpha_{2l+1}}\bar{\mathrm{D}}^2\bar{U}_{\alpha(2l))\dot\alpha(2l+1)}+\tfrac{2l+1}{(2l+2)(2l+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{2l+1}}W_{\alpha(2l+1)\dot\alpha(2l))}
\end{aligned}
\tag{107}
$$

with

$$
Y_{\alpha(2l)\dot\alpha(2l)}=\sum_{n=0}^{2l}\;(-1)^{l+n}\,\partial^{(n)}\Lambda\,\partial^{(2l-n)}\Phi\;\;. \tag{108}
$$

Now it is obvious that we can always make $\mathcal{J}_{\alpha(2l+1)\dot\alpha(2l+1)}$ real by selecting $W_{\alpha(2l+1)\dot\alpha(2l)}$ as follows:

$$
W_{\alpha(2l+1)\dot\alpha(2l)}=-\tfrac{2l+2}{2l+1}\mathrm{D}^2U_{\alpha(2l+1)\dot\alpha(2l)}-\tfrac{2l+2}{(2l+1)(2l+1)!}\mathrm{D}_{(\alpha_{2l+1}}\Big[Z^{(2l+1)}_{\alpha(2l))\dot\alpha(2l)}-\tfrac{m}{2}\Big(Y_{\alpha(2l))\dot\alpha(2l)}+\bar{Y}_{\alpha(2l))\dot\alpha(2l)}\Big)\Big] \tag{109}
$$

and the expressions for $\mathcal{J}_{\alpha(2l+1)\dot\alpha(2l+1)}$ and $\mathcal{T}_{\alpha(2l+1)\dot\alpha(2l)}$ become

$$
\begin{aligned}
\mathcal{J}_{\alpha(2l+1)\dot\alpha(2l+1)}=&\;X^{(2l+1)}_{\alpha(2l+1)\dot\alpha(2l+1)}+\tfrac{m}{2(2l+1)!^2}\Big[\bar{\mathrm{D}}_{(\dot\alpha_{2l+1}}\mathrm{D}_{(\alpha_{2l+1}}\bar{Y}_{\alpha(2l))\dot\alpha(2l))}-\mathrm{D}_{(\alpha_{2l+1}}\bar{\mathrm{D}}_{(\dot\alpha_{2l+1}}Y_{\alpha(2l))\dot\alpha(2l))}\Big] \\
&+\tfrac{1}{(2l+1)!}\Big[\mathrm{D}_{(\alpha_{2l+1}}\bar{\mathrm{D}}^2\bar{U}_{\alpha(2l))\dot\alpha(2l+1)}-\bar{\mathrm{D}}_{(\dot\alpha_{2l+1}}\mathrm{D}^2U_{\alpha(2l+1)\dot\alpha(2l))}\Big]\;\;,
\end{aligned}
\tag{110}
$$

$$
\mathcal{T}_{\alpha(2l+1)\dot\alpha(2l)}=\tfrac{1}{(2l+1)!}\mathrm{D}_{(\alpha_{2l+1}}\mathcal{T}_{\alpha(2l))\dot\alpha(2l)}\;\;, \tag{111}
$$

$$
\begin{aligned}
\mathcal{T}_{\alpha(2l)\dot\alpha(2l)}=&\;(-1)^l\partial^{(2l)}\Phi\,\bar{\Phi}-\tfrac{2(l+1)}{2l+1}Z^{(2l+1)}_{\alpha(2l)\dot\alpha(2l)}+\tfrac{m(l+1)}{2l+1}\Big(Y_{\alpha(2l)\dot\alpha(2l)}+\bar{Y}_{\alpha(2l)\dot\alpha(2l)}\Big)+\tfrac{m}{2}\Omega_{\alpha(2l)\dot\alpha(2l)} \\
&+\tfrac{2(l+1)}{2l+1}\mathrm{D}^{\alpha_{2l+1}}U_{\alpha(2l+1)\dot\alpha(2l)}+\bar{\mathrm{D}}^{\dot\alpha_{2l+1}}\bar{U}_{\alpha(2l)\dot\alpha(2l+1)}
\end{aligned}
\tag{112}
$$

where

$$\Omega_{\alpha(2l)\dot{\alpha}(2l)} = (-1)^{l+1}\,\Lambda\,\partial^{(2l)}\Phi + i\sum_{n=0}^{2l-1}\,(-1)^{l+1+n}\,\partial^{(n)}\bar{\mathrm{D}}\mathrm{D}\Lambda\,\partial^{(2l-1-n)}\Phi \quad . \tag{113}$$

The result for the variation of the $S_o + S_m$ theory is

$$\begin{aligned}\delta_g\,(S_o + S_m) =& -g\int\sum_{l=0}^{\infty}\Big\{\,\Big[\bar{\mathrm{D}}^2\ell^{\alpha(2l+1)\dot{\alpha}(2l)} - \mathrm{D}_{\alpha_{2l+2}}\lambda^{\alpha(2l+2)\dot{\alpha}(2l)}\Big]\,\mathrm{D}_{\alpha_{2l+1}}\mathcal{T}_{\alpha(2l)\dot{\alpha}(2l)} + c.c. \\ &- \tfrac{1}{(2l+1)!}\bar{\mathrm{D}}^{(\dot{\alpha}_{2l+1}}\ell^{\alpha(2l+1)\dot{\alpha}(2l))}\;\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)} + c.c.\Big\}\end{aligned} \tag{114}$$

where $\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)}$ and $\mathcal{T}_{\alpha(2l+1)\dot{\alpha}(2l)}$ are given by (110) and (112). Therefore to get the invariant theory we have to add the following higher spin, cubic interaction terms

$$\begin{aligned}S_{\text{HS-massive chiral}} = g\int\sum_{l=0}^{\infty}\Big\{\;& H^{\alpha(2l+1)\dot{\alpha}(2l+1)}\;\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)} \\ &+ \chi^{\alpha(2l+1)\dot{\alpha}(2l)}\;\mathrm{D}_{\alpha_{2l+1}}\mathcal{T}_{\alpha(2l)\dot{\alpha}(2l)} + c.c.\Big\}\quad .\end{aligned} \tag{115}$$

Apart from the various mass terms that deform the expressions for the higher spin supercurrent and supertrace, the biggest difference from the massless chiral story is that the massive chiral superfields has cubic interactions only with $(2l+2\;,\;2l+3/2)$ supermultiplets that correspond to superspin $Y = 2l + 3/2$. This includes supergravity $(l = 0)$ but not the vector supermultiplet.

10.2. Minimal Multiplet of Higher Spin Supercurrents

Similar to the massless case, expressions (110) and (112) include an arbitrary improvement term $U_{\alpha(2l+1)\dot{\alpha}(2l)}$, hence we have to check whether this freedom can be used to completely eliminate the supertrace. For the case of supergravity the *canonical* supercurrent multiplet we get is:

$$\mathcal{J}_{\alpha\dot{\alpha}} = \mathrm{D}_\alpha\Phi\,\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\Phi} + \tfrac{m}{2}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}_\alpha\,(\bar{\Lambda}\bar{\Phi}) - \tfrac{m}{2}\mathrm{D}_\alpha\bar{\mathrm{D}}_{\dot{\alpha}}\,(\Lambda\Phi) + \mathrm{D}_\alpha\bar{\mathrm{D}}^2\bar{U}_{\dot{\alpha}} - \bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2U_\alpha\,, \tag{116}$$

$$\mathcal{T} = -\Phi\bar{\Phi} + \tfrac{m}{2}\Lambda\Phi + m\bar{\Lambda}\bar{\Phi} + 2\mathrm{D}^\alpha U_\alpha + \bar{\mathrm{D}}^{\dot{\alpha}}\bar{U}_{\dot{\alpha}}\quad . \tag{117}$$

It is easy to see that there is no choice of U_α that can cancel the terms of $\mathcal{T}$ proportional to the mass. This is true not just for the case of supergravity, but for the higher spin supermultiplets as well. The higher spin supertrace $\mathcal{T}_{\alpha(2l)\dot{\alpha}(2l)}$ can not be eliminated and there is no *minimal* supercurrent multiplet for massive chirals.

However, we can use the procedure of Section 7 in order to absorb all the m independent terms of the supertrace and make it proportional to the mass. In this configuration the supercurrent will be the same as the minimal supercurrent of massless chiral (73) plus terms proportional to mass. For the case of supergravity this will give

$$\mathcal{J}_{\alpha\dot{\alpha}} = \mathcal{J}^{min}_{\alpha\dot{\alpha}} - \tfrac{m}{6}\mathrm{D}_\alpha\bar{\mathrm{D}}_{\dot{\alpha}}\,(\Lambda\Phi) + \tfrac{m}{6}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}_\alpha\,(\bar{\Lambda}\bar{\Phi})\quad , \tag{118}$$

$$\mathcal{T} = \tfrac{m}{6}\Lambda\Phi + \tfrac{m}{3}\bar{\Lambda}\bar{\Phi} \tag{119}$$

where $\mathcal{J}^{min}_{\alpha\dot{\alpha}}$ is given in (59a).

10.3. Conservation Equation

The conservation equation that the $\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)}$ and $\mathcal{T}_{\alpha(2l)\dot{\alpha}(2l)}$ satisfy on-shell is

$$\bar{\mathrm{D}}^{\dot{\alpha}_{2l+1}}\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)} = \tfrac{1}{(2l+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{2l+1}}\mathcal{T}_{\alpha(2l))\dot{\alpha}(2l)}\quad,\; l = 0, 1, 2, \ldots \tag{120}$$

and it is straight forward to show that expressions (110) and (112) do that on-shell [13]. As we did for the massless chiral, we will use this conservation equation to derive a closed form expression for the higher spin supercurrent and supertrace. Based on the previous results the general ansatz for the higher spin supercurrent and supertrace is

$$\begin{aligned}\mathcal{J}_{\alpha(s)\dot{\alpha}(s)} = \mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)} &+ m\sum_{p=0}^{s-1}\gamma_p\,\partial^{(p)}\mathrm{D}\bar{\mathrm{D}}\Lambda\,\partial^{(s-1-p)}\Phi + m\sum_{p=0}^{s-1}\delta_p\,\partial^{(p)}\bar{\mathrm{D}}\Lambda\,\partial^{(s-1-p)}\mathrm{D}\Phi \\ &- m\sum_{p=0}^{s-1}\gamma^{*}_p\,\partial^{(p)}\bar{\mathrm{D}}\mathrm{D}\bar{\Lambda}\,\partial^{(s-1-p)}\bar{\Phi} - m\sum_{p=0}^{s-1}\delta^{*}_p\,\partial^{(p)}\mathrm{D}\bar{\Lambda}\,\partial^{(s-1-p)}\bar{\mathrm{D}}\bar{\Phi}\,,\end{aligned} \tag{121}$$

$$\begin{aligned}\mathcal{T}_{\alpha(s-1)\dot{\alpha}(s-1)} = \; & m\sum_{p=0}^{s-1}\zeta_p\,\partial^{(p)}\Lambda\,\partial^{(s-1-p)}\Phi + m\sum_{p=0}^{s-1}\xi_p\,\partial^{(p)}\bar{\Lambda}\,\partial^{(s-1-p)}\bar{\Phi} \\ &+ m\sum_{p=0}^{s-2}\sigma_p\,\partial^{(p)}\bar{\mathrm{D}}\mathrm{D}\Lambda\,\partial^{(s-2-p)}\Phi\end{aligned} \tag{122}$$

with $\mathcal{J}^{min}_{\alpha(s)\dot{\alpha}(s)}$ given by (88). The conservation Equation (120) fixes the coefficients δ_p, ξ_p, ζ_p, σ_p:

$$\delta_p = -\gamma_p\,, \qquad p = 0,1,\ldots,s-1\,, \tag{123a}$$
$$\xi_p = -\tfrac{s+1}{s}\,\gamma^{*}_p\,, \qquad p = 0,1,\ldots,s-1\,, \tag{123b}$$
$$\zeta_0 = -\tfrac{1}{s}\,\gamma_0\,, \tag{123c}$$
$$\zeta_p = -\tfrac{p+1}{s}\,\gamma_p + \tfrac{s-p}{s}\,\gamma_{p-1}\,, \qquad p = 1,2,\ldots,s-1\,, \tag{123d}$$
$$\sigma_0 = -\tfrac{i}{s}\,\gamma_1 + i\,\tfrac{s-1}{s}\,\gamma_0\,, \tag{123e}$$
$$\sigma_p = (-1)^{p+1}\,\tfrac{i}{s}\,\gamma_1 + (-1)^p\,i\,\tfrac{s-1}{s}\,\gamma_0 + i\sum_{n=1}^{p}(-1)^{p+n+1}\left[\tfrac{n+1}{s}\,\gamma_{n+1} - \tfrac{s-2n-1}{s}\,\gamma_n - \tfrac{s-n}{s}\,\gamma_{n-1}\right] \quad p = 1,2,\ldots,s-2 \tag{123f}$$

and the coefficients γ_p satisfy the constraints:

$$\gamma_p + \gamma_{s-p-1} = \frac{(-1)^{s+p}\,(i)^{s+1}}{\binom{2s+1}{s+1}}\sum_{n=0}^{p}\binom{s}{n}^2\left[\tfrac{s+1}{s+1-n} + (-1)^s\,\tfrac{s+1}{n+1}\right]\,,\; p = 0,1,\ldots,s-1\,, \tag{124a}$$
$$\sigma_{s-2} = -i\,\tfrac{s-1}{s}\,\gamma_{s-1} + \tfrac{i}{s}\,\gamma_{s-2}\,. \tag{124b}$$

Notice that the left hand side of (124a) is invariant under $p \to s-1-p$, therefore we get a consistency condition

$$\left[\,1 + (-1)^s\,\right]\sum_{n=0}^{s}\binom{s}{n}^2\frac{s+1}{n+1} = 0 \tag{125}$$

which selects only the odd values of s, in agreement with (115). For $s = 2l+1$, Equation (124a) fixes γ_l

$$\gamma_l = \frac{l+1}{\binom{4l+3}{2l+2}}\sum_{n=0}^{l}\binom{2l+1}{n}^2\left[\tfrac{1}{2l+2-n} - \tfrac{1}{n+1}\right]\,. \tag{126}$$

A consequence of that is $\xi_l \neq 0$ due to (123b). Therefore the supertrace can not be zero as in the massless case. Moreover, the constraints (124a) and (124b) provide a system of $l+2$ linear equations for the $2l+1$, γ_p coefficients, so there is a freedom of choice for $l-1$ of these coefficients. This freedom

[13] Keep in mind that the on-shell equation of motion for a free massive chiral is $\bar{\mathrm{D}}^2\bar{\Phi} = m\Phi$.

corresponds to the fact that there is no unique canonical supercurrent multiplet, in contrast with the massless case where the minimal multiplet is unique. An example of a choice is to have

$$\gamma_{l+2} = \gamma_{l+3} = \cdots = \gamma_{2l} = 0\,. \tag{127}$$

11. Summary and Discussion

Let us briefly summarize and discuss the results obtained. In Section 3 we presented the most general ansatz for the transformation of a 4D, $\mathcal{N} = 1$ chiral superfield with linear terms (9). The consistence with chirality, constrained the parameters (10) and revealed structures similar to the gauge transformations of free, massless, higher-superspin theories. This was a hint that chiral superfields can have cubic interactions with higher spin superfields. Therefore, using (12) and Noether's method we:

(*i*) Proved that a single, massless, chiral superfield can have cubic interactions (52) only with the half-integer superspin $(s+1, s+1/2)$ irreducible representations of the super-Poincaré group. Moreover, despite the fact that there are two different formulations of the half-integer superspin supermultiplets, the chiral superfield has a clear preference to couple only to one of them, the one that can be lifted to $\mathcal{N} = 2$ higher spin supermultiplets.

(*ii*) Generated the *canonical* multiplet of higher spin supercurrents $\left\{\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}, \mathcal{T}_{\alpha(k)\dot{\alpha}(k)}\right\}$ (48) and (50) which satisfy conservation Equation (78) and leads to the cubic interactions

$$g\int\sum_{k=0}^{\infty}\left\{H^{\alpha(k+1)\dot{\alpha}(k+1)}\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)} + \chi^{\alpha(k+1)\dot{\alpha}(k)}\mathrm{D}_{\alpha_{k+1}}\mathcal{T}_{\alpha(k)\dot{\alpha}(k)} + \bar{\chi}^{\alpha(k)\dot{\alpha}(k+1)}\bar{\mathrm{D}}_{\dot{\alpha}_{k+1}}\bar{\mathcal{T}}_{\alpha(k)\dot{\alpha}(k)}\right\}\,. \tag{128}$$

The objects $\mathcal{J}_{\alpha(k+1)\dot{\alpha}(k+1)}$ and $\mathcal{T}_{\alpha(k)\dot{\alpha}(k)}$ are the higher spin supercurrent and higher spin supertrace respectively and are the higher spin analogues of the supercurrent and supertrace that appear in supergravity.

(*iii*) Proved that for every k, there is a unique alternative multiplet of higher spin supercurrents, called *minimal* $\left\{\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}, 0\right\}$ (73) and (88) with conservation Equation (80). The cubic interactions for the minimal multiplet have the simpler form

$$g\int\sum_{k=0}^{\infty} H^{\alpha(k+1)\dot{\alpha}(k+1)}\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}\,. \tag{129}$$

Furthermore, we presented the construction of the appropriate improvement term that will take us from the *canonical* to the *minimal* multiplet. The supercurrent $\mathcal{J}^{min}_{\alpha(k+1)\dot{\alpha}(k+1)}$ matches exactly the supercurrent generated by superconformal higher spins presented in [41].

The identification of the *minimal* multiplet with the results in [41] was expected because superconformal higher spin description does not include a compensator like $\chi_{\alpha(k+1)\dot{\alpha}(k)}$, hence the cubic interaction terms of the chiral with the superconformal higher spin supermultiplets can only take the form of (129). However, the superfield $H_{\alpha(k+1)\dot{\alpha}(k+1)}$ that appears in [41] is not the same because its dynamics involve higher derivative terms and also has different engineering dimensions.

In Section 9, we discuss the component structure of the theory and specifically we searched for the higher spin currents generated by the supercurrents. Starting from the superspace conservation equation we project down to the component level and we find:

(*iv*) An expression for the integer spin current $\mathcal{J}^{min\ (1,1)(S,S)}_{\alpha(s+1)\dot{\alpha}(s+1)}$ (95). There are two contributions to this current. The first is of the boson - boson type constructed out of a complex scalar ϕ which is defined as the the θ independent term of Φ ($\phi = \Phi|$). The second contribution is of the fermion—fermion type and is constructed out of a spinor χ_α defined as the θ term of Φ ($\chi_\alpha = \mathrm{D}_\alpha\Phi$). Both of these contributions agree with known results.

(v) An expression for the half-integer spin current $\mathcal{J}^{min\ (1,0)(S)}_{\alpha(s+1)\dot{\alpha}(s)}$ (97). This current appears for the first time in the literature because it requires both the complex scalar and the spinor, therefore non-supersymmetric theories can not be used to construct it.
(vi) An expression for an $\mathcal{R}$-symmetry current $\mathcal{J}^{min\ (0,0)}_{\alpha(s)\dot{\alpha}(s)}$ (98). This current also appears for the first time.

It is important to emphasize that in general the higher spin supercurrent and higher spin supertrace are independent quantities and the *minimal* multiplet can not always be reached. It depends on the peculiarities of the starting action and its symmetries, such as superconformal, to decide whether this can be done or not. In this work, we present a method of constructing the higher spin supercurrent and supertrace which is not restricted by these considerations. In Section 10, we discuss the higher spin supercurrent multiplet of a massive chiral superfield. Our results are:

(vii) A massive chiral can have cubic interactions only with the odd s $[s = 2l + 1]$ half-integer superspin supermultiplets $(2l + 2\,,\ 2l + 3/2)$.
($viii$) The expressions for the higher spin supercurrent $\mathcal{J}_{\alpha(2l+1)\dot{\alpha}(2l+1)}$ (110) and (121) and supertrace $\mathcal{T}_{\alpha(2l)\dot{\alpha}(2l)}$ (112) and (122) of the *canonical* multiplet. These expressions have not been obtained before.
(ix) There is no *minimal* multiplet of supercurrents for this case since the supertrace can not be adsorbed by improvement terms. However, it can be arranged to be proportional to the mass parameter, so at the massless limit we land at the *minimal* multiplet of the massless chiral superfield.

There are several directions for the further development and generalization of the superfield interaction vertices studied in the paper. Firstly, the approach under consideration can directly be applied to derivation of the cubic interaction of the higher-superspin superfield with chiral superfield on the AdS superspace background. Secondly, it would be extremely interesting to construct the supercurrent and corresponding cubic interaction vertex for 4D, $\mathcal{N} = 2$ massless higher-superspin gauge superfield. In this case the supercurrent should apparently be built from hypermultiplet superfields on the framework of harmonic superspace [83] which provides unconstrained superfiled description for 4D, $\mathcal{N} = 2$ supermultiplets. Thirdly, it would be interesting to apply this approach to other matter supermultiplets such as the complex linear.

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