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Peccei–Quinn Transformations and Black Holes: Orbit Transmutations and Entanglement Generation †

Thiago Prudêncio [1,*], Alessio Marrani [2,3] and Diego J. Cirilo-Lombardo [4,5]

1 Coordination of Science & Technology (CCCT/BICT), Universidade Federal do Maranhão (UFMA), Sao Luís-MA 65080-805, Brazil
2 Museo Storico della Fisica e Centro Studi e Ricerche "Enrico Fermi", Via Panisperna 89A, Roma I-00184 , Italy; alessio.marrani@pd.infn.it
3 Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova, and INFN, Sez. di Padova, Via Marzolo 8, Padova I-35131, Italy
4 National Institute of Plasma Physics (INFIP), Consejo Nacional de Investigaciones Científicas y Tecnicas (CONICET), Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Cuidad Universitaria, Buenos Aires 1428, Argentina; diego777jcl@gmail.com
5 Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna-Moscow 141980, Russia
* Correspondence: thprudencio@gmail.com & prudencio.thiago@ufma.br
† Open Questions in Black Hole Physics

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Abstract: In a recent paper (Mod. Phys. Lett. A 2015, 30, 1550104), the black-hole/qubit correspondence (BHQC) was exploited to define "black hole quantum circuits" allowing for a change of the supersymmetry-preserving features of electromagnetic charge configurations supporting the black hole solution. This resulted in switching from one U-duality orbit to another, or equivalently, from an element of the corresponding Freudenthal triple system with a definite rank to another one. On the supergravity side of BHQC, such quantum gates are related to particular symplectic transformations acting on the black hole charges; namely, such transformations cannot belong to the U-duality group, otherwise switching among orbits would be impossible. In this paper, we consider a particular class of such symplectic transformations, namely the ones belonging to the so-called Peccei–Quinn symplectic group, introduced some time ago within the study of very special Kähler geometries of the vector multiplets' scalar manifolds in $\mathcal{N} = 2$ supergravity in $D = 4$ spacetime dimensions.

Keywords: Peccei-Quinn; U-duality; Black hole-qubit

1. Introduction

In general, extremal black holes (EBHs) appearing as solutions of $D = 4$ supergravity theory carry electromagnetic charges that transform linearly under the U-duality[1] group (G_4), which however, due to its non-transitive action, determines a stratification of the corresponding representation space **R** [4] (cf., e.g., see [5,6] for reviews and lists of references). Due to the attractor mechanism [7–11], the Bekenstein–Hawking [12,13] entropy of an EBH is a function only of the electromagnetic charges; in most of the $\mathcal{N} = 2$ models with symmetric scalar manifolds, as well as in $\mathcal{N} > 3$-extended theories,

[1] Here, U-duality is referred to as the "continuous" symmetries of [1,2]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [3].

the entropy is given by (π times) the square root of the absolute value of the U-duality quartic invariant $\mathcal{I}_4$ [14,15] (also see, e.g., [16,17] for introductory reviews and lists of references).

Due to the existence of the Bogomol'nyi–Prasad–Sommerfield (BPS) bound, the mass of an EBH is bounded from below; for instance, in the Reissner–Nordström BH, which can be embedded in $\mathcal{N} = 2$ "pure" supergravity, as well as in $\mathcal{N} = 1$ supergravity coupled to one vector multiplet [18], $M \geq \sqrt{q^2 + p^2}$ (which is saturated in the extremal case: $M = \sqrt{q^2 + p^2}$). On the other hand, in the presence of scalars, extremality does not necessarily imply the saturation of the BPS bound: the simplest example is provided by the dilaton EBH (which has a natural embedding into $\mathcal{N} = 4$ "pure" supergravity), for which $M \geq \frac{1}{\sqrt{2}}(|Q| + |P|)$ generally holds, with Q and P denoting suitably dilaton-dressed electric and magnetic charges [19] (see also [17]); in this case, the Bekenstein–Hawking entropy reads $S = 2\pi|PQ|$.

An EBH can be conceived as the final limit state of a thermal evaporation process associated with the emission of Hawking radiation; when reaching the extremal state (characterized by vanishing temperature), the Hawking radiation ceases: the EBH is not evaporating anymore, in order to sustain the consistency of the BPS bound [19]. If the BH charge is large, the last stages of evaporation occur for a mass much greater than Planck mass M_P, but the EBH can then discharge by the creation of pairs of elementary charged particles [20,21]; for example, in the case of a dyonic Reissner–Nordström BH, the evaporation stops when the charge achieves the absolute value of Planck mass: $\sqrt{q^2 + p^2} = M_P$.

In the four-dimensional Einstein gravity theory with maximal local supersymmetry, namely $\mathcal{N} = 8$ supergravity, various degrees of BPS supersymmetry preservation are possible, each one associated with a massive representation of the $\mathcal{N} = 8$ supersymmetry algebra [22,23]: namely, 1/2-BPS, 1/4-BPS and 1/8-BPS EBHs exist [4,24] (cf. also [25] for a more recent analysis). Essentially, the classification of BPS states preserving different amounts of supersymmetry of the background algebras is similar to the stratification of timelike, light-like and spacelike vectors in a Minkowski space [4]; in such a classification, EBHs are usually named "small" or "large", depending on whether their Bekenstein–Hawking entropy vanishes or not.

The string-theoretic interpretation of BHs is usually given in terms of Dp-branes; their wrapping around compactified dimensions allows for a consistent picture, relating them to qubits (the basic quantities studied by quantum information theory (QIT)): this led to the formulation of the so-called black-hole/qubit correspondence (BHQC) [26–37]. Important achievements have been reached in this context, and some others are underway; just to name a few, the EBH entropy in the so-called $\mathcal{N} = 2$ supergravity STU model [38,39] was associated with tripartite entanglement measurement and classification in QIT [26,40], and this was further extended to the hitherto unsolved issue of the classification of four qubits entanglement [41]. Moreover, the Hilbert space for qubits was traced back to wrapped branes inside the cohomology of the extra dimensions [32,42], and exceptional groups E_7 and E_6 were respectively related to the tripartite entanglement of seven qubits [43] and to the bipartite entanglement of three qutrits [44]. Recently, "BH quantum gates" were formulated within the BHQC in [45].

The Freudenthal triple system (FTS) [46–48] plays a key role in BHQC; these are algebraic systems, whose elements can be grouped into different ranks, ranging from zero to four [49–51]. In the classification of tripartite entanglement, the ranks 0, 1, 2a, 2b, 2c, 3 and 4 are associated respectively with null, separable $a - b - c$, biseparable $a - bc$, $b - ac$ and $c - ab$, W and GHZ states [28]. By means of a set of local operations and classical communications (LOCC), the state can be operated without generating a new class of entanglement. The inter-relation between equivalent states in the same class is achieved by the possibility of interconverting one to another by means of the group of local unitaries (LU) in the set of LOCC [52]. Rather than declare equivalence when states are deterministically related to each other by LOCC, one may require only that they may be transformed into one another with some non-zero probability of success; this can be achieved by the so-called stochastic LOCC (SLOCC) [53]. Within the SLOCC paradigm, two states are identified if there is a non-zero probability that one can be converted into the other and vice versa; one can trace back the origin of such a generalization by the physically-motivated fact that any set of SLOCC equivalent states may be used to perform the same non-classical operations, only with varying likelihoods of success. In terms of FTS, with SLOCC, one is varying the representative element with the same entanglement class, corresponding to a well-defined FTS rank. As analyzed in [29], SLOCC equivalence classes of three qubit states are stratified into an entanglement hierarchy [53], which can be consistently reproduced in terms of FTS, or equivalently in terms of orbits of its automorphism group.

Going beyond the SLOCC paradigm within the BHQC, [45] investigated the possibility of establishing quantum circuits (realizing certain classes of quantum gate operations) capable of switching, within the same FTS, from one rank class to another one. On the BH (supergravity) side of BHQC, this essentially amounts to switching from one U-duality orbit to another one; in the present treatment, this phenomenon will be named orbit transmutation. In FTS terms, this amounts to switching from an FTS element with a definite rank to another FTS element with another rank, lying in another orbit of the FTS automorphism group. In this context, specific quantum circuits generating Bell states, quantum teleportation and GHZ states can be clearly associated with the protocols of actions on the charge state configurations of EBHs [45].

On the BH (supergravity) side of BHQC, the aforementioned "BH quantum gates" are related to a particular kind of symplectic transformations (which are the most general transformations acting linearly on BH electromagnetic charges [2]). Indeed, "BH quantum gate" symplectic transformations generally allow for the switching from the representative of one U-duality orbit to a representative of another U-duality orbit; the most interesting cases are the ones in which a change of FTS rank occurs, i.e., in which a transition between different levels of the entanglement hierarchy takes place. Since the U-duality orbit structure (namely, the stratification of the charge representation space $\mathbf{R}$ under the non-transitive action of the U-duality group G_4) is invariant under G_4 itself, it is clear that such symplectic transformations cannot belong to the U-duality group, otherwise switching among orbits would be impossible. Thus, the "BH quantum gate" transformations generally belong to the pseudo-Riemannian, non-symmetric coset [3].

$$\frac{Sp(2n+2,\mathbb{R})}{G_4}, \tag{1}$$

where $2n+2$ denotes the dimension of the charge representation space $\mathbf{R}$.

In the present paper, we consider a particular class of "BH quantum gate" symplectic transformations, namely the ones belonging to the so-called Peccei–Quinn (PQ) symplectic group, introduced in [56] in the study of very special (i.e., cubic) Kähler geometries in $\mathcal{N}=2$ supergravity [4]. In such a framework, the special Kähler geometry of the vector multiplets' scalar manifold is based on a holomorphic prepotential function F in the symplectic sections X^Λ. At least within models with symmetric scalar manifolds, all symplectic transformations belonging to the coset Equation (1) and, thus, a fortiori the PQ symplectic transformations, change the value of unique quartic invariant polynomial $\mathcal{I}_4$ of the charge representation space $\mathbf{R}$, and consequently, they change the EBH Bekenstein–Hawking entropy. This is of particular importance in the theory of EBH attractors, because such a phenomenon allows for orbit transmutations, namely for transitions between different charge orbits in the stratification of $\mathbf{R}$, with a corresponding change of the supersymmetry-preserving features of the corresponding EBH charge configurations [56].

The plan of the paper is as follows.

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2 symplectic transformations also provide an example of pseudo-dualities in supergravity [54]

3 this coset was recently exploited in the analysis of the so-called symplectic deformations of gauged $\mathcal{N}=8$, $D=4$ supergravity [55], later extended to other supergravity theories

4 We will always consider the "large, real charge" supergravity limit within BHQC. In the case of (dyonic) quantized charges, the analysis of FTSs is more complicated, and a full classification of U-duality orbits is not even currently available (for some advances along this venue, and lists of references, cf., e.g., [25,57]).

In Section 2, we recall the definition of the Peccei–Quinn symplectic group and introduce the corresponding operator in the Hilbert space of EBH charge configuration states. Then, in Section 3, we introduce some examples of "large" and "small" EBH charge configurations, whose transformation under the PQ operator is discussed in Section 4, with particular emphasis on the occurrence of different types of orbit transmutations. After a brief interlude dedicated to state superposition in Section 5, in Section 6, we start the analysis of general mechanisms of entanglement generation on EBH charge states, by defining the entanglement PQ operator and pointing out the necessity of switching to a complex ground field (a already stressed within BHQC; see, e.g., [28]) within the indistinguishability assumption. Some final comments are contained in Section 7.

2. Peccei–Quinn Symplectic Group and Operator

The transformations of the Peccei–Quinn (PQ) group $PQ\,(n+1)$ were introduced in [56], as given by:

$$\mathbf{W} := \begin{pmatrix} \mathbb{I}_{n+1} & 0_{n+1} \\ \mathcal{W} & \mathbb{I}_{n+1} \end{pmatrix} \in PQ\,(n+1)\,, \tag{2}$$

where $\mathbb{I}_{n+1}$ and 0_{n+1} respectively denote the identity and null $(n+1)\times(n+1)$ matrices and $\mathcal{W}$ is defined as the following $(n+1)\times(n+1)$ real matrix ($i,j=1,...,n$, $\Theta_{ij}=\Theta_{(ij)}$):

$$\mathcal{W} := \begin{pmatrix} \varrho & c_j \\ c_i & \Theta_{ij} \end{pmatrix}. \tag{3}$$

It is then easy to realize that $PQ\,(n+1)$ is an $(n+1)(n+2)/2$-dimensional Abelian group given by:

$$PQ\,(n+1) = Sp(2n+2,\mathbb{R}) \cap LUT(2n+2,\mathbb{R}), \tag{4}$$

where $Sp(2n+2,\mathbb{R})$ is the maximally non-compact (split) real form of the symplectic group whose Lie algebra is $\mathfrak{c}_{n+1}$ (in Cartan notation), and $LUT(2n+2,\mathbb{R})$ denotes the subgroup of $SL(2n+2,\mathbb{R})$ given by the $(2n+2)\times(2n+2)$ lower unitriangular matrices, which are unipotent.

The group $PQ\,(n+1)$ and its Lie algebra $\mathfrak{pq}_{n+1}$ were introduced in [56] in the study of very special Kähler geometries. However, interestingly, matrices with structure as $\mathbf{W}$ (2), and thus belonging to $PQ\,(n+1)$, made their appearance also in other contexts. For example, $\mathbf{W}$ can be regarded as a particular case of the quantum perturbative duality transformations in supersymmetric Yang–Mills theories coupled to supergravity [58]; in such a framework, Equation (2) defines the structure of quantum perturbative monodromy matrices in heterotic string compactifications with classical U-duality group $SL(2,\mathbb{R})\times SO(2,n+2)$.

In $\mathcal{N}=2$ (ungauged) supergravity theories in $D=4$ space-time dimensions, n denotes the number of (Abelian) vector multiplets coupled to the gravity multiplet. By virtue of a theorem due to Dynkin [59], the U-duality group G_4 is embedded into the symplectic group $Sp(2n+2,\mathbb{R})$ in a maximal (and generally non-symmetric) way, as follows:

$$G_4 \subsetneq Sp(2n+2,\mathbb{R}); \tag{5}$$

$$\mathbf{R} = \mathbf{2n+2}, \tag{6}$$

where **R** is the symplectic (i.e., anti-self-conjugated) representation of G_4 in which the electric and magnetic charges of the (E)BH sit. Consequently, one can define a dyonic[5] charge vector state $|\mathcal{Q}\rangle$ as follows[6]:

$$|\mathcal{Q}\rangle := p^0|0\rangle + q_0|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle + \sum_{i=1}^{n} q_i|3^i\rangle, \tag{7}$$

where:

$$\{|0\rangle, |1\rangle, |2^i\rangle, |3^i\rangle, i = 1, ..., n\} \tag{8}$$

is a symplectic basis of the Hilbert space realization of **R**. It is instructive to report here the holomorphic sections' state associated with a given holomorphic prepotential $F(X)$:

$$|\mathcal{H}\rangle = X^0|0\rangle + F_0|1\rangle + \sum_{i=1}^{n} X^i|2^i\rangle + \sum_{i=1}^{n} F_i|3^i\rangle, \tag{9}$$

where $F_i(X) := \partial F(X)/\partial X^i$.

A general operator in the Hilbert space of EBH state configurations can be defined as follows:

$$\begin{aligned}\hat{\mathcal{O}}_{BH} \quad : \quad &= \sum_{\alpha,\beta=0}^{1} s_{\alpha\beta}|m\rangle\langle n| + \sum_{i=1}^{n}\sum_{\alpha=0}^{1}\left(r_{i\alpha}|2^i\rangle\langle\alpha| + t_{i\alpha}|3^i\rangle\langle\alpha| + r_{\alpha i}|\alpha\rangle\langle 2^i| + t_{\alpha i}|\alpha\rangle\langle 3^i|\right)\\ &+ \sum_{i,j=1}^{n}\left(a_{ij}|2^j\rangle\langle 2^i| + b_{ij}|3^j\rangle\langle 3^i| + c_{ij}|2^j\rangle\langle 3^i| + d_{ij}|3^j\rangle\langle 2^i|\right).\end{aligned} \tag{10}$$

However, in the present treatment, we will focus on the class of Peccei–Quinn (PQ) operators, related to the aforementioned PQ symplectic transformations. From Equations (2) and (3), one can define the PQ operator $\mathcal{O}_{PQ}$ associated with the PQ transformation **W** as follows:

$$\hat{\mathcal{O}}_{PQ} := |0\rangle\langle 0| + |1\rangle\langle 1| + \sum_{i=1}^{n}\left(|2^i\rangle\langle 2^i| + |3^i\rangle\langle 3^i|\right) + \rho|1\rangle\langle 0| + \sum_{i=1}^{n}\left(c_i|1\rangle\langle 2^i| + c_i|3^i\rangle\langle 0|\right) + \sum_{i,j=1}^{n}\Theta_{ij}|3^i\rangle\langle 2^j|, \tag{11}$$

corresponding to:

$$\forall i, j = 1, .., n : \left\{\begin{array}{l} s_{00} = 1 = s_{11},\ \ s_{10} = \rho,\ \ s_{01} = 0; \\ r_{i0} = 0 = r_{i1},\ \ t_{i0} = c_i,\ \ t_{i1} = 0; \\ r_{0i} = c_i,\ \ r_{1i} = 0,\ \ t_{0i} = 0 = t_{1i}; \\ a_{ij} = \delta_{ij} = b_{ij},\ \ c_{ij} = 0,\ \ d_{ij} = \Theta_{ij} = \Theta_{(ij)} \end{array}\right. \tag{12}$$

in the parametrization of the general operator (10).

The PQ-transformations of States (7) and (9) formally read as:

$$|\mathcal{Q}_{PQ}\rangle \quad : \quad = \hat{\mathcal{O}}_{PQ}|\mathcal{Q}\rangle; \tag{13}$$

$$|\mathcal{H}_{PQ}\rangle \quad : \quad = \hat{\mathcal{O}}_{PQ}|\mathcal{H}\rangle. \tag{14}$$

---

5 We use the so-called "special coordinates" symplectic frame of $\mathcal{N} = 2$, $D = 4$ supergravity [60]. The PQ symplectic group was actually introduced within the so-called $4D/5D$ "special coordinates" symplectic frame, which is the most natural for the study of (projective) very special Kähler geometry [56].

6 Throughout the present investigation, we will not make use of the Einstein summation convention. Such a choice, which may result in being cumbersome for the customary supergravity treatment, is made in order to comply with the most used notation in QIT.

3. Some "Large" and "Small" Configurations

Within the considered symplectic frame, the general charge configuration state (7) is associated with the following, unique [61,62] G_4-invariant (cf., e.g., [4,15,63] and the refrences therein):

$$\mathcal{I}_4(\mathcal{Q}) = -\left(p^0 q_0 + \sum_{i=1}^{n} p^i q_i\right)^2 + 4q_0 \mathcal{I}_3(p) - 4p^0 \mathcal{I}_3(q) + 4\{\mathcal{I}_3(p), \mathcal{I}_3(q)\}, \tag{15}$$

where:

$$\mathcal{I}_3(p) \ : \ = \frac{1}{3!} \sum_{i,j,k=1}^{n} d_{ijk} p^i p^j p^k, \quad \mathcal{I}_3(q) := \frac{1}{3!} \sum_{i,j,k=1}^{n} d^{ijk} q_i q_j q_k, \tag{16}$$

$$\{\mathcal{I}_3(p), \mathcal{I}_3(q)\} \ : \ = \sum_{i=1}^{n} \frac{\partial \mathcal{I}_3(p)}{\partial p^i} \frac{\partial \mathcal{I}_3(q)}{\partial q_i} = \frac{1}{4} \sum_{i,j,k,l,m=1}^{n} d^{ijk} d_{ilm} q_j q_k p^l p^m. \tag{17}$$

As anticipated in Section 1, the Bekenstein–Hawking EBH entropy formula can be recast in terms of such an invariant as:

$$S(\mathcal{Q}) = \pi \sqrt{|\mathcal{I}_4(\mathcal{Q})|}. \tag{18}$$

Below, we consider some noteworthy EBH charge configurations, which also provide well-defined representatives of the corresponding U-duality orbits, i.e., examples of FTS elements of all possible ranks (4, 3, 2, 1).

1. Kaluza-Klein (KK) configurations:

$$|\mathcal{Q}_{KK}\rangle \ : \ = p^0|0\rangle + q_0|1\rangle; \tag{19}$$

$$\mathcal{I}_4(\mathcal{Q}_{KK}) = -\left(p^0\right)^2 q_0^2 < 0, \tag{20}$$

$$\frac{\partial \mathcal{I}_3(q)}{\partial q_i} = 0, \frac{\partial \mathcal{I}_3(p)}{\partial p^i} = 0, \ \forall i, \tag{21}$$

$$S(\mathcal{Q}_{KK}) = \pi |p^0 q_0|. \tag{22}$$

1.1] In the case with both p^0 and q_0 non-vanishing, one obtains a dyonic "large" configuration (corresponding to the maximal rank= 4 element in the related FTS), whereas for p^0 or q_0 vanishing, one has a "small" configuration (namely: 1.2] for $p^0 = 0$: $|\mathcal{Q}_{eKK}\rangle$ and 1.3] for $q_0 = 0$: $|\mathcal{Q}_{mKK}\rangle$, such that $S(\mathcal{Q}_{eKK}) = S(\mathcal{Q}_{mKK}) = 0$, corresponding to minimal rank= 1 elements in the FTS).

2. Electric (E) configurations:

$$|\mathcal{Q}_{E}\rangle \ : \ = p^0|0\rangle + \sum_{i=1}^{n} q_i |3^i\rangle; \tag{23}$$

$$\mathcal{I}_4(\mathcal{Q}_{E}) = -4p^0 \mathcal{I}_3(q) \neq 0, \tag{24}$$

$$\frac{\partial \mathcal{I}_3(p)}{\partial p^i} = 0, \tag{25}$$

$$S(\mathcal{Q}_{E}) = 2\pi \sqrt{|p^0 \mathcal{I}_3(q)|}. \tag{26}$$

2.1] In the case with both p^0 and $\mathcal{I}_3(q)$ non-vanishing, one obtains a dyonic "large" configuration (corresponding to the maximal rank= 4 element in the related FTS, i.e., to an element in the same duality orbit of the "large" dyonic KK configuration). As an example of "small" configurations ($S(\mathcal{Q}_E) = 0$), one may set $p^0 = 0$; this latter case further splits into three subcases: 2.2] $\mathcal{I}_3(q) \neq 0$, corresponding to a rank-three FTS element: $|\mathcal{Q}_{3E}\rangle$; 2.3] $\mathcal{I}_3(q) = 0$, but $\partial \mathcal{I}_3(q)/\partial q_i \neq 0$ for at least some i, corresponding to a rank-two FTS element: $|\mathcal{Q}_{2E}\rangle$; 2.4] $\partial \mathcal{I}_3(q)/\partial q_i = 0 \ \forall i$, but $q_i \neq 0$ for at least some i, corresponding to a rank-one FTS element (i.e., in the same duality orbit of the small KK configurations): $|\mathcal{Q}_{1E}\rangle$.

3. Magnetic (M) configurations:

$$
\begin{aligned}
|\mathcal{Q}_M\rangle &= q_0|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle; && (27)\\
\mathcal{I}_4(\mathcal{Q}_M) &= 4q_0\mathcal{I}_3(p) \neq 0, && (28)\\
\frac{\partial \mathcal{I}_3(q)}{\partial q_i} &= 0, && (29)\\
S(\mathcal{Q}_M) &= 2\pi\sqrt{|q_0\mathcal{I}_3(p)|}. && (30)
\end{aligned}
$$

3.1] In the case with both q_0 and $\mathcal{I}_3(p)$ non-vanishing, one obtains a dyonic " large" configuration (corresponding to the maximal rank= 4 element in the related FTS). As an example of "small" configurations ($S(\mathcal{Q}_M) = 0$), one may set $q_0 = 0$; this latter case further splits into three subcases: 3.2] $\mathcal{I}_3(p) \neq 0$, corresponding to a rank-three FTS element: $|\mathcal{Q}_{3M}\rangle$; 3.3] $\mathcal{I}_3(p) = 0$, but $\partial\mathcal{I}_3(p)/\partial p^i \neq 0$ for at least some i, corresponding to a rank-two FTS element: $|\mathcal{Q}_{2M}\rangle$; 3.4] $\partial\mathcal{I}_3(p)/\partial p^i = 0\ \forall i$, but $p^i \neq 0$ for at least some i, corresponding to a rank-one FTS element: $|\mathcal{Q}_{1M}\rangle$.

4. Peccei–Quinn Orbit Transmutations

As introduced in Section 2, a PQ symplectic transformation is given by the action of the corresponding operator $\hat{\mathcal{O}}_{PQ}$ (11). On the charge state (7), the action (13) can be explicated as follows:

$$
\begin{aligned}
|\mathcal{Q}_{PQ}\rangle \quad &: = \hat{\mathcal{O}}_{PQ}|\mathcal{Q}\rangle = p^0|0\rangle + \left(q_0 + \rho p^0 + \sum_{i=1}^{n} c_i p^i\right)|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle + \sum_{i=1}^{n}\left(q_i + c_i p^0 + \sum_{j=1}^{n} p^j\Theta_{ji}\right)|3^i\rangle\\
&= p^0|0\rangle + \tilde{q}_0|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle + \sum_{i=1}^{n}\tilde{q}_i|3^i\rangle, \qquad (31)
\end{aligned}
$$

where:

$$
\begin{aligned}
\tilde{q}_0 \quad &: = q_0 + \rho p^0 + \sum_{i=1}^{n} c_i p^i, && (32)\\
\tilde{q}_i \quad &: = q_i + c_i p^0 + \sum_{j=1}^{n} p^j\Theta_{ji}. && (33)
\end{aligned}
$$

As a consequence, PQ operators leave the EBH magnetic charges invariant, while changing the electric ones [56]:

$$
\begin{cases}
p^0 \to p^0, \\
p^i \to p^i \\
q_0 \to q_0 + \rho p^0 + \sum_{i=1}^{n} c_i p^i, \\
q_i \to q_i + c_i p^0 + \sum_{j=1}^{n} p^j\Theta_{ji}.
\end{cases} \qquad (34)
$$

Let us now analyze the effects of the application of $\hat{\mathcal{O}}_{PQ}$ on the various U-orbit representatives/FTS elements presented in Section 3 (considering before the "large" and then the "small" ones).

1.1] "Large" KK configuration:

$$
\begin{aligned}
\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{KK}\rangle &= |\mathcal{Q}'_E\rangle;\\
|\mathcal{Q}'_E\rangle \quad &: = p^0|0\rangle + \left(q_0 + \rho p^0\right)|1\rangle + p^0 c_i|3^i\rangle. \qquad (35)
\end{aligned}
$$

Consequently, after a PQ transformation, the original KK dyonic EBH state is changed to an electric state. The value of the quartic invariant, and thus of the Bekenstein–Hawking EBH entropy, generally changes:

$$\mathcal{I}_4\left(\mathcal{Q}_{KK}\right) = -\left(p^0\right)^2 q_0^2 \longrightarrow \mathcal{I}_4\left(\tilde{\mathcal{Q}}_{KK}\right) = -\left(p^0\right)^2\left(q_0+\rho p^0\right)^2 - \frac{2}{3}\left(p^0\right)^4 \sum_{i,j,k=1}^{n} d^{ijk} c_i c_j c_k \tag{36}$$

Thus, depending on whether:

$$\frac{2}{3}\sum_{i,j,k=1}^{n} d^{ijk} c_i c_j c_k \gtreqless -\left(\frac{q_0}{p^0}+\rho\right)^2, \tag{37}$$

A "large" ($\mathcal{I}_4 > 0$: BPS or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$: BPS or non-BPS) or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration is generated by the action of the Peccei–Quinn symplectic group. Equation (37) shows that the relations among the components of $\mathcal{Q}$ and the parameters of the Peccei–Quinn symplectic transformation turn out to be crucial for the properties of the resulting charge configuration.

2.1]"Large" electric configuration:

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_E\rangle = p^0|0\rangle + \rho p^0|1\rangle + \sum_i \left(q_i + p^0 c_i\right)|3^i\rangle = |\mathcal{Q}''_E\rangle + |\mathcal{Q}_{\rho eKK}\rangle, \tag{38}$$

$$|\mathcal{Q}''_E\rangle : = p^0|0\rangle + \sum_i \left(q_i + p^0 c_i\right)|3^i\rangle; \tag{39}$$

$$|\mathcal{Q}_{\rho eKK}\rangle : = \rho p^0|1\rangle. \tag{40}$$

Correspondingly, the quartic invariant changes as follows:

$$\mathcal{I}_4\left(\mathcal{Q}_E\right) = -4p^0\mathcal{I}_3(q) \longrightarrow \mathcal{I}_4\left(\mathcal{Q}''_E + \mathcal{Q}_{\rho eKK}\right) = -\rho^2\left(p^0\right)^4 - 4p^0\mathcal{I}_3(q_i + p^0 c_i), \tag{41}$$

and there are three possibilities:

$$\mathbf{i} : \mathcal{I}_4\left(\mathcal{Q}''_E + \mathcal{Q}_{\rho eKK}\right) > 0; \tag{42}$$

$$\mathbf{ii} : \mathcal{I}_4\left(\mathcal{Q}''_E + \mathcal{Q}_{\rho eKK}\right) < 0; \tag{43}$$

$$\mathbf{iii} : \mathcal{I}_4\left(\mathcal{Q}''_E + \mathcal{Q}_{\rho eKK}\right) = 0. \tag{44}$$

In Cases **i** and **ii**, the resulting FTS element has rank four , and thus, it corresponds to a "large" EBH, whereas in Case **iii**, the resulting FTS has rank three, corresponding to a "small" EBH. Therefore, depending on the sign of $p^0\mathcal{I}_3(q)$, the action of $\hat{\mathcal{O}}_{PQ}$ may result in various types of orbit transmutations: a change of orbit representative within the same U-orbit (at most, generally with a change of value of $\mathcal{I}_4$) or a change of U-orbit, with preservation of the FTS rank = 4 (namely, a change of sign, and generally, of value of $\mathcal{I}_4$), or (in Case **iii**) a change from a "large" U-orbit (rank-four) to a "small" U-orbit (rank-three).

3.1]"Large" magnetic configuration:

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_M\rangle = \left(q_0 + \sum_i^n c_i p^i\right)|1\rangle + \sum_i^n p^i|2^i\rangle + \sum_{i,j}^n p^j\Theta_{ji}|3^i\rangle = |\tilde{\mathcal{Q}}_M\rangle + |\check{\mathcal{Q}}_E\rangle, \tag{45}$$

where:

$$|\tilde{\mathcal{Q}}_M\rangle \quad : \quad = \tilde{q}_0|1\rangle + \sum_i^n p^i |2^i\rangle, \tag{46}$$

$$\tilde{q}_0 \quad : \quad = q_0 + \sum_i^n c_i p^i, \tag{47}$$

and:

$$|\check{\mathcal{Q}}_E\rangle \quad : \quad = \sum_i^n \tilde{q}_i |3^i\rangle, \tag{48}$$

$$\tilde{q}_i \quad : \quad = \sum_j^n p^j \Theta_{ji}. \tag{49}$$

In particular, $|\check{\mathcal{Q}}_E\rangle$ can correspond to one of the three subcases (2.1], 2.2] and 2.3]) of Point 2 in the Section 3:

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 3 \Leftrightarrow \sum_{i,j,k=1}^n d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k \neq 0 \tag{50}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^n p^i \tilde{q}_i\right)^2 + 4\tilde{q}_0\mathcal{I}_3(p) + 4\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}, \tag{51}$$

and there are three possibilities:

$$\mathbf{i} \quad : \quad \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) > 0; \tag{52}$$

$$\mathbf{ii} \quad : \quad \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) < 0; \tag{53}$$

$$\mathbf{iii} \quad : \quad \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) = 0. \tag{54}$$

In Cases **i** and **ii**, the resulting FTS element has rank four, and thus, it corresponds to a "large" EBH, whereas in Case **iii**, the resulting FTS has rank three, corresponding to a "small" EBH. Thus, depending on the sign of $q_0\mathcal{I}_3(p)$, the action of $\hat{\mathcal{O}}_{PQ}$ may once again result in various types of orbit transmutations: a change of orbit representative within the same U-orbit (at most, generally with a change of value of $\mathcal{I}_4$), or a change of U-orbit, with preservation of the FTS rank $= 4$ (namely, a change of sign, and generally, of value of $\mathcal{I}_4$), or (in Case **iii**) a change from a "large" U-orbit (rank-four) to a "small" U-orbit (rank-three).

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 2 \Leftrightarrow \sum_{i,j,k=1}^n d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k = 0, \text{ but } \sum_{j,k=1}^n d^{ijk}\tilde{q}_j\tilde{q}_k \neq 0 \text{ for at least some } i, \tag{55}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^n p^i \tilde{q}_i\right)^2 + 4\tilde{q}_0\mathcal{I}_3(p) + 4\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}, \tag{56}$$

and there are three possibilities (**i–iii**) as above (note that the term $\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}$ may or may not be vanishing).

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 1 \Leftrightarrow \sum_{j,k=1}^{n} d^{ijk}\tilde{q}_j\tilde{q}_k = 0\ \forall i, \text{ but } \tilde{q}_i \neq 0 \text{ for at least some } i, \tag{57}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\tilde{\mathcal{Q}}_M + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2 + 4\tilde{q}_0\mathcal{I}_3(p), \tag{58}$$

and there are three possibilities (**i–iii**) as above.

2.2]"Small" rank-three electric configuration ($\mathcal{I}_3(q) \neq 0$):

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{3E}\rangle = \sum_{i=1}^{n} q_i|3^i\rangle = |\mathcal{Q}_{3E}\rangle. \tag{59}$$

3.2]"Small" rank-three magnetic configuration ($\mathcal{I}_3(p) \neq 0$):

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{3M}\rangle = \sum_{i=1}^{n} c_i p^i|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle + \sum_{i,j=1}^{n} p^j\Theta_{ji}|3^i\rangle = |\tilde{\mathcal{Q}}_M\rangle\big|_{q_0=0} + |\check{\mathcal{Q}}_E\rangle, \tag{60}$$

where $|\tilde{\mathcal{Q}}_M\rangle\big|_{q_0=0}$ is the $q_0 = 0$ case limit of $|\tilde{\mathcal{Q}}_M\rangle$ defined in Equation (46), and $|\check{\mathcal{Q}}_E\rangle$ is given in Equations (48) and (49). There are three subcases, respectively corresponding to rank $\left(\check{\mathcal{Q}}_E\right) = 3, 2, 1$.

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 3 \Leftrightarrow \sum_{i,j,k=1}^{n} d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k \neq 0 \tag{61}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\tilde{\mathcal{Q}}_M\big|_{q_0=0} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2 + 4\sum_{i=1}^{n} c_i p^i\mathcal{I}_3(p) + 4\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}, \tag{62}$$

and there are three possibilities:

$$\mathbf{i} \;:\; \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M\big|_{q_0=0} + \check{\mathcal{Q}}_E\right) > 0; \tag{63}$$

$$\mathbf{ii} \;:\; \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M\big|_{q_0=0} + \check{\mathcal{Q}}_E\right) < 0; \tag{64}$$

$$\mathbf{iii} \;:\; \mathcal{I}_4\left(\tilde{\mathcal{Q}}_M\big|_{q_0=0} + \check{\mathcal{Q}}_E\right) = 0. \tag{65}$$

In Cases **i** and **ii**, the resulting FTS element has rank four, and thus, it corresponds to a "large" EBH, whereas in Case **iii**, the resulting FTS has rank three, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ may once again result in various types of orbit transmutations: a change from a "small" U-orbit (rank-three) to a "large" U-orbit (rank-four, with positive or negative $\mathcal{I}_4$), or (in Case **iii**) a preservation of the rank$= 3$ of the original FTS element, but generally with a change of orbit representative (depending on the real form of the theory under consideration, this may result necessarily in remaining in the same rank-three orbit, as in $\mathcal{N} = 8$ supergravity, or possibly in switching to another rank-three orbit, as in $\mathcal{N}=2$ supergravity with symmetric, very special Kähler vector multiplets' scalar manifolds [5,6]).

$$\text{rank}\left(\check{\mathcal{Q}}_E\right)=2\Leftrightarrow\sum_{i,j,k=1}^{n}d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k=0,\text{ but }\sum_{j,k=1}^{n}d^{ijk}\tilde{q}_j\tilde{q}_k\neq 0\text{ for at least some }i,\tag{66}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\left.\tilde{\mathcal{Q}}_M\right|_{q_0=0}+\check{\mathcal{Q}}_E\right)=-\left(\sum_{i=1}^{n}p^i\tilde{q}_i\right)^2+4\sum_{i=1}^{n}c_ip^i\mathcal{I}_3(p)+4\{\mathcal{I}_3(p),\mathcal{I}_3(\tilde{q})\},\tag{67}$$

and there are three possibilities **i–iii** as above (note that the term $\{\mathcal{I}_3(p),\mathcal{I}_3(\tilde{q})\}$ may or may not be vanishing).

$$\text{rank}\left(\check{\mathcal{Q}}_E\right)=1\Leftrightarrow\sum_{j,k=1}^{n}d^{ijk}\tilde{q}_j\tilde{q}_k=0\;\forall i,\text{ but }\tilde{q}_i\neq 0\text{ for at least some }i,\tag{68}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\left.\tilde{\mathcal{Q}}_M\right|_{q_0=0}+\check{\mathcal{Q}}_E\right)=-\left(\sum_{i=1}^{n}p^i\tilde{q}_i\right)^2+4\sum_{i=1}^{n}c_ip^i\mathcal{I}_3(p),\tag{69}$$

and there are three possibilities **i–iii** as above.

2.3]"Small" rank-two electric configuration ($\mathcal{I}_3(q)=0$, but $\partial\mathcal{I}_3(q)/\partial q_i\neq 0$ at least for some i):

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{2E}\rangle=\sum_{i=1}^{n}q_i|3^i\rangle=|\mathcal{Q}_{2E}\rangle.\tag{70}$$

3.3]"Small" rank-two magnetic configuration ($\mathcal{I}_3(p)=0$, but $\partial\mathcal{I}_3(p)/\partial p^i\neq 0$ at least for some i):

$$\begin{aligned}\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{2M}\rangle &= \sum_{i=1}^{n}c_ip^i|1\rangle+\sum_{i=1}^{n}p^i|2^i\rangle+\sum_{i,j=1}^{n}p^j\Theta_{ji}|3^i\rangle \quad (71)\\ &= |\mathring{\mathcal{Q}}_{2M}\rangle+|\check{\mathcal{Q}}_E\rangle, \quad (72)\end{aligned}$$

where:

$$|\mathring{\mathcal{Q}}_{2M}\rangle:=\sum_{i=1}^{n}c_ip^i|1\rangle+\sum_{i=1}^{n}p^i|2^i\rangle.\tag{73}$$

Depending on $\text{rank}\left(\check{\mathcal{Q}}_E\right)=3,2,1$, one has still three subcases.

$$\text{rank}\left(\check{\mathcal{Q}}_E\right)=3\Leftrightarrow\sum_{i,j,k=1}^{n}d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k\neq 0\tag{74}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M}+\check{\mathcal{Q}}_E\right)=-\left(\sum_{i=1}^{n}p^i\tilde{q}_i\right)^2+4\{\mathcal{I}_3(p),\mathcal{I}_3(\tilde{q})\},\tag{75}$$

and there are three possibilities:

$$\mathbf{i}\;:\;\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M}+\check{\mathcal{Q}}_E\right)>0;\tag{76}$$

$$\mathbf{ii}\;:\;\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M}+\check{\mathcal{Q}}_E\right)<0;\tag{77}$$

$$\mathbf{iii}\;:\;\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M}+\check{\mathcal{Q}}_E\right)=0.\tag{78}$$

In Cases **i** and **ii**, the resulting FTS element has rank four, and thus, it corresponds to a "large" EBH, whereas in Case **iii**, the resulting FTS has rank three, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ may once again result in various types of orbit transmutations: a change from a "small" U-orbit (rank-two) to a "large" U-orbit (rank-four, with positive or negative $\mathcal{I}_4$), or (in Case **iii**) a change from a "small" rank-two orbit to a "small" rank-three orbit.

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 2 \Leftrightarrow \sum_{i,j,k=1}^{n} d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k = 0, \text{ but } \sum_{j,k=1}^{n} d^{ijk}\tilde{q}_j\tilde{q}_k \neq 0 \text{ for at least some } i, \tag{79}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2 + 4\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}, \tag{80}$$

and there are three possibilities (note that the term $\{\mathcal{I}_3(p), \mathcal{I}_3(\tilde{q})\}$ may or may not be vanishing):

$$\mathbf{i} \;:\; \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) > 0; \tag{81}$$

$$\mathbf{ii} \;:\; \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) < 0; \tag{82}$$

$$\mathbf{iii} \;:\; \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) = 0. \tag{83}$$

Once again, in Cases **i** and **ii**, the resulting FTS element has rank four, and thus, it corresponds to a "large" EBH, whereas in Case **iii**, the resulting FTS has one of the possible ranks $3, 2, 1$, corresponding to a "small" EBH. Thus, in this case that the action of $\hat{\mathcal{O}}_{PQ}$ may once again result in various types of orbit transmutations: a change from a "small" U-orbit (rank-two) to a "large" U-orbit (rank-four, with positive or negative $\mathcal{I}_4$), or (in Case **iii**) the change from a "small" U-orbit (rank-two) to "small" orbit, of possible rank 3, 2 ,1. In the case of rank$=$ 2, generally a change of orbit representative takes place (as above, depending on the real form of the theory under consideration, this may result necessarily in remaining in the same rank-two orbit, or possibly in switching to another rank-two orbit [5,6]).

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 1 \Leftrightarrow \sum_{j,k=1}^{n} d^{ijk}\tilde{q}_j\tilde{q}_k = 0 \;\forall i, \text{ but } \tilde{q}_i \neq 0 \text{ for at least some } i, \tag{84}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2, \tag{85}$$

and there are two possibilities:

$$\mathbf{i} \;:\; \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) < 0; \tag{86}$$

$$\mathbf{ii} \;:\; \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{2M} + \check{\mathcal{Q}}_E\right) = 0. \tag{87}$$

In Case **i**, the resulting FTS element has rank four (with negative $\mathcal{I}_4$), and thus, it corresponds to a "large" EBH, whereas in Case **ii**, the resulting FTS has one of the possible ranks $3, 2, 1$, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ may once again result in various types of orbit transmutations: a change from a "small" U-orbit (rank-two) to a "large" U-orbit (rank-four, with negative $\mathcal{I}_4$), or (in Case **ii**) the change from a "small" U-orbit (rank-two) to "small" orbit, of possible rank 3, 2, 1. For the case of rank$=$ 2, generally a change of orbit representative takes place (as before, depending on the real form of the theory under consideration, this may result necessarily in remaining at the same rank-two orbit, or possibly in switching to another rank-two orbit [5,6]).

2.4]"Small" rank-one electric configuration ($\partial\mathcal{I}_3(q)/\partial q_i = 0\ \forall i$, but $q_i \neq 0$ at least for some i):

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{1E}\rangle = \sum_{i=1}^{n} q_i|3^i\rangle = |\mathcal{Q}_{1E}\rangle. \tag{88}$$

3.4]"Small" rank-one magnetic configuration ($\partial\mathcal{I}_3(p)/\partial p^i = 0\ \forall i$, but $p^i \neq 0$ at least for some i):

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{1M}\rangle = \sum_{i=1}^{n} c_i p^i|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle + \sum_{i,j=1}^{n} p^j\Theta_{ji}|3^i\rangle = |\mathring{\mathcal{Q}}_{1M}\rangle + |\check{\mathcal{Q}}_E\rangle, \tag{89}$$

where:

$$|\mathring{\mathcal{Q}}_{1M}\rangle := \sum_{i=1}^{n} c_i p^i|1\rangle + \sum_{i=1}^{n} p^i|2^i\rangle. \tag{90}$$

Depending on $\text{rank}(\check{\mathcal{Q}}_E) = 3, 2, 1$, one has still three subcases.

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 3 \Leftrightarrow \sum_{i,j,k=1}^{n} d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k \neq 0 \tag{91}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2, \tag{92}$$

and there are two possibilities:

$$\mathbf{i}\ :\ \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) < 0; \tag{93}$$

$$\mathbf{ii}\ :\ \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = 0. \tag{94}$$

In Case **i** the resulting FTS element has rank four (with negative $\mathcal{I}_4$), and thus, it corresponds to a "large" EBH, whereas in Case **ii**, the resulting FTS has one of the possible ranks 3, 2, 1, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ results in various types of orbit transmutations: a change from a "small" U-orbit (rank-one) to a "large" U-orbit (rank-four, with positive or negative $\mathcal{I}_4$), or (in Case **ii**) the change from a "small" U-orbit (rank-one) to "small" orbit, of possible rank 3, 2, 1; in the case of rank$= 1$, generally a change of orbit representative takes place, and regardless of the real form of the theory under consideration, one necessarily remains in the same rank-one orbit, which is always unique [5,6].

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 2 \Leftrightarrow \sum_{i,j,k=1}^{n} d^{ijk}\tilde{q}_i\tilde{q}_j\tilde{q}_k = 0,\ \text{but}\ \sum_{j,k=1}^{n} d^{ijk}\tilde{q}_j\tilde{q}_k \neq 0\ \text{for at least some}\ i, \tag{95}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2, \tag{96}$$

and, as above, there are two possibilities:

$$\mathbf{i}\ :\ \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) < 0; \tag{97}$$

$$\mathbf{ii}\ :\ \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = 0. \tag{98}$$

In Case **i**, the resulting FTS element has rank four, and thus, it corresponds to a "large" EBH, whereas in Case **ii**, the resulting FTS has possible ranks $3, 2, 1$, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ results in various types of orbit transmutations: a change from a "small" U-orbit (rank-one) to a "large" U-orbit (rank-four, with negative $\mathcal{I}_4$), or (in Case **ii**) the change from a "small" U-orbit (rank-one) to "small" orbit, of possible rank 3, 2, 1; in the case of rank= 1, once again, one necessarily remains in the same rank-one orbit [5,6].

$$\text{rank}\left(\check{\mathcal{Q}}_E\right) = 1 \Leftrightarrow \sum_{j,k=1}^{n} d^{ijk}\tilde{q}_j\tilde{q}_k = 0 \ \forall i, \text{ but } \tilde{q}_i \neq 0 \text{ for at least some } i, \tag{99}$$

$$\Downarrow$$

$$\mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = -\left(\sum_{i=1}^{n} p^i\tilde{q}_i\right)^2, \tag{100}$$

and there are two possibilities:

$$\mathbf{i} \quad : \quad \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) < 0; \tag{101}$$

$$\mathbf{ii} \quad : \quad \mathcal{I}_4\left(\mathring{\mathcal{Q}}_{1M} + \check{\mathcal{Q}}_E\right) = 0. \tag{102}$$

In Case **i**, the resulting FTS element has rank four (with negative $\mathcal{I}_4$), and thus, it corresponds to a "large" EBH, whereas in Case **ii**, the resulting FTS has possible rank $3, 2, 1$, corresponding to a "small" EBH. Thus, in this case, the action of $\hat{\mathcal{O}}_{PQ}$ results in various types of orbit transmutations: a change from a "small" U-orbit (rank-one) to a "large" U-orbit (rank-four, with negative $\mathcal{I}_4$), or (in Case **ii**), the change from a "small" U-orbit (rank-one) to "small" orbit, of possible rank 3, 2, 1; in the case of rank= 1, once again, one necessarily remains in the same rank-one orbit [5,6].
1.3]"Small" rank-one magnetic KK configuration:

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{mKK}\rangle = p^0|0\rangle + \rho p^0|1\rangle + p^0\sum_{i=1}^{n} c_i|3^i\rangle = |\mathcal{Q}'_E\rangle|_{q_0=0}, \tag{103}$$

where $|\mathcal{Q}'_E\rangle|_{q_0=0}$ is the $q_0 = 0$ limit of $|\mathcal{Q}'_E\rangle$ given by (35). Thus, in this case, the Peccei–Quinn symplectic transformation generates a ρ-dependent graviphoton electric charge and c_i-dependent electric charges. These latter in Type II compactifications correspond to a stack of $D2$ branes depending on the components of the second Chern class c_2 of the Calabi–Yau three-fold. The corresponding transformation of I4 reads:

$$\mathcal{I}_4(\mathcal{Q}_{mKK}) = 0 \longrightarrow \mathcal{I}_4(|\mathcal{Q}'_E\rangle|_{q_0=0}) - \left(p^0\right)^4\left(\rho^2 + \frac{2}{3}\sum_{i,j,k=1}^{n} d^{ijk}c_ic_jc_k\right) \gtreqless 0 \tag{104}$$

Thus, depending on the sign of the r.h.s. of $\mathcal{I}_4(|\mathcal{Q}'_E\rangle|_{q_0=0})$, a "large" ($\mathcal{I}_4 > 0$: BPS, or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration is generated.
1.2]"Small" rank-one electric KK configuration:

$$\hat{\mathcal{O}}_{PQ}|\mathcal{Q}_{eKK}\rangle = q_0|1\rangle = |\mathcal{Q}_{eKK}\rangle. \tag{105}$$

5. Superpositions

In the Hilbert space representation, the superposition of EBH charge configurations allows one to deal with the whole charge scenario for the state configurations of EBHs. A quite general class of superpositions can be written in the following form:

$$\begin{aligned}|\mathbf{Q}\rangle \quad : \quad &= \alpha_1|\mathcal{Q}_{KK}\rangle + \alpha_2|\mathcal{Q}_E\rangle + \alpha_3|\mathcal{Q}_M\rangle + \alpha_4|\mathcal{Q}_{3E}\rangle + \alpha_5|\mathcal{Q}_{3M}\rangle \\ &+ \alpha_6|\mathcal{Q}_{2E}\rangle + \alpha_7|\mathcal{Q}_{2M}\rangle + \alpha_8|\mathcal{Q}_{1E}\rangle + \alpha_9|\mathcal{Q}_{1E}\rangle \\ &+ \alpha_{10}|\mathcal{Q}_{mKK}\rangle + \alpha_{11}|\mathcal{Q}_{eKK}\rangle, \end{aligned} \tag{106}$$

where $\alpha_1, ..., \alpha_{11} \in \mathbb{C}$ (see the next section for a discussion of the necessary emergence of complex parameters). We can rewrite this state as:

$$\begin{aligned}|\mathbf{Q}\rangle &= (\alpha_1+\alpha_2+\alpha_{10})p^0|0\rangle + (\alpha_1+\alpha_3+\alpha_{11})q_0|1\rangle \\ &+ (\alpha_2+\alpha_4+\alpha_6+\alpha_8)\sum_{i=1}^{n} q_i|3^i\rangle + (\alpha_3+\alpha_5+\alpha_7+\alpha_9)\sum_{i=1}^{n} p^i|2^i\rangle, \end{aligned} \tag{107}$$

and the the explicit expression of the quartic invariant for such a superposed EBH charge configuration reads:

$$\begin{aligned}\mathcal{I}_4(\mathbf{Q}) &= -(\alpha_1+\alpha_2+\alpha_{10})^2(\alpha_1+\alpha_3+\alpha_{11})^2\left(p^0q_0\right)^2 \\ &\quad -(\alpha_2+\alpha_4+\alpha_6+\alpha_8)^2(\alpha_3+\alpha_5+\alpha_7+\alpha_9)^2\sum_{i=1}^{n}\left(p^iq_i\right)^2 \\ &- 2(\alpha_1+\alpha_2+\alpha_{10})(\alpha_1+\alpha_3+\alpha_{11})(\alpha_2+\alpha_4+\alpha_6+\alpha_8)(\alpha_3+\alpha_5+\alpha_7+\alpha_9)p^0q_0\sum_{i=1}^{n}p^iq_i \\ &+ 4(\alpha_1+\alpha_3+\alpha_{11})(\alpha_3+\alpha_5+\alpha_7+\alpha_9)^3q_0\mathcal{I}_3(p) \\ &\quad -4(\alpha_1+\alpha_2+\alpha_{10})(\alpha_2+\alpha_4+\alpha_6+\alpha_8)^3p^0\mathcal{I}_3(q) \\ &\quad +4(\alpha_3+\alpha_5+\alpha_7+\alpha_9)^3(\alpha_2+\alpha_4+\alpha_6+\alpha_8)^3\{\mathcal{I}_3(p),\mathcal{I}_3(q)\}, \end{aligned} \tag{108}$$

and thus, the Bekenstein–Hawking EBH entropy is given by:

$$S(\mathbf{Q}) = \pi\sqrt{|\mathcal{I}_4(\mathbf{Q})|}. \tag{109}$$

As a simple example, let us consider the superposition of a "large" KK and a "large" electric charge configuration:

$$|\mathbf{Q}_{KK,E}\rangle := \alpha_1|\mathcal{Q}_{KK}\rangle + \alpha_2|\mathcal{Q}_E\rangle,\ \alpha_1,\alpha_2 \in \mathbb{C}, \tag{110}$$

yielding to

$$\begin{aligned}\mathcal{I}_4(\mathbf{Q}_{KK,E}) &= -(\alpha_1+\alpha_2)^2\alpha_1^2\left(p^0q_0\right)^2 - 4(\alpha_1+\alpha_2)\alpha_2{}^3p^0\mathcal{I}_3(q), && (111) \\ S(\mathbf{Q}) &= \pi\sqrt{\left|(\alpha_1+\alpha_2)^2\alpha_1^2\left(p^0q_0\right)^2 + 4(\alpha_1+\alpha_2)\alpha_2{}^3p^0\mathcal{I}_3(q)\right|}. && (112)\end{aligned}$$

6. Entanglement PQ Operators and Complexification

Generalized PQ operators can be defined as tensor products of PQ operators (11). The simplest (i.e., two-fold) generalized PQ operator is defined as:

$$\hat{\mathcal{O}}^{(2)}_{PQ} := \hat{\mathcal{O}}_{PQ} \otimes \hat{\mathcal{O}}'_{PQ}, \tag{113}$$

where (recalling Equation (11)):

$$\hat{\mathcal{O}}_{PQ} = |0\rangle\langle 0| + |1\rangle\langle 1| + \rho|1\rangle\langle 0| + \sum_{i=1}^{n}\left(|2^i\rangle\langle 2^i| + |3^i\rangle\langle 3^i|\right) + \sum_{i=1}^{n}\left(c_i|1\rangle\langle 2^i| + c_i|3^i\rangle\langle 0|\right) + \sum_{i,j=1}^{n}\Theta_{ij}|3^i\rangle\langle 2^i|, \tag{114}$$

$$\hat{\mathcal{O}}'_{PQ} = |0\rangle\langle 0| + |1\rangle\langle 1| + \rho'|1\rangle\langle 0| + \sum_{i=1}^{n}\left(|2^i\rangle\langle 2^i| + |3^i\rangle\langle 3^i|\right) + \sum_{i=1}^{n}\left(c'_i|1\rangle\langle 2^i| + c'_i|3^i\rangle\langle 0|\right) + \sum_{i,j=1}^{n}\Theta'_{ij}|3^i\rangle\langle 2^i|. \tag{115}$$

For instance, the action of $\hat{\mathcal{O}}^{(2)}_{PQ}$ (with $c_i = 0$) on a composed configuration of two KK "large" EBH states (which can also be interpreted as the state of a two-centered EBH in which each center is characterized by a KK "large" charge configuration) is given by (recall Equation (35)):

$$\begin{aligned}
\hat{\mathcal{O}}^{(2)}_{PQ}\Big|_{c_i=0}\left(|\mathcal{Q}_{KK}\rangle \otimes |\mathcal{Q}'_{KK}\rangle\right) \quad :&= \hat{\mathcal{O}}_{PQ}\Big|_{c_i=0}|\mathcal{Q}_{KK}\rangle \otimes \hat{\mathcal{O}}'_{PQ}\Big|_{c_i=0}|\mathcal{Q}'_{KK}\rangle = |\tilde{\mathcal{Q}}_{KK}\rangle \otimes |\tilde{\mathcal{Q}}'_{KK}\rangle \\
&= \left(p^0|0\rangle + \left(q_0 + \rho p^0\right)|1\rangle\right) \otimes \left(p^{0\prime}|0\rangle + \left(q'_0 + \rho' p^{0\prime}\right)|1\rangle\right) \\
&= p^0 p^{0\prime}|00\rangle + p^0\left(q'_0 + \rho' p^{0\prime}\right)|01\rangle \\
&\quad + \left(q_0 + \rho p^0\right)p^{0\prime}|10\rangle + \left(q_0 + \rho p^0\right)\left(q'_0 + \rho' p^{0\prime}\right)|11\rangle,
\end{aligned} \tag{116}$$

where we introduced:

$$|\alpha\beta\rangle := |\alpha\rangle \otimes |\beta\rangle,\ \alpha, \beta = 0, 1 \tag{117}$$

as the basis of the Hilbert space of two "large" KK EBHs (or, equivalently, of a two-centered EBH in which each center is characterized by a KK "large" charge configuration).

However, in order to generate entanglement in EBH systems through PQ symplectic transformations and thus realize "EBH quantum circuits" and "EBH quantum gates" in the context of BHQC [45], one can employ simpler composed PQ operators, namely:

$$\hat{\mathcal{E}}_{PQ} := \alpha\hat{\mathbb{I}} \otimes \hat{\mathcal{O}}_{PQ} + \hat{\mathcal{O}}_{PQ} \otimes \beta\hat{\mathbb{I}}, \tag{118}$$

which we name the (two-fold) entangled PQ operator; $\hat{\mathbb{I}}$ denotes the identity operator and $\alpha, \beta \in \mathbb{C}$.

As an example, let us again consider a composed configuration of two KK "large" EBH states. The action of $\hat{\mathcal{E}}_{PQ}$ (118) (with $c_i = 0$) is given by:

$$\begin{aligned}
\hat{\mathcal{E}}_{PQ}\Big|_{c_i=0}\left(|\mathcal{Q}_{KK}\rangle \otimes |\mathcal{Q}'_{KK}\rangle\right) &= \alpha\left(p^0|0\rangle + q_0|1\rangle\right) \otimes \left(p^{0\prime}|0\rangle + \left(q'_0 + \rho p^{0\prime}\right)|1\rangle\right) \\
&+ \beta\left(p^0|0\rangle + \left(q_0 + \rho p^0\right)|1\rangle\right) \otimes p^{0\prime}|0\rangle + q'_0|1\rangle \\
&= \alpha\left(p^0 p^{0\prime}|00\rangle + p^0\left(q'_0 + \rho p^{0\prime}\right)|01\rangle + q_0 p^{0\prime}|10\rangle + q_0\left(q'_0 + \rho p^{0\prime}\right)|11\rangle\right) \\
&\quad + \beta\left(p^0 p^{0\prime}|00\rangle + p^0 q'_0|01\rangle + \left(q_0 + \rho p^0\right)p^{0\prime}|10\rangle + \left(q_0 + \rho p^0\right)q'_0|11\rangle\right) \\
&= \left(\alpha + \beta\right)p^0 p^{0\prime}|00\rangle + \left(\alpha p^0\left(q'_0 + \rho p^{0\prime}\right) + \beta p^0 q'_0\right)01\rangle \\
&\quad + \left(\alpha q_0 p^{0\prime} + \beta\left(q_0 + \rho p^0\right)p^{0\prime}\right)|10\rangle + \left(\alpha q_0\left(q'_0 + \rho p^{0\prime}\right) + \beta\left(q_0 + \rho p^0\right)q'_0\right)|11\rangle.
\end{aligned} \tag{119}$$

In the limit of indistinguishability of the KK EBH states, one can generate different states of the composed KK two-fold Hilbert space by a suitable choice of the complex parameters α and β; indeed, in such a limit, Equation (119) simplifies down to:

$$\begin{aligned}
\hat{\mathcal{E}}_{PQ}\Big|_{c_i=0}\left(|\mathcal{Q}_{KK}\rangle \otimes |\mathcal{Q}_{KK}\rangle\right) &= \left(\alpha + \beta\right)\left(p^0\right)^2|00\rangle + \left(\left(\alpha + \beta\right)p^0 q_0 + \alpha\rho\left(p^0\right)^2\right)01\rangle \\
&\quad + \left(\left(\alpha + \beta\right)p^0 q_0 + \beta\rho\left(p^0\right)^2\right)|10\rangle + \left(\alpha + \beta\right)q_0\left(q_0 + \rho p^0\right)|11\rangle.
\end{aligned} \tag{120}$$

Thence, for instance, the conditions:

$$\left\{ \begin{array}{l} \left((\alpha+\beta)\, p^0 q_0 + \alpha\rho\, \left(p^0\right)^2\right) = 0, \\ \left((\alpha+\beta)\, p^0 q_0 + \beta\rho\, \left(p^0\right)^2\right) = 0 \end{array} \right. \tag{121}$$

solved by:

$$\alpha = \beta, \ \ \rho = -2\frac{q_0}{p^0} \tag{122}$$

yield a non-normalized Bell (GHZ) state:

$$2\alpha\left(\left(p^0\right)^2 |00\rangle - q_0^2|11\rangle\right). \tag{123}$$

on which the further normalization conditions:

$$\left(p^0\right)^2 = \frac{1}{2\alpha\sqrt{2}} = -q_0^2 \tag{124}$$

have a solution only when considering the complexification $\mathbf{R}_{\mathbb{C}}$ of the G_4-representation space of the EBH electromagnetic charges. As observed, e.g., in [28], this is a crucial step in order to implement a quantum mechanical computation (such as the ones exploited by the "EBH quantum circuits" and "EBH quantum gates" [45]), since in QIT, all parameters are generally complex and not real (as instead, the electric and magnetic charges of an EBH are). This clearly implies that the PQ symplectic transformations themselves should be considered on a complex ground field $\mathbb{C}$, since they act on the complex vector space $\mathbf{R}_{\mathbb{C}}$, on which $G_4(\mathbb{C})$ acts linearly, but non-transitively. Thus, by choosing [7]:

$$\left\{ \begin{array}{l} \alpha = \beta \\ p^0_{\pm} = \pm 2^{-3/4}\alpha^{-1/2} \\ q_{0,\pm} = \pm i 2^{-3/4}\alpha^{-1/2} \\ \rho = \pm 2i \end{array} \right. \tag{125}$$

one obtains:

$$\hat{\mathcal{E}}_{PQ}\Big|_{c_i=0} \left(|\mathcal{Q}_{KK}\rangle \otimes |\mathcal{Q}_{KK}\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \tag{126}$$

namely, one can generate a normalized Bell (GHZ) state acting with the entanglement PQ operator on a suitable, complexified "large" KK EBH state.

On the other hand, the conditions:

$$\left\{ \begin{array}{l} (\alpha+\beta)\left(p^0\right)^2 = 0, \\ (\alpha+\beta)\, q_0 \left(q_0 + \rho p^0\right) = 0 \end{array} \right. \tag{127}$$

solved by:

$$\alpha = -\beta \tag{128}$$

yield a non-normalized W-state:

$$-\alpha\rho\left(p^0\right)^2\left(|10\rangle - |01\rangle\right), \tag{129}$$

---

7 Note that the "±" branches of p^0 and q_0 are independent, but the "±" branch of ρ depends on their choice, consistently with Equation (122).

which cannot be normalized, even on $\mathbb{C}$: necessarily, a phase $e^{i\pi}$ is introduced between the pure states $|10\rangle$ and $|01\rangle$. Indeed, by choosing:

$$\alpha = -\frac{1}{\sqrt{2}\rho\,(p^0)^2} = -\beta,$$

one obtains:

$$\hat{\mathcal{E}}_{PQ}\big|_{c_i=0}\left(|\mathcal{Q}_{KK}\rangle \otimes |\mathcal{Q}_{KK}\rangle\right) = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right), \tag{130}$$

namely, one can generate a normalized W-state with $e^{i\pi}$-interference acting with the entanglement PQ operator on a suitable (not necessarily complexified!) "large" KK EBH state. The presence of the interference between the pure states $|10\rangle$ and $|01\rangle$ in Equation (130) seems to be a feature of the entanglement generation through PQ symplectic transformations, in the indistinguishability limit/assumption.

However, we anticipate that when relaxing the indistinguishability assumption (i.e., when resorting to a multi-centered EBH picture), much more freedom is introduced in the above entanglement generation procedure, and more general (and less constrained) solutions are obtained .

Clearly, the presence of other EBH charge configurations/states (such as the ones considered in Sections 3 and 4) allows for a wealth of entangled EBH states. A thorough analysis of PQ entanglement generation on EBH states will be carried in a forthcoming paper.

7. Conclusions

Within the black-hole/qubit correspondence, in the present paper, we have paved the way to a consistent generation of entanglement in EBH systems, by means of an operator implementation of the so-called Peccei–Quinn (PQ) symplectic transformations.

Given an EBH state described by its supporting electromagnetic charge configuration (i.e., by an element, with a well-defined associated rank, of the corresponding Freudenthal triple system), the action of the PQ operator can result in various types of orbit transmutations, namely in various mechanisms of switching among different U-duality orbits.

In the indistinguishability assumption, the need for complexification of the U-duality representation space $\mathbf{R}$ (resulting in a complexification of the PQ transformations, as well) and the unavoidable occurrence of phases among pure EBH states has been pointed out; however, we anticipate that if one resorts to the multi-centered EBH space-time picture and relaxes the indistinguishability condition, at least some of such limits can be circumvented.

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