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A Solution of the Mitra Paradox

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Abstract: The "Mitra paradox" refers to the fact that while the de Sitter spacetime appears non-static in a freely falling reference frame, it looks static with reference to a fixed reference frame. The coordinate-independent nature of the paradox may be gauged from the fact that the relevant expansion scalar, $\theta = \sqrt{3\Lambda}$, is finite if $\Lambda > 0$. The trivial resolution of the paradox would obviously be to set $\Lambda = 0$. However, here it is assumed that $\Lambda > 0$, and the paradox is resolved by invoking the concept of "expansion of space". This is a reference-dependent concept, and it is pointed out that the solution of the Mitra paradox is obtained by taking into account the properties of the reference frame in which the coordinates are co-moving.

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1. Introduction

Abhas Mitra [1] has recently discussed an interesting problem concerning the physical interpretation of the de Sitter spacetime. He has pointed out that seemingly there is a contradiction between the static form of the de Sitter metric and the non-static, expanding representation of this spacetime as a Friedmann-Lemaitre-Robertson-Walker universe model of the steady state type. This will here be called the Mitra paradox.

Mitra writes that there is a physical or at least interpretational self-contradiction between the original static interpretation of the de Sitter metric and the present day non-static de Sitter view. Furthermore he writes that a metric represents a physical point of view, and due to the principle of covariance, the essential physical picture should not depend on the choice of coordinates. He also points out that that there has not been any attempt for physical resolution to reconcile the static and non-static versions of for example the de Sitter metric. In this paper I will provide such a reconciliation.

Since there is a similar conflict between the Minkowski spacetime and the Milne universe model [2,3], I will start the present discussion by considering the corresponding Mitra paradox for these metrics. Then the de Sitter spacetime will be considered and finally the Schwarzschild and the Schwarschild-de Sitter spacetime.

Write Schutz [4] writes that we define a *static* spacetime to be one in which we can find a time coordinate t with two properties: (i) all metric components are independent of t; and (ii) the geometry is unchanged by time reversal, $t \to -t$. A spacetime with the property (i) but not (ii) is said to be *stationary*. This definition can be formulated in terms of Killing vectors. A static spacetime is a spacetime, which has a time-like Killing vector field that is hypersurface orthogonal. This is a coordinate-independent characterization of a static spacetime. If any coordinate system exists in which none of the metric components depend upon time, there exists a time-like Killing vector in the spacetime. In this case the actual physical geometry of the spacetime is unchanging with time. Although the geometry of a stationary spacetime does not change in time, it can rotate. If the spacetime does not permit a time coordinate so that all the metric components are independent of t it is non-static and non-stationary. In this case the spacetime has no time-like Killing vector field.

The somewhat surprising fact is that even *a static spacetime can have a time-dependent metric*. The Mitra paradox is concerned with finding the proper physical meaning of this fact.

2. The Connection between the Global Minkowski Metric and the Milne Universe

It is well known that the Minkowski and Milne metrics are connected by a change of reference frame [5–8]. This will here be utilized to shed some light upon the Mitra paradox. We consider Minkowski spacetime with spherical coordinates $(R,\ \theta,\ \varphi)$ and a time coordinate T so that the line element takes the form

$$ds^2 = -\,dT^2 + dR^2 + R^2 d\Omega^2 \quad , \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{1}$$

where we have used units so that $c = 1$. Then we introduce new coordinates $(t,\ r)$ by the transformation

$$t = \sqrt{T^2 - R^2} \quad , \quad r = \frac{t_0 R}{\sqrt{T^2 - R^2}} \tag{2}$$

The inverse transformation is

$$T = t\sqrt{1 + \frac{r^2}{t_0^2}} \quad , \quad R = \frac{r\,t}{t_0} \tag{3}$$

where t_0 is the present age of the universe. It follows from this transformation that

$$R = \frac{r}{\sqrt{1 + r^2/t_0^2}}\frac{T}{t_0} \quad , \quad T^2 - R^2 = t^2 \tag{4}$$

We see that the world-lines of the reference particles defining the reference frame in which $(t,\ r)$ are co-moving, i.e., $r =$ constant, are straight lines, and the simultaneity curves $t =$ constant are hyperbolae.

It is seen that while the coordinates $(T,\ R)$ are co-moving in a static reference frame, the coordinates $(t,\ r)$ are co-moving in an expanding reference frame. A reference particle with $r =$ constant in the expanding frame has a coordinate velocity

$$v_R = \frac{r/t_0}{\sqrt{1 + r^2/t_0^2}} = \frac{R}{T} \tag{5}$$

in the rigid frame. In the expanding frame the line element of the Minkowski spacetime takes the form

$$ds^2 = -\,dt^2 + \left(\frac{t}{t_0}\right)^2 \left(\frac{dr^2}{1 + r^2/t_0^2} + r^2 d\Omega^2\right) \tag{6}$$

This is the line element of an empty, expanding universe model with negative spatial curvature—the Milne universe.

The Minkowski coordinates $(T,\ R)$ are the co-moving coordinates of a rigid inertial reference frame of an arbitrarily chosen reference particle P in the expanding cloud of particles defining the Milne universe model. The time T is the *private time* of P. The time t is measured on clocks following all of the reference particles. As seen from the first of Equation (4) the space T = constant has a finite extent, $R_{\max} = \lim\limits_{r\to\infty} R = T$. This space is the *private space* of an observer following the particle P. The space t = constant is represented by a hyperbola given in the second of Equation (4) as shown in the Minkowski diagram of the P observer. It is defined by simultaneity of the clocks carried by all the reference particles, and is called the *public space* or simply the space of the universe model. It has infinite extension in spite of the fact that the Big Bang has the character of a point event in the Milne universe model.

In the inertial and rigid Minkowski coordinate system the velocity of a reference particle with co-moving coordinates is less than c for all values of. However, in the expanding cosmic frame it is different. Here the velocity of the reference particles as defined by an observer at the origin is given by Hubble's law. Hence the reference particles have superluminal velocity at sufficiently great distances from the observer. According to special relativistic kinematics, superluminal velocity is problematic because the particles cannot move through space with a velocity greater than c. However, according to the general relativistic interpretation, the reference particles define the public space of the universe model, and there is no limit to how fast space itself can expand.

The metric (1) is static and the metric (6) not. The Mitra paradox is concerned with a reconciliation of these properties of two metrics that are connected by a coordinate transformation, and hence that represent one and the same spacetime.

The Mitra paradox makes it clear that one cannot define a static spacetime as a spacetime where the metric is independent of time. The metric is coordinate-dependent, and may be independent of time in one coordinate system, but dependent on it in another, while the static property of a spacetime is invariant.

3. Proposal for a Solution of the Mitra Paradox

An important point when we try to solve the Mitra paradox is to distinguish between coordinate-dependent quantities and coordinate-independent physical quantities. The term *metric* is usually taken to mean the functions that appear in the line element multiplied by the coordinate differentials. Hence the metric is understood to mean the *components* of the metric tensor. This means that *the metric is a coordinate-dependent quantity.* It is natural, therefore, that at least in some cases, one and the same spacetime can be represented by both a static and a non-static metric.

Another important distinction is the difference between a coordinate system and a reference frame. In four-dimensional spacetime, a *coordinate system* provides a region of spacetime with a continuum of 4-tuples so that each event in spacetime is marked with a 4-tuple, different events with different 4-tuples. A coordinate system is a mathematical quantity. A *reference frame* is a continuum of world-lines representing reference particles with specified motions. This is a physical quantity. *Co-moving coordinates* in a reference frame are coordinates so that the reference particles of the frame have constant spatial coordinates.

The Mitra paradox is not only concerned with the metric, but also with 3-space. Hence it is important to distinguish between a coordinate 3-space and a coordinate-independent physical 3-space. Here we meet an important difficulty of the Mitra paradox. The 3-space has two very different qualities. On the one hand, it is a set of simultaneous events measured by clocks at rest in the chosen reference frame. Again this is coordinate-dependent or better, reference-dependent, due to the relativity of simultaneity.

It follows from Friedmann's 1. equation for a flat universe, $H = \sqrt{8\pi G\rho/3}$, that if the Minkowski spacetime is perceived as the limit of the Friedmann-Lemaitre-Robertson-Walker universe model with empty space, the Hubble parameter vanishes, and the 3-space is static. In a similar way, the De Sitter spacetime is then the limit of empty space with a cosmological constant, Λ, having a positive, constant Hubble parameter, $H = \sqrt{\Lambda/3}$, and the 3-space of the de Sitter spacetime is non-static and expanding. However, both of these spacetimes have maximal symmetry and have a time-like hyper surface orthogonal Killing vector, meaning that these spacetimes are static. Hence it is important to note the difference between a static 3-space and a static spacetime. The Mitra paradox is concerned with 3-space.

We considered the globally flat spacetime above. In the standard coordinates co-moving with a static reference frame with time-independent distances between the reference points, the 3-space is static. But as described in terms of coordinates co-moving in an expanding reference frame, the 3-space is not static, but expands. The flat, static spacetime then looks like an expanding universe—the Milne universe model.

This seems strange. So far we have defined 3-space as a set of simultaneous events. There is no motion involved in this definition. Hence the definition of a 3-space should be supplied by a second quality permitting space to expand. We can then supply the definition of a 3-space: A 3-space is made up of a set of reference particles at a given point of time. The 3-space of a reference frame is defined by identifying the reference particles of the 3-space with the reference particles of the frame. The most simple mathematical description of the 3-space is obtained by using coordinates co-moving with the reference frame of the 3-space. *The 3-space is said to be stationary if the physical distances between the reference points does not change*. In this case the reference frame can be said to be *rigid*. If these physical distances change, the 3-space is non-stationary. If the rotation of the velocity field of the reference particles of a stationary 3-space vanishes, the 3-space is said to be *static*.

It is then clear that whether a 3-space appears static or non-static depends upon the reference frame it is associated with. This means that *the static or non-static character of a 3-space is a coordinate-dependent quality of the spacetime*. It will be made clear below that this is an important ingredient in the solution of the Mitra paradox.

One may wonder whether this means that physical 3-space does not exist. It is then important to make one more distinction: between *physical* and *invariant*. A quantity is said to be invariant if it has the same value in every reference frame or coordinate system. A physical property need not be invariant. For example a 3-velocity of a particle is a physical property, but it is not invariant. It can even be transformed away by going into the rest frame of the particle.

We should not talk about *the* 3-space of Minkowski spacetime. We should talk about *a* 3-space. Minkowski spacetime can have a static 3-space and equally well a non-static 3-space. Although the property of a 3-space of being static is a physical property, its static character is not invariant. This is the proposed solution of the Mitra paradox, which will be further worked out below.

4. Static and Expanding 3-Space

Let us first consider a static spacetime as described in the co-moving coordinates of a rigid reference frame, RF, so that the metric is static and has the form

$$ds^2 = -f\,(R)\,dt^2 + \frac{dR^2}{f\,(R)} + R^2 d\Omega^2 \tag{7}$$

Here the radial coordinate is chosen so that the invariant area of a spherical surface with radius R is equal to $4\pi R^2$. This radial coordinate is sometimes called the curvature radius or alternatively the area coordinate. The 3-space of simultaneous events as measured by clocks carried by the reference particles of RF, is static. This is the preferred 3-space of spacetimes with a localized mass distribution, such as the Schwarzschild spacetime.

Assume that there exists a surface with coordinate radius $R = R_0$ so that a particle permanently at rest on this surface has vanishing 4-acceleration, i.e., a free particle instantaneously at rest at this surface will remain at rest on the surface. The radial component of the 4-acceleration of a particle at rest in the coordinate system is according to the geodesic equation given in terms of certain Christoffel symbols and the time component of the 4-velocity as

$$a^R = \Gamma^R_{T\,T}\left(u^T\right)^2 = (1/2)\,f\prime(R) \tag{8}$$

Hence the radius R_0 is given by $f\prime\,(R_0) = 0$. Let us now consider the 3-space of simultaneous events as shown by clocks carried by *free particles* starting their movements from a state of rest at $R = R_0$. These particles make up a locally inertial frame, IF. Hence this 3-space may be called an *inertial 3-space*. This is the preferred 3-space of the relativistic universe models. In [9] it was shown that the 3-velocity of the inertial 3-space as given with respect to the orthonormal basis of an observer at rest in RF is

$$v_{3-space} = \left(\frac{d\hat{x}^R}{d\tau}\right)_{3-space} = \pm\sqrt{1 - \frac{f(R)}{f(R_0)}} \tag{9}$$

In the case of the Minkowski metric (1) this gives $v_{3-space} = 0$, and the inertial 3-space is then at rest in an arbitrary rigid frame in flat spacetime.

Let us now describe the 3-space with reference to an expanding reference frame in which the metric is of the Friedmann-Lemaitre-Robertson-Walker type. Then the reference frame consists of a set of freely moving particles expanding together with the cosmic fluid. Let t be the proper time of clocks co-moving with the reference particles of this frame, and r a co-moving radial coordinate following the cosmic fluid. Then the line element has the form

$$ds^2 = -\,dt^2 + [a(t)]^2\left(\frac{dr^2}{1 - k\,r^2} + r^2 d\Omega^2\right) \tag{10}$$

where k is a constant, which can have the values $k = \left\{-1/t_0^2\,,\,0\,,\,1/t_0^2\right\}$, and $a(t)$ is the scale factor. If it is normalized to $a(t_0) = 1$ at the present time t_0, it represents the ratio of the cosmic distances between the reference particles at an arbitrary point of time and their present distances. In this coordinate system the 3-space has an expansion, $\theta = 3\,\dot{a}/a$. Hence the 3-space is not static in this frame if $\dot{a} \neq 0$.

So the 3-space of the Minkowski spacetime may be pictured as either static or non-static depending upon the reference frame that is used. This freedom of point of view is due to the Lorentz invariance of this solution of Einstein's field equations. It is not typical of the solutions of the field equations in general. But there are two other solutions that share this property of Lorentz invariance with the Minkowski spacetime, and those are the de Sitter and anti-de Sitter spacetimes. Let us consider the de Sitter spacetime.

5. The de Sitter Spacetime

We consider the de Sitter spacetime with spherical coordinates $(R,\ \theta,\ \varphi)$ and a time coordinate T, so that the line element takes the form

$$ds^2 = -\left(1 - H^2R^2\right)dT^2 + \frac{dR^2}{1 - H^2R^2} + R^2 d\Omega^2 \tag{11}$$

for $R < R_H = 1/H$, where $H = \sqrt{\Lambda/3}$, and the cosmological constant $\Lambda = 8\pi G\rho_\Lambda$ represents the constant density ρ_Λ of a Lorentz Invariant Vacuum Energy, LIVE, with stress $p_\Lambda = -\,\rho_\Lambda$. It should be noted that the coordinate clocks showing T go with a position-independent rate equal to that of a standard clock at $R = 0$.

We introduce new coordinates $(t,\ r)$ by the transformation

$$t = T + \frac{1}{2H}\ln\sqrt{1 - H^2R^2} \quad , \quad r = \frac{R}{\sqrt{1 - H^2R^2}}e^{-\,HT} \tag{12}$$

The inverse transformation is

$$T = t - (1/2H)\,\ln\sqrt{1 - H^2r^2e^{2\,H\,t}} \quad , \quad R = r\,e^{H\,t} \tag{13}$$

Differentiating the second of Equation (12) with respect to T with constant r gives the coordinate velocity of a reference particle in the $(t,\ r)$-system with respect to the $(T,\ R)$-system

$$v_R = H\,e^{H\,t} = \sqrt{1 - H^2R^2}\,e^{H\,T} \tag{14}$$

Hence the $(t,\, r)$ coordinates are co-moving in an expanding reference frame relative to the rigid $(T,\, R)$-system. In the coordinates $(t,\, r)$ the line element has the form

$$ds^2 = -\,dt^2 + e^{2H\,t}\left(dr^2 + r^2 d\Omega^2\right) \tag{15}$$

The $(t,\, r)$-coordinates are co-moving with free particles, as is the case for all the FLRW-universe models. Equation (14) shows that the free particles in this spacetime move outwards with an accelerated motion.

It follows from Equation (8) that for the metric (11) the 4-acceleration of a particle fixed in the rigid reference frame, is

$$a^R = H^2 R \tag{16}$$

Hence a free particle at rest in the rigid reference frame must be at the position $R_0 = 0$. The 3-space made up of simultaneous events as measured by clocks carried by these particles is the inertial 3-space. In this case the velocity of inertial 3-space as given by Equation (9) is

$$v_{3-space} = H\,R \tag{17}$$

Even if the metric (11) is static, the velocity of the inertial 3-space obeys the Hubble lav. There is an expansion equal to $3H$. Hence the 3-space expands in accordance with the Hubble law. This is often called the *Hubble flow*. The inertial 3-space flows with the velocity of light at $R = R_H = 1/H$ and with superluminal velocity for $R > R_H$. There is a horizon at $R = R_H$.

The coordinate time t is shown on standard clocks following the freely falling reference particles of the 3-space, and the coordinate r is co-moving with those particles. That is the reason for the time dependence of the metric (15) in this coordinate system. Hence there is no contradiction between the static form of the line element (11) and the non-static form (15). The first form reflects the rigidity of the reference frame in which the coordinates T , R are co-moving, and the second reflects the expansion of the reference frame in which the coordinates t , r are co-moving. This solves the Mitra paradox.

However, there is something strange about the metric (11). There is a coordinate singularity at $R_H = 1/H$. Note that $\lim\limits_{R \to 1/H} r = \infty$, and that $v_{3-space}\,(R_H) = 1$. Hence the 3-space is flowing with the velocity of light at this surface.

Consider radially moving light in the metric (11). The coordinate velocity of light is the same in the inwards and outwards direction, and is equal to

$$(dR/dT)_L = \pm\left(1 - H^2 R^2\right) \tag{18}$$

which vanishes at $R = R_H$. Hence in this coordinate system the light cone collapses at $R = R_H$.

In order to have open light cones at $R = R_H$ one may introduce a new time coordinate. There are several related such coordinates, and it may be useful to compare the description of the light cones in three of them. All of them are given by an *internal coordinate transformation* in the sense that the coordinate clocks are at rest in the same reference frame as those showing T.

We first consider a light cone coordinate, $\overline{T}$, used by Spradlin, Strominger and Volovich [10], given by the coordinate transformation

$$\overline{T} = T - \frac{1}{2H}\ln\frac{1 + H\,R}{1 - H\,R} \tag{19}$$

By using L'Hopital's rule we get $\lim\limits_{H \to 0} \overline{T} = T - R$ showing that $\overline{T}$ reduces to an ordinary light cone coordinate in Minkowski spacetime. Differentiating we have

$$d\overline{T} = dT - \frac{dR}{1 - H^2 R^2} \tag{20}$$

Hence the coordinate clocks showing $\overline{T}$ have another simultaneity than those showing T. With the new time coordinate the line element of the de Sitter spacetime takes the form

$$ds^2 = -\left(1 - H^2R^2\right) d\overline{T}^2 - 2d\overline{T}\, dR + R^2 d\Omega^2 \tag{21}$$

For light moving radially we have $ds^2 = d\Omega^2 = 0$ and hence,

$$2d\overline{T}\, dR = -\left(1 - H^2R^2\right) d\overline{T}^2 \tag{22}$$

For light moving outwards $dR > 0$, which is not permitted by Equation (22). However, for light moving outwards, Equations (16) and (18) give $d\overline{T} = 0$, which *is* permitted. Hence $\overline{T}$ is a light cone coordinate for outgoing light. For light moving inwards, Equation (22) gives the coordinate velocity

$$\left(\frac{dR}{d\overline{T}}\right)_{L-} = -\frac{1}{2}\left(1 - H^2R^2\right) \tag{23}$$

which vanishes at $R = R_H$. The "inwards directed" velocity of light changes sign at $R = R_H$ and becomes outwards directed for $R > R_H$.

Another time coordinate T_P called the Painlevé-de Sitter coordinate, was used by Parikh [11] and is given by the transformation

$$T_P = T + (1/2\,H) \ln\left(1 - H^2R^2\right) \tag{24}$$

Comparing with equation the first of the Equation (12) we see that $T_P = t$. Hence the Paainlevé-de Sitter time is the same as the cosmic time, which is measured by standard clocks following freely moving particles. Differentiating gives

$$dT_P = dT - \frac{HR}{1 - H^2R^2} dR \tag{25}$$

Inserting this into Equation (11) we find that the line element takes the form

$$ds^2 = -\left(1 - H^2R^2\right) dT_P^2 - 2HR\, dT_P\, dR + dR^2 + R^2 d\Omega^2 \tag{26}$$

The coordinate velocity of outgoing and ingoing light is

$$(dR/dT_P)_+ = 1 + HR \quad , \quad (dR/dT_P)_- = -\,(1 - HR) \tag{27}$$

At the horizon $(dR/dT_P)_+ = 2 \quad , \quad (dR/dT_P)_- = 0$. The velocity of the ingoing light changes sign at the horizon, and moves outwards outside the horizon. This is an effect of the repulsive gravity due to the LIVE, which fills this spacetime and causes an accelerated expansion of the inertial 3-space. Note that in this context "accelerated" means non-vanishing 3-acceleration. The 4-acceleration of the reference particles of the inertial 3-space vanishes, since the particles are freely falling.

Finally a time coordinate corresponding to the ingoing Eddington-Finkelstein coordinate in the Schwarzshild spacetime is defined by the condition that the coordinate velocity of outgoing light is equal to 1. This was used by Braeck and Grøn [9] and is given by

$$\widetilde{T} = T + \frac{1}{2H} \ln\frac{1 - HR}{1 + HR} + R \tag{28}$$

Differentiation gives

$$d\widetilde{T} = dT - \frac{H^2R^2}{1 - H^2R^2} dR \tag{29}$$

With this time coordinate the line element takes the form

$$ds^2 = -\left(1 - H^2R^2\right) d\widetilde{T}^2 - 2H^2R^2\, d\widetilde{T}\, dR + \left(1 + H^2R^2\right) dR^2 + R^2 d\Omega^2 \tag{30}$$

The coordinate velocity of outgoing and ingoing light is

$$\left(dR/d\widetilde{T}\right)_+ = 1 \quad , \quad \left(dR/d\widetilde{T}\right)_- = -\frac{1 - H^2R^2}{1 + H^2R^2} \tag{31}$$

At the horizon $\left(dR/d\widetilde{T}\right)_+ = 1 \quad , \quad \left(dR/d\widetilde{T}\right)_- = 0$. Again the velocity of the ingoing light changes sign at the horizon, i.e., the light cones turn outwards, implying that nothing can enter the horizon from the outside region.

All of the line elements (21), (26) and (30) are stationary, although they are not static. The stationary character shows that the coordinate R is co-moving in a rigid reference frame. The reason that they are not static is that the coordinate clocks are not Einstein synchronized. Their simultaneity is not that of Einstein synchronized clocks at rest in the rigid reference frame.

The de Sitter spacetime is static since there exists a coordinate system where the metric is static and the time-like basis vector is a Killing vector.

Nevertheless this spacetime is filled with vacuum energy that expands. This energy causes repulsive gravity, which acts back upon the energy itself and makes the expansion accelerate. It should be noted that in a homogeneous universe there is no pressure gradient, so the accelerated expansion is not a pressure effect, but a gravitational effect. The negative pressure, $p = -\rho c^2$, contributes to the effective gravitational mass density, $\rho_{grav} = \rho + 3p/c^2$, making it negative, which means that gravity is repulsive [12]. Hence there is energy with accelerated expansion in this spacetime. Is it then reasonable to say that it is static?

Compare 3-space with a river, and consider the river now and an hour later, assuming that there is the same amount of water in the river at these points of time. In this situation, the river has not changed. The river is static. But the water is not static. It flows. Similarly spacetime is static, but 3-space is expanding.

In spacetime the river corresponds to the geometry of space at a certain position, and the flowing water corresponds to the flowing reference particles constituting the 3-space. In the de Sitter spacetime the geometry of space is unchanged at a fixed position in a rigid reference frame. Hence it is a static spacetime; but the 3-space is flowing. It is not static. The metric in a coordinate system co-moving with the reference particles of the 3-space is not static, but depends upon time as in the metric of Equation (15).

6. The Schwarzschild Spacetime

Outside the Schwarzschild radius the Schwarzschild spacetime has a time-like Killing vector field that is hypersurface orthogonal. Hence it is static and there exists a coordinate system in which the metric is independent of time and the line element has no product terms where a spatial differential is multiplied by a time differential. One such coordinate system is the standard so-called curvature coordinates where the invariant area of a surface with coordinate radius R around the origin is $4\pi R^2$. In this coordinate system the line element takes the form

$$ds^2 = -\left(1 - R_S/R\right) dT^2 + \frac{dR^2}{1 - R_S/R} + R^2 d\Omega^2 \tag{32}$$

where $R_S = 2GM$ is the Schwarzschild radius of the central mass. Inside the Schwarzschild radius the Killing vector field is spacelike, and in this region the Schwarzschild spacetime is not static.

Consider now an observer moving with the inertial 3-space in this spacetime, i.e., he is falling freely from a state of rest infinitely far away from the central mass. The co-moving radius of this observer is r and he carries with him a standard clock showing t. The new coordinates are given by transformation [13–15]

$$T = t + R_S \ln \frac{\sqrt{R/R_S}+1}{\sqrt{R/R_S}-1} - 2\sqrt{R_S R} \quad , \quad R^{3/2} = -\,(3/2)\sqrt{R_S}\,(t+r) \tag{33}$$

In terms of the co-moving coordinates of the inertial 3-space the line element of the Schwarzschild spacetime takes the form

$$ds^2 = -\,dt^2 + \left(\frac{2}{3}\frac{R_S}{t+r}\right)^{2/3} dr^2 + \left(\frac{3}{2}\sqrt{R_S}\,(t+r)\right)^{4/3} d\Omega^2 \tag{34}$$

An observer with $r = r_0$ has initially a large negative value of t, which increases towards $-\,r_0 - (2/3)\,R_S$ as the observer passes the Schwarzschild horizon. Hence the line element (34) corresponds to that of an inhomogeneous universe with an anisotropic and position-dependent scale factor, and the inertial 3-space expands in the radial direction and contracts in the tangential direction. In these coordinates the metric of the Schwarzschild spacetime is not static. These geometrical changes with time of the inertial 3-space are due to tidal forces becoming stronger at the position of the reference particles co-moving with the inertial 3-space, as they approach the central mass.

The coordinate transformation is well defined only for $R > R_S$. This is due to the rigid character of the reference frame in which the coordinates (T, R) are co-moving, which is physically possible only outside the Schwarzschild horizon. However the line element (34) has no coordinate singularity at the Schwarzschild horizon. The coordinates t and r are well defined in all of spacetime outside the central singularity, also inside the horizon, and the line element gives a singularity-free description of the Schwarzschild spacetime in the whole of this region. This illustration shows that a static spacetime, which is usually expressed so that 3-space is static, may also be expressed so that the 3-space is non-static.

7. The Schwarzschild-de Sitter Spacetime

This is a static spacetime in which the line element may be written

$$ds^2 = -\left(1 - \frac{R_S}{R} - H^2R^2\right) dT^2 + \frac{dR^2}{1 - \frac{R_S}{R} - H^2R^2} + R^2 d\Omega^2 \tag{35}$$

In this spacetime the inertial 3-space has a rather interesting behavior. At the surface with

$$R_0 = \left(R_S R_H^2/2\right)^{1/3} \tag{36}$$

the 4-acceleration of a particle permanently at rest vanishes [9]. Hence the reference particles of the inertial 3-space are at rest at this surface. But the inertial 3-space diverges at this surface. It expands outside this surface and contracts inside it.

8. Static Form of the FLRW Metric

Mitra has recently deduced an interesting form of the Friedmann-Lemaitre-Robertson-Walker metric in curvature coordinates [16] and used this to investigate when an expanding universe can look static [17]. The FLRW-metric is first written in the usual form (10). Mitra then found that the metric can be written in curvature coordinates as follows

$$ds^2 = -\left(\frac{\partial t}{\partial T}\right)^2 \frac{1 - k\,(R/a)^2 - (\dot{a}/a)^2 R^2}{1 - k\,(R/a)^2} dT^2 + \frac{dR^2}{1 - k\,(R/a)^2 - (\dot{a}/a)^2 R^2} + R^2 d\Omega^2 \tag{37}$$

where

$$R = r\,a\,(t) \tag{38}$$

He then showed that essentially only the Milne universe and the de Sitter and anti-de Sitter universe models can be written in static form using curvature coordinates.

As an illustrating example we will here consider only the first model studied by Mitra. It is the de Sitter universe model with negative spatial curvature, $k = -1$ and $\Lambda > 0$. For this model the solution of the Friedmann equations gives the scale factor

$$a\,(t) = (1/H)\sinh\,(H\,t) \tag{39}$$

where $H = \sqrt{\Lambda/3}$. This universe model is filled by vacuum energy with constant density and stress given by

$$p_\Lambda = -\,\rho_\Lambda = -\,\Lambda/8\pi G \tag{40}$$

Inserting Equations (38) and (39) into Equation (37), the line element takes the form

$$ds^2 = -\left(\frac{\partial t}{\partial T}\right)^2 \frac{1 - H^2R^2}{1 + r^2} dT^2 + \frac{dR^2}{1 - H^2R^2} + R^2 d\Omega^2 \tag{41}$$

Comparing with Equation (11) we obtain

$$\frac{\partial t}{\partial T} = \sqrt{1 + r^2} \tag{42}$$

Mitra has shown that Equations (38), (39) and (42) lead to the transformation

$$HR = r\sinh\,(Ht) \quad , \quad \tanh\,(HT) = \sqrt{1 + r^2}\tanh\,(Ht) \tag{43}$$

The inverse transformation may be written

$$\sinh\,(H\,t) = \sqrt{\sinh^2\,(HT) - H^2R^2\cosh^2\,(HT)} \quad , \quad r = \frac{HR}{\sqrt{\sinh^2\,(HT) - H^2R^2\cosh^2\,(HT)}} \tag{44}$$

Differentiating the first of these equations partially with respect to T and using the second equation, we get

$$\frac{\partial t}{\partial T} = \sqrt{\frac{1 - H^2R^2}{\tanh^2\,(HT) - H^2R^2}}\tanh\,(HT) = \sqrt{1 + r^2} \tag{45}$$

in agreement with Equation (42). Furthermore, by taking the differentials of the transformation (43) and inserting the expressions into Equation (11) one finds that (43) transforms the static metric (11) to the line element of the expanding de Sitter universe model with a negatively curved 3-space,

$$ds^2 = -\,dt^2 + \sinh^2\,(Ht)\left(\frac{dr^2}{1 + r^2} + r^2 d\Omega^2\right) \tag{46}$$

Equation (38) shows that the value of R increases with time for a fixed value of r. Again we see that the reconciliation of the static and non-static forms of the line element for one and the same spacetime is in recognizing the motion of the reference frames in which the radial coordinates are co-moving. The radial coordinate of the time-dependent metric is co-moving with an expanding reference frame, and the radial coordinate of the static metric is co-moving with a rigid reference frame. This is the solution of the Mitra paradox as applied to the present spacetime.

9. Energy Conservation

Writing the line element of the de Sitter spacetime in terms of coordinates co-moving with free particles, i.e., in the form (15), Mitra deduced that the vacuum energy inside a radius r is

$$U = (\Lambda/6\,G)\; r^3 e^{3Ht} \tag{47}$$

Mitra concluded: "Thus the total energy of the de Sitter model increases in an exponential manner. Such a bad violation of the "Principle of Conservation of Energy" in the co-moving frame is in sharp contrast with the corresponding nice behavior in the Schwarzschild frame". (He uses units in which the gravitational constant is $G = 1$, but I have kept G in the formulae.)

The solution of this seeming paradox is as follows. The Friedmann equations lead to

$$dU + p_\Lambda dV = 0 \tag{48}$$

where $V = (4\pi/3)\,a^3$, $U = \rho_\Lambda\,V$, and a is the scale factor of Equation (37). Equation (49) is the 1. Law of thermodynamics for adiabatic expansion as applied to a co-moving region with radius $r = 1$ around an observer. It expresses the law of energy conservation. Heat is defined as transport of energy due to temperature difference. In a homogeneous universe there are no large scale temperature differences, and this is the reason that the universe expands adiabatically.

Using $\rho_\Lambda = \Lambda/8\pi G$ and $a = e^{H\,t}$, we get $dU = \rho_\Lambda 4\pi a^2 da = (\Lambda H/2G)\,e^{3Ht}dt$, which is the same as we get by taking the differential of U in Equation (47). The volume work performed at the boundary of the region is $dW = p_\Lambda dV = -\,\rho_\Lambda 4\pi a^2 da$. Hence the energy conservation equation is obeyed in spite of the fact that the amount of vacuum energy is increasing inside the co-moving surface. The reason is that there is a negative work at the boundary, which transfers energy from the outside region to the inside region. Imagining that the region is extended so that the boundary is infinitely far from the observer, one may say that the density of the vacuum energy is kept constant in spite of the expansion by extracting energy from an infinitely far region. This shows that global energy conservation is indeed a problematic concept at least for a universe with infinitely great spatial extension.

10. Results and Discussion

Mitra has pointed out that there seems to be an interpretational self-contradiction between the static interpretation of the de Sitter metric and the non-static de Sitter universe model. This has here been called the Mitra paradox. He also writes that there has not been any attempt for a physical resolution to reconcile the static and non-static versions of for example the de Sitter metric.

Both the problem and the resolution are of a conceptual nature of great significance for a proper way of teaching the general theory of relativity. A theory is much more than some rules for calculating physical effects, making it possible to falsify the theory. The theory also provides us with concepts representing the foundations of our world picture. As said by Einstein: It is the theory that tells what we observe.

Hence it is extremely important to obtain a proper physical interpretation of the general theory of relativity, free of contradictions. This also means that interpretational problems such as that formulated by Mitra, should not be neglected. The present article has been an effort to give a constructive discussion of this problem—and to solve it. I have here provided a resolution by focusing upon the difference between 3-space and spacetime and pointing out the significance of the motion of the reference frames in which different coordinate systems are co-moving.

11. Conclusions

The Mitra paradox is concerned with the physical reconciliation of two metrics, where one is static and the other time-dependent, that are connected by a coordinate transformation, and hence that represent one and the same spacetime.

The Mitra paradox makes it clear that one cannot define a static spacetime as a spacetime where the metric is independent of time. The metric is coordinate-dependent, and may be independent of time in one coordinate system, but dependent on it in another, while the static property of a spacetime is invariant and characterized by the existence of a hypersurface orthogonal Killing vector.

The solution of the Mitra paradox lies in recognizing that the metric is not determined by the geometric properties of the spacetime. In general there are ten independent components of the metric tensor and only six independent field equations, leaving the freedom of choosing the coordinate system. By choosing coordinates in a given spacetime as co-moving with a rigid reference frame one obtains a time-independent metric—otherwise a time-dependent one.

These general properties of the solution to the Mitra paradox have been illustrated in the present paper by considering several cases, the first of which being flat spacetime. With coordinates co-moving in a rigid inertial frame one obtains the usual Minkowski metric, and with coordinates co-moving in an expanding reference frame one obtains the time-dependent Milne metric. Secondly, the de Sitter spacetime has been considered. Again, by using coordinates co-moving in a rigid frame one obtains a static metric and using coordinates co-moving with freely moving particles that make up a system that expands due to repulsive gravity in this spacetime, one obtains a time-dependent metric. Thirdly, we have discussed the Schwarzschild spacetime. Again the metric is static in a rigid frame. But using coordinates co-moving with a system of freely falling particles one obtains a time-dependent metric, still representing the Schwarzschild spacetime.

Finally, as shown by Mitra, and interpreted physically here, a similar result is obtained for just three different Friedmann-Lemaitre Robertson-Walker universe models, the Milne universe and the de Sitter and anti-de Sitter universe models. All these models are solutions of Einstein's equations for empty space, the first one without a cosmological constant and the two latter ones with a positive and negative cosmological constant, respectively.

The universe models with matter or radiation energy are solutions of the field equations with a time-dependent energy-momentum tensor. Hence Mitra's result implies that universe models with a time-dependent energy-momentum tensor cannot be represented globally by a line element with a time-independent metric. This should be formulated in a coordinate-independent way.

For a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} + p g^{\mu\nu} \tag{49}$$

we may define the energy-momentum scalar

$$T^{\mu\nu} T_{\mu\nu} = \rho^2 + 3p^2 \tag{50}$$

Hence we may conclude by formulating Mitra's result for the FLRW-universe models in the following way: The line element cannot be written in a globally time-independent way for a universe model with a time-dependent energy-momentum scalar.

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