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Scalarized Extremal Black Holes in the Einstein–Maxwell-Scalar Theory with Two U(1) Fields

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Abstract

We study scalarized extremal black holes in the Einstein–Maxwell-scalar (EMS) theory with two different scalar couplings to two U(1) fields. This theory is inspired by the dyonic EMS theory. Two scalarized extremal black holes are found with constant scalar hair. We confirm that these are precisely obtained from the extremal scalarization and entropy function approaches. This may imply that it is not easy to find extremal black holes with primary scalar hair.

Keywords: scalarized black holes; extremal black holes; scalar hair

1. Introduction

First of all, we would like to mention that the black holes with primary scalar hair were found from the (beyond) Horndeski theory, where the shift symmetry of the scalar ($\phi \to \phi+$ const.) exists [1–3]. In this case, the primary scalar charge is an independent one and is related to the shift-symmetric charge.

A nonminimal scalar coupling to the Gauss–Bonnet (GB) curvature induced the instability of Schwarzschild black holes and thus led to spontaneous scalarization triggered by a tachyonic scalar [4–6]. This is a modern way to obtain black holes with secondary scalar hair and it plays a key role in understanding the interaction between gravity and the scalar [7]. Here, one usually finds secondary scalar charges because spontaneous scalarization requires the regularity of the horizon scalar and the asymptotic condition, leading to it being dependent on the known hairs and action variables. Furthermore, a nonminimal coupling to the Maxwell invariant demonstrated spontaneous scalarization of Reissner–Nodström (RN) black holes in the Einstein–Maxwell-scalar (EMS) theory. It led to scalarized non-extremal black holes with secondary scalar hair [8] because the spontaneous scalarization needed to impose the asymptotic condition.

A comparative analysis of dilaton and scalar couplings revealed that two cases have provided charged black holes with secondary hair [9]. A difference is that the former did not accommodate any extremal black holes with scalar hair, while the latter was allowed to possess scalarized extremal black holes when extending to a dyon with electric and magnetic charges. However, we wish to point out that their extremal black hole solution with secondary scalar hair does not seem to be a promising one because the extremal condition for a metric function ($\delta(r) = 0$) is not imposed on finding the scalarized extremal black holes.

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Up to now, finding a scalarized extremal black hole with secondary scalar hair has been considered a quite difficult task analytically and numerically. It is clear that one known solution is the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) black hole found from the Einstein-conformally coupled scalar (ECS) theory [10,11]. It takes a compact form of the secondary scalar hair $\phi(r) = m/(r-m)$ with the black hole mass m, even though it blows up at the horizon [12]. It is worth noting that the BBMB black hole is regarded as a unique static and asymptotically flat solution to the ECS theory, evading no hair theorem [13]. Furthermore, this BBMB solution is recovered in a nontrivial way from the numerical series solution when imposing an asymptotically flat condition [14,15]. Before imposing this condition, the scalar charge is primary.

On the other hand, spontaneous scalarization of RN black holes was investigated when approaching extremality by considering the Einstein–Maxwell–Gauss–Bonnet-scalar theory with a quadratic scalar coupling to the GB term [16]. Two branches of scalarized black holes appeared when approaching extremality when using the Bertotti–Robinson geometry ($AdS_2 \times S^2$) as the near-horizon approximation to the extremal black hole. Also, there exist some examples of using entropy function approach [17] to find scalarized extremal black holes [9,18–20]. At this stage, we wish to point out its limitation: this approach is not suitable for finding extremal black holes with secondary scalar hair because it uses an attractor mechanism with a constant scalar existing on the degenerate horizon.

In this work, we will introduce the EMS theory with two different scalar couplings to two U(1) fields to find scalarized extremal black holes. This action is closely related to the bosonic sector of $N = 4$ supergravity which has admitted the dilatonic black hole located at $r = r_+ = 0$ with secondary dilaton hair. Unfortunately, it does not include an extremal black hole with any dilaton hair [21,22]. We note that this model is also used to explore spontaneous scalarization of non-extremal (RN) black holes by choosing the exponential coupling function $f(\phi) = e^{\alpha\phi^2}$ [23]. Actually, this model is very similar to the dyonic EMS theory, which was previously used to find scalarized extremal black holes [9,24]. We find two scalarized extremal black holes with constant scalar hair. It is worth noting that these solutions are precisely recovered from both the extremal scalarization and entropy function approaches.

2. EMS Theory with U(1) Fields

First of all, we mention briefly the bosonic sector for $N = 4$ supergravity [21,22,25]

$$S_{\rm N4} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[R - 2\partial_\mu\phi\partial^\mu\phi - e^{-2\phi}F^2 - e^{2\phi}H^2\Big], \tag{1}$$

where ϕ is the dilaton and $F = dA$ and $H = dB$ correspond to two U(1) field strengths. An analytic black hole solution was found to be

$$ds^2_{\rm N4} = -\frac{dt^2}{H_1H_2} + H_1H_2\Big(dr^2 + r^2d\Omega_2^2\Big), \quad e^{2\bar\phi} = \frac{H_2}{H_1}, \ \bar F = \frac{1}{\sqrt{2}}dH_1^{-1}\wedge dt, \ \bar H = \frac{1}{\sqrt{2}}dH_2^{-1}\wedge dt \tag{2}$$

with two harmonic functions $H_1 = 1 + \frac{\sqrt{2}Q}{r}$ and $H_2 = 1 + \frac{\sqrt{2}P}{r}$. The event horizon is unfortunately located at $r = r_+ = 0$. The dilaton $\bar\phi$ is given by the secondary hair $\bar\phi = \frac{1}{2}\ln[H_2(r)/H_1(r)]$ and its horizon dilaton takes the form $\lim_{r\to 0}\bar\phi = \frac{1}{2}\ln[\frac{P}{Q}]$. In case of $P = Q$ (becoming extremal black hole), however, one finds that $\bar\phi = 0$. This implies that the extremal black hole could not have any dilaton hair. This is not a desired case.

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Hence, we introduce a new action of the EMS theory with two U(1) fields by replacing the dilaton coupling $e^{2\phi}$ with a scalar coupling function $f(\phi)$ in Equation (1)

$$S_{\rm EMS} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[R - 2\partial_\mu\phi\partial^\mu\phi - \frac{F^2}{f(\phi)} - f(\phi)H^2\Big]. \tag{3}$$

Then, the Einstein equation is derived as

$$G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2 g_{\mu\nu} + 2T^{U(1)}_{\mu\nu}, \tag{4}$$

where

$$T^{U(1)}_{\mu\nu} = \frac{1}{f(\phi)}\Big(F_{\mu\rho}F_\nu{}^\rho - \frac{F^2}{4}g_{\mu\nu}\Big) + f(\phi)\Big(H_{\mu\rho}H_\nu{}^\rho - \frac{H^2}{4}g_{\mu\nu}\Big). \tag{5}$$

Two Maxwell equations are given by

$$\partial_\mu\Big(\sqrt{-g}\frac{F^{\mu\nu}}{f(\phi)}\Big) = 0, \quad \partial_\mu\Big(\sqrt{-g}f(\phi)H^{\mu\nu}\Big) = 0. \tag{6}$$

Finally, the scalar equation takes the form

$$\Box\phi + \frac{f'(\phi)}{4f(\phi)}\Big(-\frac{F^2}{f(\phi)} + f(\phi)H^2\Big) = 0. \tag{7}$$

3. Scalarized Extremal Black Holes

To obtain scalarized black holes, we introduce the metric and fields as [8]

$$\begin{aligned} ds^2_{\rm SCBH} &= \bar{g}_{\mu\nu}dx^\mu dx^\nu = -N(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ N(r) &= 1 - \frac{2m(r)}{r}, \quad \bar{\phi} = \phi(r), \quad \bar{A}_t = v_Q(r), \quad \bar{B}_t = v_P(r). \end{aligned} \tag{8}$$

Substituting Equation (8) into Equations (4)–(7) with $v'_Q = \frac{e^{-\delta}Qf(\phi)}{r^2}$ and $v'_P = \frac{e^{-\delta}P}{r^2 f(\phi)}$, one has three ordinary differential equations

$$m'(r) = \frac{1}{2}r^2 N\phi'^2(r) + \frac{1}{2r^2}\Big(f(\phi)Q^2 + \frac{P^2}{f(\phi)}\Big), \tag{9}$$

$$\delta'(r) = -r\phi'^2(r), \tag{10}$$

$$(e^{-\delta}r^2 N\phi'(r))' + \frac{e^{-\delta}}{2r^2}\frac{f'(\phi)}{f(\phi)}\Big(f(\phi)Q^2 - \frac{P^2}{f(\phi)}\Big) = 0, \tag{11}$$

where the prime ($'$) denotes the derivative with respect to its argument. It is desirable to note that the second term of Equation (11) gives a hint for obtaining a constant scalar solution when imposing either $f'(\phi) = 0$ or $f(\phi) = \frac{P}{Q}$.

Here, we could obtain three non-extremal and extremal black hole solutions depending the scalar hair ϕ. Firstly, one finds the dyonic black hole and its extremal black hole for $\phi = 0$ and $f(0) = 1$:

$$N(r) = 1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}, \quad \delta(r) = 0, \quad N_e(r) = \Big(1 - \frac{M}{r}\Big)^2 \quad \text{for} \quad M = \sqrt{Q^2+P^2}. \tag{12}$$

Secondly, we obtain one scalarized non-extremal black hole and its extremal black hole for $\phi = \phi_c$ and $f(\phi_c) = \frac{P}{Q}$:

$$N_a(r) = 1 - \frac{2M}{r} + \frac{2QP}{r^2}, \quad \delta(r) = 0, \quad N_{ae}(r) = \Big(1 - \frac{M}{r}\Big)^2 \quad \text{for} \quad M = \sqrt{2QP} = r_e. \tag{13}$$

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Finally, the other scalarized non-extremal black hole and its extremal black hole are found for $\phi = \tilde{\phi}_c$ and $f'(\tilde{\phi}_c) = 0$:

$$\begin{aligned} N_b(r) &= 1 - \frac{2M}{r} + \frac{Q^2 f(\tilde{\phi}_c) + \frac{P^2}{f(\tilde{\phi}_c)}}{r^2}, \quad \delta(r) = 0, \\ N_{be}(r) &= \left(1 - \frac{M}{r}\right)^2 \quad \text{for} \quad M = \sqrt{Q^2 f(\tilde{\phi}_c) + \frac{P^2}{f(\tilde{\phi}_c)}} \equiv r_e. \end{aligned} \tag{14}$$

We note that for $Q = P$, the second solution becomes the first one without scalar hair. Also, the extremal condition of $\delta(r) = 0$ plays an important role in obtaining scalarized (extremal) black holes with constant scalar hair.

4. Extremal Scalarization Approach

Before we proceed, we would like to mention that spontaneous scalarization of non-extremal black holes was performed with exponential coupling $e^{\alpha\phi^2}$ [23], where the tachyonic scalar was used to trigger scalarization. In this case, we found infinite branches of scalarized non-extremal black holes.

In this section, we use the extremal scalarization approach to find scalarized extremal black holes. For this purpose, we need to introduce the constraint equation [9]

$$\frac{N''}{2} - N\delta'' + N'\left(\frac{1}{r} - \frac{3\delta'}{2}\right) + N\delta'\left(\delta' - \frac{1}{r}\right) + N\phi'^2 = \frac{1}{r^4}\left(f(\phi)Q^2 + \frac{P^2}{f(\phi)}\right), \tag{15}$$

which was obtained from combining Equations (9)–(11) with their first derivatives. In this approach, it is convenient to express Equation (9) in terms of the metric function N by exploiting a relation of $m'(r) = \frac{1-N-rN'}{2}$.

Now, we consider the near-horizon forms which are suitable for scalarized extremal black holes

$$N(r) = N_2(r - r_+)^2 + N_3(r - r_+)^3 \cdots, \quad \delta(r) = \delta_0 + \delta_1(r - r_+) + \cdots, \tag{16}$$

$$v_Q(r) = v_{Q1}(r - r_+) + \cdots, \qquad v_P(r) = v_{P1}(r - r_+) + \cdots, \tag{17}$$

$$\phi(r) = \phi_0 + \phi_1(r - r_+) + \cdots, \tag{18}$$

where

$$v_{Q1} = \frac{e^{-\delta_0} Q f(\phi_0)}{r_+^2}, \quad v_{P1} = \frac{e^{-\delta_0} P}{r_+^2 f(\phi_0)}. \tag{19}$$

We note that δ_0 is considered a free parameter and ϕ_0 denotes the horizon scalar. Two equations, (9) and (10), determine, respectively,

$$r_+^2 = f(\phi_0)Q^2 + \frac{P^2}{f(\phi_0)}, \quad \delta_1 = -r_+\phi_1^2. \tag{20}$$

On the other hand, Equation (11) implies

$$f'(\phi_0) = 0, \quad f(\phi_0) = \frac{P}{Q}, \tag{21}$$

while its first derivative indicates the ϕ_1 equation as

$$\frac{f'(\phi_0)}{2r_+}\Big(Q^2-\frac{P^2}{f^2(\phi_0)}\Big)\phi_1^2+ \Big[2r_+^2N_2+\frac{f''(\phi_0)}{2r_+^2}\Big(Q^2-\frac{P^2}{f^2(\phi_0)}\Big)+\frac{f'^2(\phi_0)P^2}{r_+^2f^3(\phi_0)}\Big]\phi_1-\frac{f'^2(\phi_0)}{r_+^3}\Big(Q^2-\frac{P^2}{f^2(\phi_0)}\Big)=0. \tag{22}$$

Finally, Equation (15) leads to

$$N_2=\frac{1}{r_+^4}\Big(f(\phi_0)Q^2+\frac{P^2}{f(\phi_0)}\Big). \tag{23}$$

For $f'(\phi_0)\neq 0$ and $N_2\neq -\frac{f'^2(\phi_0)P^2}{2r_+^4f^3(\phi_0)}$, Equations (20)–(23) determine

$$f(\phi_0)=\frac{P}{Q},\quad r_+=\sqrt{2QP},\quad \phi_1=\delta_1=0,\quad N_2=\frac{1}{r_+^2}. \tag{24}$$

Also, for $f'(\phi_0)=0$ ($f(\phi_0)\neq P/Q$), Equations (20)–(23) imply, unless $N_2=-\frac{f''(\phi_0)}{4r_+^4}\Big(Q^2-\frac{P^2}{f^2(\phi_0)}\Big)$,

$$r_+=\sqrt{f(\phi_0)Q^2+\frac{P^2}{f(\phi_0)}},\quad \phi_1=\delta_1=0,\quad N_2=\frac{1}{r_+^2}. \tag{25}$$

At this stage, one has to match Equations (16)–(18) with the asymptotic forms in the far region

$$\begin{aligned} &m(r)=M-\frac{f(\phi_\infty)Q^2+P^2/f(\phi_\infty)+Q_s^2}{2r}+\cdots,\quad \delta(r)=\frac{Q_s^2}{2r^2}+\cdots,\\ &v_Q(r)=\Phi_Q-\frac{Q}{r}+\cdots, \qquad\qquad v_P(r)=\Phi_P-\frac{P}{r}+\cdots,\\ &\phi(r)=\phi_\infty+\frac{Q_s}{r}+\cdots, \end{aligned} \tag{26}$$

where Q_s, Φ_Q, Φ_P, and ϕ_∞ denote the scalar charge, the electrostatic potentials at infinity, and the scalar field at infinity, in addition to the ADM mass M and two electric charges (Q,P). It is reseonable to choose $\phi_\infty=\phi_0$. We stress that the key ingredient to obtain an extremal black hole with scalar hair is to impose an extremal condition of $\delta(r)=0$. This suggests strongly that $\delta_0=\delta_1=0$ and $Q_s=0$.

For any $f'(\phi_0)\neq 0$, considering Equations (24) and (26), one recovers the scalarized extremal black hole Equation (13) exactly when choosing $\phi_\infty=\phi_0=\phi_c$.

For $f'(\phi_0)=0$, taking into account Equations (25) and (26), we find the scalarized extremal black hole Equation (14) for $\phi_\infty=\phi_0=\tilde{\phi}_c$.

Finally, we wish to note that the scalarized extremal black hole solution found in [9] does not seem to be a promising one because the extremal condition for a metric function ($\delta(r)=0$) might not be imposed on finding scalarized extremal black holes. If this condition is further imposed, their solution may lead to our solutions.

5. Entropy Function Approach

The previous construction of scalarized extremal black holes has showed extremal black holes with constant scalar hair. In this section, we wish to adopt the entropy function approach to finding scalarized extremal black holes by considering Bertotti–Robinson

geometry $AdS_2 \times S^2$ as the near-horizon approximation of an extremal black hole. For this purpose, we introduce the line element with two unknown parameters v_0 and v_1 as

$$ds^2_{\rm BR} = v_0\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_1(d\theta^2 + \sin^2\theta d\varphi^2) \tag{27}$$

as well as the matter fields ansatz on the horizon as

$$\phi = \phi_0, \quad A = erdt, \quad B = prdt. \tag{28}$$

Here, we note that there is no room to accommodate $\delta(r) \neq 0$ when considering the near-horizon approximation of the extremal black hole. Hence, we may choose $\delta(r) = 0$ as an extremal condition. Five parameters $\{v_0, v_1, \phi_0, e, p\}$ satisfy a set of algebraic relations which result from Equations (4), (6) and (7). Here, instead of attempting to solve these, we wish to determine these parameters by making use of the entropy function approach [17,26,27]. This approach allows us to also compute the black hole entropy which is considered as the only physical quantity to describe scalarized extremal black holes.

For this purpose, we introduce the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{16\pi}\int d\theta d\varphi\sqrt{-g}\Big[R - 2\partial_\mu\phi\partial^\mu\phi - \frac{F^2}{f(\phi)} - f(\phi)H^2\Big] \\ &= \frac{1}{2}\Big[v_0 - v_1 + \frac{v_1}{v_0}\Big(\frac{e^2}{f(\phi)} + f(\phi)p^2\Big)\Big].\end{aligned} \tag{29}$$

Now, we are in a position to define the entropy function $\mathcal{E}$ by taking the Legendre transform of the above density with respect to two electric charges Q and P as

$$\mathcal{E} = 2\pi\Big(eQ + pP - \mathcal{L}\Big). \tag{30}$$

The equations for five parameters $\{v_0, v_1, \phi_0, e, p\}$ are given by

$$\frac{\partial\mathcal{E}}{\partial v_0} = 0 \;\rightarrow\; -1 + \frac{v_1}{v_0^2}\Big(\frac{e^2}{f(\phi_0)} + f(\phi_0)p^2\Big) = 0, \tag{31}$$

$$\frac{\partial\mathcal{E}}{\partial v_1} = 0 \;\rightarrow\; 1 - \frac{1}{v_0}\Big(\frac{e^2}{f(\phi_0)} + f(\phi_0)p^2\Big) = 0, \tag{32}$$

$$\frac{\partial\mathcal{E}}{\partial \phi_0} = 0 \;\rightarrow\; \Big(-\frac{e^2}{f^2(\phi_0)} + p^2\Big)f'(\phi_0) = 0, \tag{33}$$

$$\frac{\partial\mathcal{E}}{\partial e} = 0 \;\rightarrow\; Q = e\frac{v_1}{v_0}\frac{1}{f(\phi_0)}, \tag{34}$$

$$\frac{\partial\mathcal{E}}{\partial p} = 0 \;\rightarrow\; P = p\frac{v_1}{v_0}f(\phi_0). \tag{35}$$

Summation of Equations (31) and (32) leads to the relation

$$v_0 = v_1 = \frac{e^2}{f(\phi_0)} + f(\phi_0)p^2. \tag{36}$$

Then, two equations, (34) and (35), indicate

$$Q = \frac{e}{f(\phi_0)}, \quad P = pf(\phi_0). \tag{37}$$

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On the other hand, Equation (33) implies two different conditions

$$f'(\phi_0) = 0, \quad f(\phi_0) = \frac{e}{p}. \tag{38}$$

The latter condition implies

$$v_0 = v_1(\equiv r_e^2) = 2ep = 2QP, \tag{39}$$

which corresponds to the scalarized extremal black hole found in Equation (13) for $\phi_0 = \phi_c$. The former condition leads to

$$v_0 = v_1(\equiv r_e^2) = \frac{e^2}{f(\phi_0)} + f(\phi_0)p^2 = Q^2 f(\phi_0) + \frac{P^2}{f(\phi_0)}, \tag{40}$$

which is the same condition as in Equation (14) when choosing $\phi_0 = \tilde{\phi}_c$. Hence, we confirm that two scalarized extremal black hole solutions are recovered from the entropy function approach.

Finally, let us compute their entropy. For an exponential coupling $f(\phi) = e^{\alpha\phi^2}$, one finds that the entropy of an scalarized extremal black hole is given by the Bekenstein–Hawking entropy

$$\mathcal{E}_{f=e^{\alpha\phi^2}} = 2\pi\Big(eQ + pP - \frac{v_1}{2}\Big) = \pi v_1 = \pi(2QP) < \pi(Q^2 + P^2), \tag{41}$$

which shows that the entropy of a scalarized extremal black hole is less than that of a dyonic extrmal black hole for $Q \neq P$. In this case, the constant scalar hair takes the form

$$\phi_0(Q, P, \alpha) = \frac{1}{\sqrt{\alpha}}\sqrt{\ln\Big[\frac{P}{Q}\Big]}, \tag{42}$$

which is secondary because it is expressed in terms of all known parameters of Q, P, and α. This constant scalar hair is similar to the horizon dilaton ($\bar{\phi} = \frac{1}{2}\ln\Big[\frac{P}{Q}\Big]$) for a dilatonic non-extremal black hole. Furthermore, the constant scalar in Equation (42) is considered a fixed scalar because its value on the degenerate horizon is fixed by two U(1) charges [21,22,25].

For a polynomial form of $f(\phi) = 1 + \alpha\phi^2 - \beta\phi^4$, one obtains the constant scalar hair from the condition of $f'(\phi_0) = 0$ as

$$\phi_0(\alpha, \beta) = \sqrt{\frac{\alpha}{2\beta}}, \tag{43}$$

which is still secondary hair since it is fixed by two coupling parameters α and β. For $f(\phi_0) = 1 + \frac{\alpha^2}{4\beta}$, the entropy of the scalarized extremal black hole is given by

$$\mathcal{E}_{f=1+\alpha\phi^2-\beta\phi^4} = 2\pi\Big(eQ + pP - \frac{v_1}{2}\Big) = \pi v_1 = \pi\Big[f(\phi_0)Q^2 + \frac{P^2}{f(\phi_0)}\Big]. \tag{44}$$

We find two inequalities such that $\mathcal{E}_{f=1+\alpha\phi^2-\beta\phi^4} > \mathcal{E}_{dbh}$ for $Q > P$, while $\mathcal{E}_{f=1+\alpha\phi^2-\beta\phi^4} < \mathcal{E}_{dbh}$ for $Q < P$, with the entropy of dyonic extremal black hole $\mathcal{E}_{dbh} = \pi(Q^2 + P^2)$. Finally, considering a quadratic coupling function $f(\phi) = 1 + \alpha\phi^2$ used for spontaneous scalarization, one could not obtain non-zero constant scalar hair ϕ_0.

6. Discussion

It is known that one famous solution for scalarized extremal black holes is the BBMB black hole found from the ECS theory [10,11]. Its secondary scalar hair takes the form of

$\phi(r) = m/(r-m)$ with m black hole mass, even though it blows up at the horizon. It is believed that it is not easy to find a scalarized extremal black hole with primary scalar hair (independent scalar charge Q_s) because this requires an asymptotically flat condition changing a primary scalar into a secondary one [14,15]. Hence, this allows us to find scalarized extremal black holes with either secondary scalar hair or constant scalar hair.

In the present work, we obtained two scalarized extremal black holes with constant scalar hair in the EMS theory with two different scalar coupling functions to two U(1) fields. It is worth noting that these solutions are precisely recovered from the extremal scalarization and entropy function approaches. We note that similar scalarized extremal black hole solutions can be found from the dyonic EMS theory with a single scalar coupling [9,24]. Actually, these will be obtained from our result by exchanging "$Q \leftrightarrow P$".

Finally, we would like to mention that it is not easy to find scalarized (extremal) black holes with primary scalar hair unless its action possesses a shift symmetry, like internal symmetry in the (beyond) Horndeski theory [1,3].

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References

1. Babichev, E.; Charmousis, C. Dressing a black hole with a time-dependent Galileon. *J. High Energy Phys.* **2014**, *2014*, 106. [CrossRef]
2. Herdeiro, C.A.R.; Radu, E. Asymptotically flat black holes with scalar hair: A review. *Int. J. Mod. Phys. D* **2015**, *24*, 1542014. [CrossRef]
3. Bakopoulos, A.; Charmousis, C.; Kanti, P.; Lecoeur, N.; Nakas, T. Black holes with primary scalar hair. *Phys. Rev. D* **2024**, *109*, 024032. [CrossRef]
4. Doneva, D.D.; Yazadjiev, S.S. New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Scalar-Tensor Theories. *Phys. Rev. Lett.* **2018**, *120*, 131103. [CrossRef]
5. Silva, H.O.; Sakstein, J.; Gualtieri, L.; Sotiriou, T.P.; Berti, E. Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling. *Phys. Rev. Lett.* **2018**, *120*, 131104. [CrossRef]
6. Antoniou, G.; Bakopoulos, A.; Kanti, P. Evasion of No-Hair Theorems and Novel Black-Hole Solutions in Gauss-Bonnet Theories. *Phys. Rev. Lett.* **2018**, *120*, 131102. [CrossRef]
7. Doneva, D.D.; Ramazanoğlu, F.M.; Silva, H.O.; Sotiriou, T.P.; Yazadjiev, S.S. Spontaneous scalarization. *Rev. Mod. Phys.* **2024**, *96*, 015004. [CrossRef]
8. Herdeiro, C.A.R.; Radu, E.; Sanchis-Gual, N.; Font, J.A. Spontaneous Scalarization of Charged Black Holes. *Phys. Rev. Lett.* **2018**, *121*, 101102. [CrossRef]
9. Astefanesei, D.; Herdeiro, C.; Pombo, A.; Radu, E. Einstein-Maxwell-scalar black holes: Classes of solutions, dyons and extremality. *J. High Energy Phys.* **2019**, *2019*, 78. [CrossRef]
10. Bocharova, N.M.; Bronnikov, K.A.; Melnikov, V.N. On an exact solution of the Einstein system of equations with a massless scalar field. *Vestn. Mosk. Univ. Ser. III Fiz. Astron.* **1970**, *6*, 706. [CrossRef]
11. Bekenstein, J.D. Exact solutions of Einstein conformal scalar equations. *Ann. Phys.* **1974**, *82*, 535–547. [CrossRef]
12. Charmousis, C.; Iosifidis, D. Self tuning scalar tensor black holes. *J. Phys. Conf. Ser.* **2015**, *600*, 012003. [CrossRef]
13. Xanthopoulos, B.C.; Dialynas, T.E. Einstein gravity coupled to a massless conformal scalar field in arbitrary space-time dimensions. *J. Math. Phys.* **1992**, *33*, 1463–1471. [CrossRef]
14. Myung, Y.S.; Zou, D.C. Black holes in new massive conformal gravity. *Phys. Rev. D* **2019**, *100*, 064057. [CrossRef]

15. Zou, D.C.; Myung, Y.S. Scalar hairy black holes in Einstein-Maxwell-conformally coupled scalar theory. *Phys. Lett. B* **2020**, *803*, 135332. [CrossRef]
16. Brihaye, Y.; Hartmann, B. Spontaneous scalarization of charged black holes at the approach to extremality. *Phys. Lett. B* **2019**, *792*, 244–250. [CrossRef]
17. Sen, A. Black hole entropy function and the attractor mechanism in higher derivative gravity. *J. High Energy Phys.* **2005**, *2005*, 38. [CrossRef]
18. Marrani, A.; Miskovic, O.; Leon, P.Q. Extremal Black Holes, Stueckelberg Scalars and Phase Transitions. *J. High Energy Phys.* **2018**, *2018*, 80. [CrossRef]
19. Marrani, A.; Miskovic, O.; Leon, P.Q. Spontaneous scalarization in (A)dS gravity at zero temperature. *J. High Energy Phys.* **2022**, *2022*, 100. [CrossRef]
20. Myung, Y.S. Charged qOS-extremal black hole and its scalarization by entropy function approach. *Eur. Phys. J. C* **2026**, *86*, 292. [CrossRef]
21. Kol, B.; Rajaraman, A. Fixed scalars and suppression of Hawking evaporation. *Phys. Rev. D* **1997**, *56*, 983–986. [CrossRef]
22. Krasnitz, M.; Klebanov, I.R. Testing effective string models of black holes with fixed scalars. *Phys. Rev. D* **1997**, *56*, 2173–2179. [CrossRef]
23. Myung, Y.S.; Zou, D.C. Scalarized charged black holes in the Einstein-Maxwell-Scalar theory with two U(1) fields. *Phys. Lett. B* **2020**, *811*, 135905. [CrossRef]
24. Chen, S.; Chew, X.Y.; Kunz, J. Dyonic Einstein–Maxwell-scalar black holes: The cold, the hot and the plunge. *Eur. Phys. J. C* **2026**, *86*, 820. [CrossRef]
25. Lee, H.W.; Myung, Y.S.; Kim, J.Y. Absorption of fixed scalar in scattering off 4-D N = 4 black holes. *Phys. Lett. B* **1997**, *410*, 6–12. [CrossRef]
26. Astefanesei, D.; Yavartanoo, H. Stationary black holes and attractor mechanism. *Nucl. Phys. B* **2008**, *794*, 13–27. [CrossRef]
27. Sen, A. Black Hole Entropy Function, Attractors and Precision Counting of Microstates. *Gen. Relativ. Gravit.* **2008**, *40*, 2249–2431. [CrossRef]

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