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Static Solitons in an Expanding Universe

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Abstract

We show, analytically, that a static sine-Gordon soliton cannot exist in 1 + 1 non-dynamical de Sitter spacetime if $\alpha := (m/H)^2 < 2$, where m is the mass parameter of the sine-Gordon theory and H is the Hubble constant. Conversely, we also show that static sine-Gordon solitons exist in 1 + 1 non-dynamical de Sitter spacetime if $\alpha > 2$. The above threshold is explained—qualitatively and to within an O(1) factor—using a heuristic argument involving the interplay of tensile force in the Lorentzian sine-Gordon soliton and the tidal force in de Sitter spacetime. A similar heuristic argument, which remains to be confirmed analytically, also suggests the existence of a threshold, $(m_V/H)^2 \sim O(1)$, below which the tidal forces are too strong to permit the existence of a static 't Hooft–Polyakov monopole in non-dynamical 3 + 1 de Sitter spacetime; m_V is the mass of the vector boson. Linde has suggested that new inflation could have triggered secondary inflation at the core of a GUT (Grand Unified Theory) monopole even if the Hubble constant at or after the GUT phase transition was significantly smaller than the mass of the X boson. We present a heuristic argument, which suggests that the $SO(3)$ 't Hooft–Polyakov monopole does not allow secondary inflation at its core when the inflationary background is weak. Based on the above, as yet analytically unconfirmed, heuristic argument for the $SO(3)$ 't Hooft–Polyakov monopole, we conjecture that secondary inflation at the core of a GUT monopole, as suggested by Linde, is infeasible.

Keywords: solitons; sine-Gordon; monopoles; de Sitter spacetime; inflation

1. Introduction

Following 't Hooft [1] and Polyakov's [2] construction of a monopole solution in a gauge theory with spontaneously broken symmetry in Lorentzian spacetime, monopole solutions in non-Lorentzian backgrounds have been studied by several researchers. For example, Niewenhuizen, Wilkinson and Perry [3] describe a 't Hooft–Polyakov-like monopole solution that couples to the background metric generated by the monopole itself. Using the dependence of a monopole's mass on the Higgs vacuum expectation value, it follows that, for a sufficiently high Higgs vacuum expectation value, a monopole would collapse into a black hole [4,5]. For a sufficiently high Higgs vacuum expectation value, it was shown that non-singular monopole solutions do not exist [4,6]. The stability of various types of magnetically charged solutions coupled to gravity has been studied in [7–9]. Monopoles in non-Lorentzian backgrounds are of interest since monopoles, if they exist, are believed to have been created during the GUT (Grand Unified Theory) phase transition in the early universe when the background was non-Lorentzian [10–12].

While [3–5] considered self-gravitating monopoles—that is, monopoles coupled to the background metric generated by the monopole itself—Linde [13] considered the interaction of a GUT monopole's core with an inflationary background. Linde argued that, although the

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Academic Editors: Andrea Lapi and Roman Pasechnik

Received: 27 December 2025
Revised: 27 March 2026
Accepted: 8 April 2026
Published: 20 April 2026

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Hubble constant of $\sim 10^{10}$ GeV in the inflationary phase, circa the GUT phase transition, is several orders of magnitude smaller than the mass of the X boson, which is believed to have a mass of $\sim 10^{15}$ GeV, the inflationary background could trigger secondary inflation at the core of a GUT monopole. A similar suggestion was also made by Vilenkin [14], who called it topological inflation. We revisit Linde's argument in Section 4.

In this paper we analytically solve the problem of the existence of static soliton solutions in 1 + 1 sine-Gordon theory that is coupled minimally to a non-dynamical de Sitter background. The associated differential equation is singular, making the construction of well-behaved soliton solutions a rather intricate problem. We prove that in the de Sitter background static sine-Gordon solitons exist if $\alpha := (m/H)^2 > 2$ and do not exist if $\alpha < 2$, where m is the mass parameter of the sine-Gordon theory and H is the Hubble constant of the de Sitter background.

A similar problem for the Mexican hat potential was studied numerically by Basu and Vilenkin [15]. They report that they could not numerically find any nontrivial solution for $\alpha^{BV} \leqslant 2$, where α^{BV} is the square of the ratio of the size of the de Sitter horizon to the flat-space thickness of the domain wall. For $\alpha^{BV} > 2$, they display solutions, which are obtained numerically. The threshold $\alpha^{BV} = 2$ that they report, based on numerical studies, is consistent with the analytic threshold $\alpha = 2$ that we derive.

Chen, Cheng, Li and Zhai [16] have attempted to analytically justify Basu and Vilenkin's numerical work. However, their argument has an error. They attempt a series expansion of the solution around their equation's singular point, implicitly assuming that the solution is analytic at the singular point. As Equation (A51) in our discussion shows, the third and higher-order derivatives of a solution of a second-order singular differential equation diverge at the equation's singular point, making a series expansion around a singular point meaningless. Instead, an effective approach to studying the existence of bounded solutions in the vicinity of a singular point is to use integral equations, as we show in Lemmas 2 and 3 (Section 2.3.3). Also, their argument hinges on the incorrect claim that their boundary condition (8) requires $a_1 > 0$.

The thresholds, such as $\alpha = 2$ that we derive, have been explained in the past by comparing the 'size' of the soliton with the length scale of the de Sitter horizon. A soliton is a non-dissipative solution that is held together as a compact configuration by the internal tensile forces. It is instructive to try to understand the above threshold in terms of the interplay between de Sitter background's tidal forces, which tear apart extended objects such as solitons, and the soliton's internal tensile forces, which resist the tidal forces to keep the soliton together as a compact object.

Following the analytic derivation of the threshold $\alpha = 2$, we present a heuristic qualitative explanation of the threshold in terms of the interplay of tensile and tidal forces. The heuristic argument suggests the existence of a threshold, which agrees with the exact threshold to within an O(1) factor.

We present a similar heuristic argument involving the interplay of the tensile force in a Lorentzian 't Hooft–Polyakov monopole and the tidal force in a 3 + 1 non-dynamical de Sitter spacetime. Our heuristic argument, which is yet to be confirmed analytically, suggests that a static 't Hooft–Polyakov monopole cannot exist in a de Sitter background when the tidal force is too strong. Based on our heuristic analysis, we propose a conjecture pertaining to the existence of static 't Hooft–Polyakov monopoles in a non-dynamical de Sitter background.

Linde [13,17] has suggested that even a 'weak' inflationary background could have inflated away the field gradients at the core of a GUT monopole and could have induced secondary inflation at the monopole's core. We present a heuristic argument, which suggests that when the inflationary background is weak the tidal forces cannot overcome

the tensile forces at the core of a Lorentzian $SO(3)$ 't Hooft–Polyakov monopole and, hence, the background cannot induce secondary inflation at the monopole's core. Based on the above heuristic argument, which remains to be confirmed analytically, we propose a conjecture that *the relative weakness of the inflationary background at and after the GUT phase transition makes secondary inflation at the core of a GUT monopole infeasible*. Our conjecture can be settled through an exact analysis of the GUT monopole in the inflationary background that prevailed during the GUT phase transition.

The paper is organized as follows. In Section 2 we formulate and analytically solve the problem of the existence of static solitons in a non-dynamical 1 + 1 de Sitter background. The main results are stated in Section 2, and the proofs are deferred to Appendices B–E. At the end of Section 2, we also discuss the back reaction of the soliton on the de Sitter background. In Section 3 we present heuristic arguments involving the interplay of tensile and tidal forces. In Section 3.1 we present a heuristic estimate of the tensile force in the sine-Gordon soliton in Lorentzian background. Comparing the tensile force to the tidal force of the 1 + 1 de Sitter background we heuristically derive a threshold that agrees with the exact threshold, obtained in Section 2, to within an O(1) factor. In Section 3.2 we compare a heuristic estimate of the tensile force in a static 't Hooft–Polyakov monopole in Lorentzian background with the tidal force of a 3 + 1 de Sitter background to derive a heuristic threshold for the non-existence of a 't Hooft–Polyakov monopole; based on the heuristic threshold, we propose a conjecture regarding the existence of a static $SO(3)$ 't Hooft–Polyakov monopole in a de Sitter background. Addressing Linde's suggestion, in Section 4, we present a heuristic analysis of the extent to which a weak inflationary background can stretch a pre-existing $SO(3)$ 't Hooft–Polyakov monopole. Based on our heuristic analysis, we propose a conjecture that the inflationary background that prevailed at the GUT phase transition could not have induced secondary inflation at the core of the GUT monopole. Finally, in Section 5, we present some remarks about the stability of Sine-Gordon solitons, the behavior of solitons at the end of inflation, the impact of quantum corrections and the precision of the numerical results presented in the paper.

2. Static Sine-Gordon Solitons in de Sitter Spacetime

The exact result, presented in this section and in Appendices C–E, pertains to a singular differential equation and, hence, is rather intricate. In Section 2.2, we present an overview of the result to ensure that the mathematical arguments that follow do not obscure the essence of the result. We begin by formulating the key differential equation.

2.1. Lagrangian and Its Symmetries

Consider the 1 + 1 sine-Gordon Lagrangian minimally coupled to background de Sitter spacetime. The action is

$$S[\varphi] = \int \sqrt{-g}\left\{\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{\beta^2}(1-\cos(\beta\phi))\right\}d\chi\, dt; \quad g^{\mu\nu} = diag\left(1, -e^{-2Ht}\right) \tag{1}$$

where m is the mass parameter of the sine-Gordon theory, β the sine-Gordon coupling constant, g the determinant of the metric and H the Hubble constant. We ignore the back reaction of the field on the background metric. The corresponding Euler–Lagrange equation is

$$\phi_{tt} + H\phi_t - e^{-2Ht}\phi_{\chi\chi} = -\frac{m^2}{\beta}\sin(\beta\phi). \tag{2}$$

The physical spatial coordinate u is related to the comoving spatial coordinate χ as $u = e^{Ht}\chi$. We look for solutions of (2) of the form $\phi(t,\chi) = \psi(u)$ that are 'static' in the sense that they depend only on the physical spatial coordinate.

A 'static' solution is not stretched by the expansion of the de Sitter universe; it bucks the tidal forces of inflationary expansion, its physical 'dimensions'—as measured by a physical ruler—remaining unaltered as the universe expands. As an example, consider a 'static' solution $\psi(u)$ that has a zero at $u_0 > 0$. As the universe expands, the location of the zero—as measured by a physical ruler—remains at a distance u_0 from $u = 0$, although the comoving coordinate of the above zero, denoted $\chi_0(t)$, decreases exponentially as $\chi_0(t) = e^{-Ht}u_0$. A distant observer at a fixed comoving coordinate, who is to the right of u_0 at $t = 0$, would see the comoving coordinate of the zero decreasing exponentially as the universe expands.

Setting $z := Hu$, $\varphi(z) := \beta\,\psi(z/H)$, and $\alpha := \frac{m^2}{H^2}$, we see that a static solution of the Euler–Lagrange equation satisfies the following singular differential equation:

$$(z^2 - 1)\varphi_{zz}(z) + 2\,z\,\varphi_z(z) + \alpha \sin(\varphi(z)) = 0; \tag{3}$$

Equation (3) is singular because the coefficient of the highest derivative vanishes at certain values of the independent variable z. In Equation (3), the coefficient of φ_{zz} vanishes at $z = \pm 1$, which corresponds to the horizon at $u = \pm\frac{1}{H}$. We will call z the *static coordinate*.

The vacua of the theory are at $\varphi = 2n\pi$ for integer values of n. A topologically nontrivial static soliton—hereafter abbreviated to just soliton—is a solution of (3) that approaches different vacua as $z \to \pm\infty$. The topological charge $C[\varphi]$ of a soliton φ is

$$C[\varphi] = \frac{1}{2\pi}\left\{\lim_{z\to\infty}\varphi(z) - \lim_{z\to-\infty}\varphi(z)\right\} \tag{4}$$

We note that $\varphi \to -\varphi$ and $\varphi \to \varphi + 2m\pi$ are symmetries of the Lagrangian for every integer m. Further, if $\varphi(z)$ is a solution of Equation (3), then so is $\varphi(-z)$, implying that, if a soliton with charge Q exists, then so does a soliton with charge $-Q$.

In order to prove the existence of a soliton solution of Equation (3) with charge $Q = \pm 1$ it is sufficient to prove the existence of a bounded solution for $z \geqslant 0$ with the following boundary conditions:

$$\varphi(0) = \pi, \qquad \lim_{z\to\infty}\varphi(z) = 0, \tag{5}$$

As noted earlier, Equation (3) is singular at the horizon $z = \pm 1$. We present results about the behavior of bounded solutions near the horizon ($z \approx 1$), outside the horizon ($z > 1$) and inside the horizon ($0 \leqslant z < 1$) in Sections 2.3–2.5, respectively. We start by presenting an overview of the discussion that follows.

2.2. Overview

In this section we present an overview of our analytical results pertaining to the sine-Gordon solitons in a de Sitter background.

2.2.1. Solutions That Are Bounded at Singularities

Equation (3) has singularities at $z = \pm 1$, and its solutions can become unbounded in the vicinity of the singularities. In Corollary 1 (Section 2.3.2), we show that a necessary condition for a solution φ_b to remain bounded at $z = 1$ is that

$$\varphi_b'(1) = -\frac{\alpha}{2}\,\sin(\varphi_b(1)).$$

Thus, for bounded solutions of Equation (3), the initial conditions at $z = 1$, namely $\varphi_b(1)$ and $\varphi_b'(1)$, cannot be specified independently. Solutions of Equation (3), bounded at $z = 1$, form a 1-parameter family, parametrized by $\varphi_b(1)$.

We need to show that solutions that are bounded in the vicinity of $z = 1$ actually exist. Rewriting Equation (3) in terms of the *hypergeometric coordinate*, $x = (1 - z)/2$, we show, in Lemma 2 (Section 2.3.3), that the solution of a certain integral equation gives a solution of Equation (3) in a neighborhood of $z = 1$ ($x = 0$). We then show, in Lemma 3 (Section 2.3.3), that the aforementioned integral equation actually has a bounded solution in the vicinity of $z = 1$ for every $0 < \varphi(1) < \pi$, thereby proving that the 1-parameter family of bounded solutions of Equation (3), mentioned above, does exist.

2.2.2. Form of the Solitons

In Lemma A6 (Appendix D), we show that, if $\varphi(1) = n\pi$, then $\varphi(z) \equiv n\pi$ for all z. Therefore, based on the discussion preceding Equation (5), without loss of generality, we can assume that $0 < \varphi(1) < \pi$ for topologically nontrivial solutions.

Figure 1 shows illustrative examples of the behavior of bounded solutions for $z > 1$. If $0 < \varphi(1) < \pi$, then, regardless of the value of α, the solutions vanish as $z \to \infty$. The decay is quite slow and, hence, the plots are shown with a logarithmic scale.

In Lemma 4 (Section 2.4), we show, rigorously, that the bounded solutions with $0 < \varphi(1) < \pi$ vanish as $z \to \infty$. Therefore, in order to prove the existence of a topologically nontrivial solution, it suffices to find a solution that starts at $0 < \varphi(1) < \pi$ and reaches $\varphi(0) = \pi$ when we integrate Equation (3) from $z = 1$ to $z = 0$.

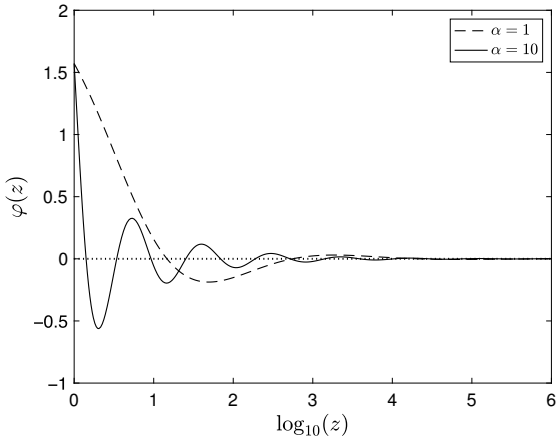


Figure 1. Sample solutions of Equation (3) in the range $1^+ < z < 10^6$, where $1^+ = 1 + 10^{-14}$. The boundary conditions are $\varphi(1^+) = \pi/2$, $\varphi'(1^+) = -(\alpha/2)\sin(\varphi(1^+))$. The dotted line represents $\varphi(z) \equiv 0$.

2.2.3. $\alpha < 2$

The behavior of solutions for $\alpha < 2$ is illustrated in Figure 2. In the graphs, we have started at $z = 1^- := 1 - 10^{-15}$, and integrated the differential equation back to $z = 0$. The noteworthy feature in Figure 2 is that, regardless of what initial value $0 < \varphi(1^-) < \pi$ is chosen at $z = 1^-$, $\varphi(z)$ remains strictly between 0 and π in the range $0 \leqslant z \leqslant 1^-$. For $\alpha < 2$, a bounded solution with $0 < \varphi(1) < \pi$ cannot reach $\varphi(0) = \pi$. Therefore, a solution satisfying the conditions in (5) does not exist for $\alpha < 2$.

Figure 3 illustrates the behavior of a solution that is bounded in the vicinity of $z = 1$ but does not reach $\varphi(0) = \pi$. If we integrate Equation (3) from $z = 1^- = 1 - 10^{-15}$ past $z = 0$ to $z = -1^+ = -1 + 10^{-14}$, the solution becomes unbounded near $z = -1$ since it does not satisfy the condition analogous to (5), namely $\varphi'(-1) = \alpha/2\sin(\varphi(-1))$, at $z = -1$.

The above discussion represents the content of Theorem 1 (Section 2.5.1), in which we show that, for $\alpha < 2$, every bounded solution of Equation (3) is topologically trivial. There are no solitons for $\alpha < 2$.

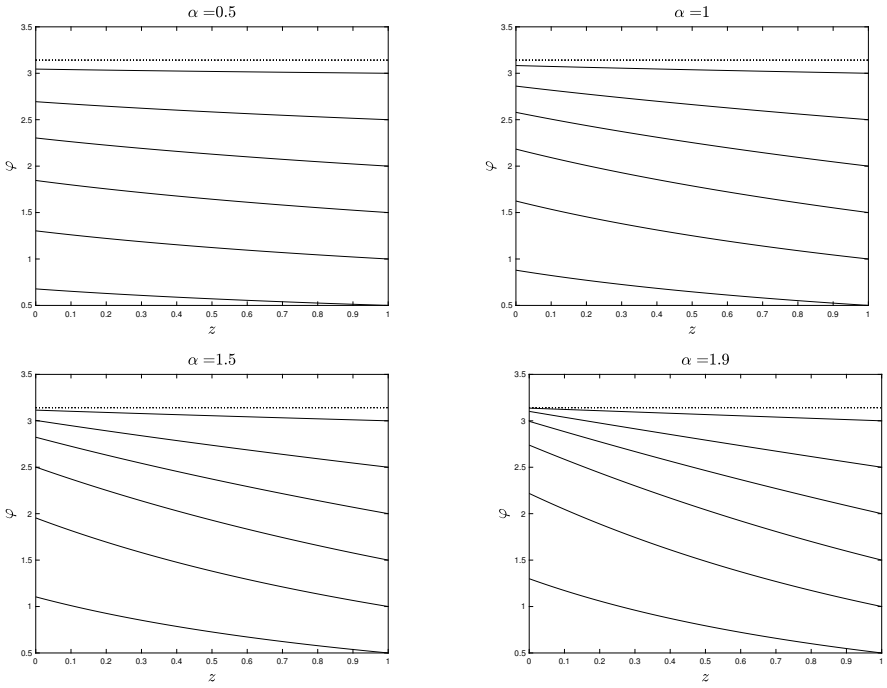


Figure 2. Solutions (solid lines) for different values of $\alpha < 2$ in the range $0 \leqslant z \leqslant 1^- := 1 - 10^{-15}$. The shown solutions correspond to $\varphi(1^-) = 0.5, 1, 1.5, 2, 2.5, 3$. $\varphi(0) < \pi$ for all of the shown solutions. The dotted line represents $\varphi(z) \equiv \pi$.

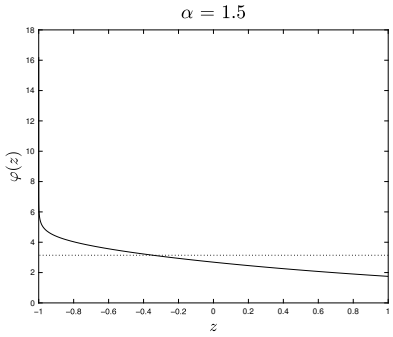


Figure 3. Solution (solid line) for $\alpha = 1.5$ and $\varphi(1 - 10^{-15}) = 1.75$, in the range $-1 + 10^{-14} \leqslant z \leqslant 1 + 10^{-15}$. The dotted line represents $\varphi(z) \equiv \pi$.

2.2.4. $\alpha > 2$

Figure 4 illustrates the behavior of solutions at a sample value of $\alpha > 2$, namely $\alpha = 5$. For $\varphi(1^-) = 1.14218$, $1^- = 1 - 10^{-15}$, we have $\varphi(0) > \pi$; for $\varphi(1^-) = 0.14218$, $\varphi(0) < \pi$.

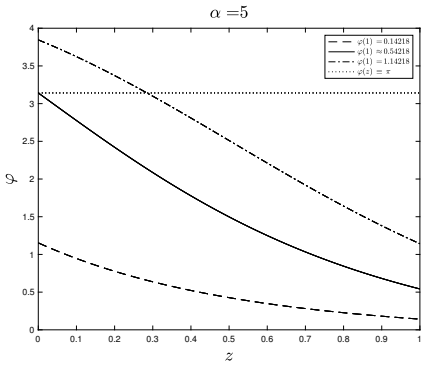


Figure 4. The three plots, in the range $0 \leqslant z \leqslant 1^- := 1 - 10^{-15}$, show that $\varphi(0) > \pi$ for $\varphi(1^-) = 1.14218$; $\varphi(0) < \pi$ for $\varphi(1^-) = 0.14218$; $\varphi(0) = \pi$ for $\varphi(1^-) \approx 0.54218$.

In Lemma A9 (Appendix E), we show that $\varphi(0)$ varies continuously with $\varphi(1)$ for a bounded solution. As we vary $\varphi(1^-)$ from 0.14218 to 1.14218, at an intermediate value of $\varphi(1^-)$—specifically at $\varphi(1^-) \approx 0.54218$—we reach $\varphi(0) = \pi$. Taken together with the asymptotic behavior depicted in Figure 1, Figure 4 shows that topologically nontrivial soliton exists for $\alpha = 5$.

Soliton solutions for some values of $\alpha > 2$ are displayed in Figure 5. The range is limited to $0 \leqslant z \leqslant 10$ to be able to clearly display the behavior inside the horizon ($0 \leqslant z \leqslant 1$). Since the initial condition has to be imposed at $z = 1$, to keep the solution bounded, the integration was done separately in two intervals, $0 \leqslant z \leqslant 1 - 10^{-15}$ and $1 + 10^{-15} \leqslant z \leqslant 10$, imposing the initial conditions at 1 ± 10^{-15}. The choice of 10^{-15} is discussed in Section 5.4.

For comparison, in Figure 6, we have also plotted the Lorentzian sine-Gordon antikink $\varphi_{SG}(x)$ that interpolates between the vacuum $\varphi = 2\pi$ at $x \to -\infty$ and the vacuum $\varphi = 0$ at $x \to \infty$.

In the following subsections of this section, we state the main results, deferring the proofs to Appendices B–E. Equation (3) has singularities at $z = \pm 1$, which form the horizon of the region $-1 < z < 1$. In Section 2.3 we present results that prove the existence of bounded solutions in a neighborhood of the singularity, $z = 1$. In Section 2.4, we present results, which show that a solution, $\varphi(z)$, approaches the vacuum that is closest to $\varphi(1)$ as $z \to \infty$. Finally, in Section 2.5, we focus on the region inside the horizon, $-1 < z < 1$, and present results, which show that solitons exist for $\alpha > 2$ and do not exist for $\alpha < 2$.

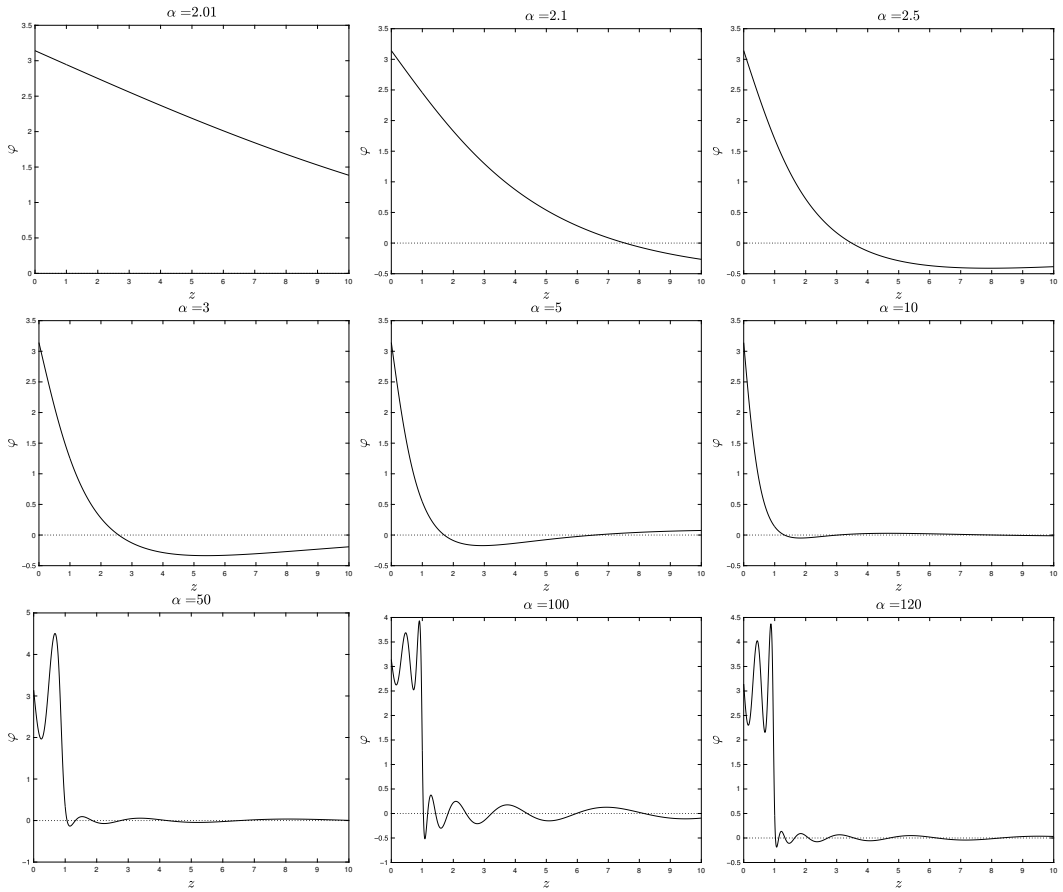


Figure 5. Soliton solutions (solid lines) for different values of $\alpha > 2$ in the interval $0 \leqslant z \leqslant 10$. All of the solutions satisfy $\varphi(0) = \pi$ and $\lim_{z\to\infty} \varphi(z) = 0$ and are bounded at the singularity $z = 1$. The plots were obtained by integrating Equation (3) separately in the intervals $[0, 1 - \epsilon]$ and $[1 + \epsilon, 10]$, where $\epsilon = 10^{-15}$. The value $\varphi(1 - \epsilon)$ was chosen to ensure $\varphi(0) = \pi$. The dotted lines represent $\varphi(z) \equiv 0$.

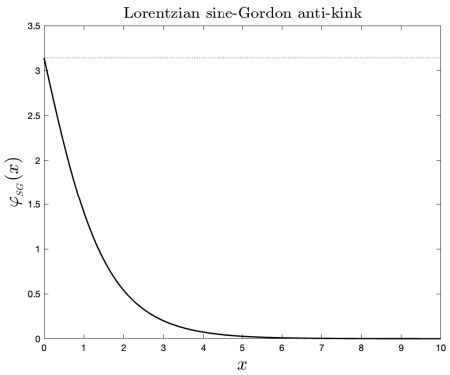


Figure 6. The Lorentzian sine-Gordon antikink $\varphi_{SG}(x) = 4\tan^{-1}(e^x)$ corresponding to $m = 1$ (solid line). The dotted line represents $\varphi(z) \equiv \pi$.

2.3. Bounded Solutions in a Neighborhood of a Singularity

Equation (3) is closely related to the hypergeometric differential equation. We use the known solutions of the hypergeometric differential equations to establish the existence of bounded solutions of Equation (3). In Section 2.3.1 below, we describe the relation between Equation (3) and the hypergeometric equation. In Section 2.3.2, we establish a necessary condition that must be satisfied by a solution at the singularity $z = 1$ if it is to be bounded in a neighborhood of the singularity. Finally, in Section 2.3.3, we prove the existence of a 1-parameter family of bounded solutions of Equation (3) in a neighborhood of the singularity $z = 1$.

2.3.1. Associated Hypergeometric Equation

The hypergeometric differential equation

$$x(x-1)\varphi_{xx} + ((a+b+1)x - c)\varphi_x + ab\varphi = 0 \tag{6}$$

was first formulated by Euler and was studied extensively by Gauss and Riemann. The relation between Equations (3) and (6) can be seen by transforming from the *static coordinate* z to the *hypergeometric coordinate* x, as $x = (1-z)/2$. In hypergeometric coordinate Equation (3) becomes

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha\varphi = f(\varphi) \qquad f(\varphi) := \alpha(\varphi - \sin(\varphi)) \tag{7}$$

With a slight abuse of notation we use φ to denote both the solution $\varphi(z)$ of Equation (3) as well as the solution $\Psi(x) = \varphi(1-2x)$ of Equation (7).

For $a + b = c = 1$, $ab = \alpha$ in Equation (6), the left-hand side of Equation (6) coincides with the left-hand side of Equation (7). Thus Equation (7) is a nonlinear relative of the hypergeometric differential equation Equation (6). The singularity $z = 1$ in Equation (3) corresponds to the singularity $x = 0$ in Equation (7).

It is well known that the hypergeometric equation Equation (6) has two linearly independent solutions in a (punctured) neighborhood of $x = 0$. For the special values of a, b, c of interest to us, namely $a + b = c = 1$, $ab = \alpha$, the two linearly independent solutions of Equation (6) are denoted $\varphi_1(x)$ and $\varphi_2(x)$. The first solution, $\varphi_1(x)$, is analytic in the neighborhood $(-1, 1)$ and is described in Equation (A4) in Appendix A. The second solution is

$$\varphi_2(x) = \varphi_1(x)\log(|x|) + h(x), \qquad x \in (-\epsilon, \epsilon) \setminus \{0\} \tag{8}$$

where $h(x)$ is an analytic function in $(-\rho_h, \rho_h)$ for some $0 < \rho_h < 1$. The form of $\varphi_1(x)$ and the construction of $\varphi_2(x)$ are described in Appendix A.

2.3.2. A Necessary Condition for Bounded Solutions

First, we show that, if a solution of Equation (7) is to be bounded at the singularity $x = 0$, then it must satisfy the boundary condition described in Lemma 1, stated below.

Lemma 1. *Let $\varphi_b(x)$ be a bounded solution of*

$$x(x-1)\varphi_b'' + (2x-1)\varphi_b' + \alpha \sin(\varphi_b) = 0 \tag{9}$$

in a neighborhood $(-\epsilon, \epsilon)$, $\epsilon > 0$. Then,

$$\lim_{x \to 0} \varphi_b'(x) = \alpha \sin(\varphi_b(0)) \tag{10}$$

The proof of Lemma 1 is presented in Appendix B. Translating Lemma 1 from hypergeometric coordinate to static coordinate, we obtain the following corollary.

Corollary 1. *Let $\varphi_b(z)$ be a bounded continuous solution of*

$$(z^2-1)\varphi_b'' + 2z\varphi_b' + \alpha \sin(\varphi_b) = 0 \tag{11}$$

in a neighborhood $(1-\epsilon, 1+\epsilon)$, $\epsilon > 0$. Then,

$$\lim_{z \to 1} \varphi_b'(z) = -\frac{\alpha}{2} \sin(\varphi_b(1)) \tag{12}$$

2.3.3. Proof of Existence of Bounded Solutions in a Neighborhood of Singularities

We show that one can obtain a solution of the differential equation Equation (7) by solving a related integral equation, as shown in Lemma 2 below.

Lemma 2. *Let $\varphi_b(x)$ be a bounded continuous solution of the integral equation*

$$\varphi(x) = \varphi_2(x) \int_0^x \left[\frac{\varphi_1(y) f(\varphi(y))}{(y-1)\, y\, W(y)} \right] dy - \varphi_1(x) \int_0^x \left[\frac{\varphi_2(y) f(\varphi(y))}{(y-1)\, y\, W(y)} \right] dy + \gamma \varphi_1(x) \tag{13}$$

in some sufficiently small interval $(-\epsilon, \epsilon)$, where $0 < \epsilon < \rho_h$. Then, in $(-\epsilon, \epsilon)$, $\varphi_b(x)$ satisfies the differential equation

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + a \sin(\varphi) = 0 \tag{14}$$

and the boundary condition described in Lemma 1.

$\varphi_1(x)$ and $\varphi_2(x)$ are described in Equations (A4) and (8). ρ_h is defined after Equation (8). $W(x)$ is the Wronskian of φ_1 and φ_2. $f(\varphi) = \alpha(\varphi - \sin(\varphi))$ and γ is an arbitrary constant.

Thus, to obtain a solution of Equation (7) around $x = 0$, it is sufficient to construct a continuous bounded solution of the the integral equation Equation (13) in a neighborhood of $x = 0$. The following lemma establishes the existence of a 1-parameter family of such solutions for integral equation Equation (13).

Lemma 3. *There exists an interval $(-\epsilon, \epsilon)$, $\epsilon > 0$ in which the integral equation*

$$\varphi(x) = \varphi_2(x) \int_0^x \left[\frac{\varphi_1(y) f(\varphi(y))}{(y-1)\, y\, W(y)} \right] dy - \varphi_1(x) \int_0^x \left[\frac{\varphi_2(y) f(\varphi(y))}{(y-1)\, y\, W(y)} \right] dy + \gamma \varphi_1(x) \tag{15}$$

has a 1-parameter family of continuous bounded solutions, $\varphi_b(x; \gamma)$, labeled by the parameter γ.

Lemmas 2 and 3 are proved in Appendix B. The proofs of Lemmas 2 and 3 rely on technical Lemmas A1–A4, which are stated and proved in Appendix B.

Lemmas 2 and 3 together show that there is a 1-parameter family of bounded solutions, $\varphi(x;\gamma)$, of the differential Equation (7) in a neighborhood of the singularity $x = 0$. Equivalently, the lemmas also show that there is a 1-parameter family of bounded solutions $\varphi(z;\gamma)$ of the differential equation Equation (3) in a neighborhood of the singularity $z = 1$.

2.4. Bounded Solutions Outside the Horizon

As shown in Figure 1, bounded solutions approach the vacuum that is closest to $\varphi(1)$ as $z \to \infty$ regardless of the value of α. In Lemma 4 below, we prove that bounded solutions, for all $\alpha > 0$, behave as shown in Figure 1.

Lemma 4. *Let $\varphi(z)$ be a solution of*

$$(z^2-1)\varphi_{zz} + 2z\varphi_z + \alpha\sin(\varphi) = 0, \qquad \alpha > 0 \tag{16}$$

satisfying the boundary conditions

$$0 < \varphi(1) < \pi, \qquad \varphi_z(1) = -\frac{\alpha}{2}\sin(\varphi(1)). \tag{17}$$

Then, for all $z \geqslant 1$, we have $-\pi < \varphi(z) < \pi$ and

$$\lim_{z\to\infty} \varphi(z) = 0$$

Lemma 4 is proved in Appendix C. Note that, if φ is a solution of Equation (16), then so are $-\varphi$ and $\varphi + 2n\pi$ for all $n \in \mathbb{Z}$. Therefore, we obtain the following corollary.

Corollary 2. *Let $\varphi(z)$ be a solution of*

$$(z^2-1)\varphi_{zz} + 2z\varphi_z + \alpha\sin(\varphi) = 0, \qquad \alpha > 0 \tag{18}$$

satisfying the following boundary conditions. For some $n \in \mathbb{Z}$,

$$(2n-1)\pi < \varphi(1) < (2n+1)\pi, \qquad \varphi_z(1) = -\frac{\alpha}{2}\sin(\varphi(1)).$$

Then, for all $z \geqslant 1$, we have $(2n-1)\pi < \varphi(z) < (2n+1)\pi$ and

$$\lim_{z\to\infty} \varphi(z) = 2n\pi$$

2.5. Bounded Solutions Inside the Horizon

Equation (7) has two length scales—the length scale m^{-1} specified by m, the 'mass' parameter of the field, and the length scale H^{-1} of the background spacetime specified by the Hubble parameter H. We say a soliton is 'large' if $m^{-1} > H^{-1}/\sqrt{2}$ (that is, if $\alpha := (m/H)^2 < 2$) and 'small' if $\alpha > 2$.

Inside the horizon, the behavior of bounded solutions depends on whether $\alpha < 2$ or $\alpha > 2$. We consider the two cases separately, below.

2.5.1. Large Static Solitons Are Forbidden

For $\alpha < 2$, solutions that are bounded near $z = 1$ with $0 < \varphi(1) < \pi$ cannot attain $\varphi(0) = \pi$, as shown by the examples in Figure 2, and blow up at $z = -1$, as shown by the example in Figure 3. The following theorem shows that topologically nontrivial (soliton) bounded solutions do not exist for $\alpha < 2$.

Theorem 1. *For $\alpha < 2$, every bounded solution of*

$$(z^2-1)\varphi_{zz} + 2z\varphi_z + \alpha \sin(\varphi) = 0 \tag{19}$$

is topologically trivial (i.e., $\lim_{z\to\infty} \varphi(z) = \lim_{z\to-\infty} \varphi(z)$).

The proof of the theorem is lengthy and is presented in Appendix D. The proof relies on technical Lemmas A5–A8, which are stated and proved in Appendix D. We describe the outline of the proof of Theorem 1, below.

As discussed in Section 2.1, without loss of generality, we can assume that a solitonic field configuration with charge Q satisfies the following conditions:

$$\lim_{z\to-\infty} \varphi(z) = -2\pi Q, \qquad \varphi(0) = -\pi Q, \qquad \lim_{z\to\infty} \varphi(z) = 0$$

Further, given the symmetries of the Lagrangian, we can restrict attention to $z \geqslant 0$. As Corollary 2 shows, if we insist that $\lim_{z\to\infty} \varphi(z) = 0$, then we must have $-\pi < \varphi(1) < \pi$.

We prove Theorem 1 by showing that, for $\alpha < 2$ and $-\pi < \varphi(1) < \pi$, every bounded solution satisfies $-\pi < \varphi(0) < \pi$. But a topologically nontrivial soliton, satisfying the above condition, can have integer nonzero charge Q only if $\varphi(0) = \pm\pi Q$. Thus, when $\alpha < 2$, topologically nontrivial static solitons cannot exist.

2.5.2. Existence of Small Static Solitons

The behavior of bounded solutions inside the horizon is illustrated for $\alpha = 5$ in Figure 4. The behavior of bounded solutions for all other $\alpha > 2$ is qualitatively similar.

In Lemma A9 (Appendix E) we confirm that $\varphi(0)$ is a continuous function of $\varphi(1)$. Using the continuity property, in Lemmas A10 and A11, we establish that, for $\alpha > 2$, one can find a constant $0 < \gamma^* < \pi$ such that, if $\varphi(1) = \gamma^*$, then $\varphi(0) = \pi$. Lemmas A10 and A11, together with Corollary 2, prove the following theorem.

Theorem 2. *For every $\alpha > 2$, the equation*

$$(z^2-1)\varphi_{zz} + 2z\varphi_z + \alpha \sin(\varphi) = 0$$

has a bounded topologically nontrivial soliton solution.

2.6. Back Reaction

In the above analysis we assumed that the de Sitter background was non-dynamical and we ignored the back reaction of the soliton on the background. In this section, we describe two approaches to incorporating the back reaction of a soliton in order to make the analysis self-consistent. The latter approach, which we adopt, makes the analysis self-consistent and also justifies our assumption that the background de Sitter metric is unaffected by the static soliton for all values of α.

It is well known that, due to its symmetries, the Riemann tensor has only one independent component in 1 + 1 dimensional spacetime and is given by

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

where R is the Ricci scalar and $g_{\mu\nu}$ the metric. As a result, the Einstein tensor $G_{\mu\nu}$ vanishes *for every metric* in 1 + 1 dimensional spacetime ($R_{\beta\nu} = \frac{1}{2}Rg_{\beta\nu} \implies G_{\beta\nu} = R_{\beta\nu} - \frac{1}{2}Rg_{\beta\nu} = 0$). Einstein's equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = \frac{8\pi G_N}{c^4}$$

where Λ is the cosmological constant and $T_{\mu\nu}$ the energy momentum tensor of matter, reduces to

$$\Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{20}$$

Since the Einstein tensor vanishes for every metric, one consistent approach in 1 + 1 spacetime would be to take the metric of spacetime to be the solution of Equation (20). If $T_{\mu\nu}$ in Equation (20) comes from the sine-Gordon field, ϕ, then $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$. Therefore,

$$g_{\mu\nu} = \left(\frac{\kappa}{\Lambda}\right)\left[\,\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\,\right] \quad \text{or } g_{\mu\nu} = \frac{\left(\frac{\kappa}{\Lambda}\right)\partial_\mu\phi\partial_\nu\phi}{1+\left(\frac{\kappa}{\Lambda}\right)\mathcal{L}}$$

Using the sine-Gordon lagrangian (1), we conclude that the metric $g_{\mu\nu}$ is the solution to

$$g_{\mu\nu} = \frac{\left(\frac{\kappa}{\Lambda}\right)\partial_\mu\phi\partial_\nu\phi}{1+\left(\frac{\kappa}{\Lambda}\right)\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \left(\frac{m^2}{\beta^2}\right)(1-\cos(\beta\phi))\right]}. \tag{21}$$

In the action (1), one would use the metric derived as a solution of (21), instead of the de Sitter metric, and attempt to construct a static soliton solution. Even if the nonlinear Equation (21) could be solved, it is not clear, *a priori*, that, in such an approach the solution of (21) would yield a de Sitter metric or even an approximation to a de Sitter metric (if a soliton interpolates between two vacua, then $T_{\mu\nu} \to 0$ at spatial infinity; therefore, $g_{\mu\nu} \to 0$ asymptotically; such a $g_{\mu\nu}$ would not be a de Sitter metric).

The alternative approach, which we adopt, is to embed sine-Gordon theory that is minimally coupled to gravity, in the so-called Jackiw–Teitelboim (JT) theory of gravity in 1 + 1 spacetime [18–21]. The JT theory has a term

$$S_{JT} = \int d^2x\,\sqrt{-g}\,\psi\,(R-\Lambda)$$

that contributes to the bulk action, where ψ is the dilaton field, R, the Ricci scalar and Λ, the cosmological constant. There is no kinetic term for ψ in the JT theory. The equation of motion of the dilaton field yields the constraint

$$R = \Lambda$$

Variation of the total action with respect to the metric yields a dynamical equation for the dilaton field, ψ, in which the energy-momentum tensor of the sine-Gordon soliton acts as a source term; the dynamical equation for ψ is obtained by varying the metric, in both S_{JT} and in the sine-Gordon action, S_{SG}, shown in Equation (1), followed by integration by parts to transfer the covariant derivative operators to act on the dilaton field ψ. Thus, a matter field, such as the sine-Gordon field, does not impact the metric in JT gravity but, instead, impacts the dynamics of the dilaton field. The metric can be stipulated to be de Sitter and is not impacted by the presence of the sine-Gordon soliton. There is no back reaction of the sine-Gordon soliton on the metric for any value of α, and the conclusions of the previous sections, including the threshold $\alpha = 2$, remain unchanged.

Taking the metric to be de Sitter in the JT gravity, we can compare the energy in the static soliton inside the horizon with the energy resident in the cosmological constant inside

the horizon. A straightforward derivation of the Riemann tensor, starting with the de Sitter metric, shows that the Riemann tensor in de Sitter spacetime is

$$R_{\alpha\beta\mu\nu} = H^2(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}).$$

Hence, we conclude that $\Lambda = 2H^2$.

Starting with the action (1), and using the previous convention $\varphi(z) := \beta\,\psi(z/H)$, where $\phi(t,\chi) := \psi(u)$, $u = e^{Ht}\,\chi$, a straightforward calculation yields the energy density

$$T^{00} = \frac{H^2}{\beta^2}\left\{\frac{1}{2}(z^2+1)\varphi_z^2 + \alpha(1-\cos(\varphi))\right\} = \frac{\Lambda}{2\beta^2}\left\{\frac{1}{2}(z^2+1)\varphi_z^2 + \alpha(1-\cos(\varphi))\right\}$$

As is well known [22], the weak coupling regime corresponds to $\beta^2 < 8\pi$. So, we set $\beta^2 = \beta_0^2 := 8\pi - 10^{-6}$. Focusing on the regime $\alpha \sim 2$, we take $\alpha = \alpha_0 := 2.0001$. Then, the ratio of the soliton's energy within the horizon, denoted E_S, to the energy due to the cosmological constant within the horizon, denoted E_Λ, in units where $8\pi G = c = 1$, is obtained numerically as

$$\frac{E_S}{E_\Lambda} = \frac{\displaystyle\int_{-1}^{1} \frac{\Lambda}{2\beta_0^2}\left\{\frac{1}{2}(z^2+1)\varphi_z^2 + \alpha_0(1-\cos(\varphi))\right\} dz}{2\Lambda} \approx 0.0796 \tag{22}$$

The integral in (22) was evaluated as a Riemann sum with $\sim 2\times 10^9$ mesh points in the range $(-1,1)$ using MATLAB R2024b. The computation was done using double precision. Equation (3) was integrated using `ode45`, a built-in function in MATLAB R2024b. The computation was started at $z = 1 - 10^{-9}$ and integrated backwards to $z = -1 + 10^{-9}$; the displacement of 10^{-9} was determined by memory constraints on the size of the grid. The integrity of the computation was verified by confirming that $\varphi_z(1-10^{-9}) = \varphi_z(-1+10^{-9})$. Within the horizon, the energy in the soliton field is $\sim$8% of the energy due to the cosmological constant for the above choice of parameter values in the weak coupling regime.

3. Heuristic Estimates Based on Tensile and Tidal Forces

In this section, we present heuristic arguments based on the interplay of tensile force and tidal force. On the one hand, the tensile force of a soliton resists inflationary stretching of the soliton and acts to preserve a soliton as a compact field configuration. On the other hand, the tidal force of the inflationary background tends to stretch the soliton. If a soliton is to be static, that is, if its physical dimensions are to buck inflationary stretching, then its internal tensile force must be strong enough to counteract the tidal stretching of the soliton by the background. The arguments we present are crude. They have only a qualitative heuristic value.

In Section 3.1, we use a heuristic argument to obtain an estimate of the tensile force in a Lorentzian sine-Gordon soliton and, using the estimate, a threshold for existence of a static sine-Gordon soliton in a de Sitter background; the estimate we obtain with the crude heuristic argument agrees with the exact estimate $\alpha = 2$ to within an $O(1)$ factor.

In Section 3.2, we use a similar heuristic argument to obtain an estimate of the tensile force in a Lorentzian $SO(3)$ 't Hooft–Polyakov monopole and, using the estimate, a threshold for existence of static $SO(3)$ 't Hooft–Polyakov monopole in a de Sitter background. The existence of the threshold, suggested by our heuristic argument, remains to be confirmed by an analytical argument. Guided by the heuristic argument, we propose a conjecture that there is an $O(1)$ threshold for the existence of a static $SO(3)$ 't Hooft–Polyakov monopole in non-dynamical de Sitter background.

3.1. Sine-Gordon Solitons

In spacetime with nonzero Riemann tensor the separation between two neighboring, initially parallel, geodesics accelerates as one moves along the fiducial geodesic. Since a soliton is an extended object two neighboring points in a soliton move along different geodesics. In de Sitter spacetime the separation between two neighboring points grows with time; in other words, the de Sitter background pulls the points apart. If the soliton is to remain a compact static object, the internal tensile forces of the soliton must be strong enough to resist the tidal force of the background spacetime that seeks to tear the soliton apart.

A rigorous study of the interplay between the internal tensile force of a soliton and the tidal force of the background requires analysis of the soliton configuration in the curved background. We consider a simpler problem. We estimate the tensile force in a sine-Gordon soliton in 1 + 1 Lorentzian spacetime and compare it with the tidal force operative in de Sitter spacetime. We show that even the rather simple analysis is able to reproduce the rigorously derived threshold to within an $O(1)$ factor.

In a Lorentzian background the sine-Gordon Lagrangian is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - U(\varphi), \qquad U(\varphi) := m^2(1-\cos(\varphi)) \tag{23}$$

(The Lorentzian sine-Gordon field φ has been redefined to absorb the coupling constant β; the resulting overall multiplicative factor, β^{-2}, is irrelevant to our argument; see Equation (30)). Time-independent solutions of the Euler–Lagrange equation satisfy

$$\varphi_{xx}(x) = U'(\varphi) \tag{24}$$

As Coleman observed, if we interpret x as time and φ as the position of a particle, then Equation (24) describes the behavior of a particle of unit mass moving in potential $-U(\varphi)$. A soliton solution that interpolates between two adjacent minima of $U(\varphi)$ corresponds to the trajectory of the particle moving, from infinite past to infinite future, between two adjacent maxima of $-U(\varphi)$. Since the energy of the particle is conserved, and, assuming the potential energy as well as the kinetic energy of the particle at the local maximum of $-U(\varphi)$ is zero, we have

$$\frac{1}{2}\,(\varphi'(x))^2 = U(\varphi) \tag{25}$$

One can verify that the time-independent topologically nontrivial soliton solution of the sine-Gordon theory, $\tilde{\varphi}(x)$, given by

$$\tilde{\varphi}(x) = 4\tan^{-1}(e^{mx}) \tag{26}$$

satisfies Equation (25). From Equation (25) we see that the energy density of the soliton $\tilde{\varphi}$ is given by

$$\mathcal{E}(x) = \frac{1}{2}(\tilde{\varphi}'(x))^2 + U(\tilde{\varphi}(x)) \;=\; 2\,U(\tilde{\varphi}(x)) \;=\; 16\,m^2\left[\frac{e^{2mx}}{(1+e^{2mx})^2}\right]$$

Thus, most of the energy of the sine-Gordon soliton is contained within a distance of $l \sim \xi m^{-1}$ from the center, where $\xi \sim O(1)$. We define a system S to be the soliton configuration within a fiducial box $[-l, l]$ and the energy contained in the system S as

$$E(l) = = \; 2\int_0^l \mathcal{E}(x)\,dx \;=\; 8m\left[\frac{e^{2ml}-1}{e^{2ml}+1}\right]; \qquad E(\infty) - E(l) \;=\; \frac{16m}{e^{2ml}+1} \tag{27}$$

Consider stretching the soliton as $\tilde{\varphi}_\lambda(x) = \tilde{\varphi}(x/\lambda)$, $\lambda \geqslant 1$, where $\tilde{\varphi}_\lambda(x)$ represents the stretched configuration. As a result of the stretching, the fiducial segment of the soliton in the interval $[-l, l]$ stretches to span $[-\lambda l, \lambda l]$, and the energy in the stretched segment, denoted $E_\lambda(\lambda l)$, is given by

$$E_\lambda(\lambda l) = 2\int_0^{\lambda l}\left[\frac{1}{2}(\tilde{\varphi}'_\lambda(x))^2 + U(\tilde{\varphi}_\lambda(x))\right]dx = \left(\frac{1}{2}\right)[\lambda + (1/\lambda)]\,E(l)$$

The increase in the internal energy in the fiducial system as a result of the stretching is

$$\Delta E_\lambda = E_\lambda(\lambda l) - E(l) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda} - 2\right)E(l) \tag{28}$$

Clearly $E_\lambda(\lambda l)$ is minimized at $\lambda = 1$, and the increase in the internal energy in the system S can be interpreted as the work done on the system against its internal tensile force, which resists stretching. We define $F(\lambda)$ as the centripetal tensile force acting at the boundary of the stretched fiducial segment that has been stretched by a factor λ. Denoting the change in the size of the system as $q := 2(\lambda l - l)$, we have

$$\Delta E_\lambda = 2\int_1^\lambda F(v)\, l\, dv$$

from which we obtain

$$F(\lambda) = \left(\frac{1}{2l}\right)\frac{d}{d\lambda}\Delta E_\lambda = \left(\frac{1}{4l}\right)E(l)\left(1 - \frac{1}{\lambda^2}\right) \tag{29}$$

$F(\lambda)$ is the centripetal force that the system S exerts at $x = \lambda l$ on its complement, which is the segment of the stretched soliton beyond λl. The mass of the complement, denoted m_c in units in which the velocity of light in vacuum, $c = 1$, is

$$m_C := \frac{1}{2}\left(E_\lambda(\infty) - E_l(\lambda l)\right) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)(E(\infty) - E(l))$$

Assuming that the complement of the stretched system behaves like a rigid body, the centripetal acceleration at $x = \lambda l$ that arises when the external stretching force is turned off is

$$a_\lambda = \frac{F(\lambda)}{m_C} = \frac{e^{2ml} - 1}{4l}\left[\frac{\lambda^2 - 1}{\lambda(\lambda^2 + 1)}\right] \tag{30}$$

(The overall factor β^{-2} that arises from the redefinition of the field (see the remark after Equation (23)) appears in both $F(\lambda)$ and m_C, and is hence irrelevant to the calculation of a_λ in Equation (30), as we remarked earlier.)

The centrifugal tidal acceleration of the displacement χ^α is given by [23]

$$a_t^\alpha = \nabla_u \nabla_u \chi^\alpha = -R^\alpha_{\beta\mu\nu} u^\beta u^\nu \chi^\mu$$

where u^β and u^ν are the tangents to the two geodesics and χ^μ the displacement between the two geodesics. It should be noted that the expression for tidal acceleration holds only for small displacement. We show below that the small-displacement assumption is satisfied in our argument; see the discussion preceding and following Equation (35). We take $u^\beta = u^\nu = (1, 0)$. Therefore, the components of the Riemann tensor of interest to us are

$R^{\alpha}_{0\mu 0}$. Calculation of the Riemann tensor shows the only nonzero component of $R^{\alpha}_{0\mu 0}$ is $R^{1}_{010} = -H^2$. Taking $\chi^{\alpha} = (0, \lambda l)$, the tidal acceleration is

$$a_t = H^2 \lambda l \tag{31}$$

Consider stretching the sine-Gordon soliton using external force by a factor λ in Lorentzian spacetime, as described above. As a result of the stretching, centripetal tensile forces arise in the soliton, as described in Equation (29). At $t = 0$ we turn off the external stretching force and simultaneously turn on the expansion of the background de Sitter spacetime with Hubble parameter H. The de Sitter background induces a centrifugal tidal acceleration in the soliton. The question we are interested in is: *For what value of λ are the centripetal tensile acceleration of the soliton and the centrifugal tidal acceleration balanced at the boundary of the stretched fiducial system?*

We take the size of the fiducial system S to be $l = m^{-1}$, the characteristic size of the sine-Gordon soliton. Equating the tensile and tidal accelerations at $l = m^{-1}$ using Equations (30) and (31) and setting $\alpha = (m/H)^2$, we obtain

$$\frac{\lambda^2 - 1}{\lambda^2(\lambda^2 + 1)} = q, \qquad q := \left[\frac{4}{e^2 - 1}\right]\frac{1}{\alpha} \approx \frac{0.6261}{\alpha}$$

Setting $w = \lambda^2$, we obtain the equation

$$q w^2 + (q - 1)w + 1 = 0 \tag{32}$$

We need at least one of the solutions of the above quadratic equation to be positive, which can happen only if $q < 1$, that is, $\alpha \gtrsim 0.6261$. In addition, since $w = \lambda^2$ must be real, we must have $(q-1)^2 - 4q \geqslant 0$, or, in terms of $\alpha \approx 0.6261/q$, we must have

$$(\alpha - 0.1074)(\alpha - 3.6490) \;\geqslant\; 0 \tag{33}$$

The inequality (33) is satisfied if $\alpha \lesssim 0.1074$ or $\alpha \gtrsim 3.6490$. Since positivity of w requires $\alpha \gtrsim 0.6261$, we conclude that a real λ exists only if

$$\alpha \gtrsim 3.6490 \tag{34}$$

For $\alpha \lesssim 3.6490$, λ would be complex, which means that tensile and tidal accelerations cannot be at equilibrium at the boundary of the fiducial system S at any real λ; in other words, the system cannot be a static configuration. It is worth comparing the inequality (34) with the result of the exact analysis in Section 2, which showed that static soliton exists only if $\alpha > 2$.

In using the expression for tidal acceleration, we made the assumption that the displacement λl was 'small'. That is, in terms of the characteristic length scale of the background, we assumed $\lambda l / H^{-1} \lesssim 1$. For $l \sim m^{-1}$, the assumption we made was

$$\lambda \lesssim \sqrt{\alpha}. \tag{35}$$

As verified below, the assumption is always satisfied by the smaller root of Equation (32), denoted λ_1. At λ_1, the inequality (35) can be written as

$$1 - q - 2\,q\,\alpha \;<\; \sqrt{(1-q)^2 - 4q} \tag{36}$$

Since $2\,q\,\alpha \approx 1.2521 > 1$ and $q > 0$, the left-hand side of (36) is negative and inequality (36) is always satisfied for $\alpha > 3.6490$.

The above analysis is crude since it estimates the tensile force of the soliton in Lorentzian and not de Sitter spacetime.

3.2. 't Hooft–Polyakov Monopole

In this section we compare a heuristic estimate of the tensile force within the Lorentzian 't Hooft–Polyakov monopole, in the Prasad–Sommerfeld limit, and the tidal force the monopole experiences in de Sitter spacetime to obtain a heuristic threshold for the existence of static 't Hooft–Polyakov monopole in de Sitter spacetime. 't Hooft–Polyakov monopoles are solitons of a Yang–Mills gauge theory in which the gauge fields couple to a triplet of Higgs scalars. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}(G^{\mu\nu})^a + \frac{1}{2}D_\mu\varphi^a D^\mu\varphi^a - \frac{\xi}{4}(\varphi^a\varphi^a - F^2)^2, \qquad a = 1,2,3 \tag{37}$$

where F is the minimum of the Higgs potential, and

$$(D_\mu\varphi)^a = \partial_\mu\varphi^a + g\,\epsilon^{abc}A^b_\mu\varphi^c, \quad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\epsilon^{abc}A^b_\mu A^c_\nu, \quad a,b,c = 1,2,3$$

The equations of motion are

$$(D_\mu D^\mu\varphi)^a = -\xi(\varphi^2 - F^2)\,\varphi^a, \qquad D_\mu(G^{\mu\nu})^a = g\epsilon^{abc}(D^\nu\varphi)^b\varphi^c$$

If we choose the temporal gauge $A^a_0 = 0$ and consider a time-independent solution, then the energy is given by

$$E = \int d^3x\left\{\frac{1}{4}(G_{ij})^a(G_{ij})^a + \frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a + \frac{\xi}{4}(\varphi^2 - F^2)^2\right\}$$

where the sum over repeated indices is implied; that is,

$$(G_{ij})^a(G_{ij})^a = \sum_{i,j,a=1}^{3}(G^a_{ij})^2, \quad (D_i\varphi)^a(D_i\varphi)^a = \sum_{i,a=1}^{3}(D_i\varphi^a)^2.$$

In the Prasad–Sommerfeld limit [24], in which the Higgs self-coupling vanishes, the equations of motion are

$$(D_iD^i\varphi)^a = 0, \qquad D_i(G^{ij})^a = g\epsilon^{abc}(D^j\varphi)^b\varphi^c \tag{38}$$

and the energy is

$$E = \int d^3x\,\mathcal{H}, \qquad \mathcal{H} = \left\{\frac{1}{4}(G_{ij})^a(G_{ij})^a + \frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a\right\} \tag{39}$$

Prasad and Sommerfeld showed that the equations of motion Equation (38) are satisfied by

$$\begin{aligned}\varphi^a &= x^aK(r), \qquad K(r) = \left(\tfrac{1}{gr^2}\right)[\,m_V\,r\coth(m_Vr) - 1\,] \\ A^a_i &= \epsilon_{aij}x^jW(r), \qquad W(r) = \left(\tfrac{1}{gr^2}\right)[\,1 - m_Vr/\sinh(m_Vr)\,]\end{aligned}, \qquad m_V = gF; \tag{40}$$

Further, the solution shown in Equation (40) satisfies the Bogomolny equation

$$\frac{1}{2}\epsilon^{ijk}G^a_{ij} = -(D^k\varphi)^a$$

from which we obtain the equipartition of energy density between the Higgs and gauge fields

$$\frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a = \frac{1}{4}G^a_{ij}(G_{ij})^a \tag{41}$$

Using the notation of Equations (39) and (40) we see that the energy density of the soliton is spherically symmetric

$$\mathcal{H}(r) = \mathcal{H}_{A}(r) + \mathcal{H}_{\varphi}(r), \quad \mathcal{H}_{\varphi}(r) \;=\; \frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a, \quad \mathcal{H}_{A}(r) \;=\; \frac{1}{4}G^a_{ij}(G_{ij})^a$$

The characteristic length scale of the soliton is $\rho \sim m_V^{-1}$. We define the fiducial system, S, as the Higgs and gauge field configurations in the static soliton within a sphere of radius $\rho \sim m_V^{-1}$ centered at the origin. The energy in the system and its complement are

$$E(\rho) = 4\pi\int_0^{\rho} r^2\,\mathcal{H}(r)\,dr, \quad E^{(c)}(\rho) \;=\; 4\pi\int_{\rho}^{\infty} r^2\,\mathcal{H}(r)\,dr, \tag{42}$$

Following the argument for the sine-Gordon soliton, we stretch the soliton with a dilatation $\vec{y} \;=\; \lambda\vec{x}$, $\lambda > 1$. The Higgs and gauge fields in the new coordinate system are

$$\tilde{\varphi}(\vec{y}) = \varphi(\vec{y}/\lambda), \qquad \tilde{A}^a_i(\vec{y}) \;=\; \frac{1}{\lambda}A^a_i(\vec{y}/\lambda)$$

The energy densities of the Higgs and gauge fields in the stretched system are denoted $\tilde{\mathcal{H}}_{\tilde{\varphi}}(r)$ and $\tilde{\mathcal{H}}_{\tilde{A}}(r)$. The energy in the stretched system is

$$\tilde{E}(\lambda\rho) = 4\pi\int_0^{\lambda\rho} r^2\left[\,\tilde{\mathcal{H}}_{\tilde{A}}(r) + \tilde{\mathcal{H}}_{\tilde{\varphi}}(r)\,\right]dr \;=\; 4\pi\lambda^3\int_0^{\rho} s^2\left[\left(\frac{1}{\lambda}\right)^4\mathcal{H}_{A}(s) + \left(\frac{1}{\lambda}\right)^2\mathcal{H}_{\varphi}(s)\right]ds$$

Using the equipartition of energies shown in Equation (41) we obtain

$$\tilde{E}(\lambda\rho) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)E(\rho)$$

The increase in the internal energy of the system, as a result of stretching, is the work done against the negative pressure $P(\lambda)$ in the system and is

$$\Delta E = \tilde{E}(\lambda\rho) - E(\rho) \;=\; \frac{1}{2}\left(\lambda + \frac{1}{\lambda} - 2\right)E(\rho) \;=\; \int_1^{\lambda} P(\lambda')\,dV \;=\; \int_1^{\lambda} f(\lambda')\,\rho\,d(\lambda')$$

where $f(\lambda')$, the total centripetal restoring force arising at the boundary of the system stretched by a factor λ', is given by

$$f(\lambda) = \frac{1}{\rho}\frac{d\Delta E}{d\lambda} \;=\; \frac{1}{2\rho}\left(1 - \frac{1}{\lambda^2}\right)E(\rho) \tag{43}$$

We assume that the centripetal force at the boundary of the stretched system acts on the complement of the stretched system, which we assume behaves as a rigid body of mass $\bar{m}$ given by

$$\bar{m} = 4\pi\int_{\lambda\rho}^{\infty} r^2\left[\,\tilde{\mathcal{H}}_{\tilde{\varphi}}(r) + \tilde{\mathcal{H}}_{\tilde{A}}(r)\,\right]dr \;=\; \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)E^{(c)}(\rho)$$

where $E^{(c)}(\rho)$ is defined in Equation (42). The magnitude of the centripetal tensile acceleration, as a function of scale factor λ, is then

$$a_\lambda = \frac{(\lambda^2 - 1)}{\lambda\rho(\lambda^2 + 1)} \frac{E(\rho)}{E^{(c)}(\rho)} \tag{44}$$

To determine the tidal force in 3 + 1 de Sitter, again we take the tangents of the fiducial and a nearby spatially separated geodesic to be $u = (1, 0, 0, 0)$. The spatial separation between the geodesics is taken to be $\chi = (0, \vec{\chi})$. Then the relevant components of the Riemann tensor are $R^i{}_{0j0}$. Straightforward calculation yields

$$R^i{}_{0j0} = -\delta^i_j H^2$$

where H is the Hubble constant. Then the magnitude of the centrifugal tidal acceleration at the boundary of the stretched system, that is, at a distance $\lambda\rho = \lambda/m_V$ from the origin, is

$$a_t = H^2 \lambda \rho \;=\; H^2 \left(\frac{\lambda}{m_V}\right) \tag{45}$$

The tensile and tidal accelerations are balanced at the boundary of the stretched system if, at $r = \lambda/m_V$, we have, using Equations (44) and (45),

$$\frac{(\lambda^2 - 1)}{\lambda^2 (\lambda^2 + 1)} = \frac{1}{\kappa\alpha^{(tP)}} \tag{46}$$

where $\alpha^{(tP)} := (m_V/H)^2$, and $\kappa := E(m_V^{-1})/E^{(c)}(m_V^{-1})$. Setting $w := \lambda^2$, the above equation becomes

$$w^2 + (1 - \kappa\alpha^{(tP)})w + \kappa\alpha^{(tP)} = 0 \tag{47}$$

Using Equations (40) and (41) it can be verified that

$$\mathcal{H}(r) = (r\,K'(r) + K(r))^2 + 2\,K(r)^2\,(1 - g\,r^2\,W(r))^2 \tag{48}$$

Since $\mathcal{H}(r) \geqslant 0$, from (42) and the definition of κ, it follows that $\kappa \geqslant 0$. Using (48) and (42), κ can be obtained by numerical integration to be $\kappa \approx 1.4820$. Recalling that $\alpha^{(tP)} \geqslant 0$, consistency of (46) requires that $\lambda^2 \geqslant 1$. From (46), it then follows that $\kappa\alpha^{(tP)} \geqslant 1$ or

$$\alpha^{(tP)} \geqslant \frac{1}{\kappa} \;\approx\; 0.6748 \tag{49}$$

Further, for $w = \lambda^2$ to be real, we must have

$$(1 - \kappa\alpha^{(tP)})^2 - 4\kappa\alpha^{(tP)} \;=\; \left(\kappa\alpha^{(tP)} - (3 + 2\sqrt{2})\right)\left(\kappa\alpha^{(tP)} - (3 - 2\sqrt{2})\right) \geqslant 0$$

or

$$\alpha^{(tP)} \;\leqslant\; 0.1158, \qquad \text{or} \qquad \alpha^{(tP)} \geqslant 3.9328 \tag{50}$$

From (49) and (50) we conclude that, if $\alpha^{(tP)} = (m_V/H)^2 < 3.9328$, then there is no (real) scale factor at which the tidal and tensile forces can reach equilibrium at $r = \lambda/m_V$.

Figure 7 shows a plot of the ratio a_λ/a_t as a function of λ, for $\alpha^{(tP)} = 1, \ldots, 6$. As the plots show, when $\alpha^{(tP)} \lesssim 3.9328$, the tidal force is stronger the tensile force at $r = \lambda/m_V$ for every value of scale factor λ, and the strong tidal forces do not permit the existence of static 't Hooft–Polyakov monopoles.

The existence of the threshold, suggested by the above heuristic argument, remains to be confirmed analytically. However, it serves as a plausibility argument for the following conjecture.

Conjecture 1. *Static 't Hooft-Polyakov monopole solution exists in non-dynamical de Sitter spacetime, in the Prasad-Sommerfeld limit, if and only if* $\alpha := (m_V/H)^2 \geqslant \alpha_0$, *where* $\alpha_0 \sim O(1)$ *is a constant,* m_V *is the mass of the vector boson and* H *is the Hubble constant.*

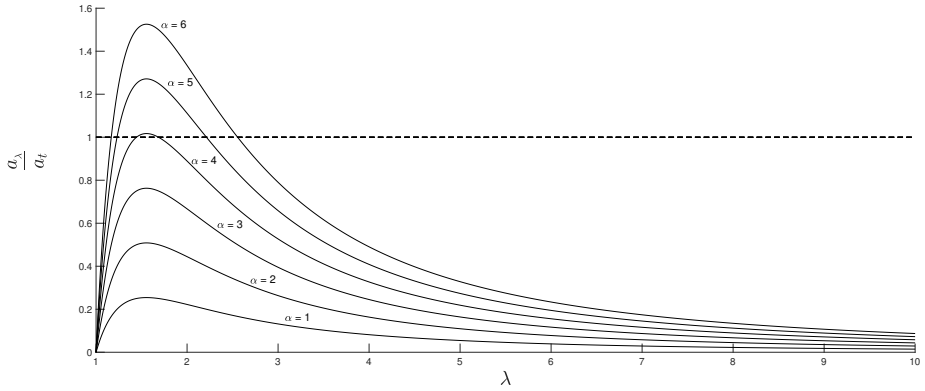


Figure 7. a_λ/a_t as a function of λ for $\alpha^{(tP)} = 1, \ldots, 6$. $\alpha^{(tP)}$ is abbreviated to α in the labels shown in the plots. The plots (solid lines) show that, for $\alpha^{(tP)} < 3.9328$, the tidal forces are too strong—$a_\lambda/a_t < 1$—and do not allow static 't Hooft–Polyakov monopole to exist in de Sitter spacetime. The dashed line represents $a_\lambda/a_t \equiv 1$.

4. Secondary Inflation

Linde [13,17] has suggested that, although the Hubble constant H in new inflation is $\sim 10^{10}$ GeV , which is about five orders of magnitude smaller than the mass of the X boson, $m_X \sim 10^{15}$ GeV, making $\alpha^{(GUT)} := (m_X/H)^2 \sim 10^{10}$, the inflationary background could trigger secondary inflation at a GUT monopole's core. Linde's argument is that, near the core, the X boson is nearly massless and the field gradients near the core of the monopole are rapidly decreased once inflation starts in the background, creating conditions suitable for the onset of secondary inflation at the core of the monopole.

In Section 4.1, we present a heuristic argument, which suggests that a weak inflationary background cannot initiate secondary inflation at the core of an $SO(3)$ 't Hooft–Polyakov monopole. Based on the heuristic analysis of secondary inflation in the 't Hooft–Polyakov monopole, we conjecture, in Section 4.2, that the secondary inflation in the GUT monopole, suggested by Linde, is infeasible.

4.1. Secondary Inflation Inside 't Hooft–Polyakov Monopole

This section presents a heuristic answer to the following question: *Consider placing an initial field configuration corresponding to the Lorentzian 't Hooft–Polyakov monopole in an inflating background. Under what conditions does the inflating background drive the initial field gradients near the core of the monopole to zero?*

If inflation is to drive the field gradients near the core in the initial monopole configuration to zero, then, as the monopole is stretched by inflation, the tidal force of the de Sitter background must exceed the tensile force of the monopole near the core of the monopole. Specifically, at some $\lambda\rho = \lambda\xi m_V^{-1}$, where $0 < \xi < O(1)$ and $\lambda > 1$, the centrifugal tidal acceleration must exceed the centripetal tensile acceleration.

Using the arguments that preceded Equations (44) and (45), and defining the fiducial system to be the Higgs and gauge fields inside a sphere of radius $\rho = \xi m_V^{-1}$, the magnitudes

of the centripetal tensile acceleration and centrifugal tidal acceleration, at $\lambda\rho = \lambda\xi\, m_V^{-1}$, denoted $a_\lambda(\xi/m_V)$ and $a_t(\xi/m_V)$, after the monopole has been stretched by a factor λ, are

$$a_\lambda(\xi/m_V) = \frac{\lambda^2-1}{\lambda(\lambda^2+1)}\left(\frac{m_V}{\xi}\right)\frac{E(\xi m_V^{-1})}{E^{(c)}(\xi m_V^{-1})}$$
$$a_t(\xi/m_V) = H^2\left(\tfrac{\lambda\xi}{m_V}\right)$$

If the tidal force dominates at a distance $\lambda\rho = \lambda\xi/m_V$ from the center of the monopole for some $0 < \xi < O(1)$, $\lambda > 1$, then we must have

$$a_\lambda(\xi/m_V) < a_t(\xi/m_V), \implies \left(\frac{m_V}{H}\right)^2\left[\frac{E(\xi m_V^{-1})}{E^{(c)}(\xi m_V^{-1})}\right]\frac{1}{\xi^2} < \frac{\lambda^2(\lambda^2+1)}{\lambda^2-1} \tag{51}$$

Defining

$$q(\lambda) := \frac{\lambda^2(\lambda^2+1)}{\lambda^2-1}, \qquad k(\xi) := \frac{1}{\xi^2}\left[\frac{E(\xi m_V^{-1})}{E^{(c)}(\xi m_V^{-1})}\right] \tag{52}$$

Equation (51) becomes

$$\alpha^{(tP)}\, k(\xi) < q(\lambda). \tag{53}$$

Inequality (53) must be satisfied for the tidal acceleration to dominate tensile acceleration at a distance $\lambda\rho = \lambda\xi m_V$ when the monopole has been stretched by a factor $\lambda > 1$.

Figure 8 shows the plot of $q(\lambda)$ and $k(\xi)$. $q(\lambda)$ attains its minimum $q_{min} \sim 5.8285$ at $\lambda^* \sim 1.55$ and $k(\xi)$ attains its minimum value $k_{min} \sim 1.4687$ at $\xi^* \sim 1.1300$.

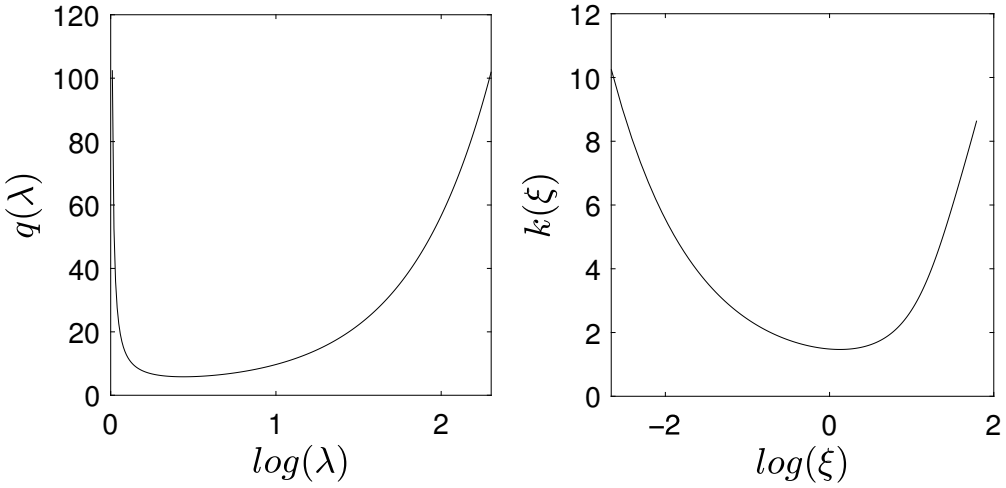


Figure 8. Plots of $k(\xi)$ and $q(\lambda)$.

Let $\alpha_0^{(tP)} = q_{min}/k_{min} \approx 3.9685$. If $\alpha^{(tP)} < \alpha_0^{(tP)}$, then

$$\alpha^{(tP)} < \alpha_0^{(tP)} = \frac{q_{min}}{k_{min}} \leqslant \frac{q(\lambda)}{k_{min}}, \text{ for all } \lambda \geqslant 1 \text{ at } \xi^* \sim 1.13$$

That is, at a distance $\rho = \xi^* m_V$ from the center, tidal acceleration is at least as large as the tensile acceleration for all λ. In other words, there is at least one distance from the center $(\xi^* m_V^{-1})$ in the unstretched monopole at which the tidal force is at least as large as the tensile force for all $\lambda > 1$.

The more interesting case is when $\alpha^{(tP)} > \alpha_0^{(tP)}$. We note that $q(\lambda)$ diverges as $\lambda \to 1$. As λ increases from 1, $q(\lambda)$ decreases until λ reaches λ^*. Since $\alpha^{(tP)} > \alpha_0^{(tP)}$, there is a $\lambda_0 \in (1, \lambda^*)$ at which

$$q(\lambda_0) = \alpha^{(tP)} k_{min}$$

For any $\lambda \in (\lambda_0, \lambda^*)$,

$$q(\lambda) < q(\lambda_0) \; = \; \alpha^{(tP)} k_{min} \leqslant \alpha^{(tP)} k(\xi), \text{ for all } \xi.$$

That is, beyond the stretching factor λ_0, inequality (53) is not satisfied at any ξ. In other words, beyond stretching factor λ_0, the tensile acceleration exceeds the tidal acceleration at the monopole's core and stretching of the monopole by the inflationary background ceases. Specifically,

$$\lambda_0(\alpha^{(tP)}) = \frac{1}{\sqrt{2}} \left[(1.4687\alpha^{(tP)} - 1) - \sqrt{2.1571(\alpha^{(tP)})^2 - 8.8122\alpha^{(tP)} + 1} \right]^{\frac{1}{2}}. \tag{54}$$

For $\alpha^{(tP)} \geqslant 628$, $\lambda_0(\alpha^{(tP)}) < 1$, which does not represent stretching. $\lambda_0(\alpha^{(tP)})$ decreases monotonically as $\alpha^{(tP)}$ increases from 4 to 627, with $\lambda_0(627) \; = \; 1 + 2.12 \times 10^{-6}$.

The discussion in this section is a suggestive heuristic argument, which is useful only as a plausibility argument for the conjecture stated in Section 4.2.

4.2. Secondary Inflation in GUT Monopole

Unlike the simple $SO(3)$ gauge theory of Higgs and gauge fields, used in the construction of the 't Hooft–Polyakov monopole, the $SU(5)$ gauge theory, underlying new inflation, has Higgs, gauge and fermion fields; monopole solution arises when $SU(5)$ is broken at the GUT scale. $SU(5)$ monopole has been discussed in the literature [25]. In the following discussion, we suggest that a heuristic argument, analogous to that presented in Section 4.1, can be developed for $SU(5)$ monopoles. Again, we restrict attention to the Prasad–Sommerfeld limit in which the coupling constant in the Higgs potential—which is independent of the unified gauge coupling constant of $SU(5)$—goes to zero.

Specifically, we can consider a fiducial system comprising the fermion, gauge and Higgs fields of an $SU(5)$ monopole, within a sphere S of radius $\rho = m_X^{-1}$ from the center of the monopole, where m_X is the mass of the X boson that emerges when the $SU(5)$ symmetry is broken. An estimate of the tensile force in the monopole can be obtained by regarding the increase in energy when the fiducial system is stretched by a dilatation factor λ as the work done against a centripetal tensile force acting on the fields outside the sphere of radius $\lambda\, m_X^{-1}$. The tidal acceleration of the de Sitter spacetime is given by Equation (45).

The heuristic argument in Section 4.1 suggests that, for an $SO(3)$ monopole, there is a $\alpha_0^{(tP)} \approx 3.9685 \sim O(1)$, and, for

$$\alpha^{(tP)} := \left(\frac{m_V}{H}\right)^2 > \alpha_0^{(tP)}$$

the tidal force of the inflationary background is too weak to continue stretching the core of the monopole beyond some small dilatation factor λ_0. In other words, after the soliton is stretched by a factor λ_0, the centripetal force dominates the tidal force, preventing further stretching; see the discussion leading up to and following Equation (54).

It seems plausible that, for the GUT monopole too, there is a $\alpha_0^{GUT} \sim O(1)$, and, for

$$\alpha^{GUT} := \left(\frac{m_X}{H}\right)^2 > \alpha_0^{GUT}$$

the tidal force of the inflationary background is too weak to continue stretching the core of an $SU(5)$ monopole beyond some small dilatation factor $\lambda_0^{(GUT)}$.

Based on the heuristic argument for the $SO(3)$ 't Hooft–Polyakov monopole and the speculated validity of an analogous argument for GUT monopole, we state the following conjecture.

Conjecture 2. *For $(m_X/H)^2 \gtrsim O(1)$, where m_X is the mass of the X boson and H the Hubble constant of inflationary background, secondary inflation cannot occur at the core of a GUT monopole.*

Our conjecture is incompatible with Linde's suggestion that, even at $\alpha^{GUT} \approx 10^{10}$, the inflationary background can stretch the core of a GUT monopole sufficiently to make the field gradients vanish near the core. Our conjecture can be settled with an exact analysis of the evolution of a GUT monopole in the de Sitter background that may have prevailed during the GUT phase transition.

5. Discussion

Besides the two conjectures stated above, we mention two other open problems in Sections 5.1 and 5.2. The first problem pertains to the stability of static solitons of de Sitter spacetime. The second problem pertains to the behavior of static solitons of de Sitter spacetime when inflation ends. We also discuss below, the impact of quantum corrections and the precision of numerical calculations presented in the paper.

5.1. Stability of the Soliton

The stability of the soliton solution described in Section 2 is an open problem. The equation satisfied by $\delta\phi(\chi, t)$, a small perturbation of the soliton solution, can be derived starting with Equation (2); not surprisingly, the equation for $\delta\phi(\chi, t)$ depends on the soliton solution, which is not available in closed form. Establishing the stability of the soliton involves showing that $\delta\phi(\chi, t)$ decays for arbitrary initial conditions.

5.2. Behavior of Solitons at the End of Inflation

When inflation ends, $H \to 0$, $\alpha \to \infty$. If spacetime becomes Lorentzian after inflation ends, it is natural to ask if a sine-Gordon soliton solution in de Sitter spacetime approaches the sine-Gordon soliton solution of flat spacetime as $H \to 0$. The answer may be nuanced. In the $H \to 0$ limit, the equation for static solutions in de Sitter spacetime—Equation (3)—does not approach the equation for static solutions in flat spacetime. At every $H \neq 0$, however small H is, Equation (3) has two singularities at $z = \pm 1$ or $u = \pm H^{-1}$. At $H = 0$, the number of singularities of the equation for static solutions drops *discontinuously* to zero. Given the discontinuity, we may not expect the static solitons in de Sitter spacetime to morph continuously into the static solitons of flat spacetime as $H \to 0$. The behavior of static solitons in de Sitter spacetime when inflation ends warrants closer examination.

5.3. Quantum Corrections

Quantization of soliton solutions has been discussed in [26]. In the regime $\alpha := (m/H)^2 < 2$, where quantum fluctuations are important, a classical static soliton solution does not exist. In the regime $\alpha \gtrsim 2$, the static soliton solution in de Sittter spacetime, denoted $\phi_s(\chi, t)$, can be regarded as a classical background for quantum fluctuations $\zeta(\chi, t)$; the classical solution discussed in Section 2 can then be retained unchanged. The equation for $\zeta(\chi, t)$, derived from Equation (2), is linear—in the small $|\zeta|$ limit—but with a damping term, which is related to the suppression of quantum fluctuations by the inflation in the background.

5.4. Numerical Calculations

The numerical calculations reported in this paper were done in MATLAB R2024b using its default double precision arithmetic. The double precision's machine epsilon is $\delta_0 = 2.22 \times 10^{-16}$; that is, δ_0 is the smallest number for which 1 and $1 + \delta_0$ are regarded as different numbers in double precision arithmetic. The machine epsilon determines how closely one can approach the singularity $z = 1$ in numerical investigations. We solved the differential equation using the built-in routine, `ode45`, which expects the second-order differential equation, such as Equation (3), to be presented as a system of first-order equations, such as

$$\begin{aligned} \varphi'(z) &= \psi(z) \\ \psi'(z) &= -\left[\tfrac{2z\psi(z)+\alpha \sin(\varphi(z))}{(z+1)(z-1)} \right] \end{aligned}$$

If $|z-1| < \delta_0$, then, numerically, it is treated as $z - 1 = 0$, and the numerical calculation leads to an overflow error. Therefore, we integrated the differential equation only up to $z = 1 \pm 10^{-15}$ to ensure that the numerical calculations remained within the limits set by the machine epsilon. The differential Equation (3) was solved separately on either side of $z = 1$.

None of the results presented in this paper relies on the exactness of our numerical calculations. The results of the numerical calculations have only heuristic value and serve only as plausibility arguments for the conjectures we propose.

Funding: This research was supported by the Office of Naval Research grant number N-00014-96-1-0281.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: I thank Alan Guth for the many helpful conversations while this work was being done [27]. The research reported in the paper was done mostly at the Center for Theoretical Physics (CTP), MIT. I thank CTP for its support. I also thank Sergiu Moroianu for helpful conversations. I thank the three anonymous referees for their many helpful comments, questions and suggestions, which have improved both the content and the presentation of the paper.

Conflicts of Interest: The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A. Solutions of Hypergeometric Differential Equation

This appendix presents the construction of the two linearly independent solutions of the hypergeometric equation

$$x(x-1)\varphi_{xx} + ((a+b+1)x - c)\varphi_x + ab\varphi = 0 \tag{A1}$$

in a neighborhood of $x = 0$ for the special case in which $a + b = 1$, $ab = \alpha$, $c = 1$ or

$$a = \frac{1+\sqrt{1-4\alpha}}{2}, \quad b = \frac{1-\sqrt{1-4\alpha}}{2}, \quad c = 1 \tag{A2}$$

For the special values of a, b, c shown in (A2), Equation (A1) becomes

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha\varphi = 0 \tag{A3}$$

The first solution of Equation (A1), which that is analytic at $x = 0$, is given by the hypergeometric series [28] (§14.2, p. 283)

$$F(x;a,b,c) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n}\, x^n, \qquad (p)_n := a(a+1)\ldots(a+n-1), \quad (p)_0 = 1$$

which converges for $|x| < 1$. We denote the solution of Equation (7) that is analytic at $x = 0$ as

$$\varphi_1(x) := F\left(x; \frac{1+\sqrt{1-4\alpha}}{2}, \frac{1-\sqrt{1-4\alpha}}{2}, 1\right) \tag{A4}$$

We construct the second solution of Equation (7), denoted $\varphi_2(x)$, as follows (see [28] (§10.32)). The $\varphi_2(x)$ we construct will be linearly independent with respect to $\varphi_1(x)$ and analytic in a punctured interval $P_\delta := [-\epsilon, \epsilon]\backslash\{0\}$ for some $\epsilon > 0$.

In a punctured interval of $x = 0$ that excludes $x = 0$ we write $\varphi_2(x)$ as

$$\varphi_2(x) = v(x)\,\varphi_1(x) \tag{A5}$$

where $v(x)$ is a non-constant function of x. Substituting (A5) into Equation (A3), noting that φ_1 satisfies Equation (A3) and setting $w(x) := v'(x)$, we get, for the special values of a, b, c shown in (A2),

$$[\,x(x-1)\,\varphi_1\,]w' + \left[\,2x(x-1)\varphi_1' + ((2x-1)\varphi_1\,\right]w = 0 \tag{A6}$$

The differential Equation (A6) is singular at $x = 0$. Let z^* denote the zero of $\varphi_1(x)$ that is closest to the origin. We choose a δ such that $0 < \delta < \min(z^*, 1)$. In the punctured interval $(-\delta, \delta) \setminus \{0\}$ the solution of Equation (A6) is

$$w(x) = \frac{c}{x(1-x)\varphi_1^2(x)}, \qquad c \neq 0 \tag{A7}$$

where c is a constant. If $c = 0$, then $v(x)$ would be a constant and $\varphi_2(x)$ would not be linearly independent of $\varphi_1(x)$. Since we are constructing the function $v(x)$, we can set $c = 1$.

In the interval $I_\delta := (-\delta, \delta)$, $\left[\,(1-x)\,\varphi_1^2(x)\,\right]^{-1}$ is an analytic function. Further, $\varphi_1(0) = 1$. Therefore we can expand the function $\left[\,(1-x)\,\varphi_1^2(x)\,\right]^{-1}$ in Taylor series around $x = 0$ within the interval I_δ as

$$\frac{1}{(1-x)\,\varphi_1^2(x)} = 1 + \sum_{n=1}^{\infty} q_n x^n, := 1 + q(x) \qquad -\delta < x < \delta \tag{A8}$$

where $q(x)$, the function represented by the series, is analytic in I_δ.

From Equations (A7) and (A8) we have (recalling that we set $c = 1$)

$$w(x) = v'(x) = \frac{1}{x} + \sum_{n=1}^{\infty} q_n x^{n-1} \tag{A9}$$

Since the series shown in (A9) converges in I_δ, it converges uniformly and absolutely in a compact sub-interval $K := [-\epsilon, \epsilon] \subset (-\delta, \delta)$, and therefore the series can be integrated term

by term in K. Since v' is singular at $x = 0$, we consider the differential equation separately in the two sub-intervals $[-\epsilon, 0)$ and $(0, \epsilon]$. Integrating (A9) we obtain

$$v(x) = \begin{cases} \log(|x|) + \tilde{q}(x) + c_1, & x \in [-\epsilon, 0) \\ \\ \log(|x|) + \tilde{q}(x) + c_2, & x \in (0, \epsilon] \end{cases}, \qquad \tilde{q}(x) = \sum_{n=1}^{\infty} \left(\frac{q_n}{n}\right) x^n \tag{A10}$$

Since we are interested in $\varphi_2(x)$, which is linearly independent of $\varphi_1(x)$, we can ignore the constants c_1, c_2 in Equation (A10) and we have

$$\varphi_2(x) = \log(|x|)\, \varphi_1(x) + h(x), \qquad x \in [-\epsilon, \epsilon] \setminus \{0\} \tag{A11}$$

where $h(x) = \tilde{q}(x)\, \varphi_1(x)$, and $\tilde{q}(x)$ is derived from the function $1/(1-x)\, \varphi_1^2(x)$ as described above. Finally, the condition for $\varphi_2(x)$ to be linearly independent of $\varphi_1(x)$ is that the Wronskian $\varphi_1 \varphi_2' - \varphi_1' \varphi_2 = v' \varphi_1^2 \neq 0$, which holds if $c \neq 0$ in Equation (A7), as mentioned above.

Appendix B. Neighborhood of the Horizon

Proof of Lemma 1. Rewriting Equation (9) and using Mean Value Theorem we have for $x \in (-\epsilon, \epsilon) \setminus \{0\}$

$$\varphi_b'(x) = -\left[\frac{1}{x(x-1)}\right] \int_0^x \alpha \, \sin(\varphi_b(y))\, dy = \frac{\alpha \, \sin(\varphi_b(x'))}{1-x} \tag{A12}$$

where $0 < |x'| < |x|$. From (A12) we have

$$\lim_{x \to 0} \varphi_b'(x) = \alpha \, \sin(\varphi_b(0))$$

□

We need the following Lemmas A1–A4 to prove Lemmas 2 and 3. Lemma A1 is standard lore, and proofs for it can be found in many books on advanced calculus. We state and prove it below to make it easier to reference it repeatedly in later arguments.

Lemma A1. *If a function $v(x)$ is continuous in $(-\epsilon, \epsilon)\setminus\{0\}$, not defined at $x = 0$ and $\lim_{x\to 0} v(x) = L$ exists, then defining $v(0) := L$ makes $v(x)$ well defined and continuous in $(-\epsilon, \epsilon)$.*

Secondly, if $v(y)$ is a continuous bounded function in the interval $(-\epsilon, \epsilon)$, then

$$I(x) := \int_0^x v(y)\, dy$$

is a continuous function at every $x \in (-\epsilon, \epsilon)$.

Proof of Lemma A1. Since L is the limit of $v(x)$ as $x \to 0$, given a $\delta > 0$, there exists a $\beta > 0$ such that, for $|x| < \beta$, $|L - v(x)| = |v(0) - v(x)| < \delta$.

For any given $\delta > 0$ we can find a $\beta > 0$ such that, for all z satisfying $|x - z| < \beta$, $|I(x) - I(z)| < \delta$ because $\left|\int_z^x v(y)\, dy\right| < M|x - z| < M\beta$, where $|v(y)| < M$ for $y \in (-\epsilon, \epsilon)$. The claim follows by choosing $0 < \beta < \delta/M$. □

Lemma A2. *Let $\varphi_1(x)$ and $\varphi_2(x)$ be the two linearly independent solutions of the hypergeometric equation*

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha\varphi = 0 \tag{A13}$$

shown in Equations (A4) *and* (A11). *Let* $W(x) = \varphi_1 \varphi_2' - \varphi_1' \varphi_2$ *denote their Wronskian. Define* $g(x) := (x-1)\, x\, W(x)$. *Let the function* $h(x)$ *in Equation* (A11) *be analytic in* $(-\rho_h, \rho_h)$, $0 < \rho_h < 1$.

Then, $g(0) = -1$, *and there exists a* $0 < \delta < \rho_h$, *and, for* $x \in (-\delta, \delta)$, $g(x)$ *is an analytic function and* $|g(x)| \geqslant \frac{1}{2}$.

Proof. Differentiating φ_1 and φ_2, $g(x)$ can be written as

$$g(x) = (x-1)\left[\, \varphi_1^2(x) + x\, \varphi_1(x)\, h'(x) - x\, \varphi_1'(x)\, h(x) \,\right]$$

We note that $x\varphi_2'(x)$ contains a term of the form $q(x) := x\, \dfrac{d\log(|x|)}{dx}\, \varphi_1^2(x)$, which has a *removable singularity* at $x = 0$. By defining $q(0) := 1$, the function q can be made analytic throughout $(-\delta, \delta)$, including at $x = 0$. See [29] (§68) for details. Since $\varphi_1, \varphi_1', h, h'$ are analytic in $(-\rho_h, \rho_h)$, it follows that $g(x)$ is analytic in the same neighborhood. Noting that $\varphi_1(0) = 1$ we conclude that $g(0) = -1$. Since $g(0) = -1$ and g is a continuous function in $(-\rho_h, \rho_h)$ there exists an interval $(-\epsilon, \epsilon)$, $0 < \epsilon < \rho_h < 1$ in which $|g(x)| \geqslant \frac{1}{2}$. □

Lemma A3. *Define*

$$V(x;\varphi) := \varphi_2(x) \int_0^x \left[\, \frac{\varphi_1(y)\, f(\varphi(y))}{y\,(y-1)\, W(y)} \,\right] dy - \varphi_1(x) \int_0^x \left[\, \frac{\varphi_2(y)\, f(\varphi(y))}{y\,(y-1)\, W(y)} \,\right] dy \tag{A14}$$

where $f(\varphi(y)) := \alpha(\varphi(y) - \sin(\varphi(y)))$. *Consider an interval* $(-\epsilon, \epsilon)$, $0 < \epsilon < \rho_h < 1$ *in which* $|y\,(y-1)\, W(y)| \geqslant \frac{1}{2}$ (*see Lemma A2*). *If* $\varphi(y)$ *is a continuous bounded function in* $(-\epsilon, \epsilon)$, *then, in the interval* $(-\epsilon, \epsilon)$, $V(x;\varphi)$ *can be written as*

$$V(x;\varphi) = h(x) \int_0^x \varphi_1(y)\, q(y)\, dy - \varphi_1(x) \int_0^x h(y)\, q(y)\, dy + \varphi_1(x) \int_0^x \frac{1}{y} \left[\int_0^y \varphi_1(u)\, q(u)\, du \right] dy$$

where

$$q(y) := \frac{\alpha(\varphi(y) - \sin(\varphi(y)))}{y\,(y-1)\, W(y)}$$

Proof. First, we observe that $q(y)$ is a continuous bounded function of y in $(-\epsilon, \epsilon)$. Using the form of φ_2 shown in Equation (A11) and performing the differentiation and partial integration, we obtain

$$\begin{aligned} V(x;\varphi) = {} & \{\varphi_1(x)\, \log(|x|) + h(x)\} \textstyle\int_0^x \varphi_1(y)\, q(y)\, dy - \varphi_1(x) \left\{ \log(|y|) \int_0^y \varphi_1(u)\, q(u)\, du \right\} \Big|_0^x \\ & + \varphi_1(x) \textstyle\int_0^x \frac{1}{y} \left[\int_0^y \varphi_1(u)\, q(u)\, du \right] dy - \varphi_1(x) \int_0^x h(y)\, q(y)\, dy \end{aligned} \tag{A15}$$

If $v(x)$ is a continuous function in an interval $(-\epsilon, \epsilon)$, then, using the Mean Value Theorem [30] (Chapter 5), we have

$$\lim_{x \to 0} \log(|x|) \int_0^x v(y)\, dy = 0 \tag{A16}$$

The claim in the lemma follows from applying Equation (A16) in Equation (A15). □

Lemma A4. *Let* $\tilde{\varphi}(x)$ *be an analytic function in the interval* $(-\rho, \rho)$ *for some* $\rho > 0$. *Then*

$$V(x;\tilde{\varphi}) = \varphi_2(x) \int_0^x \frac{\varphi_1(y)\, f(\tilde{\varphi}(y))}{(y-1)\, y\, W(y)}\, dy - \varphi_1(x) \int_0^x \frac{\varphi_2(y)\, f(\tilde{\varphi}(y))}{(y-1)\, y\, W(y)}\, dy$$

is an analytic function in the interval $(-\delta, \delta)$ *for some* $0 < \delta < \min(\rho, \rho_h)$.

Proof. From Lemma A2 we know that there exists an interval $(-\delta, \delta)$, $0 < \delta < \rho_h < 1$ in which $(y-1)\, y\, W(y)$ is analytic and nonzero. Choosing a δ, satisfying $0 < \delta < \min(\rho, \rho_h)$, we conclude that

$$q(y) := \frac{f(\tilde{\varphi}(y))}{(y-1)\, y\, W(y)}$$

is analytic in $(-\delta, \delta)$. We can expand $\varphi_1(x)\, q(x)$ in Taylor series around $x = 0$ within its domain of analyticity as

$$\varphi_1(x)\, q(x) = \sum_{n=0}^{\infty} \left(\frac{a_n}{n!}\right) x^n, \qquad x \in (-\delta, \delta) \tag{A17}$$

Since a power series converges uniformly and absolutely in a sub-interval of its interval of convergence [30] (Theorem 9, Chapter 11), we can integrate the power series term by term to get the integral of the power series [30] (Theorem 7, Chapter 11) within a sub-interval $[-\delta', \delta'] \subset (-\delta, \delta)$. Thus,

$$\frac{1}{x} \int_0^x \varphi_1(x)\, q(x)\, dy = \sum_{n=0}^{\infty} \left[\frac{a_n}{(n+1)!}\right] x^n, \qquad x \in [-\delta', \delta'] \tag{A18}$$

Since series (A17) converges in $[-\delta', \delta']$, it satisfies the Cauchy convergence criterion, and, for any $\epsilon > 0$, we can find an $n_0 > 1$ such that, for $m, n > n_0$ and $x \in [-\delta', \delta']$,

$$\left| \sum_{k=m}^{n} \left[\frac{a_k}{(k+1)!}\right] x^k \right| < \left| \sum_{k=m}^{n} \left[\frac{a_k}{k!}\right] x^k \right| < \epsilon$$

showing that the series (A18) converges in $[-\delta', \delta']$ by the Cauchy criterion.

Next, we define a function

$$p(x) := \sum_{n=0}^{\infty} \left[\frac{a_n}{n+1}\right] \frac{x^n}{n!}, \qquad x \in [-\delta', \delta'], \tag{A19}$$

If a power series S converges with an interval of convergence I (e.g., $(-\delta, \delta)$), then we know that the series S' obtained by differentiating S term by term, as well as the series $\tilde{S}$ obtained by integrating S term by term, also converge and have the same interval of convergence I (see [30] (Chapter 11) for a proof of the claim about S'; the claim about $\tilde{S}$ can be established using the comparison test). Therefore, we have

$$\frac{d^k\, p(0)}{dx^k} = \frac{a_k}{k+1}$$

and (A19) is a Taylor expansion of $p(x)$ about $x = 0$, making $p(x)$ an analytic function in $[-\delta', \delta']$. That is,

$$\frac{1}{x} \int_0^x \varphi_1(y)\, q(y)\, dy$$

is an analytic function in $[-\delta', \delta']$. The claim in the lemma follows from Lemma A3. □

Proof of Lemma 2. We define

$$A_j(x; \varphi) := \int_0^x \frac{\varphi_j(y)\, f(\varphi(y)}{(y-1)\, y\, W(y)}\, dy, \qquad j = 1, 2$$

Then Equation (13) can be written as

$$\varphi(x) = \varphi_2(x)A_1(x;\varphi) - \varphi_1(x)A_2(x;\varphi) + \gamma\varphi_1(x)$$

and

$$\begin{aligned}\varphi_b' &= \varphi_2' A_1(x;\varphi_b) - \varphi_1' A_2(x;\varphi_b) + \gamma\varphi_1',\\ \varphi_b'' &= \varphi_2'' A_1(x;\varphi_b) - \varphi_1'' A_2(x;\varphi_b) + \gamma\varphi_1'' + \frac{f(\varphi_b)}{x(x-1)}\end{aligned}$$

Defining the differential operator

$$D := x(x-1)\frac{d^2}{dx^2} + (2x-1)\frac{d}{dx} + \alpha$$

and differentiating at $x \in (-\epsilon,\epsilon)\setminus\{0\}$ we obtain

$$D[\varphi_b(x)] = D[\varphi_2(x)]A_1(x;\varphi_b) - D[\varphi_1(x)]\{A_2(x;\varphi_b) - \gamma\} + f(\varphi_b(x)) \tag{A20}$$

φ_1 and φ_2 being solutions of the hypergeometric equation $D[\varphi_2] = D[\varphi_1] = 0$, and the first two terms vanish and we get the equation

$$D[\varphi_b(x)] = f(\varphi_b(x)), \qquad x \in (-\epsilon,\epsilon)\setminus\{0\} \tag{A21}$$

which is Equation (14). Defining

$$F(x) := x(x-1)\varphi_b''(x) + (2x-1)\varphi_b'(x) + \alpha\sin(\varphi_b(x))$$

Equation (A21) also shows that $F(x) = 0$ for $x \in (-\epsilon,\epsilon)\setminus\{0\}$. Therefore,

$$\lim_{x\to 0} F(x) = 0$$

Therefore we can define $F(0) := 0$, and Equation (14) is satisfied for all $x \in (-\epsilon,\epsilon)$ by $\varphi_b(x)$.

Next, we show that φ_b' is well defined and continuous in $(-\epsilon,\epsilon)$. As shown above, φ_b satisfies Equation (14), which can be rewritten as

$$\varphi_b'(x) = -\left[\frac{1}{x(x-1)}\right]\int_0^x \alpha\,\sin(\varphi_b(y))\,dy \tag{A22}$$

From Lemma A1 we conclude that $\varphi_b'(x)$ is well defined and continuous at every $x \in (-\epsilon,\epsilon)\setminus\{0\}$. Further recall that, if a function $v(x)$ is continuous in $(-\epsilon,\epsilon)\setminus\{0\}$ and $\lim_{x\to 0} v(x) = L$, then defining $v(0) = L$ makes $v(x)$ well defined and continuous. (Given a $\delta > 0$, there exists a $0 < \beta < \epsilon$, and, for $|x| < \beta$, $\delta > |L - v(x)| = |v(0) - v(x)|$.) Thus, to establish the continuity of $\varphi_b'(x)$ over $(-\epsilon,\epsilon)$, it is sufficient to show that $\lim_{x\to 0}\varphi'(x)$ exists. Using Mean Value Theorem we have

$$\begin{aligned}\lim_{x\to 0}\varphi_b'(x) &= \lim_{x\to 0}\left\{-\left[\frac{1}{x(x-1)}\right]\int_0^x \alpha\,\sin(\varphi_b(y))\,dy\right\}\\ = \lim_{x\to 0}\left\{-\left[\frac{1}{x(x-1)}\right]x\,\alpha\,\sin(\varphi_b(x')),\right\} &= \alpha\,\sin(\varphi_b(0)) \qquad x' \in (0,x)\end{aligned}$$

Thus we define

$$\varphi_b'(0) := \lim_{x\to 0}\varphi_b'(x) = \alpha\,\sin(\varphi_b(0))$$

and with that definition $\varphi_b'(x)$ is well defined and continuous for all $x \in (-\epsilon,\epsilon)$. □

Proof of Lemma 3. Using the notation in Lemma A3 we can write Equation (15) as

$$\varphi(x) = V(x;\varphi) + \gamma\varphi_1(x) \tag{A23}$$

We show that Equation (A23) has the required solution by showing that the following recurrence equation converges uniformly to a fixpoint, starting with an analytic function $\varphi^{(0)}(x)$.

$$\varphi^{(n)}(x) = V(x;\varphi^{(n-1)}) + \gamma\varphi_1(x) \tag{A24}$$

From Lemma A4 we know that, if $\varphi^{(n-1)}(x)$ is analytic in some interval $(-\rho,\rho)$, where $0 < \rho < \rho_h < 1$, then $\varphi^{(n)}(x)$ is also analytic in the same interval. It then follows that, if we start with a function $\varphi^{(0)}(x)$ that is analytic in some $(-\rho,\rho)$, with $0 < \rho < \rho_h < 1$, then all of the functions $\varphi^{(n)}(x)$, $n = 1, 2, \ldots$ generated by the recurrence equation Equation (A24) will also be analytic in $(-\rho,\rho)$. Since $\varphi_1(x)$ is analytic in $(-1,1)$ we choose our $\varphi^{(0)}(x) := \varphi_1(x)$.

Since the sequence of functions $\varphi^{(n)}(x)$, $n = 0, 1, 2, \ldots$ are analytic in $(-\rho,\rho)$, they are bounded in every closed sub-interval $[-\epsilon,\epsilon] \subset (-\rho,\rho)$, $0 < \epsilon < \rho$. We fix such an ϵ and define

$$M_n(\epsilon) := \max_{-\epsilon\leqslant x\leqslant\epsilon}\left|\varphi^{(n)}(x) - \varphi^{(n-1)}(x)\right| = \max_{-\epsilon\leqslant x\leqslant\epsilon}\left|V(x;\varphi^{(n-1)}) - V(x;\varphi^{(n-2)})\right| \tag{A25}$$

Further, $h(x)$ and $\varphi_1(x)$ and $1/x(x-1)W(x)$ are analytic functions in $(-\rho,\rho)$, and, if ρ is chosen to be sufficiently small as we assume it is, then, from Lemma A2, we know that $1/x(x-1)W(x)$ is analytic in $(-\rho,\rho)$ as well. Therefore, $h(x)$, $\varphi_1(x)$ and $1/x(x-1)W(x)$ are also bounded in $[-\epsilon,\epsilon]$, and we define

$$C_h(\epsilon) = \max_{-\epsilon\leqslant x\leqslant\epsilon}|h(x)|,\ C_p(\epsilon) = \max_{-\epsilon\leqslant x\leqslant\epsilon}|\varphi_1(x)|,\ C_w(\epsilon) = \max_{-\epsilon\leqslant x\leqslant\epsilon}\left|\frac{1}{(x-1)\,x\,W(x)}\right| \tag{A26}$$

We note that

$$\begin{aligned}\left|f(\varphi^{(n)}) - f(\varphi^{(n-1)})\right| &= \alpha\left|\varphi^{(n)} - \sin(\varphi^{(n)}) - \varphi^{(n-1)} + \sin(\varphi^{(n-1)})\right| \\ &\leqslant \alpha\left|\varphi^{(n)} - \varphi^{(n-1)}\right| + \alpha\left|\sin(\varphi^{(n)}) - \sin(\varphi^{(n-1)})\right| \\ &= \alpha\left|\varphi^{(n)} - \varphi^{(n-1)}\right| + \alpha\left|\cos(\varphi^{(n,n-1)})\left[\,\varphi^{(n)} - \varphi^{(n-1)}\,\right]\right| \\ &\leqslant 2\alpha\left|\varphi^{(n)} - \varphi^{(n-1)}\right|\end{aligned} \tag{A27}$$

In the third step, we have used the Mean Value Theorem, and $\varphi^{(n,n-1)} = (1-\mu)\,\varphi^{(n)} + \mu\,\varphi^{(n-1)}$ for some $0 \leqslant \mu \leqslant 1$.

Using Lemma A3 and Equations (A25)–(A27), we have

$$\begin{aligned}M_n(\epsilon) &\leqslant 2C_h(\epsilon)C_p(\epsilon)C_w(\epsilon)\left[\,2\alpha M_{n-1}(\epsilon)\,\right]\epsilon + C_p(\epsilon)^2C_w(\epsilon)\left[\,2\alpha M_{n-1}(\epsilon)\,\right]\epsilon \\ &= \left\{\left[\,4C_h(\epsilon)C_p(\epsilon)C_w(\epsilon)\,\alpha + 2C_p(\epsilon)^2C_w(\epsilon)\,\alpha\,\right]\epsilon\right\}M_{n-1}\end{aligned}$$

As ϵ decreases, so do $C_h(\epsilon), C_p(\epsilon), C_w(\epsilon)$. Therefore

$$\lim_{\epsilon\to 0}\left[\,4C_h(\epsilon)C_p(\epsilon)C_w(\epsilon)\,\alpha + 2C_p(\epsilon)^2C_w(\epsilon)\,\alpha\,\right]\epsilon = 0$$

which means that, if we choose a small $\epsilon^* > 0$, then

$$\left[\,4C_h(\epsilon^*)C_p(\epsilon^*)C_w(\epsilon^*)\,\alpha + 2C_p(\epsilon^*)^2C_w(\epsilon^*)\,\alpha\,\right]\epsilon^* < 1$$

and, as $n \to \infty$, $M_n(\epsilon^*) \to 0$, which means that, in the interval $[-\epsilon^*, \epsilon^*]$, the sequence of analytic functions $\varphi^{(0)}, \varphi^{(1)}, \ldots$ converges uniformly to a fixpoint of the recurrence relation Equation (A24).

Since analytic functions are continuous, the uniform limit of a sequence of analytic functions over $[-\epsilon^*, \epsilon^*]$ is a continuous function over $[-\epsilon^*, \epsilon^*]$ [31] (§20.3, Theorem III). Further, since all of the functions $\varphi^{(0)}, \varphi^{(1)}, \ldots$ are bounded over $[-\epsilon^*, \epsilon^*]$, so is their uniform limit φ since, for any $\delta > 0$, we can find a n_0 such that, for all $n \geqslant n_0$

$$\max_{-\epsilon^* \leqslant x \leqslant \epsilon^*} \left| \varphi^{(n_0)}(x) - \varphi(x) \right| < \delta,$$

The boundedness of $\varphi(x)$ over $[-\epsilon^*, \epsilon^*]$ then follows from the boundedness of $\varphi^{(n_0)}$ over the interval.

Thus, we conclude that the uniform limit of the sequence of analytic functions bounded over $[-\epsilon^*, \epsilon^*]$ yields a continuous bounded function, which being a fixpoint of Equation (A24) is a solution of Equation (15). Since we have demonstrated the existence of a solution of Equation (15) for every value of $\gamma \in \mathbb{R}$, we have proved the existence of a 1-parameter family of solutions for Equation (15). □

Appendix C. Beyond the Horizon

Proof of Lemma 4. Consider the function

$$V(z; \varphi, \varphi_z) := \frac{\varphi_z^2}{2} + \frac{\alpha(1 - \cos(\varphi))}{z^2 - 1}.$$

Differentiating V and using Equation (16) we get

$$\frac{dV}{dz} = \frac{\partial V}{\partial \varphi} \cdot \varphi_z + \frac{\partial V}{\partial \varphi_z} \cdot \varphi_{zz} + \frac{\partial V}{\partial z} = -\frac{2z}{z^2 - 1} \left[\frac{\varphi_z^2}{2} + V \right]. \tag{A28}$$

Using the boundary condition (17), we see that

$$B(z; \varphi, \varphi_z) := (z^2 - 1) V(z; \varphi, \varphi_z) = \frac{z^2 - 1}{2} \cdot \varphi_z^2 + \alpha \cdot (1 - \cos(\varphi))$$

is well defined at $z = 1$. Using Equations (A28) and (16) we have

$$\frac{dB}{dz} = -z \varphi_z^2. \tag{A29}$$

which shows that $B(z; \varphi, \varphi_z)$ is a monotonically decreasing function of z for $z > 1$. Since $0 < \varphi(1) < \pi$, we have, for $z > 1$,

$$B(z; \varphi, \varphi_z) \leqslant B(1; \varphi, \varphi_z) = \alpha \cdot (1 - \cos(\varphi(1))) < 2\alpha \tag{A30}$$

Next, we show that $-\pi < \varphi(z) < \pi$ at every finite $z > 1$. Arguing by contradiction, let us assume that $\varphi(z_*) = \pi$ (or $\varphi(z_*) = -\pi$) at some finite $z_* > 1$. Then

$$B(z^*; \varphi, \varphi_z) = \frac{(z_*^2 - 1) \varphi_z^2(z_*)}{2} + \alpha \cdot (1 - \cos(\varphi(z_*))) \geqslant 2\alpha$$

which contradicts (A30). Therefore, $-\pi < \varphi(z) < \pi$ for all $z > 1$.

Finally we show that $\varphi(z)$ vanishes as $z \to \infty$. For $z \gg 1$, Equation (16) behaves effectively as

$$z^2 \varphi_{zz} + 2z \varphi_z + \alpha \sin(\varphi) = 0 \tag{A31}$$

Defining $t := \ln z$ and $X(t) := \varphi(e^t)$, Equation (A31) can be rewritten as

$$X_{tt}(t) + X_t(t) + \alpha \sin(X(t)) = 0 \tag{A32}$$

If we interpret $X(t)$ as the position of a particle of unit mass at time t, then Equation (A32) describes motion of the particle in a potential $\alpha(1 - \cos(X))$ in the presence of frictional force. If the particle starts at $0 \leqslant X(0) = \gamma < \pi$, then the preceding argument shows that $-\pi < X(t) < \pi$ for all $t > 0$. Owing to the frictional force the particle eventually comes to rest at the local minimum of the potential, $\varphi = 0$, as $t, z \to \infty$. □

Appendix D. Large Solitons

Lemma A5. *Let $\psi(t)$ be a solution of*

$$\ddot{\psi} = \alpha e^{-t} \psi; \qquad \alpha \in \mathbf{R} \tag{A33}$$

that satisfies the boundary condition $\psi(t_0) \neq 0$ for some $t_0 > 0$. Then

$$\lim_{t \to \infty} \psi(t) \neq 0$$

Proof. The lemma is trivial when $\alpha = 0$. We consider the two cases $\alpha < 0$ and $\alpha > 0$ separately.

Let $\alpha < 0$. Define

$$V(\psi, \dot{\psi}, t) := \frac{\dot{\psi}^2}{2} + \frac{|\alpha|\, \psi^2\, e^{-t}}{2}$$

Using Equation (A33) we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial \dot{\psi}} \cdot \ddot{\psi} + \frac{\partial V}{\partial \psi} \cdot \dot{\psi} + \frac{\partial V}{\partial t} = -V + \frac{\dot{\psi}^2}{2}$$

Multiplying both sides with e^t and rearranging we get

$$\frac{d\left[\, e^t\, V\right]}{dt} = \frac{e^t\, \dot{\psi}^2}{2}$$

which shows that $e^t\, V$ increases monotonically with t. Therefore, if ψ oscillates about zero, with $\dot{\psi}$ vanishing at $t_1 < t_2 < \ldots$, then

$$|\psi(t_1)| < |\psi(t_2)| < \ldots \tag{A34}$$

That is, if ψ oscillates about zero, then the oscillation is divergent. We show that, if $\psi(t_0) \neq 0$ and $\lim_{t\to\infty} \psi(t) = 0$, then ψ must oscillate about zero. Then, from (A34), we obtain a contradiction, which will prove the claim.

First we consider the case $\psi(t_0) > 0$. If $\lim_{t\to\infty} \psi(t) = 0$, then there exists a $\tilde{t} \in [t_0, \infty)$ at which $\psi(\tilde{t}) > 0$ and $\dot{\psi}(\tilde{t}) < 0$. Since $\alpha < 0$, Equation (A33) implies that $\ddot{\psi} < 0$ when $\psi > 0$. Therefore, $\psi(t') = 0$ for some $\tilde{t} < t' < \tilde{t} + \frac{\psi(\tilde{t})}{|\dot{\psi}(\tilde{t})|}$. Further, $|\dot{\psi}(t')| > |\dot{\psi}(t_0)|$. If $\lim_{t\to\infty} \psi(t) = 0$, then we must have $\psi(t'') < 0$, $\dot{\psi}(t'') > 0$ at some $t'' > t'$ and subsequently $\psi(t''') = 0$ for some $t''' > t''$. Again, $|\dot{\psi}(t''')| > |\dot{\psi}(t'')|$. Repeating the above argument we conclude that ψ must oscillate about zero, and, from (A34), we conclude that the oscillations are divergent, which contradicts the assumption that ψ approaches zero asymptotically. A similar argument shows that ψ oscillates if $\psi(t_0) < 0$ as well, completing the proof of the lemma for $\alpha < 0$.

Let $\alpha > 0$. We will restrict the following argument to the case $\psi(t_0) > 0$. The argument for $\psi(t_0) < 0$ is similar to the one presented below.

We start by showing that, if ψ is to vanish asymptotically, then $\dot{\psi}(t_0) < 0$, and, for $t > t_0$, $\psi(t)$ must be monotonic. If $\dot{\psi}(t_0) \geqslant 0$, then Equation (A33) implies that $\dot{\psi}(t) > 0$ for all $t > t_0$, contradicting the assumption that ψ vanishes asymptotically. To prove the second claim, we argue by contradiction. If possible, let $\psi(t') = 0$ for some $t' > t_0$. Further assume that we have chosen the smallest $t' > t_0$ at which ψ vanishes. Clearly $\dot{\psi}(t') \leqslant 0$. If $\dot{\psi}(t') = 0$, the solution to Equation (A33) with initial conditions $\psi(t') = \dot{\psi}(t') = 0$ would not be unique in the neighborhood of t' (since $\psi \equiv 0$ is a solution). On the other hand, if $\dot{\psi}(t') < 0$, then $\ddot{\psi}(t) < 0$ for all $t > t'$, making it impossible for $\psi \to 0$ as $t \to \infty$. Therefore, we conclude that t' cannot exist and ψ must decay to zero monotonically.

Next we show that, for all $t > t_0$,

$$|\dot{\psi}(t)| \leqslant \alpha\, e^{-t}\, |\psi(t)| \tag{A35}$$

Using Equation (A33) we have, for any $t' > t$,

$$\psi(t') = \int_t^{t'} \int_t^{\tau} \alpha\, e^{-\lambda}\, \psi(\lambda)\, d\lambda\, d\tau + (t' - t)\, \dot{\psi}(t) + \psi(t)$$

If $\psi(t_0) > 0$, then, using the monotonic decay of ψ established above, for any $t' > t$,

$$\begin{aligned} 0 \;<\; & \psi(t') < \alpha\, \psi(t) \int_t^{t'} \int_t^{\tau} e^{-\lambda}\, d\lambda\, d\tau + (t' - t)\, \dot{\psi}(t) + \psi(t) \\ \leqslant\; & \alpha\, \psi(t)\, e^{-t} \int_t^{t'} d\tau + (t' - t)\dot{\psi}(t) + \psi(t) \\ <\; & \left\{\alpha\, \psi(t)\, e^{-t} + \dot{\psi}(t)\right\}(t' - t) + \psi(t) \end{aligned}$$

Recalling that $\dot{\psi}(t) < 0$, the claim follows.

Integrating inequality (A35) we get

$$-\int_{\psi(t_0)}^{\psi(t)} \frac{d\psi}{\psi} \leqslant \alpha \int_{t_0}^{t} e^{-\tau}\, d\tau$$

or

$$\psi(t) \geqslant \left\{\psi(t_0) e^{-\alpha\, e^{-t_0}}\right\} e^{\alpha\, e^{-t}} \tag{A36}$$

from which we obtain a lower bound

$$|\psi(t)| \geqslant \left\{|\psi(t_0)| e^{-\alpha\, e^{-t_0}}\right\}$$

for all $t > t_0$, contradicting the assumption that ψ vanishes asymptotically. The contradiction completes the proof.

The preceding argument can be used to establish inequalities (A35) and (A36) even when $\psi(t_0) < 0$. Therefore a solution satisfying $\psi(t_0) < 0$ cannot vanish asymptotically either. □

Lemma A6. *Let φ be a solution of*

$$(z^2 - 1)\varphi_{zz} + 2z\varphi_z + \alpha \sin \varphi = 0 \tag{A37}$$

that satisfies the boundary condition $\varphi(1) = n\pi$. Then $\varphi(z) \equiv n\pi$.

Proof. Clearly $\varphi(z) \equiv n\pi$ is a solution of Equation (A37) that satisfies the boundary condition $\varphi(1) = n\pi$. Therefore all we need to show is that $\varphi(z) \equiv n\pi$ is the only solution satisfying the given boundary condition. If $\varphi(z)$ is a solution of Equation (A38), then so is $\varphi(z) + 2m\pi$, where m is an integer. Therefore it is sufficient to establish uniqueness of the solution $\varphi(z) \equiv n\pi$ for $n = 1$ and $n = 0$.

It is convenient to work with the hypergeometric coordinate $x = \frac{1-z}{2}$, in terms of which Equation (A37) can be rewritten as

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha \sin\varphi = 0. \tag{A38}$$

The singularity at $z = 1$ in Equation (A37) corresponds to the singularity at $x = 0$ in Equation (A38). We prove the uniqueness of the solution $\varphi \equiv n\pi$ in a neighborhood $(-\epsilon, \epsilon)$ for some sufficiently small $\epsilon > 0$ by showing that, if $\varphi(\epsilon) \neq 0$ (resp. $\varphi(-\epsilon) \neq 0$), then $\varphi(0) \neq 0$.

First we consider the limit $x \to 0^+$. It is convenient to change the coordinate to $w = -\log(x)$. As $x \to 0^+$, $w \to \infty$. In terms of variable w, Equation (A38) can be rewritten as

$$\frac{d}{dw}\left[(1 - e^{-w})\frac{d\varphi}{dw}\right] = \alpha\, e^{-w} \sin(\varphi)$$

If ϵ is chosen to be sufficiently small, then $w_\epsilon := -\log(\epsilon) \gg 1$ and the above differential equation effectively becomes

$$\frac{d^2\varphi}{dw^2} = \alpha e^{-w} \sin(\varphi) \tag{A39}$$

Arguing by contradiction, if $\varphi(w) \to 0$, as $w \to \infty$, then, for any $\delta > 0$, it is possible to find a $w_\delta > w_\epsilon$ such that $|\varphi(w)| < \delta$ for all $w \geqslant w_\delta$. If δ is chosen to be sufficiently small, then $\sin(\varphi) \approx \varphi$ for $w \geqslant w_\delta$, and the above differential equation effectively becomes

$$\frac{d^2\varphi}{dw^2} = \alpha\, e^{-w}\varphi \tag{A40}$$

If $\varphi(w_\epsilon) \neq 0$, then there can be no finite $w' \geqslant w_\epsilon$ such that $\varphi(w) = 0$ for all $w > w'$; if such a w' exists, then the solution to Equation (A40) would not be unique in an open neighborhood of the smallest such w'. Therefore, there must be a $w^* > w_\delta$ at which $\varphi(w^*) \neq 0$. Lemma A5 then implies that $\varphi \not\to 0$ as $w \to \infty$, contradicting our assumption. The preceding argument shows that, if $\varphi(x = 0) = 0$, then $\varphi(x) = 0$ for $0 \leqslant x \leqslant \epsilon$, establishing uniqueness in $[0, \epsilon)$ for $n = 0$.

Uniqueness in $[0, \epsilon)$ for $n = 1$ follows immediately by observing that, if φ is a solution of Equation (A39), then $\psi := \pi - \varphi$ satisfies

$$\frac{d^2\psi}{dw^2} = -\alpha e^{-w} \sin(\psi) \tag{A41}$$

If $\varphi(\epsilon) \neq \pi$ and $\varphi(w) \to \pi$ as $w \to \infty$, then $\psi(\epsilon) \neq 0$ and $\psi(w) \to 0$ as $w \to \infty$. Arguing as we did above, the Equation (A41) can be replaced by

$$\frac{d^2\psi}{dw^2} = -\alpha e^{-w}\psi$$

for $w > w_\delta$. As we showed above, we can find a $w^* > w_\delta$ at which $\psi(w^*) \neq 0$. Since $\psi(w^*) \neq 0$ for some finite w^*, Lemma A5 then implies that $\psi(w) \not\to 0$ as $w \to \infty$.

In order to investigate the limit $x \to 0^-$ it is convenient to set $w = -\log(-x)$. In terms of w, Equation (A38) can be rewritten as

$$\frac{d}{dw}\left[(1+e^{-w})\frac{d\varphi}{dw}\right] = -\alpha e^{-w}\sin(\varphi)$$

Choosing ϵ to be sufficiently small as we did before, $w_\epsilon := -\log(\epsilon)$ can be made sufficiently large, and, in the interval (w_ϵ, ∞), the above equation effectively becomes

$$\frac{d^2\varphi}{dw^2} = -\alpha e^{-w}\sin(\varphi)$$

The arguments presented for $n = 0$ and $n = 1$ over the interval $[0, \epsilon)$ establish uniqueness for $n = 1$ and $n = 0$, respectively, over the interval $(-\epsilon, 0]$. □

Lemma A7. *Let $\varphi(x)$ be a solution of*

$$x(x-1)\varphi_{xx}(x) + (2x-1)\varphi_x(x) + \alpha\,\sin(\varphi(x)) = 0 \tag{A42}$$

satisfying the boundary condition $0 < \varphi(0) < \pi$. If $\varphi(x) < \pi$ at every $x \in [0, a] \subseteq [0, \frac{1}{2}]$, then $\varphi(x) > 0$ at every $x \in [0, a]$.

The lemma asserts that φ cannot reach 0 before it reaches π. That is, the behavior shown in Figure A1 is forbidden by Equation (A42).

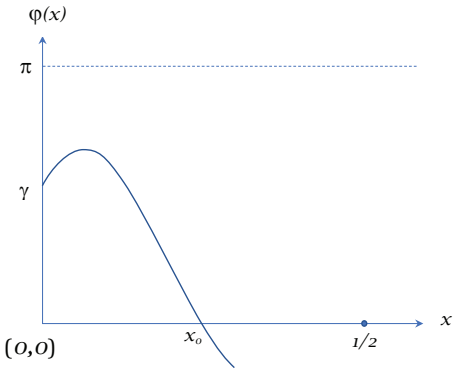


Figure A1. The behavior illustrated by the solid line is forbidden for a solution of Equation (A42). The dotted line represents $\varphi(x) \equiv \pi$.

Proof. Assume to the contrary that there is a $x_0 \in (0, \frac{1}{2}]$ at which $\varphi(x_0) = 0$ and $0 < \varphi(x) < \pi$ for all $x \in [0, x_0)$. Then, rewriting Equation (A42), we see that

$$\varphi'(x) = \frac{1}{x(1-x)}\int_0^x \alpha\,\sin(\varphi(y))\,dy > 0, \qquad x \in (0, x_0]$$

Since $\varphi' > 0$ at all $x \in (0, x_0]$ and $\varphi(0) > 0$, we conclude that $\varphi(x_0) > \varphi(0) > 0$, which contradicts the assumption that $\varphi(x_0) = 0$. The contradiction proves the claim. □

Restating Lemma A7 in static coordinates we obtain the following corollary.

Corollary A1. *Let $\varphi(z)$ be a solution of*

$$(z^2-1)\varphi_{zz}(z) + 2z\,\varphi_z(z) + \alpha, \sin(\varphi(z)) = 0 \tag{A43}$$

satisfying the boundary condition $0 < \varphi(1) < \pi$. If $\varphi(z) < \pi$ at every $z \in [a, 1] \subseteq [0, 1]$, then $\varphi(z) > 0$ at every $x \in [a, 1]$.

Lemma A8. *For every bounded solution of*

$$(z^2-1)\varphi'' + 2z\varphi' + \alpha \sin(\varphi(z)) = 0 \tag{A44}$$

satisfying the boundary condition $0 < \varphi(1) < \pi$, *if* $0 < \varphi(z) < \pi$, *at every* $z \in [0,1]$, *then Equation (A44) does not have a topologically nontrivial solution.*

Proof. Equation (A44) has the following property: if $\varphi(z)$ is a bounded solution of Equation (A44), then $-\varphi(z)$, $\varphi(-z)$ and $\varphi(z) + 2n\pi$, $n \in \mathbb{Z}$ are also bounded solutions of Equation (A44). Using the above properties of Equation (A44) and the assumptions in the lemma, we obtain two conclusions. For a bounded solution of Equation (A44),

1. if $n\pi < \varphi(1) < (n+1)\pi$ for $n \in \mathbb{Z}$, then $n\pi < \varphi(z) < (n+1)\pi$ at every $z \in [0,1]$;
2. if $n\pi < \varphi(-1) < (n+1)\pi$ for $n \in \mathbb{Z}$, then $n\pi < \varphi(z) < (n+1)\pi$ at every $z \in [-1,0]$.

To prove the lemma, assume to the contrary that there is a topologically nontrivial solution $\tilde{\varphi}$ with charge $C(\tilde{\varphi}) = Q \neq 0$. From Lemma A6, we know that $\tilde{\varphi}(-1) \neq m\pi$ for any $m \in \mathbb{Z}$ and $\tilde{\varphi}(1) \neq n\pi$ for any $n \in \mathbb{Z}$. Therefore, assume that, for some $m, n \in \mathbb{Z}$,

$$m\pi < \tilde{\varphi}(-1) < (m+1)\pi, \qquad n\pi < \tilde{\varphi}(1) < (n+1)\pi$$

But, from the two conclusions above, we have

$$m\pi < \tilde{\varphi}(0) < (m+1)\pi, \qquad n\pi < \tilde{\varphi}(0) < (n+1)\pi$$

which leads us to conclude that $m = n$. From Lemma 4 we conclude that, if m is even,

$$\lim_{z\to-\infty} \tilde{\varphi}(z) = \lim_{z\to\infty} \tilde{\varphi}(z) = m\pi$$

and, if m is odd,

$$\lim_{z\to-\infty} \tilde{\varphi}(z) = \lim_{z\to\infty} \tilde{\varphi}(z) = (m+1)\pi$$

In either case, $\tilde{\varphi}$ is a topologically trivial solution, contradicting our assumption that $\tilde{\varphi}$ is topologically nontrivial. □

Proof of Theorem 1. From Lemmas A6 and A8 it follows that, to prove the theorem, it is sufficient to show that, if a bounded solution φ satisfies $0 < \varphi(1) < \pi$, then $0 < \varphi(z) < \pi$ for all $z \in [0,1]$ for $\alpha < 2$. We prove the above statement in hypergeometric coordinates. Specifically, we will show (in hypergeometric coordinate x) that, if φ is a bounded solution of

$$x(x-1)\varphi'' + (2x-1)\varphi' + \alpha \sin(\varphi) = 0, \qquad \alpha < 2 \tag{A45}$$

and $0 < \varphi(0) < \pi$, then $0 < \varphi(x) < \pi$ at all $x \in [0, \frac{1}{2}]$. We set $\gamma = \varphi(0)$. We prove the following claims.

Claim 1. *If* $0 < \varphi(0) < \pi$ *and* $\varphi''(x) \leqslant 0$ *at all* $x \in (0, \frac{1}{2}]$, *then*

$$0 < \varphi(x) < \pi, \qquad \text{at every } x \in (0, \frac{1}{2}]$$

Proof. Assume $\varphi''(x) \leqslant 0$ throughout $(0, \frac{1}{2}]$. Then, using Lemma 1, we have at $x \in (0, \frac{1}{2}]$

$$\varphi'(x) = \lim_{\epsilon \to 0} \left[\int_{\epsilon}^{x} \varphi''(y) dy + \varphi'(\epsilon) \right] \leqslant \alpha \sin(\gamma)$$

Therefore, at every $x \in (0, \frac{1}{2}]$, recalling that $\alpha < 2$, we have

$$\varphi(x) = \gamma + \int_0^x \varphi'(y)\, dy \;\leqslant\; \gamma + x\,\alpha\, \sin(\gamma) \;\leqslant\; \gamma + \frac{\alpha}{2} \sin(\gamma) < \gamma + \sin(\gamma) \tag{A46}$$

If $\varphi(x) = \pi$ at some $x \in (0, \frac{1}{2}]$, then, from (A46), we have

$$\pi < \gamma + \sin(\gamma) \implies \frac{\sin(\pi - \gamma)}{\pi - \gamma} > 1 \tag{A47}$$

which is impossible. The contradiction proves that $\varphi(x) < \pi$ at all $x \in \left(0, \frac{1}{2}\right]$. From Lemma A7 we conclude that $\varphi(x) > 0$ at all $x \in \left(0, \frac{1}{2}\right]$. □

Therefore, in the remainder of the proof we will assume that $\varphi''(x) > 0$ at some $x \in (0, \frac{1}{2}]$. Define

$$x_1 = \inf_{x \in (0, \frac{1}{2}]} \{x \,|\, \varphi''(x) > 0\} \tag{A48}$$

It is possible that $x_1 = 0 \notin (0, \frac{1}{2}]$.

We prove the theorem by contradiction. If possible, let x_3, defined below, exist.

$$x_3 = \inf_{x \in (0, \frac{1}{2}]} \{x \,|\, \varphi(x) = \pi\} \tag{A49}$$

If x_3 exists then we note that $x_3 \in (0, \frac{1}{2}]$. x_1 and x_3 are illustrated in Figure A2.

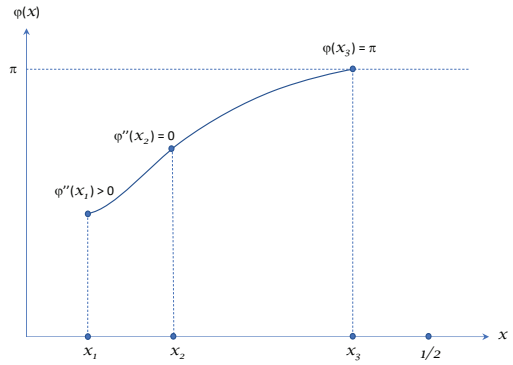


Figure A2. Illustration of the salient values x_1, x_2 and x_3 used in the proof. The solid line represents a solution of Equation (A45). The dotted line represents $\varphi(x) \equiv \pi$.

Claim 2. *At every* $x \in [0, x_1]$ *we have* $0 < \varphi(x) < \pi$.

Proof. If $x_1 = 0$, then, from $\varphi(x_1) = \gamma < \pi$, the claim holds. On the other hand, if $0 < x_1 \leqslant \frac{1}{2}$, then, from the definition of x_1, in (A48), $\varphi''(x) \leqslant 0$ at every $x \in (0, x_1]$. Therefore, recalling that $\alpha < 2$, at every $x \in [0, x_1]$,

$$\varphi(x) = \gamma + \int_0^x \lim_{\epsilon \to 0} \left[\int_{\epsilon}^{y} \varphi''(z)\, dz + \varphi'(\epsilon) \right] dy \leqslant \gamma + x\,\alpha\, \sin(\gamma) < \gamma + \sin(\gamma) < \pi$$

The last inequality holds since $0 < \gamma < \pi$ and $\sin(\gamma)/(\pi - \gamma) < 1$ at $\gamma \neq \pi$. From Lemma A7 we conclude that $\varphi(x) > 0$ at every $x \in [\,0, x_1\,]$. □

Claim 3. *At every* $x \in (x_1, x_3]$, $\varphi'(x) > 0$.

Proof. Rewriting Equation (A45) and using Lemma 1, we have, for $x \in (x_1, x_3]$,

$$\varphi'(x) = \frac{1}{x\,(1-x)} \int_0^x \alpha\, \sin(\varphi(y))\, dy \tag{A50}$$

From Claim 2 above, Lemma A7, and the definition of x_3, we know that, at every $x \in [0, x_3]$, $0 < \varphi(x) \leqslant \pi$. From Equation (A50) we conclude that $\varphi'(x) > 0$ at all $x \in [0, x_3]$. □

Claim 4. *At every* $x \in (x_1, x_3]$, $\varphi''(x) > 0$.

Proof. Assume to the contrary that $\varphi''(x_2) = 0$ at some $x_2 \in (x_1, x_3]$. See Figure A2. We will assume that x_2 is the smallest value of $x \in (x_1, x_3]$ at which φ'' vanishes. Therefore $\varphi''(x) > 0$ at all $x \in (x_1, x_2)$, and, since $\varphi''(x_2) = 0$, we must have $\varphi'''(x_2) \leqslant 0$. However, we will show that $\varphi'''(x_2) > 0$. The contradiction will prove the claim.

Differentiating Equation (A45) we obtain

$$\varphi'''(x) = \frac{1}{x(1-x)} \left[\, (4x-2)\varphi'' + (2 + \alpha\, \cos(\varphi))\varphi' \,\right] \tag{A51}$$

We see from Equation (A51) that φ''' is well defined in $(x_1, x_3]$. For $\alpha < 2$, we have $2 + \alpha \cos(\varphi) > 0$. Since $\varphi''(x_2) = 0$, noting that $x_2\,(1 - x_2) > 0$ and using Claim 3, we have

$$\varphi'''(x_2) = \frac{(2 + \alpha\, \cos(\varphi(x_2)))\, \varphi'(x_2)}{x_2\,(1-x_2)} \;>\; 0$$

The contradiction shows that x_2 does not exist. Therefore $\varphi''(x) > 0$ at all $x \in (x_1, x_3]$. □

From Claim 4 and Equation (A45) we have at every $x \in (x_1, x_3]$,

$$\varphi''(x) = \frac{1}{x(1-x)} \left[\, (2x-1)\, \varphi'(x) + \alpha\, \sin(\varphi(x)) \,\right] > 0$$

In particular, at x_3, we have

$$0 \;=\; \alpha\, \sin(\varphi(x_3)) > (1-2x)\, \varphi'(x_3) \geqslant 0$$

The contradiction shows that x_3 does not exist and $0 < \varphi(x) < \pi$ at every $x \in [0, \frac{1}{2}]$. From Lemma A8 it follows that every bounded solution of Equation (A45) is topologically trivial. □

Appendix E. Small Solitons

Lemma A9. *For* $0 \leqslant \gamma \leqslant \pi$, *let* $\varphi(x;\gamma)$ *be a continuous and bounded solution of*

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha \sin(\varphi) = 0 \tag{A52}$$

in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, *satisfying the boundary conditions*

$$\varphi(0;\gamma) = \gamma, \qquad \varphi'(0;\gamma) = \alpha \sin(\gamma). \tag{A53}$$

Then, at any $\tilde{x} \in \left[0, \frac{1}{2}\right]$, $\varphi(\tilde{x};\gamma)$ *and* $\varphi'(\tilde{x};\gamma)$ *are continuous functions of* γ.

Proof. Observe that, if we choose an ϵ such that $0 < \epsilon \leqslant \tilde{x} \leqslant \frac{1}{2}$, then the differential Equation (A52) has no singularities in the interval $[\epsilon, \tilde{x}]$. Therefore, $\varphi(\tilde{x};\gamma)$ depends continuously on the initial conditions $\varphi(\epsilon;\gamma)$ and $\varphi'(\epsilon;\gamma)$. To prove the lemma it is therefore sufficient to show that $\varphi(\epsilon;\gamma)$ and $\varphi'(\epsilon;\gamma)$ are continuous functions of γ at some $0 < \epsilon \leqslant \tilde{x} \leqslant \frac{1}{2}$.

If $\varphi(x;\gamma)$ is a solution of Equation (A52), satisfying the boundary condition (A53), then $\varphi(x;\gamma)$ is also a solution of the integral equation

$$\varphi(x;\gamma) = V(x;\varphi(x;\gamma)) + \gamma \cdot \varphi_1(x) \tag{A54}$$

where $V(x;\varphi(x;\gamma))$ is defined in (A14) and $\varphi_1(x)$ is a solution of the homogeneous hypergeometric Equation (A13).

To show that $\varphi(x;\gamma)$ is a continuous function of γ at x, we show that, for any given $\rho > 0$, there is a $\delta > 0$ such that, if $|\gamma - \gamma'| < \delta$, then $|\varphi(x;\gamma) - \varphi(x;\gamma')| < \rho$. Using Equation (A54) we have

$$|\varphi(x;\gamma) - \varphi(x;\gamma')| \quad \leqslant \quad |V(x;\varphi(x;\gamma)) - V(x;\varphi(x;\gamma'))| + |\varphi_1(x)| \cdot |\gamma - \gamma'| \tag{A55}$$

In order to derive an upper bound on the right-hand side of the above inequality, we note, using Lemma A3, that

$$V(x;\varphi) = h(x) \int_0^x \varphi_1(y) q(y) dy + \varphi_1(x) \int_0^x \frac{1}{y} \int_0^y \varphi_1(u) q(u)\, du\, dy - \varphi_1(x) \int_0^x h(y) q(y) dy \tag{A56}$$

$h(x)$, described in the discussion following Equation (A11), $xW(x)$ and $\varphi_1(x)$ are analytic in $[-\epsilon, \epsilon]$ for sufficiently small $\epsilon \in \left(0, \frac{1}{2}\right)$; see Lemma A2. Therefore, we can find constants $k_{\varphi_1}(\epsilon), k_h(\epsilon), k_w(\epsilon)$ such that, for all $0 \leqslant x \leqslant \epsilon$, $|\varphi_1(x)| < k_{\varphi_1}(\epsilon)$, $|h(x)| < k_h(\epsilon)$ and $|x(x-1)W(x)|^{-1} < k_w(\epsilon)$.

We bound the variation of $f(\varphi(x;\gamma)) = \alpha(\varphi(x;\gamma) - \sin(\varphi(x;\gamma))$ with γ. Observe that $|\sin(\theta) - \sin(\theta')| \leqslant |\cos(\theta^*)||\theta - \theta'| \leqslant |\theta - \theta'|$, where θ^* is some number between θ and θ'. Therefore, for any $x \in [-\epsilon, \epsilon]$,

$$|f(\varphi(x;\gamma)) - f(\varphi(x;\gamma'))| \quad \leqslant \quad 2\alpha |\varphi(x;\gamma) - \varphi(x;\gamma')| \leqslant 2\alpha M(\epsilon, \gamma, \gamma') \tag{A57}$$

where

$$M(\epsilon, \gamma, \gamma') := \max_{x \in [-\epsilon, \epsilon]} \{|\varphi(x;\gamma) - \varphi(x;\gamma')|\} \tag{A58}$$

Using Equation (A56) and the bounds on $|h(x)|, |\varphi_1(x)|$ and $|x(x-1)W(x)|^{-1}$ in $[-\epsilon, \epsilon]$, we obtain

$$|V(x, \varphi(x;\gamma)) - V(x, \varphi(x;\gamma'))| \leqslant C(\epsilon) \cdot \epsilon \cdot M(\epsilon, \gamma, \gamma') \tag{A59}$$

where $C(\epsilon) = 4k_h(\epsilon)k_{\varphi_1}(\epsilon)k_w(\epsilon)\alpha + 2 \cdot (k_{\varphi_1}(\epsilon))^2 k_w(\epsilon)\alpha$.

Using (A55) and (A59) we have

$$|\varphi(x;\gamma) - \varphi(x;\gamma')| \leqslant C(\epsilon) \cdot \epsilon \cdot M(\epsilon, \gamma, \gamma') + k_{\varphi_1}(\epsilon)|\gamma - \gamma'|$$

The above inequality holds at every $x \in [-\epsilon, \epsilon]$ for a sufficiently small ϵ. Therefore,

$$M(\epsilon, \gamma, \gamma') \leqslant C(\epsilon) \cdot \epsilon \cdot M(\epsilon, \gamma, \gamma') + k_{\varphi_1}(\epsilon)|\gamma - \gamma'|$$

Choosing a sufficiently small $\epsilon_1 \in \left(0, \frac{1}{2}\right)$ we have

$$M(\epsilon_1, \gamma, \gamma') \leqslant \frac{k_{\varphi_1}(\epsilon_1)|\gamma - \gamma'|}{1 - \epsilon_1\, C(\epsilon_1)} \tag{A60}$$

Observe that, for every $x \in [0, \epsilon_1]$,

$$M(x, \gamma, \gamma') \leqslant M(\epsilon_1, \gamma, \gamma') \leqslant \frac{k_{\varphi_1}(\epsilon_1) \cdot |\gamma - \gamma'|}{1 - \epsilon_1 \cdot C(\epsilon_1)};$$

Given any $\rho > 0$, we choose $\delta = \dfrac{\rho(1 - \epsilon_1 \cdot C(\epsilon_1))}{k_{\varphi_1}(\epsilon_1)}$. Then, at any $x \in [0, \epsilon_1]$,

$$|\varphi(x;\gamma) - \varphi(x;\gamma')| \leqslant M(x) \leqslant M(\epsilon_1) < \rho, \qquad \text{if } |\gamma - \gamma'| < \delta$$

showing that $\varphi(x;\gamma)$ is a continuous function of γ at every $x \in [0, \epsilon_1]$.

Using the integral Equation (A54) and the simplified form of V shown in Equation (A56) we can similarly show that, at every $x \in [0, \epsilon_2]$, for some sufficiently small ϵ_2,

$$|\varphi'(x;\gamma) - \varphi'(x;\gamma')| \leqslant K(\epsilon_2) \cdot \left[\, 2\alpha M(\epsilon_2, \gamma, \gamma') \,\right]$$

where

$$K(\epsilon_2) := k_w \Big[k'_h k_{\varphi_1} \epsilon_2 + k_h k_{\varphi_1} + k'_{\varphi_1} k_{\varphi_1} + (k_{\varphi_1})^2 + k'_{\varphi_1} k_h \epsilon_2 + k_{\varphi_1} k_h \Big]$$

and and the bounds on $h, h', \varphi_1, \varphi'_1, \{x(x-1)W(x)\}^{-1}$ in the interval $[0, \epsilon_2]$ are $|h(x)| < k_h$, $|h'(x)| < k'_h$, $|\varphi_1(x)| < k_{\varphi_1}$, $|\varphi'_1(x)| < k'_{\varphi_1}$, $[x(x-1)W(x)]^{-1} < k_w$. Since $M(\epsilon_2, \gamma, \gamma') \to 0$ as $\gamma \to \gamma'$, for a given $\rho > 0$, we can make $M(\epsilon_2, \gamma, \gamma') < \dfrac{\rho}{2\alpha\, K(\epsilon_2)}$ by choosing $|\gamma - \gamma'| < \delta$ for some $\delta > 0$. Then,

$$|\varphi'(x;\gamma) - \varphi'(x;\gamma')| < K(\epsilon_2) \cdot M(\epsilon_2, \gamma, \gamma') < \rho, \qquad \text{if } |\gamma - \gamma'| < \delta$$

which shows that $\varphi'(x;\gamma)$ is also a continuous function of γ at any $x \in [0, \epsilon]$. □

Lemmas A10 and A11 use the continuity property established in Lemma A9 to show, respectively, that there exists a γ for which the solution overshoots $\varphi = \pi$ in the interval $\left[0, \frac{1}{2}\right]$ and a γ for which the solution reaches $\varphi = \pi$ at $x = \frac{1}{2}$. A similar argument has been used to study dynamics of Yang–Mills fields in asymptotically hyperbolic spacetime [32].

Lemma A10. *Let $\varphi(x;\gamma)$ be a continuous bounded solution of*

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha \sin(\varphi) = 0, \qquad \alpha > 2 \tag{A61}$$

in the interval $\left[0, \frac{1}{2}\right]$, satisfying the initial conditions

$$0 < \varphi(0;\gamma) = \gamma < \pi; \qquad \varphi_x(0;\gamma) = \alpha \sin(\gamma). \tag{A62}$$

Then there exists a $\gamma \in (0, \pi)$ for which

$$M(\gamma) := \max_{0 \leqslant x \leqslant \frac{1}{2}} \varphi(x;\gamma) \geqslant \pi$$

Proof. For $\alpha > 2$, there exists a $\bar{\gamma} \in \left(\frac{\pi}{2}, \pi\right)$ such that, for every $\gamma^* \in (\bar{\gamma}, \pi)$,

$$\alpha \cos(\gamma^*) + 2 < 0 \tag{A63}$$

We will show that, for every $\gamma^* \in (\bar{\gamma}, \pi)$, $M(\gamma^*) \geqslant \pi$.

Pick a $\gamma^* \in (\bar{\gamma}, \pi)$ and assume to the contrary that $M(\gamma^*) < \pi$. First, we claim that, if $M(\gamma^*) < \pi$, then, at every $x \in \left(0, \frac{1}{2}\right]$, (*i*) $\varphi_x(x) > 0$, and (*ii*) $\varphi_{xx}(x) < 0$.

We begin by noting that, if $M(\gamma^*) < \pi$, then, from Lemma A7, $\varphi(x; \gamma^*) > 0$ at every $x \in \left[0, \frac{1}{2}\right]$. To prove the first claim, assume to the contrary that $\varphi_x(x') = 0$ at some $x' \in (0, \frac{1}{2}]$. Further, let x' be the smallest such x'. Since $0 < \varphi(x; \gamma^*) < \pi$ for $0 < x < x'$,

$$\varphi_x(x') = \alpha \sin(\gamma) + \frac{1}{x'(1-x')} \int_0^{x'} \alpha \sin(\varphi(x))\, dx \geqslant \alpha \sin(\gamma) > 0$$

The contradiction proves the claim. To prove the second claim observe that

$$\begin{aligned} \varphi_{xx}(0) &= \lim_{x \to 0} \left(\frac{1}{1-x}\right) \left\{ \frac{(2x-1)\varphi_x(x) + \alpha \sin(\varphi(x))}{x} \right\} \\ &= \lim_{x \to 0} (2x-1)\varphi_{xx}(x) + (2 + \alpha \cos(\varphi(x)))\varphi_x(x) \end{aligned} \tag{A64}$$

The second equality is obtained using *'l Hospital's* rule. Taking the limit and rearranging, we get

$$\varphi_{xx}(0) = \frac{1}{2} \cdot (2 + \alpha \cos(\gamma^*)) \alpha \sin(\gamma^*) < 0 \tag{A65}$$

The last inequality follows from (A63). We see from (A65) that the limit in (A64) exists, and therefore there exists a $\lambda > 0$ such that, at every $\tilde{x} \in (-\lambda, \lambda)$, $\varphi_{xx}(\tilde{x}) < 0$. Therefore $\varphi_{xx}(\tilde{x}) \neq 0$ for $\tilde{x} \in (-\lambda, \lambda)$.

Again, arguing by contradiction, if possible, let φ_{xx} change sign in a small neighborhood of $\bar{x}$, with $\varphi_{xx}(\bar{x}) = 0$ and $\varphi_{xxx}(\bar{x}) > 0$, at some $\bar{x} \in \left(0, \frac{1}{2}\right]$. As we noted above, $\varphi_{xx}(x)$ is nonzero in the interval $(-\lambda, \lambda)$, so $\bar{x} \neq 0$. Differentiating Equation (A61), we obtain

$$\varphi_{xxx}(\bar{x}) = \frac{1}{\bar{x}(1-\bar{x})} \cdot \{(2 + \alpha \cos(\varphi(\bar{x})))\varphi_x(\bar{x})\} < 0 \tag{A66}$$

Note that $\varphi_{xxx}(x)$ is well defined at $\bar{x} \in \left(0, \frac{1}{2}\right]$. The inequality in (A66) follows from (A63), and, by observing that $\varphi_x(x) > 0$ for $x \in \left[0, \frac{1}{2}\right]$, $\varphi(\bar{x}) \geqslant \varphi(0) = \gamma^* > 0$. The contradiction proves that, if $M(\gamma^*) < \pi$, then $\varphi_{xx}(x) < 0$ for $x \in \left(0, \frac{1}{2}\right]$.

But,

$$\varphi_{xx}\left(\frac{1}{2}\right) = 4\left\{\alpha \sin\left(\varphi\left(\frac{1}{2}\right)\right)\right\} > 0,$$

since $0 < \varphi\left(\frac{1}{2}\right) < M(\gamma^*) < \pi$. The contradiction proves that $M(\gamma^*) \geqslant \pi$. □

Lemma A11. *Let $\varphi(x; \gamma)$ be a continuous bounded solution of*

$$x(x-1)\varphi_{xx} + (2x-1)\varphi_x + \alpha \sin(\varphi) = 0, \qquad \alpha > 2 \tag{A67}$$

in the interval $\left[0, \frac{1}{2}\right]$, *satisfying the initial conditions*

$$\varphi(0;\gamma) = \gamma; \qquad \varphi_x(0;\gamma) = \alpha \sin(\gamma). \tag{A68}$$

Then there exists a $\gamma^* \in (0,\pi)$, *for which* $\varphi\left(\frac{1}{2};\gamma^*\right) = \pi$.

Proof. From Lemma A10 we know that there exists at least one $\gamma \in (0,\pi)$ for which $M(\gamma) \geqslant \pi$. Therefore, one can define γ^* as follows.

$$\gamma^* := \inf_{0<\gamma<\pi} \{\gamma \,|\, M(\gamma) \geqslant \pi\}.$$

If $M(\gamma^*) > \pi$, then, from Lemma A9, we can find a δ such that $M(\gamma) > \pi$ for all $\gamma \in (\gamma^* - \delta, \gamma^* + \delta)$, contradicting the definition of γ^*. Therefore, we conclude that $M(\gamma^*) = \pi$.

We note that $\varphi(0;\gamma^*) = \gamma^* < \pi$. Let x^* be the leftmost point at which $\varphi(x^*;\gamma^*) = \pi$. If $x^* \in \left(0, \frac{1}{2}\right)$, then $\varphi_x(x^*;\gamma^*) = 0$. From Equation (A67) $\varphi_{xx}(x^*;\gamma^*) = 0$. Repeated differentiation of Equation (A67) shows that all the higher derivatives of $\varphi(x;\gamma^*)$ must also vanish at x^*. But, by uniqueness of the solution at x^*, $\varphi(x;\gamma^*) \equiv \pi$ in a small neighborhood of x^*, which contradicts the assumption that x^* is the leftmost point at which $\varphi(x;\gamma^*) = \pi$. Therefore, we conclude that $\varphi(x;\gamma^*) < \pi$ for $0 \leqslant x < \frac{1}{2}$, and $\varphi\left(\frac{1}{2};\gamma^*\right) = \pi$. □

Lemma A12. *For* $\alpha > 2$, *there exists a solution of*

$$(z^2-1)\varphi_{zz} + 2z\varphi_z + \alpha \sin(\varphi) = 0 \tag{A69}$$

satisfying the boundary conditions

$$\lim_{z\to-\infty} \varphi(z) = 2\pi; \qquad \lim_{z\to\infty} \varphi(z) = 0 \tag{A70}$$

Proof. From Lemmas A11 and 4 we know that Equation (A69) has a solution $\varphi(z)$ in $[0,\infty)$ with the following properties:

$$\varphi(0) = \pi; \qquad 0 < \varphi(1) < \pi; \qquad \varphi(\infty) = 0$$

The symmetries of Equation (A69), $\varphi \to \varphi + 2n\pi$, $n \in \mathbb{Z}$, $\varphi \to -\varphi$ and $z \to -z$, imply the existence of the solution claimed in the theorem. □

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