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Null Rigidity and Wave-Sector Reduction of Conformally Deformed Black Holes

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Abstract

Several minimal-length constructions deform the spacetime metric conformally at leading order, $\tilde{g}_{\mu\nu} = C(x)g_{\mu\nu}$. We ask which observables can register such a deformation. The two structural results rest on standard properties of conformal rescalings; what is worked out here is their consequence for this class of metrics, carried to closed form on a black-hole background. For an arbitrary conformal factor the unparametrized null geodesics are unchanged, so the critical impact parameter, the shadow angular radius seen from infinity, the deflection angles and the Shapiro delay keep their undeformed values at all orders, and the photon sphere keeps its coordinate location although its areal radius changes. A minimally coupled massless test scalar on $\tilde{g}$ obeys, exactly, a wave equation on g with the single potential $V_\omega = \square\sqrt{C}/\sqrt{C} = -\frac{1}{6}(C\tilde{R} - R)$, into which the Weyl invariant does not enter; a conformally coupled massless scalar does not see the deformation at all. For a Reissner–Nordström background and the power-law family $\eta = \varepsilon_0(M/r)^p$, with p as a model input rather than a prediction, we obtain the leading large- l scalar quasinormal shift $p(p-3)\varepsilon_0/[12 \cdot 3^p l(l+1)]$, suppressed as l^{-2} , and closed forms for the ISCO radius, the disk radiative efficiency and the orbital frequency. Within this family the channels fall silent at the distinct exponents $p = 3, 2$ and $5/2$. All results are checked by computer algebra.

Keywords: conformal transformations; null geodesics; black-hole shadow; quasinormal modes; innermost stable circular orbit; curvature invariants; Reissner–Nordström black hole; minimal length

1. Introduction

Approaches to quantum gravity that agree on very little agree that there is a shortest resolvable length near the Planck scale. Written on the commutators this becomes a generalized uncertainty principle (GUP) [1–10], or equivalently a noncommutative deformation of the phase-space algebra [11,12]. Confronting such deformations with data has become a program in its own right [13–17], with bounds drawn from mechanical oscillators, from cosmology and from gravitational waves [18,19].

There is more than one route from a deformed algebra to a deformed geometry, and they converge on the same leading structure. Carrying a relativistic GUP (RGUP) onto a Finsler or Hamilton bundle and reading the fundamental tensor off the Hessian of the fundamental function [20–27] produces, after matching back to pseudo-Riemannian form, a metric whose leading deformation is conformal,

$$\tilde{g}_{\mu\nu} = C(x)g_{\mu\nu}, \quad C > 0. \quad (1)$$



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Caianiello's maximal-acceleration geometry gives the same structure with C fixed by the proper acceleration [28–32]—so do Weyl-integrable and scalar–tensor effective metrics in the Jordan frame [33,34], and so does the quantum-conditioned metric of Refs. [35–37], whose curvature has been worked out on several backgrounds including the charged one used here [38]. We do not derive Equation (1) and we do not adopt any one of these derivations. We take the conformal form as the object of study and ask which observables can distinguish the conformal sector—a question that, as far as we are aware, has not been treated systematically for this class.

That question has an unexpectedly sharp answer, and the sharpness is the reason for this paper. Two facts pull in opposite directions. Conformally related metrics have identical null cones, so light rays cannot distinguish $\tilde{g}$ from g , but the massless wave equation is not conformally invariant in four dimensions unless the field is conformally coupled, so waves can. The gap between these two statements is where the entire observable content of a conformal quantum correction lives, and it turns out to be narrow, quantifiable and, in a precise sense, complete.

1.1. What Is Established Here

We prove two theorems and then work out what they cost and buy on a Reissner–Nordström (RN) benchmark.

- (N1) **Null rigidity, and a no-go for the shadow program.** Theorem 1 shows that for an *arbitrary* conformal factor—not merely a radial or small one—the null geodesic congruence of $\tilde{g}$ coincides with that of g up to reparametrization, so the critical impact parameter, shadow angular radius, deflection angle, Shapiro delay and lensing image positions are exactly equal to their values for the undeformed metric at all orders, and the photon sphere keeps its coordinate location. Quantum-corrected shadows are now a large part of the literature [39–47]; the theorem says that whatever those calculations detect, it cannot be the conformal part of the deformation. The blindness is exact rather than approximate, which turns it into a null test: any measured shadow or lensing deviation in this class of models diagnoses a departure from conformality.
- (N2) **A reduction theorem for the wave sector.** Theorem 2 shows that a minimally coupled massless scalar on $\tilde{g}$ satisfies, exactly and without approximation, a wave equation on the *undeformed* metric carrying a single scalar potential,

$$V_\omega = \frac{\square\sqrt{C}}{\sqrt{C}} = -\frac{1}{6}(C\tilde{R} - R). \quad (2)$$

Three things follow. The wave-sector response of the deformation, whatever the tensorial structure that produced it, collapses onto one function of position. That function is built from the induced Ricci scalar, and the Weyl invariant—which carries most of the curvature response—does not enter it. And a massless scalar with the four-dimensional conformal coupling $\xi = \frac{1}{6}$ is exactly insensitive to the deformation (Corollary 2), so what is seen here is fixed by the matter coupling as much as by the geometry.

- (N3) **Eikonal suppression of the ringdown response.** For a minimally coupled test scalar—not a gravitational-wave ringdown, whose perturbation equations the conformal ansatz does not fix—the first-order shift of a quasinormal frequency is $O(l^{-2})$, Equation (38), and therefore vanishes in the eikonal limit exactly as Theorem 1 requires since eikonal modes are governed by the photon sphere [48,49]. In the minimally coupled scalar sector the correction is therefore a non-eikonal effect, with its leading contribution suppressed as l^{-2} and its weight in the low- l , low-overtone part

of the spectrum. On RN with $\eta = \varepsilon_0(M/r)^p$ the shift is given in closed form by Equation (40).

- (N4) **Closed-form timelike response and a complementarity result.** The ISCO radius, the radiative efficiency of a thin disk and the ISCO orbital frequency all shift at first order, Equations (47)–(49), but with different p -dependences and therefore with different zeros. These zeros belong to the power-law family $\eta \propto r^{-p}$ with p chosen by hand, and are properties of that family rather than predictions of a fundamental theory. Ringdown silence occurs only at $p = 3$, disk-continuum silence only at $p = 2$ and disk-timing silence only at $p = 5/2$; the three blind spots are pairwise distinct, so no profile in the family is silent in more than one channel (Proposition 2).
- (N5) **A curvature index with a physical warrant.** Splitting the Kretschmann response into Weyl and Ricci channels gives a magnitude index $\mathcal{D} \in [0, 1]$, amplitude-free at leading order, with the closed form $\mathcal{D}_\infty = p(p+3)q^2/[p(p+3)q^2 + 24]$. What it measures is the size of the Ricci-sector response relative to the Weyl-sector one. Theorem 2 is what gives that comparison a physical edge: the potential a propagating scalar feels is built from the induced Ricci scalar and contains no Weyl contribution at all, so a response that sits in the Weyl channel cannot reach the wave sector, while one in the Ricci sector shares a sector with the part that does. The correspondence is not an identity— ΔK_{Ric} involves the full Ricci tensor and V_ω only its trace—and we do not read $\mathcal{D}$ as a fraction of observable response.

1.2. Relation to Known Results

The ingredients are standard and we claim none of them. That conformally related metrics share null geodesics is textbook [50,51]; the conformal transformation laws for the connection, the Ricci tensor and the Weyl tensor are long established [33,34,52]; and the substitution that turns a wave equation in one conformal frame into a wave equation with a potential in another is familiar from scalar–tensor theory [33,53]. The contribution here lies in the use made of them: the statement that these facts settle the null and scalar-wave channels together, the identification of the resulting potential with the induced Ricci scalar in the form (2), the eikonal-suppression law that reconciles the two theorems, and the closed-form response and blind-spot structure on RN. We are not aware of a systematic treatment of this class along these lines. One immediate use of Theorem 1 is that a shadow deviation reported for a deformed metric cannot be attributed to a purely conformal factor, whatever else may produce it.

Section 2 fixes the conformal setting. Section 3 develops the invariant decomposition of the curvature response and Section 4 evaluates it on RN; Section 5 treats the horizon. Section 6 proves null rigidity, Section 7 the reduction theorem and its ringdown consequence, and Section 8 the timelike sector, the sensitivity analysis and the complementarity result. Section 9 is numerical, Sections 10 and 11 contain the discussion and conclusions, and Appendix A documents the computer-algebra checks. Every closed-form result quoted below—the transformation laws, the reduction theorem, the Regge–Wheeler potential, the ringdown coefficient and the three ISCO coefficients—was recomputed from the metric components alone by a computer-algebra system, with no transformation law supplied as an input. We use $G = c = 1$, signature $(-, +, +, +)$, and the dimensionless variables $x = r/M$, $q = Q/M$, following Refs. [50,51,54–56].

2. Conformal Effective Geometry

2.1. Setting and Notation

Write the effective metric as

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} = e^{2\omega(x)} g_{\mu\nu}, \quad \omega = \frac{1}{2} \ln C, \quad C = \Omega^2 > 0. \quad (3)$$

Different constructions fill in C differently and we keep all of them in view. In the GUP-on-a-bundle route, C collects the deformation function $\phi(p_0) = 1 + \beta p_0^\rho p_{0\rho}$, a maximal-force normalization and a contribution from the tangent covectors [35,37]; in Caianiello-type models, it is fixed by the proper acceleration [28–30,32]; and in a Jordan-frame scalar–tensor or Weyl-integrable setting it is the scalar itself [33,34]. Nothing below uses any of these origins. Every statement depends on ω alone, and therefore holds for any model whose leading deformation is conformal—which is the reason the results can be read as constraints on a class rather than on a scheme.

The Levi–Civita connection transforms exactly as [33,34,50,51]

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \delta_\mu^\lambda \partial_\nu \omega + \delta_\nu^\lambda \partial_\mu \omega - g_{\mu\nu} \nabla^\lambda \omega. \quad (4)$$

A form used elsewhere in this literature writes the same connection as [38]

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \frac{\mathcal{F}_{,\gamma}}{2C} \left(\delta_\mu^\lambda + \delta_\nu^\lambda - g^{\lambda\gamma} g_{\mu\nu} \right). \quad (5)$$

Comparing the two forms, and using $\partial_\gamma \omega = \partial_\gamma C / (2C)$, identifies the deformation one-form with the gradient of the conformal factor,

$$\boxed{\mathcal{F}_{,\gamma} = \partial_\gamma C} \quad (6)$$

We adopt Equation (6) throughout. It is convenient because the whole input of the analysis then reduces to one radial profile $C(r)$: once that is specified, both the conformal factor and the connection deformation follow.

2.2. Where the Geometric Content Sits

The following elementary observation organizes the rest of the paper.

Lemma 1 (Homothety). *If C is constant then $\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda$ and $\tilde{R}^\alpha_{\beta\mu\nu} = R^\alpha_{\beta\mu\nu}$, $\tilde{R}_{\mu\nu} = R_{\mu\nu}$, so that*

$$\tilde{R} = C^{-1} R, \quad \tilde{\mathcal{R}}_2 = C^{-2} \mathcal{R}_2, \quad \tilde{W}^2 = C^{-2} W^2, \quad \tilde{K} = C^{-2} K. \quad (7)$$

This follows at once from Equation (4) with $\partial_\mu \omega = 0$. The scalars then differ only through the number of indices raised with $\tilde{g}^{\mu\nu} = C^{-1} g^{\mu\nu}$.

Lemma 1 tells us where the geometric content of the deformation lives. A homothety is a rescaling of the unit of length: dimensional invariants pick up the corresponding power of C , as Equation (7) shows, but geodesics remain geodesics, and the causal structure, every dimensionless curvature ratio and the Petrov type are untouched [52]. Any new connection and curvature structure generated by the deformation must therefore be carried by $\partial_\mu \omega$ and $\nabla_\mu \nabla_\nu \omega$, which by Equation (6) is the same as saying that it sits in the radial profile of C rather than in its value.

Corollary 1. *A quantitative treatment requires promoting C from a parameter to a field, $C \rightarrow C(r)$. That is the level at which we work below.*

This fits with the earlier treatments rather than against them. The non-vanishing congruence expansion reported in Ref. [38] is exactly the kind of effect that survives Lemma 1 since geodesic focusing responds to gradients. Corollary 1 only makes explicit which input a sharper calculation needs, and Section 9 supplies two concrete choices.

3. Invariant Decomposition of the Curvature Response

3.1. The Two Channels

Curvature is normally assessed with the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and that is a sensible choice: it is a coordinate-independent quadratic invariant, and whether it stays finite tells us whether a curvature is intrinsic or removable [57–60]. In four dimensions, however, K is a sum. The Riemann tensor splits into a Weyl piece and a Ricci piece,

$$K = W^2 + 2\mathcal{R}_2 - \frac{1}{3}R^2, \quad W^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \quad \mathcal{R}_2 \equiv R_{\mu\nu}R^{\mu\nu}, \quad (8)$$

and the two pieces react to a conformal deformation by different mechanisms, so a shift in K alone does not say which of them moved. Separating them is the business of this section.

For $\tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}$ in four dimensions the Ricci tensor and scalar transform as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \omega - g_{\mu\nu} \square \omega + 2\nabla_\mu \omega \nabla_\nu \omega - 2g_{\mu\nu} (\nabla \omega)^2, \quad (9)$$

$$\tilde{R} = e^{-2\omega} [R - 6\square \omega - 6(\nabla \omega)^2], \quad (10)$$

while the Weyl tensor with all indices lowered scales homogeneously, $\tilde{C}_{\alpha\beta\gamma\delta} = e^{2\omega}C_{\alpha\beta\gamma\delta}$, so that

$$\tilde{W}^2 = e^{-4\omega} W^2 \quad (11)$$

Equations (9)–(11) are standard results [33,34,50]. We do not assume them: they are re-derived from scratch for the RN background, together with every invariant used below, in Appendix A.

What matters for us is how differently the two behave. Equation (11) contains no derivative of ω at all, so the Weyl invariant responds algebraically and the ratio $\tilde{W}^2/W^2 = e^{-4\omega}$ reads off the local conformal amplitude. That amplitude is a property of ω , and only indirectly of whatever quantum parameter produced it. In the Ricci sector the background term $R_{\mu\nu}$ is of course carried over unchanged; what the deformation *adds* in Equation (9) is built entirely from $\nabla_\mu \nabla_\nu \omega$ and $\nabla_\mu \omega \nabla_\nu \omega$. The induced Ricci contribution is therefore differential in ω . Table 1 collects the comparison.

Table 1. The two curvature channels of a conformal deformation $\tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}$ in four dimensions.

	Weyl Channel	Ricci Channel
invariant	$W^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$	$2\mathcal{R}_2 - \frac{1}{3}R^2$
transformation	$e^{-4\omega}W^2$ (exact, all orders)	non-homogeneous, Equation (9)
induced contribution	set by the local value of ω	governed by ω' and ω''
constant ω	rescaled by C^{-2}	rescaled by C^{-2}
new connection curvature	no	yes, if $\partial_\mu \omega \neq 0$
physical content	tidal/free gravitational field	Ricci curvature/effective-source sector

3.2. The Curvature-Response Index

Split the total response of the Kretschmann scalar into the two channel responses,

$$\Delta K = \Delta W^2 + \Delta K_{\text{Ric}}, \quad \Delta W^2 \equiv \tilde{W}^2 - W^2, \quad \Delta K_{\text{Ric}} \equiv \left(2\tilde{\mathcal{R}}_2 - \frac{1}{3}\tilde{R}^2\right) - \left(2\mathcal{R}_2 - \frac{1}{3}R^2\right), \quad (12)$$

and define the *magnitude-based Ricci-response fraction*

$$\mathcal{D}(r) = \frac{|\Delta K_{\text{Ric}}|}{|\Delta K_{\text{Ric}}| + |\Delta W^2|} \in [0, 1]. \quad (13)$$

The absolute values are intentional, and should be read with care. Either channel response can have either sign, and the two can partly cancel in the total ΔK . What $\mathcal{D}$ compares is the *size* of the two responses; it is not an algebraic partition of ΔK . So $\mathcal{D} \rightarrow 0$ means the deformation acts on the tidal field much more strongly than on the Ricci sector, $\mathcal{D} \rightarrow 1$ means the reverse, and $\mathcal{D} = \frac{1}{2}$ marks equal absolute channel responses rather than equal signed contributions to K . For a diagnostic whose job is to say which sector moved, this is the right reading, whether or not the two movements happen to add or subtract in the composite invariant.

A definition of this kind needs a warrant, and it does not have one yet: any number of ratios could be built out of the same two channels. Part of the answer arrives in Section 7. There the potential felt by a propagating massless scalar turns out to be built from the induced Ricci scalar, with no Weyl contribution, which tells us that the two channels are not on the same footing observationally: a deformation whose response is entirely tidal cannot reach the wave sector at all. We are careful about how far that carries. The propagating potential depends on the trace of the deformed Ricci tensor, whereas ΔK_{Ric} depends on the full tensor, so $\mathcal{D}$ ranks the two sectors rather than counting an observable fraction. Two technical properties support the choice. First, $\mathcal{D}$ is a ratio of differences of invariants, so it needs no regulator; unlike a normalization by the background $\mathcal{R}_2$ it stays finite in the Ricci-flat limit. Second, $\mathcal{D}$ is amplitude-free at leading order whenever the background Ricci tensor does not vanish—by itself only the observation that a ratio of two first-order quantities loses its dependence on the amplitude but useful here because it makes the sector ranking a property of the shape of the deformation. Both ΔW^2 and ΔK_{Ric} are then first order in the deformation, and the overall strength cancels in the ratio. That is what makes the index predictive in a framework whose amplitude is set by an unmeasured minimal-length parameter [5,19]. Section 4 establishes both properties in closed form.

4. The Reissner–Nordström Benchmark

4.1. Invariants of the Reference Metric

The RN line element [61–64] is

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (14)$$

In the dimensionless variables $x = r/M$ and $q = Q/M$ its invariants are

$$R = 0, \quad M^4 \mathcal{R}_2 = \frac{4q^4}{x^8}, \quad M^4 W^2 = \frac{48(x - q^2)^2}{x^8}, \quad M^4 K = \frac{48}{x^6} - \frac{96q^2}{x^7} + \frac{56q^4}{x^8}, \quad (15)$$

in agreement with the standard results [59,60] and reproduced independently in Appendix A. The balance between the channels in the undeformed geometry is the rational function

$$\mathcal{W}(q, x) \equiv \frac{W^2}{\mathcal{R}_2} = \frac{12(x - q^2)^2}{q^4}, \quad (16)$$

plotted in Figure 1. RN is Weyl-dominated everywhere except near $x = q^2$, which lies inside the inner horizon, and the charge controls how strongly the Ricci channel is populated

in the first place. This is what makes RN, rather than a Ricci-flat background, the right benchmark for our question: a Schwarzschild exterior switches the Ricci channel off entirely, and there is then nothing to resolve. RN is used here as a controlled benchmark and not as a claim that astrophysical black holes carry significant charge; q is the parameter that turns the Ricci channel on, and that is what it is for.

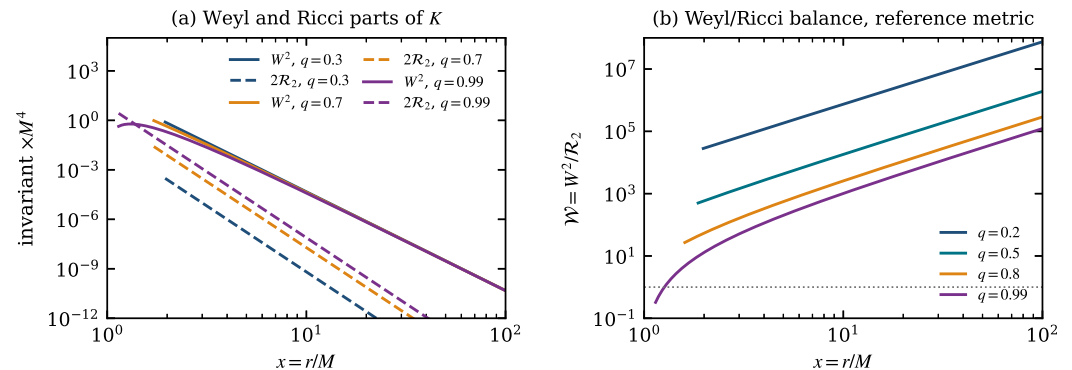


Figure 1. Undeformed RN geometry. (a) The Weyl part W^2 (solid) and the Ricci part $2\mathcal{R}_2$ (dashed) of the Kretschmann scalar. (b) The channel ratio $\mathcal{W} = W^2/\mathcal{R}_2$ of Equation (16); the dotted line marks equality. The charge-to-mass ratio q is the only control parameter.

4.2. Radial Conformal Field

For $\omega = \omega(r)$ on the background (14),

$$(\nabla\omega)^2 = f(\omega')^2, \quad \square\omega = \frac{1}{r^2} \frac{d}{dr} [r^2 f \omega'] = f\omega'' + \left(\frac{2f}{r} + f' \right) \omega', \quad (17)$$

and since $R = 0$ for RN,

$$\tilde{R} = -6e^{-2\omega} \left[f\omega'' + \left(\frac{2f}{r} + f' \right) \omega' + f(\omega')^2 \right]. \quad (18)$$

The undeformed RN geometry is Ricci-scalar flat while the deformed one is not. A non-vanishing $\tilde{R}$ is thus the simplest single signature that the deformation does more than rescale.

It helps to parametrize the deformation as

$$C(r) = 1 + \eta(r), \quad \omega = \frac{1}{2} \ln[1 + \eta(r)], \quad |\eta| \ll 1, \quad (19)$$

where η carries the small parameter of the underlying quantum model, typically $\eta \sim \beta_0 \ell_P^2 \mathcal{A}^2$ with $\mathcal{A}$ a curvature or acceleration scale. To first order $\omega \simeq \eta/2$, and Equations (11) and (18) give

$$\tilde{W}^2 \simeq (1 - 2\eta) W^2, \quad \tilde{R} \simeq -3 \square\eta. \quad (20)$$

The Weyl channel measures the local conformal amplitude; the induced scalar curvature measures its second derivative. A small deformation that varies over a short length can therefore generate a larger Ricci response than a large one that is nearly uniform. Define the gradient scale

$$L_q^{-2} \equiv \left| \frac{\square\eta}{\eta} \right|, \quad \text{so that} \quad |\tilde{R}| \simeq 3 |\eta| L_q^{-2}, \quad (21)$$

which separates the amplitude η from the geometric gradient scale L_q as two independent characteristics of a quantum-conditioned geometry.

4.3. Closed-Form First-Order Response

Let the deformation decay as a power at large radius,

$$\eta(x) = \varepsilon_0 x^{-p}, \quad \varepsilon_0 \ll 1, \quad (22)$$

which is how curvature- and acceleration-sourced profiles alike behave asymptotically. Expanding the exact invariants (Appendix A) gives, to first order in ε_0 ,

$$M^4 \Delta W^2 = -\frac{96 \varepsilon_0}{x^{p+8}} (x - q^2)^2 = -2\eta M^4 W^2, \quad (23)$$

$$M^4 \Delta K_{\text{Ric}} = \frac{4 q^2 \varepsilon_0}{x^{p+8}} \left[p(p+3) x^2 - 2p(p+4) x + (p^2 + 5p - 4) q^2 \right]. \quad (24)$$

We now read off three consequences.

4.4. Charge Activation of the Leading-Order Ricci Contribution to K

The statement below is about the Ricci-sector contribution to the *Kretschmann response*, not about the induced Ricci curvature itself. The two are different, as the remark after the proposition makes clear.

Proposition 1 (Charge activation). *To first order in the deformation, $\Delta K_{\text{Ric}} \propto Q^2$. On a Ricci-flat background ($Q = 0$, Schwarzschild) the Ricci-channel contribution to K vanishes at first order, so $\Delta K = \Delta W^2 + \mathcal{O}(\eta^2)$ and $\mathcal{D} = \mathcal{O}(\eta) \rightarrow 0$.*

The reason is structural. If $R_{\mu\nu} = 0$ then the deformed Ricci tensor is itself first order, $\tilde{R}_{\mu\nu} = \delta R_{\mu\nu} = \mathcal{O}(\eta)$, so both $\tilde{R}_2$ and $\tilde{R}^2$ are second order. The cross term $2R^{\mu\nu} \delta R_{\mu\nu}$, which supplies the first-order Ricci response, needs a non-vanishing background Ricci tensor, that is, charge. Note that $\tilde{R}$ itself is first order and non-zero even on Schwarzschild, Equation (20); it is its contribution to K that is second order. The charge is thus a genuine control parameter in this model space: only a background that already carries Ricci curvature converts a conformal quantum deformation into an Einstein-sector response at leading order.

4.5. The Weyl-Pure Radius

The bracket in Equation (24) is a quadratic in x with a root outside the horizon,

$$x_W = \frac{p(p+4) + \sqrt{p^2(p+4)^2 - p(p+3)(p^2 + 5p - 4)q^2}}{p(p+3)}, \quad (25)$$

where the leading Ricci-channel response vanishes and the deformation is, to first order, purely Weyl-like. We have checked numerically that for $p \in [0.5, 8]$ and $q \in [0, 1]$ the discriminant in Equation (25) stays positive and $x_W > r_+/M$ throughout, with a minimum margin of 0.17, while the second root always lies inside the outer horizon. The index (13) accordingly develops a sharp node at x_W , which the numerical evaluation of Section 9 displays. Since x_W depends only on p and q , its position predicts the deformation *profile* rather than its strength. Numerical values are given there.

4.6. Amplitude-Free Asymptotic Index

Taking $x \rightarrow \infty$ in Equations (23) and (24), the leading terms are $M^4 \Delta W^2 \rightarrow -96\varepsilon_0 x^{-(p+6)}$ and $M^4 \Delta K_{\text{Ric}} \rightarrow 4p(p+3)q^2\varepsilon_0 x^{-(p+6)}$, so

$$\mathcal{D}_\infty = \frac{p(p+3)q^2}{p(p+3)q^2 + 24} \quad (26)$$

The amplitude ε_0 , which carries the minimal-length parameter $\beta_0 \ell_P^2$ and is at present only loosely bounded [6,18,19], has canceled exactly. The asymptotic response index depends on the charge-to-mass ratio and on the falloff exponent alone. Equation (26) inverts to

$$p(p+3) = \frac{24 \mathcal{D}_\infty}{q^2 (1 - \mathcal{D}_\infty)}, \quad (27)$$

so measuring $\mathcal{D}_\infty$ on a background of known q fixes the radial profile of the quantum deformation without any knowledge of its magnitude.

The Ricci channel dominates asymptotically, $\mathcal{D}_\infty > \frac{1}{2}$, if and only if $p(p+3)q^2 > 24$. Since $q \leq 1$ for a black hole, this needs

$$p > p_{\text{crit}} = \frac{1}{2}(\sqrt{105} - 3) \simeq 3.6235, \quad (28)$$

which is reached only near extremality. For the two profiles of Section 9 the condition reads $q > \sqrt{6/7} \simeq 0.926$ at $p = 4$, while at $p = 3$ it would need $q > 2/\sqrt{3} \simeq 1.155$ and so is never met. Deformations sourced by the curvature scale, $\eta \propto \sqrt{K} \sim r^{-3}$ with $p = 3$, therefore stay Weyl-dominated asymptotically on any RN background, while acceleration-sourced ones, $\eta \propto a^2 \sim r^{-4}$ with $p = 4$, can become Ricci-dominated close to extremality. Two model classes that look the same at the level of K are thus sharply separated.

5. Horizon and Extremal Limit

At a non-extremal horizon $f(r_+) = 0$ with $f'(r_+) \neq 0$, so Equation (17) collapses to $\square\omega|_{r_+} = f'(r_+) \omega'(r_+)$ for regular ω , and

$$\tilde{R}(r_+) = -6e^{-2\omega_+} f'(r_+) \omega'(r_+) = -12\kappa e^{-2\omega_+} \omega'(r_+) = -24\pi T_H e^{-2\omega_+} \omega'(r_+), \quad (29)$$

where $\kappa = f'(r_+)/2 = \sqrt{M^2 - Q^2}/r_+^2$ is the surface gravity and $T_H = \kappa/2\pi$ the Hawking temperature [65,66]. The scalar-curvature response at the horizon is therefore not an independent quantity: it is the surface gravity times the radial gradient of the conformal field.

Equation (29) rests on the regularity assumption above, and that assumption is not automatic. It holds for a conformal profile built from a curvature invariant, which is finite on the horizon. It fails for a profile tied to the proper acceleration of *static* observers, because that congruence is itself singular at r_+ . Of the two representative profiles used in Section 9, only model A may therefore be carried to the horizon. Model B is excluded from the horizon limit, and the near-horizon behavior of its curves in the figures of Section 9 is a breakdown of the weak-deformation expansion, not a physical effect.

Two limits follow. For $Q \rightarrow 0$ the Ricci sector of the reference metric vanishes but $\tilde{R}(r_+) = -3e^{-2\omega_+} \omega'(r_+)/M \neq 0$, so quantum-induced Ricci curvature exists on Schwarzschild even though, by Proposition 1, it leaves no first-order trace in K . For $Q \rightarrow M$ the surface gravity vanishes and

$$\tilde{R}(r_+) \propto \sqrt{1 - q^2} \rightarrow 0, \quad (30)$$

so the leading horizon scalar response is suppressed at extremality. The extremal limit is delicate for independent reasons: extremal RN horizons support the Aretakis instability for linear scalar perturbations, while the non-extremal exterior is stable [67–69]. The suppression we find refers only to the leading scalar-curvature response of the conformal deformation. Both statements are checked numerically in Section 9. There $\tilde{R}(r_+)$ is obtained from the Riemann tensor of $\tilde{g}$ reconstructed from the metric components, not from Equation (29), and over the six decades in $\sqrt{1 - q^2}$ shown the two differ by at most 1×10^{-44} in relative terms, a figure set by how close to the horizon the reconstruction is evaluated rather than

by the relation. A horizon relation of this kind connects the present analysis to the wider discussion of non-singular and quantum-corrected black-hole interiors [70–72].

6. Null Rigidity

Modified black holes are now tested with light more often than with anything else: photon spheres, shadows and lensing set against Event Horizon Telescope data [39–44,47,73–78]. For the conformal sector that entire program returns nothing, and it does so exactly.

Theorem 1 (Null rigidity). *Let $\tilde{g}_{\mu\nu} = C(x)g_{\mu\nu}$ with $C > 0$ smooth and otherwise arbitrary. Then the null geodesics of $\tilde{g}$ and of g coincide as point sets, differing only by reparametrization of the affine parameter. Consequently every observable that is a functional of the unparametrized congruence alone is identical in the two geometries. Quantities that reference the affine parameter itself, the optical scalars among them, are not covered by the statement and do change. For a static, spherically symmetric g with $C \rightarrow 1$ at spatial infinity, and with the time coordinate given the same asymptotic normalization in both geometries, the invariant quantities include the coordinate location r_{ph} of the photon sphere, the critical impact parameter, the shadow angular radius measured by a distant observer, the weak- and strong-field deflection angles, the lensing image positions and the Shapiro time delay. The areal radius of the photon sphere, $\sqrt{C(r_{\text{ph}})} r_{\text{ph}}$, is not invariant.*

Proof. Under Equation (4) a null vector k^μ obeys $k^\nu \tilde{\nabla}_\nu k^\mu = k^\nu \nabla_\nu k^\mu + 2(k^\nu \partial_\nu \omega)k^\mu - (k \cdot k) \nabla^\mu \omega$, and the last term vanishes because k is null with respect to g and therefore also with respect to $\tilde{g}$. So a g -geodesic null curve is a $\tilde{g}$ -geodesic null curve up to the reparametrization $d\tilde{\lambda} = C d\lambda$, and vice versa. The first sentence follows; the second is immediate since the listed quantities are built from the trajectories and from ratios of conserved quantities along them.

For the explicit spherically symmetric statements write $\tilde{g} = -\tilde{A} dt^2 + \tilde{B} dr^2 + \tilde{C} d\Omega^2$ with $\tilde{A} = Cf$, $\tilde{B} = C/f$, $\tilde{C} = Cr^2$. The photon sphere is the stationary point of $\tilde{C}/\tilde{A}$, and

$$\frac{\tilde{C}}{\tilde{A}} = \frac{r^2}{f(r)}, \quad (31)$$

with C canceling identically, so the coordinate radius r_{ph} , the critical impact parameter $b_c = r_{\text{ph}}/\sqrt{f(r_{\text{ph}})}$ and the shadow radius $R_{\text{sh}} = b_c$ seen from infinity are unchanged. The sphere itself is not the same object in the two geometries: its areal radius is $\sqrt{C(r_{\text{ph}})} r_{\text{ph}}$ in $\tilde{g}$, so what the conformal factor leaves alone is where the orbit sits in the coordinates shared by the two metrics and what a distant observer measures, not the proper size of the sphere. The null orbit equation reads

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{E^2 r^4}{L^2} + 2Mr - Q^2 - r^2, \quad (32)$$

again free of C , so the deflection integral is unchanged in both the Gauss–Bonnet [78] and Tsukamoto [40] treatments; along a null ray $ds^2 = 0$ gives $(dt/dr)^2 = f^{-2}(1 - b^2 f/r^2)^{-1}$, so the Shapiro delay [79,80] is unchanged, the coordinate t carrying the same normalization at infinity in both geometries because $C \rightarrow 1$ there. All three are verified symbolically in Appendix A. $\square$

Two consequences deserve to be stated without hedging. The first is negative and practical. EHT measurements of M87* and Sgr A* [73–75], solar light bending [81] and the Cassini Shapiro delay [80] constrain the strictly conformal sector not just weakly but

not at all however precise they become; improving them does not help. The qualifier matters, and we keep it everywhere: the statement is about the conformal part of the deformation, not about quantum-deformed metrics in general, and the second consequence below turns exactly on that distinction. This is worth saying plainly because the literature on quantum-corrected shadows continues to grow, and a shadow computed from a metric of the form (1) reproduces the answer for the undeformed metric.

The second is a usable test. Because the statement is exact, Equation (31) is a null test of conformality itself: in the underlying constructions the conformal form is only the leading behavior, and the full deformed metric contains a term built from tangent covectors that is not proportional to $g_{\mu\nu}$ [35]. Any measured deviation of a shadow or a deflection angle from its undeformed value is therefore a measurement of the non-conformal part alone, cleanly separated from everything the conformal factor does.

A corollary that will matter in the next section: eikonal quasinormal modes are controlled by the photon sphere and its Lyapunov exponent [48,49], so Theorem 1 already implies that the deformation cannot shift the ringdown at large multipole. Section 7 confirms this by direct computation and, more usefully, says how fast the response switches off.

7. The Wave Sector: A Reduction Theorem

Null rigidity is a statement about rays, which is to say about the leading term of a geometric-optics expansion. Waves at finite frequency are a different matter because the massless scalar equation is not conformally invariant in four dimensions. What follows makes that precise, and the outcome is stronger than a correction: the entire deformation can be traded for one scalar function on the undeformed background.

Theorem 2 (Wave-sector reduction). *Let $\tilde{g}_{\mu\nu} = C(x)g_{\mu\nu} = e^{2\omega}g_{\mu\nu}$ in four dimensions and let φ satisfy the minimally coupled massless equation $\tilde{\square}\varphi = 0$ on $\tilde{g}$. Then $\psi \equiv e^\omega\varphi = \sqrt{C}\varphi$ satisfies, exactly,*

$$(\square - V_\omega)\psi = 0, \quad V_\omega \equiv \square\omega + (\nabla\omega)^2 = \frac{\square\sqrt{C}}{\sqrt{C}}, \quad (33)$$

where $\square$ and ∇ are those of the undeformed metric g . Moreover

$$V_\omega = -\frac{1}{6}(C\tilde{R} - R), \quad (34)$$

with $\tilde{R}$ as the Ricci scalar of $\tilde{g}$ and R as that of g .

Proof. In n dimensions $\tilde{\square}\varphi = e^{-2\omega}[\square\varphi + (n-2)\nabla^\mu\omega\nabla_\mu\varphi]$, so at $n = 4$ the equation is $\square\varphi + 2\nabla\omega \cdot \nabla\varphi = 0$. Substituting $\varphi = e^{-\omega}\psi$ and using $\square e^{-\omega} = e^{-\omega}[(\nabla\omega)^2 - \square\omega]$ gives

$$\square\varphi + 2\nabla\omega \cdot \nabla\varphi = e^{-\omega}[\square\psi - (\square\omega + (\nabla\omega)^2)\psi], \quad (35)$$

which is Equation (33); the cross terms in $\nabla\omega \cdot \nabla\psi$ cancel identically, which is the whole content of the substitution. Equation (34) then follows from the conformal law (10), $\tilde{R} = e^{-2\omega}[R - 6\square\omega - 6(\nabla\omega)^2]$, on solving for $\square\omega + (\nabla\omega)^2$. Both identities are verified from the metric components in Appendix A. $\square$

Equation (34) is the result we regard as central, and it is worth reading slowly. The deformation is a symmetric tensor field with, in principle, ten functions of position; conformality reduces that to one function C ; and Theorem 2 shows that as far as wave propagation is concerned even C enters only through the combination $-\frac{1}{6}(C\tilde{R} - R)$. That combination is, up to the conformal weight, the induced Ricci scalar. So the Weyl invariant—which on

any black-hole background carries almost all of the curvature, and all of it when $Q = 0$ —does not enter the potential a massless scalar propagates in. The sector-resolved index of Section 3 thereby acquires a physical edge it does not have on its own. It is worth being exact about how much: V_ω depends on the trace of the deformed Ricci tensor alone, while ΔK_{Ric} is built from the whole tensor, so $\mathcal{D}$ tells us which sector the response sits in, not what fraction of it a wave can see.

Corollary 2 (Coupling dependence). *A massless scalar carrying the four-dimensional conformal coupling, so that $(\tilde{\square} - \frac{1}{6}\tilde{R})\varphi = 0$, is exactly insensitive to the deformation: after the field redefinition $\psi = \sqrt{C}\varphi$ it obeys $(\square - \frac{1}{6}R)\psi = 0$ on the undeformed metric. What a scalar field sees is therefore controlled by the departure of its coupling from $\xi = \frac{1}{6}$ as much as by the geometry.*

This is the sharpest statement we can make about where the effect resides. It also fixes the limits of the analysis honestly: nothing here bears on gravitational perturbations, whose equations are not determined by the effective metric alone but by the field equations of the underlying theory, which the conformal ansatz does not supply. We treat the massless scalar as the standard proxy for the ringdown as is usual in this literature [49,82], and say so again in Section 10.5.

Ringdown Response and Its Eikonal Suppression

Everything in this subsection concerns the minimally coupled massless scalar of Theorem 2, which is the standard proxy for a ringdown and not a substitute for one. Gravitational perturbations obey equations the conformal ansatz does not determine, and by Corollary 2 even a scalar responds differently if its coupling differs; the shifts below should be read accordingly.

Take g static and spherically symmetric with $\omega = \omega(r)$, and separate $\psi = e^{-i\Omega t} Y_{lm}(\theta, \phi) u(r)/r$. Equation (33) becomes a Regge–Wheeler problem [53,82] in the tortoise coordinate of the undeformed metric, $dr_* = dr/f$,

$$\frac{d^2 u}{dr_*^2} + [\Omega^2 - V_{\text{eff}}(r)] u = 0, \quad V_{\text{eff}} = f \left[\frac{l(l+1)}{r^2} + \frac{f'}{r} \right] + f V_\omega, \quad (36)$$

verified in Appendix A. The deformation enters only through the last term. Because the undeformed potential is stationary at its peak r_0 , first-order perturbation theory in the WKB scheme of Schutz, Will and Iyer [83,84] gives the frequency shift without needing the shift of r_0 , which contributes at second order:

$$\delta(\Omega^2) = f(r_0) V_\omega(r_0) + \mathcal{O}(\varepsilon_0^2), \quad \frac{\delta\Omega}{\Omega_0} = \frac{f(r_0) V_\omega(r_0)}{2\Omega_0^2}. \quad (37)$$

At large multipole $r_0 \rightarrow r_{\text{ph}}$ and $\Omega_0^2 \rightarrow l(l+1)f(r_{\text{ph}})/r_{\text{ph}}^2$, so

$$\frac{\delta\Omega}{\Omega_0} = \frac{r_{\text{ph}}^2 V_\omega(r_{\text{ph}})}{2l(l+1)} + \mathcal{O}(l^{-3}). \quad (38)$$

The relative shift falls off as l^{-2} . This is not an accident of the WKB approximation but the requirement of Theorem 1: eikonal modes are the photon sphere in disguise [48,49], the photon sphere is rigid, so the response must switch off in that limit. What Equation (38) adds is the rate. For this field the correction is a non-eikonal effect, carried by the lowest multipoles, and a search that leans on the high- l part of a spectrum is looking where the leading contribution is guaranteed to be small.

For the power-law profile (22) the potential is, to first order,

$$V_\omega = \frac{p \varepsilon_0}{2r^4} \left(\frac{M}{r} \right)^p \left[(p-1)r^2 - 2Mpr + (p+1)Q^2 \right] = \frac{1}{2} \square \eta, \quad (39)$$

and on Schwarzschild, where $r_{\text{ph}} = 3M$,

$$\frac{\delta\Omega}{\Omega_0} = \frac{p(p-3)}{12 \cdot 3^p l(l+1)} \varepsilon_0 + \mathcal{O}(\varepsilon_0^2, l^{-3}). \quad (40)$$

Figure 2 shows both the profile dependence and the multipole falloff. Three features are worth extracting. The numerical prefactor is small and falls steeply with p : at $l = 2$ it is $\varepsilon_0/324$ for $p = 4$ and $\varepsilon_0/162$ for $p = 5$. The sign changes at $p = 3$. And at $p = 3$ exactly the shift vanishes, because the photon sphere then sits at the zero of V_ω : from Equation (39) the bracket vanishes at $r = 2Mp/(p-1)$, which equals $3M$ when $p = 3$. The curvature-sourced profile $\eta \propto \sqrt{K} \sim r^{-3}$ of Section 9, the most natural invariant choice, is thus *ringdown silent* at leading order.

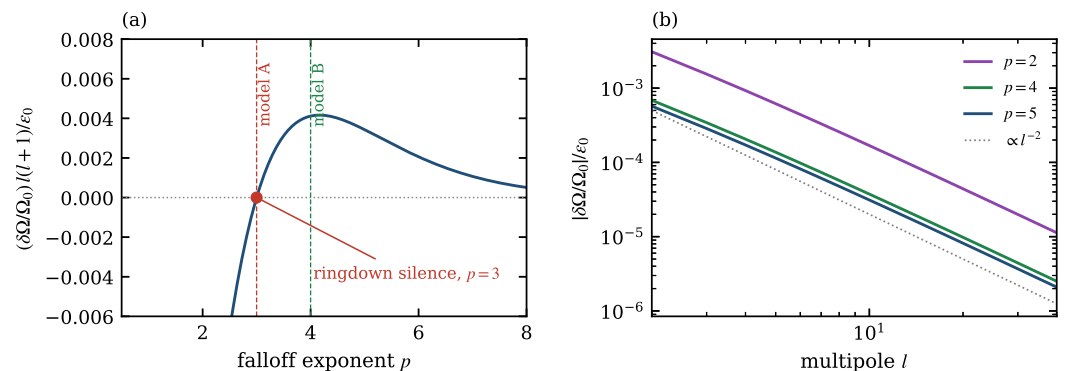


Figure 2. Ringdown response of the conformal deformation on a Schwarzschild background. (a) The multipole-independent coefficient $(\delta\Omega/\Omega_0) l(l+1)/\varepsilon_0 = p(p-3)/(12 \cdot 3^p)$ of Equation (40) against the falloff exponent. It changes sign at $p = 3$, where the photon sphere coincides with the zero of V_ω and the leading response vanishes: the curvature-sourced profile of Equation (51) is ringdown silent. Dashed lines mark the two profiles of Section 9; model B is an illustrative benchmark, not a candidate physical model. (b) The same shift against multipole for three exponents, with the l^{-2} reference of Equation (38) dotted. The eikonal suppression required by Theorem 1 is visible as the common slope: a conformal correction is a low- l effect and is absent from the high-multipole part of a ringdown.

On RN the silence condition generalizes to

$$(p-1)r_{\text{ph}}^2 - 2Mpr_{\text{ph}} + (p+1)Q^2 = 0, \quad r_{\text{ph}} = \frac{1}{2} \left(3M + \sqrt{9M^2 - 8Q^2} \right), \quad (41)$$

which for $q \in [0, 1]$ selects p between 3 and 3.30. That a whole one-parameter family can hide from the ringdown is what makes the timelike channel of the next section necessary rather than merely complementary.

8. The Timelike Sector

Theorem 1 closes the null channel and Theorem 2 opens the wave channel. The third and last is the timelike one, where the deformation acts directly on the trajectories rather than on a field propagating along them. We treat the weak field first because it is where the existing bounds live, and then the strong field, where the response is larger and the observables are the ones an accretion disk actually produces.

8.1. Orbit Equation

Timelike geodesics are not conformally invariant, and this is where the deformation becomes visible. Bound orbits have already been used to probe nonperturbative quantum corrections to the RN geometry [85], which makes them the natural instrument here as well. For the deformed metric with radial $\eta(r)$, and with $u = 1/r$, the conserved energy E and angular momentum L give the exact orbit equation

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{L^2} [E^2 - f(1 + \eta) - fL^2u^2], \quad (42)$$

whose derivative is

$$u'' + u = \frac{M}{L^2} + 3Mu^2 - \frac{Q^2u}{L^2} - 2Q^2u^3 - \frac{1}{2L^2} [f_{,u}\eta + f\eta_{,u}]. \quad (43)$$

Setting $\eta = 0$ recovers the RN orbit equation of the reference metric, and $Q = 0$ in addition recovers the Schwarzschild case with its standard perihelion advance $\Delta\phi_{\text{GR}} = 6\pi M^2/L^2$ [50,51,86].

8.2. Perihelion Precession

On a Schwarzschild background the two representative profiles of Section 9 are $\eta_A = 4\sqrt{3}\varepsilon_0(Mu)^3$ and $\eta_B = \varepsilon_0(Mu)^4/f$, and the last term of Equation (43) is $-(2L^2)^{-1}d(f\eta)/du$. For model B this is exactly $-2\varepsilon_0M^4u^3/L^2$; for model A it is $-6\sqrt{3}\varepsilon_0M^3u^2/L^2$ plus a term smaller by a factor of order Mu . Each is a perturbation $g(u)$ of the Newtonian equation $u'' + u = M/L^2$, and to the first order in g , the advance of the perihelion per orbit is given by the standard result for a perturbed Kepler orbit [51,56]

$$\Delta\phi = \frac{L^2}{Me} \int_0^{2\pi} g(u_0(\phi)) \cos\phi \, d\phi, \quad u_0 = \frac{M}{L^2}(1 + e \cos\phi), \quad (44)$$

which for $g = 3Mu^2$ returns $\Delta\phi_{\text{GR}} = 6\pi M^2/L^2$ for any eccentricity. No near-circular expansion is involved; the approximation is first order in ε_0 and leading order in the compactness $M^2/L^2 = M/[a(1 - e^2)]$, with a the semi-major axis. Writing $\Delta\phi = \Delta\phi_{\text{GR}}(1 + \delta_{\text{prec}})$,

$$\delta_{\text{prec}}^{(A)} = -2\sqrt{3}\varepsilon_0 \frac{M^2}{L^2} \quad (p = 3), \quad \delta_{\text{prec}}^{(B)} = -\varepsilon_0 \left(\frac{M^2}{L^2}\right)^2 \left(1 + \frac{e^2}{4}\right) \quad (p = 4), \quad (45)$$

each up to relative corrections of order M^2/L^2 . The model-A perturbation has the same power of u as the general-relativistic term, so its coefficient is independent of e ; the model-B perturbation goes as u^3 and picks up the factor $1 + e^2/4$, which is 1.011 for Mercury and 1.196 for the highly eccentric S2 orbit and must be kept there. In both cases ε_0 denotes the same physical quantity, $\beta_0\ell_p^2/M^2$; only the radial shape of η differs. The two expressions scale differently with the compactness, and that is what lets the test tell the profiles apart at a fixed deformation strength. Both were checked against the direct integration of Equation (43) over the full range of eccentricities used here (Appendix A).

8.3. Bounds from Mercury and from the S2 Orbit

We use two systems. Mercury's perihelion advance is measured as $(42.9794 \pm 0.0030)''$ per century against the general-relativistic $42.9814''$ [86], a fractional tolerance of 7.0×10^{-5} , with $M^2/L^2 = 2.66 \times 10^{-8}$. In the strong-field regime, the GRAVITY collaboration has detected the Schwarzschild precession of the star S2 orbiting Sgr A* with $f_{\text{SP}} = 1.10 \pm$

0.19 [87,88], a fractional tolerance of 0.19 but a far larger compactness, $M^2/L^2 = 1.94 \times 10^{-4}$. The bounds are collected in Table 2 and shown in Figure 3.

Table 2. Constraints on the dimensionless deformation amplitude ε_0 from timelike observables, together with the null observables that are exactly insensitive to it within the strictly conformal sector (Theorem 1). Each “none” entry is a direct consequence of that theorem: the observable is built from null geodesics alone, which a conformal factor leaves unchanged, so no precision of measurement can constrain ε_0 through it. The statement concerns the strictly conformal sector, not quantum-deformed metrics in general. Model A has $p = 3$; model B has $p = 4$ and is an observer-dependent illustrative benchmark rather than a candidate physical realization (Section 9). The model-B bounds include the eccentricity factor of Equation (45).

Observable	System	Model A ($p = 3$)	Model B ($p = 4$)
Perihelion precession	Mercury	$\varepsilon_0 \leq 7.6 \times 10^2$	$\varepsilon_0 \leq 9.8 \times 10^{10}$
Perihelion precession	S2 / Sgr A*	$\varepsilon_0 \leq 2.8 \times 10^2$	$\varepsilon_0 \leq 4.2 \times 10^6$
Light deflection	Sun	none (Theorem 1)	none (Theorem 1)
Shapiro delay	Cassini	none (Theorem 1)	none (Theorem 1)
Shadow diameter	M87*, Sgr A*	none (Theorem 1)	none (Theorem 1)

Three points deserve comment. The strong-field system gives the better bound for the more slowly decaying profile despite its much larger observational uncertainty, because the compactness enters linearly; this is a concrete argument for the value of Galactic Center astrometry in constraining minimal-length deformations. The two profiles are separated by four orders of magnitude in the S2 bound, so the observable discriminates between profiles rather than merely limiting an amplitude. Most importantly, these bounds are far weaker than the theoretical expectation $\varepsilon_0 \sim \beta_0 \ell_P^2 / M^2 \lesssim 10^{-76}$ for any astrophysical mass; in terms of the RGUP parameter the Mercury bound reads $\beta_0 \leq 6.3 \times 10^{78}$, of the order of earlier Solar System estimates and far weaker than laboratory bounds [6,18,19]. The value of the analysis is not that it excludes the deformation. It is that it identifies which observables could in principle do so, and shows that the structure of the response, Equation (26), is fixed independently of this poorly constrained amplitude.

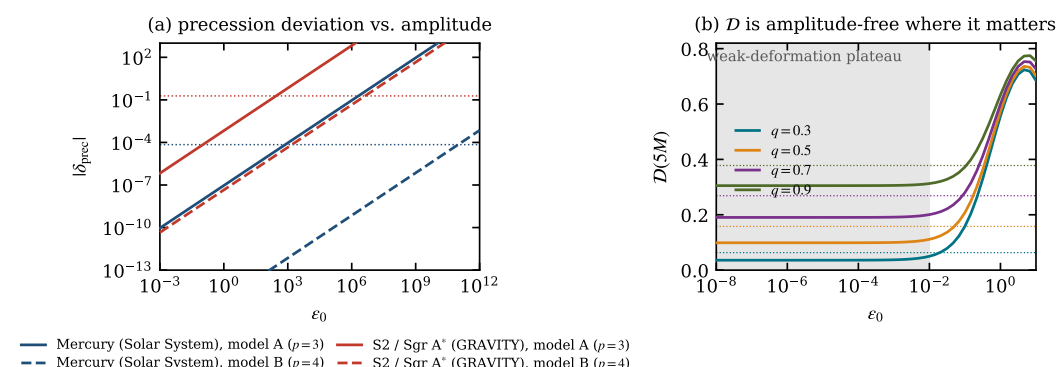


Figure 3. (a) Fractional deviation of the perihelion precession, Equation (45), against the deformation amplitude, for the two profiles and the two systems; dotted horizontal lines are the observational tolerances, and the model-B curves include the eccentricity factor $1 + e^2/4$. Model B is shown as an illustrative benchmark only. (b) The response index at $x = 5$ as a function of ε_0 . The shaded band marks the weak-deformation plateau, in which $\mathcal{D}$ is independent of the amplitude; the entire physically expected range, $\varepsilon_0 \sim \beta_0 \ell_P^2 / M^2$, lies far inside it. Dotted lines are the analytic asymptotes of Equation (26).

8.4. The Innermost Stable Circular Orbit and the Disk

Perihelion precession is a weak-field probe and, as Table 2 shows, not a competitive one. The strong-field timelike observables are better placed, and they are also the ones that carry the accretion-disk phenomenology. Circular orbits of $\tilde{g}$ are governed by

$$E^2 = U(r), \quad U(r) = \frac{fL^2}{r^2} + Cf, \quad (46)$$

which follows from Equation (42) on setting $du/d\phi = 0$; the factors of C^{-2} multiplying $\dot{r}^2$ drop out of the conditions $U' = U'' = 0$ because $E^2 = U$ there. For $\eta = \varepsilon_0(M/r)^p$ on a Schwarzschild background, solving those two conditions to first order gives

$$r_{\text{ISCO}} = 6M \left[1 - \frac{p(p-2)}{6^p} \varepsilon_0 \right], \quad (47)$$

and, for the specific energy at the ISCO and hence for the radiative efficiency $\epsilon_{\text{rad}} = 1 - E_{\text{ISCO}}$ of a thin disk [89–91],

$$\delta\epsilon_{\text{rad}} = \frac{\sqrt{2}(p-2)}{6^{p+1}} \varepsilon_0, \quad \epsilon_{\text{rad}}^{\text{GR}} = 1 - \frac{2\sqrt{2}}{3} \simeq 0.0572. \quad (48)$$

The orbital frequency measured at infinity, $\Omega_\phi = Lf/(Er^2)$, evaluated at the shifted orbit, gives a third and independent combination,

$$\frac{\delta\Omega_{\text{ISCO}}}{\Omega_{\text{ISCO}}} = \frac{3p(2p-5)}{4 \cdot 6^p} \varepsilon_0, \quad M\Omega_{\text{ISCO}}^{\text{GR}} = 6^{-3/2}. \quad (49)$$

These three were obtained by implicit differentiation of $U' = U'' = 0$ at $\varepsilon_0 = 0$ rather than by series expansion, because a series in ε_0 of a profile carrying a symbolic exponent is easy to truncate incorrectly; each was then confirmed against an exact numerical root (Appendix A).

For the two profiles of Section 9 the numbers are $\delta r_{\text{ISCO}} = -\varepsilon_0 M/12$, $\delta\epsilon_{\text{rad}} = +\sqrt{2}\varepsilon_0/1296$ and $\delta\Omega/\Omega = +\varepsilon_0/96$ at $p = 3$; and $\delta r_{\text{ISCO}} = -\varepsilon_0 M/27$, $\delta\epsilon_{\text{rad}} = +\sqrt{2}\varepsilon_0/3888$ and $\delta\Omega/\Omega = +\varepsilon_0/144$ at $p = 4$. A positive deformation with $p > 2$ pulls the inner edge of the disk inwards and, consistently, raises the binding energy released, so the disk becomes marginally brighter for a given accretion rate: $\delta\epsilon_{\text{rad}}/\epsilon_{\text{rad}}^{\text{GR}} = 1.9\% \varepsilon_0$ at $p = 3$ and $0.64\% \varepsilon_0$ at $p = 4$.

What makes the timelike sector useful is that these three responses are not proportional to one another. The radius and the efficiency share a zero at $p = 2$, since both are governed by $(p-2)$, but the frequency does not: it vanishes at $p = 5/2$ instead, and it changes sign there. Continuum-fitting measurements, which are sensitive to r_{ISCO} and ϵ_{rad} , and timing measurements of quasi-periodic oscillations tied to the ISCO frequency [92,93], therefore constrain different combinations of the same deformation, and a profile invisible to one of them is visible to the other. Together with the ringdown silence at $p = 3$ this gives a small completeness statement.

Proposition 2 (Complementarity). *Within the family $\eta = \varepsilon_0(M/r)^p$, $p > 0$, on a Schwarzschild background, the leading ringdown response (40) vanishes only at $p = 3$; the leading disk-continuum response (47) and (48) only at $p = 2$; and the leading disk-timing response (49) only at $p = 5/2$. The three sets are pairwise disjoint, so no profile in the family is silent in more than one channel, and every profile produces a first-order signature in at least two of the three, while none produces any signature at all in the null channel.*

Figure 4 makes the disjointness visible. Naming the three responsive channels once, they are the scalar ringdown, the disk continuum (r_{ISCO} and ϵ_{rad} together, since both carry the same factor $p - 2$) and the disk timing. The proposition is elementary but it is the practical summary of the paper: within the observables treated here, the conformal sector has one closed door and three open ones that cannot be shut together.

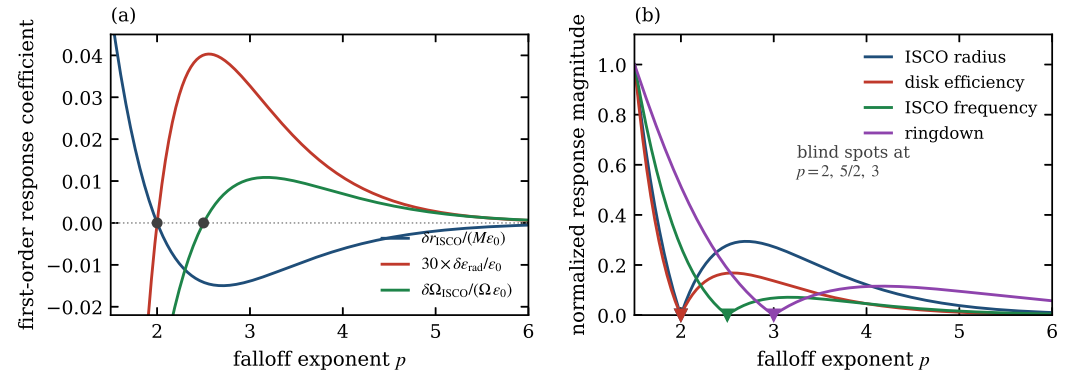


Figure 4. Complementarity of the responsive channels. (a) The three first-order timelike coefficients of Equations (47)–(49) against p ; the efficiency curve is scaled by 30 to make it visible on the same axes. Gray markers show the zeros. The ISCO radius and the disk efficiency share the factor $(p - 2)$ and vanish together, while the orbital frequency instead vanishes at $p = 5/2$ and changes sign there. (b) The magnitudes of all four responsive coefficients, each normalized to its own maximum on the plotted range, with the blind spots marked by triangles. The zeros at $p = 2$, $p = 5/2$ and $p = 3$ are pairwise distinct, which is the content of Proposition 2: no profile in the family is silent in more than one channel. The null channel, which would be identically zero on these axes for every p , is not plotted.

8.5. What Precision Would Be Needed

It is better to say what the bounds of Table 2 cannot do than to present them as constraints. With $\epsilon_0 \sim \beta_0 \ell_p^2/M^2$, any solar-mass or supermassive object gives $\epsilon_0 \lesssim 10^{-76}$, so no observable discussed here is within seventy orders of magnitude of the effect. The useful question is the inverse one: at what compactness would each channel reach a given β_0 ?

Inverting Equations (40), (48) and (49) at $p = 4$, a fractional measurement precision σ probes

$$\beta_0 \lesssim \sigma \frac{M^2}{\ell_p^2} \times \begin{cases} 81 l(l+1), & \text{ringdown at multipole } l, \\ 6^5 \epsilon_{\text{rad}}^{\text{GR}}/(2\sqrt{2}) \simeq 1.6 \times 10^2, & \text{disk efficiency,} \\ 4 \cdot 6^4/(3 \cdot 4 \cdot 3) \simeq 1.4 \times 10^2, & \text{ISCO frequency.} \end{cases} \quad (50)$$

For a stellar-mass hole, $M^2/\ell_p^2 \sim 10^{76}$, so even $\sigma = 10^{-3}$ returns $\beta_0 \lesssim 10^{75}$ —worse by sixty-eight orders than the mechanical oscillator bound $\beta_0 \lesssim 5 \times 10^6$ of Ref. [18]. The scaling $\beta_0 \propto M^2$ says where the situation reverses: a hole light enough for $M \sim 10^3 \ell_p$, that is a primordial relic of order $10^{-5} g$, would put $\beta_0 \sim 10^6$ within reach of a percent-level measurement. No such object is observed, and we do not suggest that one will be. The point of Equation (50) is that the obstruction is a mass scale and not a missing observable, which is a different and more tractable statement than the one the precession bounds appear to make.

9. Numerical Evaluation

9.1. Two Deformation Profiles

To make things concrete without committing to a particular quantization we evaluate two radial profiles obeying Equation (19):

$$\text{Model A (curvature-sourced):} \quad \eta_A(r) = \varepsilon_0 M^2 \sqrt{K(r)} \sim r^{-3}, \quad p = 3, \quad (51)$$

$$\text{Model B (acceleration-sourced):} \quad \eta_B(r) = \varepsilon_0 M^2 a^2(r), \quad a^2 = \frac{[f'(r)]^2}{4f(r)}, \quad p = 4. \quad (52)$$

Model A is a simple curvature-invariant choice—one of several that one could build from R , $\mathcal{R}_2$, W^2 or K —with the practical merits that the deformation is set by the local curvature scale, it is regular on the horizon, and it needs no observer. Model B is the Caianiello-type realization [28–32], in which the deformation follows the proper acceleration of a static observer. We do not offer it as an invariant realization of the quantum deformation since a metric should not depend on a preferred family of observers. It is included as an observer-dependent phenomenological benchmark, because its steeper falloff ($p = 4$ rather than $p = 3$) exercises the profile dependence of the diagnostic. It diverges at the horizon, where the static congruence ceases to exist, and is excluded from the horizon analysis of Section 5. Figure 5 shows the two profiles and the gradients of ω that drive the Ricci channel.

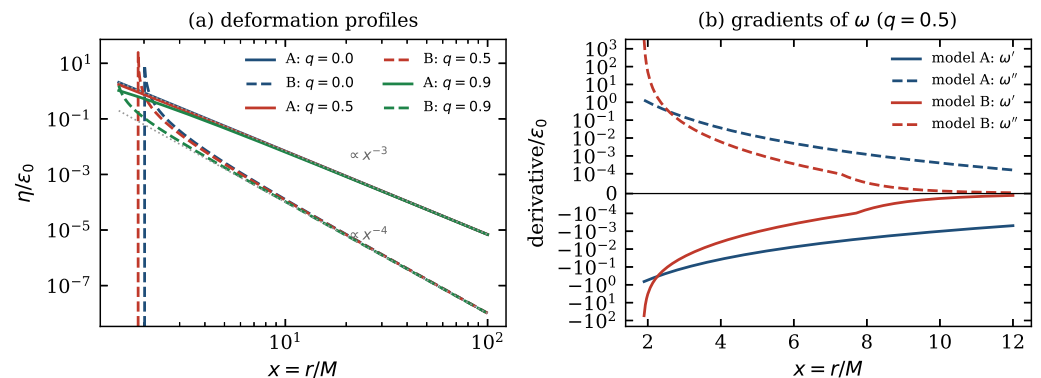


Figure 5. (a) The two deformation profiles of Equations (51) and (52), normalized by the amplitude ε_0 ; gray dotted lines show the asymptotic power laws x^{-3} and x^{-4} . (b) The first and second radial derivatives of the conformal field $\omega = \frac{1}{2} \ln(1 + \eta)$ for $q = 0.5$; these, and not ω itself, generate the Ricci-channel response. Model B depends on a family of static observers and diverges at the horizon; it is carried along to exercise the profile dependence, not as a physical proposal.

9.2. Channel Responses and the Index

Figure 6 shows $|\Delta W^2|$ and $|\Delta K_{\text{Ric}}|$ computed from the exact invariants. The Weyl response is a smooth power law across the whole exterior. The Ricci response develops the node at x_W predicted by Equation (25), and is suppressed relative to the Weyl response by a factor that grows as $q \rightarrow 0$, as Proposition 1 requires.

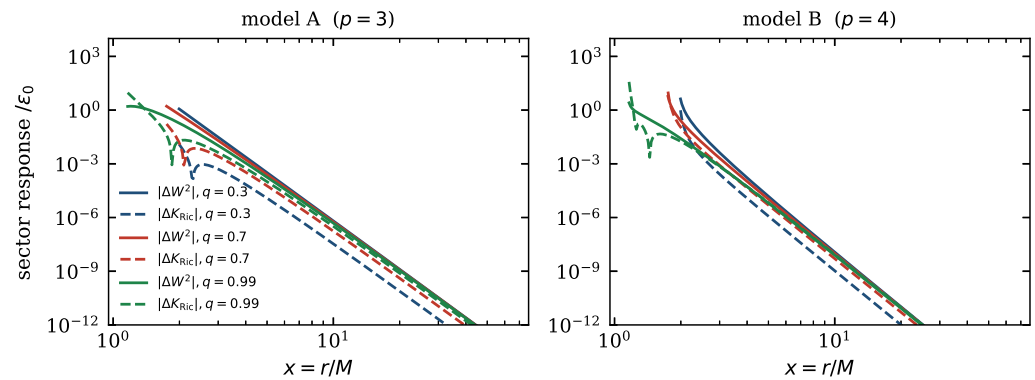


Figure 6. Absolute channel responses $|\Delta W^2|$ (solid) and $|\Delta K_{\text{Ric}}|$ (dashed), normalized by the amplitude ε_0 , for the two deformation profiles. The sharp dips in the dashed curves mark the Weyl-pure radius x_W of Equation (25), where the leading Ricci response passes through zero and changes sign; the plotted quantity is a magnitude, so on a logarithmic axis the zero appears as a narrow downward excursion. As elsewhere, model B is an illustrative, observer-dependent benchmark.

The response index is shown in Figure 7. Its shape is fully accounted for by the two analytic results of Section 4: a node at x_W and a plateau at $\mathcal{D}_\infty$ from Equation (26), which the numerics reproduce to seven significant figures (Table 3). The near-horizon part of the model-B curves is drawn for completeness only: there $\eta \rightarrow \infty$, the weak-deformation expansion has broken down and the static-observer prescription no longer defines a congruence, so that region lies outside the regime in which the profile is admissible and we do not interpret it.

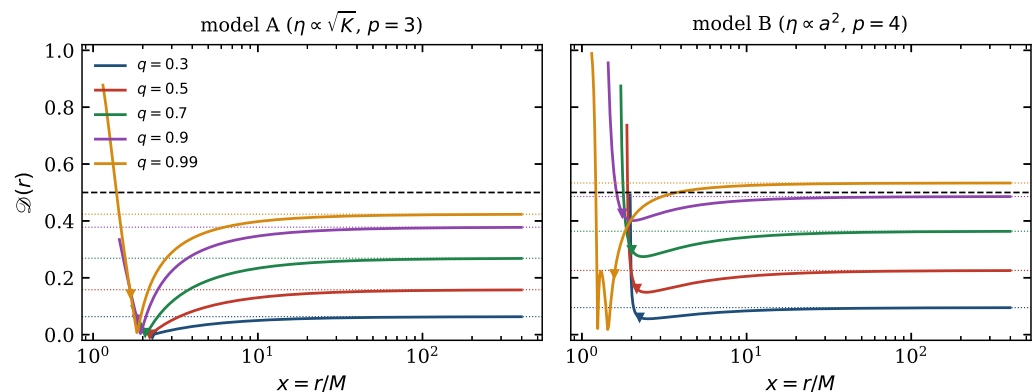


Figure 7. The curvature-response index $\mathcal{D}(r)$ of Equation (13) for the two deformation profiles and several charge-to-mass ratios, at $\varepsilon_0 = 10^{-6}$. Dotted horizontal lines are the analytic asymptotes $\mathcal{D}_\infty$ of Equation (26); triangles mark the Weyl-pure radius x_W of Equation (25). The dashed line at $\mathcal{D} = \frac{1}{2}$ separates Weyl-dominated from Ricci-dominated response. The model-B curves are illustrative; their near-horizon rise lies outside the regime in which that profile is admissible.

9.3. Amplitude Independence

Table 4 shows the property anticipated in Section 3. For $q \neq 0$ the index converges to a finite, ε_0 -independent limit as $\varepsilon_0 \rightarrow 0$, with a residual dependence of order ε_0 that is invisible for any realistic amplitude. For $q = 0$, by contrast, $\mathcal{D} \propto \varepsilon_0$, exactly as Proposition 1 requires. The index therefore carries information about the deformation profile that survives complete ignorance of the minimal-length scale.

Table 3. Asymptotic response index $\mathcal{D}_\infty$ from Equation (26) and Weyl-pure radius x_W from Equation (25), for the two deformation profiles. The rightmost columns compare the analytic $\mathcal{D}_\infty$ with the value obtained by numerical evaluation of the exact invariants at $x = 10^6$; the check was run for every row. Model B is an illustrative, observer-dependent benchmark (Section 9).

q	r_+/M	Model A ($p = 3$)		Model B ($p = 4$)		Numerical $\mathcal{D}_\infty$	
		$\mathcal{D}_\infty$	x_W	$\mathcal{D}_\infty$	x_W	A	B
0.10	1.9950	0.0074	2.3286	0.0115	2.2807	0.0074442	0.0115321
0.30	1.9539	0.0632	2.2897	0.0950	2.2398	0.0632317	0.0950225
0.50	1.8660	0.1579	2.2075	0.2258	2.1530	0.1578945	0.2258063
0.70	1.7141	0.2687	2.0704	0.3637	2.0066	0.2687383	0.3637326
0.90	1.4359	0.3779	1.8457	0.4859	1.7596	0.3779157	0.4858611
0.99	1.1411	0.4237	1.6883	0.5335	1.5741	0.4236558	0.5334623

Table 4. Amplitude independence of the response index, model A at $x = 5$. For $q \neq 0$ the limit $\varepsilon_0 \rightarrow 0$ is finite; for $q = 0$ the index is proportional to ε_0 , in accordance with Proposition 1.

ε_0	$\mathcal{D}$ at $q = 0.5$	$\mathcal{D}$ at $q = 0$
10^{-2}	0.1113196	1.511×10^{-2}
10^{-3}	0.1001004	1.535×10^{-3}
10^{-4}	0.0989602	1.538×10^{-4}
10^{-6}	0.0988346	1.538×10^{-6}
10^{-9}	0.0988333	1.538×10^{-9}
10^{-12}	0.0988333	1.538×10^{-12}

9.4. Parameter-Space Map and Horizon Behavior

Figure 8 maps $\mathcal{D}_\infty$ over the (p, q) plane and shows the critical contour $\mathcal{D}_\infty = \frac{1}{2}$, that is $p(p+3)q^2 = 24$, whose intersection with $q = 1$ gives Equation (28). Figure 9 shows the induced scalar curvature $\tilde{R}(r)$ and confirms both the horizon relation (29) and the extremal suppression (30).

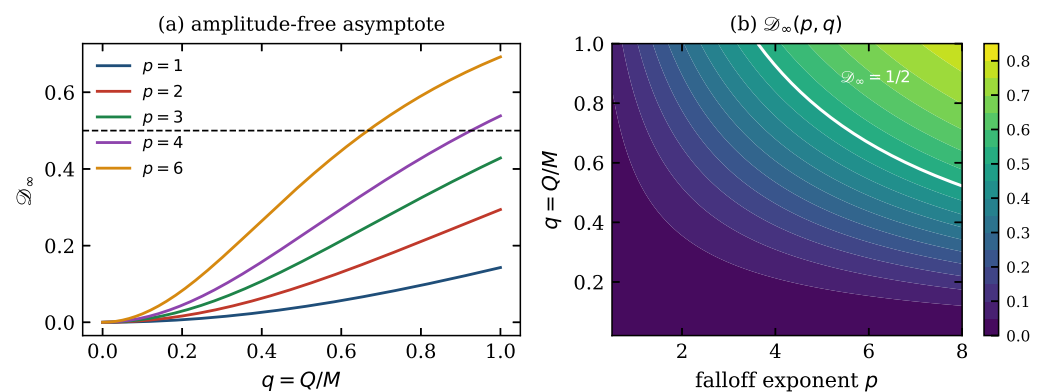


Figure 8. (a) The amplitude-free asymptotic index $\mathcal{D}_\infty$ of Equation (26) as a function of the charge-to-mass ratio, for several falloff exponents; the horizontal dashed line marks $\mathcal{D}_\infty = \frac{1}{2}$, the value at which the two channels respond equally. (b) $\mathcal{D}_\infty$ over the (p, q) plane; the faint dashed curves are contours of $\mathcal{D}_\infty$ at the levels of the color bar and the white contour is $\mathcal{D}_\infty = \frac{1}{2}$, that is $p(p+3)q^2 = 24$. Ricci-dominated asymptotic response needs both a steeply varying deformation and a large charge.

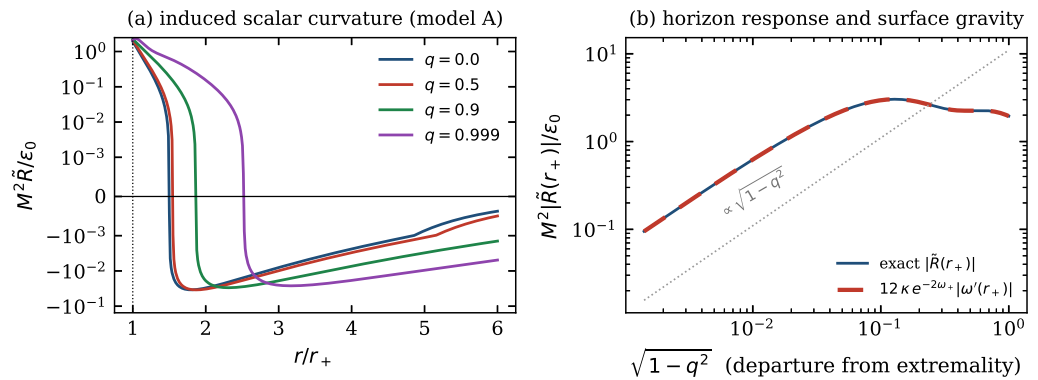


Figure 9. (a) Quantum-induced scalar curvature $\tilde{R}(r)$ (model A) outside the outer horizon; for the reference metric $R = 0$ everywhere. (b) The horizon value: $|\tilde{R}(r_+)|$ from the Riemann tensor of $\tilde{g}$ (solid) against the surface-gravity expression (29) (dashed). The maximum relative difference over the range shown is 1×10^{-44} , and both fall as $\sqrt{1-q^2}$ toward extremality (gray dotted reference).

10. Discussion

10.1. What the Two Theorems Buy

The reason to separate rays from waves is that the two behave oppositely and for the same underlying reason. A conformal deformation is invisible to the first and visible to the second, and the second sees it only through one scalar. For the scalar field treated here, everything one might want to compute about the observability of Equation (1) is therefore contained in $V_\omega = -\frac{1}{6}(C\tilde{R} - R)$ and in the rate l^{-2} at which its effect switches off. That is a small amount of information, which is exactly the finding: the conformal sector is hard to see for a structural reason, and the reason can be stated in one line. The electromagnetic and gravitational sectors, massive fields and other couplings are not covered by it, and we say what that leaves open in Section 10.5.

This also settles what the curvature analysis of Sections 3 and 4 is for. Taken alone, a Weyl/Ricci index is a bookkeeping device, and one may reasonably ask why that particular ratio. Theorem 2 supplies the asymmetry that makes the ranking worth having: the potential a scalar propagates in is built from the induced Ricci scalar and contains no Weyl contribution, so the two sectors differ in kind and not only in size. The closed form (26) then says which sector a given profile loads, which is a coarser statement than an observable fraction and one we can actually support. The amplitude-freedom of $\mathcal{D}$, by itself a triviality of first-order perturbation theory, matters for the same downstream reason: it makes the sector ranking depend on the shape of the deformation only.

10.2. Profile, Not Amplitude

The channel carrying the response is fixed by the shape of the deformation and not by its strength, and this now shows up three times over. Asymptotically, Equation (26) depends on (p, q) alone. In the ringdown, Equation (40) depends on p through $p(p-3)3^{-p}$ and vanishes at $p = 3$. In the disk, Equations (47)–(49) depend on p through $p(p-2)6^{-p}$, $(p-2)6^{-p}$ and $p(2p-5)6^{-p}$, vanishing at $p = 2$, $p = 2$ and $p = 5/2$ respectively. The amplitude ϵ_0 is a common factor everywhere and cancels in every ratio. Since ϵ_0 is the one quantity the framework cannot fix while p follows from the prescription used to build η , the model-discriminating content of the whole analysis sits in quantities that do not involve the minimal-length parameter at all.

We should be careful about how much this proves. The exponent p is an input: it is a property of the choice of curvature or acceleration scale entering η , not a prediction. What the results establish is a map from that choice to observable signatures, with an explicit and

rather rigid structure—three zeros, at $p = 2$, $p = 5/2$ and $p = 3$, in three different channels. That map is falsifiable given a prescription; it does not by itself select one.

10.3. Which Instruments Could Reach the Surviving Channels

The three responsive channels of Section 8 and Section 7 are not equally accessible, and it is worth being explicit about which measurement would test which statement, if only to record how far the required sensitivities are from the amplitudes of Section 8.5.

The null result is the one that can be tested now. Theorem 1 predicts that the shadow diameter and the photon-ring structure seen from afar take their undeformed values exactly, so the next-generation Event Horizon Telescope, with its longer baselines and its access to the $n = 1$ photon ring [39,75], together with the astrometric monitoring of S-stars around Sgr A* that already supplies the precession bound of Table 2, provides a direct test of conformality rather than of the deformation amplitude. A deviation would not measure ε_0 ; it would falsify Equation (1) as the leading form.

The eikonal-suppression law (38) asks for something different: not more sensitivity at high multipole, where the response is guaranteed to vanish, but the resolution of individual low- l modes. That is the regime in which a space-based interferometer operating on massive black-hole mergers, and third-generation ground interferometers with high-signal-to-noise ringdowns, are expected to separate the fundamental quadrupole from its subdominant partners [49,82]. The l^{-2} falloff of Figure 2b is then itself the observable signature: a deformation of the conformal class must show up as a departure that *shrinks* with multipole, which distinguishes it from non-conformal modifications, whose eikonal modes move with the photon sphere. We note in passing that pulsar-timing arrays, sensitive to the nanohertz background from supermassive binaries, do not probe this regime at all; the relevant band is that of individual ringdowns.

The disk channel is the one where the two coefficients of Equations (48) and (49) are separated in practice. Continuum fitting returns r_{ISCO} and the radiative efficiency; X-ray timing of quasi-periodic oscillations returns the orbital frequency [92,93]. Because the two respond with different p -dependence, and vanish at different exponents, a joint spectral-timing measurement of a single source constrains the profile in a way that neither does alone.

None of this brings the amplitude within reach: Equation (50) is unambiguous on that point, and we do not claim otherwise. What the instrument list establishes is that each of the three statements corresponds to a distinct and identifiable measurement, so that the framework is falsifiable in principle by channel and not merely constrained in bulk.

10.4. Relation to Other Approaches

The conformal effective metric belongs to a family of quantum-corrected black-hole geometries under active study, alongside higher-curvature and scalar-tensor deformations [42,43], Palatini and Born-Infeld non-singular models [70–72] and quantum-potential cosmologies [94,95]. What distinguishes it is exact conformality, and Theorem 1 shows this is a liability as much as a simplification: it removes the observational channel that most of that literature uses. The comparison also indicates where the family is genuinely distinguishable. A higher-curvature deformation is not conformal, so it moves the photon sphere; a conformal one cannot. A shadow measurement that is precise enough to detect any deviation at all is thus, within this family, a measurement of non-conformality.

The horizon relation (29) places the induced scalar curvature in correspondence with the surface gravity and hence with the thermodynamic quantities [65,66]. Through Equation (34) that relation now reads as a statement about the wave potential at the horizon as well, since V_ω and $\tilde{R}$ differ there only by the conformal weight. We have not pursued the

consequences for greybody factors and Hawking emission, which follow from the same potential and are the natural next calculation.

10.5. Limitations

There are five, stated plainly. The effective metric is taken from existing frameworks and the analysis inherits their assumptions; nothing here is a fundamental quantum theory of gravity. The deformation is treated as strictly conformal, which is a leading approximation whose validity Theorem 1 can test but not establish. The radial profile is model input, as just discussed. The ringdown analysis uses a minimally coupled massless scalar as a proxy: gravitational perturbations are governed by the field equations of the underlying theory, which the conformal ansatz does not determine, and Corollary 2 shows that the answer depends on the coupling, so Equation (40) should be read as the response of a test field and not as a prediction for an observed ringdown. Finally, a semiclassical metric description cannot be expected to survive at Planckian curvature [96,97]; our statements concern the exterior and are not extrapolated inward.

11. Conclusions

We asked which observations could register a conformal deformation $\tilde{g}_{\mu\nu} = C(x)g_{\mu\nu}$ of a black-hole spacetime, and the answer turned out to be sharply limited and completely characterizable.

1. **Null rigidity.** For an arbitrary conformal factor the unparametrized null congruence is unchanged, so the critical impact parameter, the shadow seen from infinity, deflection angle, lensing image positions and Shapiro delay take exactly their undeformed values at all orders, and the photon sphere keeps its coordinate location while its areal radius changes (Theorem 1). The shadow-and-lensing program cannot constrain this sector, and a measured deviation in it would be a measurement of non-conformality instead.
2. **Wave-sector reduction.** A minimally coupled massless test scalar on $\tilde{g}$ obeys exactly a wave equation on g with the single potential $V_\omega = \square\sqrt{C}/\sqrt{C} = -\frac{1}{6}(C\tilde{R} - R)$ (Theorem 2). The potential is built from the induced Ricci scalar, and the Weyl invariant does not enter it; a massless scalar with $\xi = \frac{1}{6}$ is insensitive to the deformation altogether (Corollary 2), so what is seen depends on the matter coupling.
3. **Eikonal suppression.** For a minimally coupled massless scalar, the leading first-order WKB shift of a quasinormal frequency is $\delta\Omega/\Omega_0 = r_{\text{ph}}^2 V_\omega(r_{\text{ph}})/[2l(l+1)]$, falling as l^{-2} in accord with null rigidity. On Schwarzschild $\delta\Omega/\Omega_0 = p(p-3)\epsilon_0/[12 \cdot 3^p l(l+1)]$, which vanishes for the curvature-sourced profile $p = 3$.
4. **Timelike response.** For the power-law family, $r_{\text{ISCO}} = 6M[1 - p(p-2)\epsilon_0 6^{-p}]$, $\delta\epsilon_{\text{rad}} = \sqrt{2}(p-2)\epsilon_0/6^{p+1}$ and $\delta\Omega_{\text{ISCO}}/\Omega_{\text{ISCO}} = 3p(2p-5)\epsilon_0/(4 \cdot 6^p)$. Disk continuum and disk timing respond to different combinations of the deformation.
5. **Complementarity.** Ringdown silence occurs only at $p = 3$, disk-continuum silence only at $p = 2$ and disk-timing silence only at $p = 5/2$; the three sets are pairwise disjoint, so no profile in the family is silent in more than one channel (Proposition 2).
6. **Curvature bookkeeping, and what it costs to test.** The Kretschmann response splits into Weyl and Ricci channels with the index $\mathcal{D}$ of Equation (13) and the amplitude-free asymptotic value $\mathcal{D}_\infty = p(p+3)q^2/[p(p+3)q^2 + 24]$; the Ricci contribution is switched on by charge (Proposition 1), vanishes at the Weyl-pure radius (25), and at a non-extremal horizon the induced scalar curvature is set by the surface gravity, Equation (29). The index is a geometric diagnostic that ranks the two sectors; it is not an observable fraction. Its link to observation is the one Theorem 2 supplies, namely

that a response confined to the Weyl sector cannot reach the scalar wave equation at all.

7. **How far this is from a test.** For astrophysical masses the amplitude is $\varepsilon_0 \sim \beta_0 \ell_P^2 / M^2 \lesssim 10^{-76}$, some seventy orders of magnitude below what any channel considered here can reach, now or foreseeably. The obstruction is the mass scale rather than the absence of an observable: Equation (50) gives the reach of each channel, and none becomes competitive with laboratory bounds on β_0 except at relic masses.

Two calculations follow directly. Greybody factors and the Hawking spectrum are governed by the same potential V_ω and require no new input. And because Corollary 2 ties visibility to the coupling ξ , a massive or non-minimally coupled field will respond differently, which turns the coupling constant into an additional handle on the deformation. Extending the analysis to rotating backgrounds is straightforward in principle, with the caveat that the separability of Equation (33) on Kerr requires V_ω to respect the Carter structure.

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Data Availability Statement: No new datasets were generated or analyzed. All curvature quantities reported here were reconstructed from the metric components using a computer algebra system, as described in Appendix A, and the closed forms carrying the observational conclusions were checked again by independent numerical integration. The verification code reproduces every zero listed in Appendix A and is available from the author on reasonable request. Symbolic verification establishes the internal consistency of the derivations; it does not bear on the validity of the underlying physical model.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Computer-Algebra Verification

All curvature quantities were computed with SymPy [98] from the metric components alone. For each of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = e^{2\omega(r)} g_{\mu\nu}$ we reconstructed the Christoffel symbols, the full Riemann tensor, the Ricci tensor and scalar, the quadratic Ricci invariant and the Kretschmann scalar directly from the definitions. No conformal transformation law was supplied as an input; the standard laws quoted in the text were recovered from that reconstruction instead, which is what makes the check independent. The Weyl invariant was then obtained from the four-dimensional identity (8). The following returned exact zeros:

- The RN invariants of the reference metric, Equation (15).
- The conformal Ricci-scalar law, Equation (10), and the conformal Weyl law. Equation (11), to all orders in ω .
- **The reduction theorem**, Equation (33): the residual $\tilde{\square}(e^{-\omega}\psi) - e^{-3\omega}[\square\psi - (\square\omega + (\nabla\omega)^2)\psi]$ vanishes identically for an unspecified $\psi(t, r, \theta, \phi)$ and an unspecified $\omega(r)$ on the RN background, with $\tilde{\square}$ built from $\tilde{g}$ by the same reconstruction.
- **The potential identity**, Equation (34): the residual $V_\omega + \frac{1}{6}(C\tilde{R} - R)$ vanishes identically, with $\tilde{R}$ obtained from the Riemann tensor of $\tilde{g}$ and not from the transformation law.

- **The Regge–Wheeler reduction**, Equation (36), obtained by separating $\psi = e^{-i\Omega t} Y_{lm} u(r)/r$ in Equation (33) and comparing with $f d_r(f d_r u) + (\Omega^2 - V_{\text{eff}})u$; the residual vanishes away from the coordinate singularity $\sin \theta = 0$.
- **The explicit potential**, Equation (39), and its identity with $\frac{1}{2}\square\eta$ at first order.
- **The ringdown shift**, Equations (38)–(40), and the silence condition (41).
- **The ISCO results**, Equations (47)–(49). These were obtained by implicit differentiation of $U' = U'' = 0$ with respect to ε_0 at $\varepsilon_0 = 0$, not by series expansion: expanding $(M/r)^p$ about a shifted radius with a symbolic exponent is unreliable, and an earlier series-based evaluation gave incorrect coefficients for Equations (48) and (49). All three were then confirmed against exact numerical roots of $U'' = 0$ at $\varepsilon_0 = 10^{-4}$ and 10^{-5} for $p \in \{2, 3, 4\}$, with the finite-difference estimates of $\partial r_{\text{ISCO}}/\partial \varepsilon_0$, $\partial \epsilon_{\text{rad}}/\partial \varepsilon_0$ and $\partial \ln \Omega_{\text{ISCO}}/\partial \varepsilon_0$ reproducing the analytic coefficients to seven significant digits and the residuals scaling as ε_0 .
- The first-order channel responses, Equations (23) and (24), and the asymptotic index, Equation (26).
- The three invariance statements of Theorem 1: equality of $\tilde{C}/\tilde{A}$ with r^2/f , of the null orbit equation and of $(dt/dr)^2$ with their counterparts for g .
- The timelike orbit equation (42), checked by recovering the standard Schwarzschild advance $6\pi M^2/L^2$ at $\eta = 0$.

Numerical evaluation used arbitrary-precision arithmetic (mpmath, 60 decimal digits) because ΔW^2 and ΔK_{Ric} are differences of nearly equal quantities and double precision is insufficient beyond $x \sim 10^2$. Analytic derivatives of ω were used throughout, never finite differences. The horizon relation (29) was checked against $\tilde{R}$ from the reconstructed Riemann tensor over six decades in $\sqrt{1 - q^2}$ in 80-digit arithmetic, with a maximum relative difference of 1×10^{-44} , and the analytic $\mathcal{D}_\infty$ was reproduced to the precision quoted in Table 3, the residuals being limited by the finite evaluation radius $x = 10^6$ rather than by the arithmetic.

Three relations carrying physical conclusions were checked again by independent methods. The Ricci-channel closed form (24) was evaluated directly from the exact invariants at finite $\varepsilon_0 = 10^{-25}$ and compared with the analytic bracket for $q \in \{0.3, 0.7, 0.99\}$, $p \in \{3, 4, 5\}$ and $x \in \{3, 7, 25\}$, agreeing to 25 significant digits. The perihelion coefficients (45) were checked by integrating Equation (43) numerically and locating successive perihelia. For $p = 3$ at the Mercury eccentricity the relative discrepancy falls as 3.9×10^{-3} , 9.8×10^{-4} , 2.4×10^{-4} for $L/M = 30, 60, 120$, and changes by less than 5% across eccentricities from 0.2 to 0.8847. For $p = 4$ at the S2 eccentricity $e = 0.8847$, it falls as 1.0×10^{-2} , 2.5×10^{-3} , 6.3×10^{-4} over the same separations once the factor $1 + e^2/4$ is included; without it, the discrepancy would stay near 0.2. Both runs confirm the coefficients and the order of the expansion. The null invariance of Theorem 1 was checked by quadrature of the deflection integral for conformal amplitudes differing by more than an order of magnitude, with the deflection angle and critical impact parameter unchanged to ten significant digits. Figures were produced with NumPy [99] and Matplotlib [100].

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