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Birth of an Isotropic and Homogeneous Universe with a Running Cosmological Constant

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Abstract

The present work discusses the birth of the Universe via quantum tunneling through a potential barrier, based on quantum cosmology, taking a running cosmological constant into account. We consider the Friedmann–Lemaître–Robertson–Walker (FLRW) metric with positively curved spatial sections ($k = 1$) and the matter's content is a dust perfect fluid. The model was quantized by the Dirac formalism, leading to a Wheeler–DeWitt equation. We solve that equation both numerically and using a WKB approximation. We study the behavior of tunneling probabilities TP_{WKB} and TP_{int} by varying the energy E of the dust perfect fluid, the phenomenological parameter ν , the present value of the Hubble function H_0 , and the constant energy density ρ_Λ^0 , with the last three parameters all being associated with the running cosmological constant. We observe that both tunneling probabilities, TP_{WKB} and TP_{int} , decrease as one increases ν . We also note that TP_{WKB} and TP_{int} grow as E increases, indicating that the Universe is more likely to be born with higher dust energy E values. The same is observed for the parameter ρ_Λ^0 , that is, TP_{WKB} and TP_{int} are larger for higher values of ρ_Λ^0 . Finally, the tunneling probabilities decrease as one increases the value of H_0 . Therefore, the best conditions for the Universe to be born, in the present model, would be to have the highest possible values for E and Λ and the lowest possible values for ν and H_0 .



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1. Introduction

One of the main problems faced by the Standard Model of Cosmology is the origin of spacetime through a big bang singularity. The existence of this singularity in the solutions to Einstein's equations, when applied to cosmological models, highlights the importance of seeking a quantum gravity theory to eliminate it. Since the early Universe had much smaller dimensions than atomic nuclei, it is valid to suggest that it was governed by the laws of quantum mechanics. The first attempt to find a quantum gravity theory was the quantization of General Relativity, which gave rise to the Wheeler–DeWitt equation [1,2]. The application of this quantum theory to cosmology produced Quantum Cosmology (QC) [3]. Although it is widely accepted that QC is not the fundamental theory for describing the very early universe, many important results have already been obtained from the study of QC models. One such result is the birth of the universe through a quantum tunneling process. Using this process, it is possible to show that the universe could originate without the big bang singularity [4–11].

In the literature, there are many works using the quantum tunneling process in different quantum cosmology models [12–21]. In addition to the investigation of the quantum tunneling process, we can find several other recent works on the different aspects of QC. In [22], the authors discuss time in QC. It is shown that the reparametrization invariance is not guaranteed after one quantizes the model. Reference [23] shows a different quantization process, where the authors use what they call a “super field” consisting of the union of the spacetime wavefunction and the matter fields. The authors of [24] study how a form of dark energy called “phantom energy” affects the wavefunction of the universe, which provides a solution to the appropriate Wheeler–DeWitt equation. One of the most important results is that the higher the energy content in the model, the greater the probability that the universe will be born with a determined size, avoiding the initial singularity in $a = 0$. References [25–27] present a novel approach in which the authors try to use fractional calculus in QC. In [28], the author reviews the DeBroglie–Bohm quantum theory. This is a theory that dispenses with the collapse of the wavefunction and, therefore, is quite appropriate for application to QC. In [29], the authors also address the issue of time not being an exactly defined notion in QC and present two solutions with applications focused on QC. In reference [30], the authors employ the fractional Riesz derivative in the Wheeler–DeWitt equation for a closed de Sitter geometry. The authors of [31] study a quantum cosmology model using electromagnetic radiation as a matter content. Reference [32] presents the first study of QC using quantum computers. There, the authors solve a Wheeler–DeWitt equation and show that the use of quantum computers allows one to reach solutions with a high degree of precision. In reference [33], the authors explore a two-dimensional cosmological model of quantum gravity. They do this, initially, by writing the classical sector with the aid of the ADM formalism, and then they solve the Wheeler–DeWitt equation in order to describe the quantum sector. In [34], the authors study approximate solutions to the Wheeler–DeWitt equation and compare the results with cosmological perturbative theories. In Reference [35], the author works on the quantization of quantum cosmology models, introducing the concepts of mini-superspace, perfect fluids, and scalar fields, bringing up conceptual problems such as the evolution of time and unitarity, as well as physical interpretations. These are some examples showing that Quantum Cosmology is an active field that is growing over time.

Since the seminal work by P. A. M. Dirac in 1937 [36], arguing about the possibility that some of the fundamental constants of nature may vary with the age of the Universe, many works have been conducted exploring that possibility. Although not mentioned in Dirac’s work, the cosmological constant (CC) is now understood as a very important constant of nature because it is a possible candidate to explain the present accelerated expansion of our Universe [37,38]. Some recent observational evidence shows that the expansion of the universe accelerated faster in the past than it is now [39]. This opens up the possibility, among others, of a running CC. In fact, even before this recent observational evidence, several researchers considered the possibility of a running CC [40–47]. A promising line of investigation regarding a running CC is based on the study, using quantum field theory in curved spacetime, of the renormalization group (RG) [45,48–52]. Our motivation to use the running CC provided by the RG method here is due to the fact that this running CC has a very solid physical and mathematical foundation. It is based on the renormalization procedure of a quantum field theory in curved spacetime. More precisely, when one considers the Standard Particle Model coupled with classical gravity, the spontaneous symmetry-breaking of the electroweak gauge symmetry and the presence of the non-perturbative QCD vacuum condensate both contribute to the vacuum energy density. That contribution induces a running cosmological constant whose precise expression is obtained with the aid of the Renormalization Group equation [48]. Another important motivation

to study the running CC provided by the RG method is that there are already several attempts to explain different cosmological processes using that specific running CC. Some of those attempts will be testable in the near future [50,53]. Also, when using the running CC given by the RG method it is possible, with the aid of some cosmological observations, to impose phenomenological limits on a certain parameter of the method [51,54]. This certainly improves the predictions of this method. To ensure completeness, we want to mention that although many authors have considered the possibility of a running CC, it is possible that a scale-dependent CC violates general covariance [55]. Since there is a dispute with respect to that result [56,57], in the present paper we assume that a running CC is not forbidden and explore its consequences.

In this work, we study the Universe in its early stages through quantum cosmology. For this purpose, the tunneling probabilities for the emergence of the Universe through a potential barrier are calculated. We achieve this with the help of the solutions to the Wheeler–DeWitt equation, obtained in two different ways: the numerical solution and the WKB approximation. The Universe is described by a Friedmann–Lemaître–Robertson–Walker (FLRW) model with positive curvature of the spatial sections, coupled with a dust perfect fluid and a running cosmological constant, described by the RG method. The latter is introduced to describe the dark energy present in the Universe. Our motivation to use positively curved spatial sections ($k = 1$) is that this creates a cosmological model with a potential barrier which allows us to study the birth of the universe through the quantum tunneling process. The other types of curved spatial sections ($k = 0, -1$) do not produce cosmological models with potential barriers. Cosmological inflation is supposed to have occurred very early in the universe’s history, specifically between 10^{-36} and 10^{-32} seconds after its birth [58–60]. Due to the presence of the running CC in our model, it is expected that the universe described by the model will undergo an inflationary expansion. Unfortunately, since our model is very simple, it will never end the inflationary expansion in the appropriate way [61]. Then, we shall not discuss this important stage of the early Universe dynamics in the present paper. We hope to return to this important issue in future work. It is also important to mention that, in general, we cannot use the real values of the physical quantities. Doing so would be extremely difficult to perform the calculations or would even make them impossible using our present computational resources. Therefore, we chose, throughout the paper, some values of the physical quantities that allow us to obtain numerical results as examples. We believe that these results will provide a qualitative idea about the behavior of the model at the quantum level. We hope to solve this limitation in the future by improving our computational resources.

Taking into account the papers mentioned above, in which the birth of the universe through a tunneling mechanism in quantum cosmology has already been studied, we can note some similarities and differences between them and our present paper. The similarities between these papers are as follows: the FLRW spacetime, the presence of a radiation perfect fluid, and a time-independent physical quantity representing the dark energy sector. Also, in most of the studies, the authors chose general relativity as the gravitational theory and positively curved spatial sections ($k = 1$) to obtain a potential barrier. In most studies, the physical quantities representing the dark energy sector are different. We can note the following quantities representing the dark energy sector: a positive cosmological constant [13], a Chaplygin gas [14], and a generalized Chaplygin gas [12,15]. There is also a paper where, instead of radiation, the authors considered a dust perfect fluid and the physical quantity representing the dark energy sector was a Chaplygin gas [17]. In other papers, the authors chose Hořava–Lifshitz as the gravitational theory without the need for a physical quantity representing the dark energy sector [19,20]. An interesting result appeared when the authors introduced an ad hoc potential. Due

to this new contribution, it was possible to construct potential barriers for models with flat ($k = 0$) and negatively curved spatial sections ($k = -1$), in addition to models with positively curved spatial sections ($k = 1$) [16,18]. A common result obtained by all these models that agrees with the model studied here is that TP s increase when the value of the physical quantity representing the dark energy sector increases. The novelty of the present paper, in comparison with previous papers, is that, for the first time, the physical quantity representing the dark energy sector is a running CC, which varies with time. During the calculations, we observed that the time-dependent part of the running CC joined the kinetic sector of the model. Therefore, the resulting Wheeler–DeWitt equation has a time-independent potential which depends on three parameters. This means a more complicated behavior was obtained for the TP s here than in previous models.

In Section 2, we derive the total Hamiltonian, draw a phase portrait, and solve the Hamilton equations for a FLRW cosmological model with constant positive spatial sections and coupled with a dust perfect fluid and a running cosmological constant. In Section 3, we quantize the model and solve the resulting Wheeler–DeWitt equation, using a numerical method and the WKB approximation. In Section 4, we compute the WKB (TP_{WKB}) and the integrated (TP_{int}) tunneling probabilities for the universe to tunnel through the potential barrier. We investigate how TP_{WKB} and TP_{int} depend on several parameters of the model. Finally, in Section 5, we summarize the main points and results of the paper. In Appendix A, we compute, in detail, with the aid of the Schutz variational formalism, the total Hamiltonian for the dust fluid.

2. The Classical Model

The classical cosmology, based on the FLRW model, has its foundations in Einstein’s General Theory of Relativity (GR) [62], accurately elucidating large-scale phenomena such as black holes, gravitational waves, light deflection, and the precession of Mercury’s orbit. In this section, the ADM formalism [63,64] was used to derive the Hamiltonian density of GR, thus making it possible to quantize it.

We consider an isotropic and homogeneous universe described by the Friedmann–Lemaître–Robertson–Walker metric:

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (1)$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor, and k determines the curvature of the spatial sections and can assume the values $k = -1, 0$, and $+1$, resulting in an open, flat, or closed geometry, respectively. In this work, we use $k = 1$. We also use the natural unit system, where we have $c = 8\pi G = \hbar = 1$. (r, θ, ϕ) are the usual spherical coordinates. We assume that the matter content of the model is provided by a perfect fluid that has a four-velocity described by $U^\mu = \delta_0^\mu$ in the co-moving coordinate system [65]. The energy-momentum tensor is written as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - pg_{\mu\nu}, \quad (2)$$

where p and ρ are the pressure and energy density of the fluid, respectively, and the Greek indices μ and ν range from zero to three. The equation of state for the perfect fluid is

$$p = \alpha\rho, \quad (3)$$

where α is a constant associated with the fluid.

The action integral, written in terms of the Lagrangian density of matter ($\mathcal{L}_m$) and the Ricci scalar (R), is given by,

$$S = \int_M \left(\frac{R}{2} + \mathcal{L}_m \right) \sqrt{-g} d^4x, \quad (4)$$

where g is the determinant of the metric, M is the four-dimensional spacetime, $\mathcal{L}_m$ is defined as the sum of the contributions from the running CC, $\mathcal{L}_\Lambda = p_\Lambda$ (where p_Λ is the pressure associated with the running CC), and the perfect fluid $\mathcal{L}_\alpha = p_\alpha$ (where p_α is the pressure associated to the fluid). The Ricci scalar is

$$R = -\frac{6\dot{N}\dot{a}}{aN^3} + \frac{6\ddot{a}}{aN^2} + \frac{6\dot{a}^2}{a^2N^2} + \frac{k}{a^2}, \quad (5)$$

where the dot means the derivative with respect to time.

Through inserting the value of R Equation (5) and $\sqrt{-g}$, computed for the FLRW metric Equation (1) with positively curved spatial sections, into Equation (4), we obtain

$$S = 3V_0 \int_M \left(\frac{\ddot{a}a^2}{N} + \frac{\dot{a}^2a}{N} - \frac{\dot{N}\dot{a}a^2}{N^2} + Na - \frac{Na^3\rho_\Lambda}{3} + \frac{Na^3\rho_\alpha}{3} \right) dt, \quad (6)$$

where ρ_Λ is the running CC energy density (we used that $p_\Lambda = -\rho_\Lambda$), ρ_α is the fluid energy density (we used Equation (3)), and V_0 is defined as, due to our choice of $k = 1$,

$$V_0 = \int_V \frac{r^2 \sin \theta}{(1-r^2)^{1/2}} dr d\theta d\phi, \quad (7)$$

where V is the spatial volume of M . In order to avoid the singularity when $r \rightarrow 1$ in Equations (1) and (7), one makes the coordinate transformation [62]: $r = \sin \chi$. Then, V_0 Equation (7) becomes

$$V_0 = \int_V \sin^2 \chi \sin \theta d\chi d\theta d\phi, \quad (8)$$

where the coordinates vary in the following ranges: $0 \leq \chi \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$.

The running cosmological constant energy density, ρ_Λ , has the following expression [45,48–52]:

$$\rho_\Lambda = \rho_\Lambda^0 + \nu(H^2 - H_0^2), \quad (9)$$

where ρ_Λ^0 represents the vacuum energy density at the present time, H is the Hubble function, defined as $H = \dot{a}/a$, H_0 is a constant that provides the Hubble function at the present time, and the parameter ν is a dimensionless constant that emerges from the renormalization process and characterizes the magnitude of quantum effects on the vacuum energy density. This is defined as

$$\nu = \frac{\sigma}{12\pi} \frac{M^2}{M_P^2}, \quad (10)$$

where σ represents the predominance of bosons ($\sigma = +1$) or fermions ($\sigma = -1$) at high energies, M is a weighted sum of the effective masses of massive virtual particles, and M_P is the Planck mass ($M_P \sim 1.22 \times 10^{19} \text{GeV}$). Since $M \ll M_P$, the parameter ν is typically very small, in the order of 10^{-6} [51,54]. This value reflects the low quantum contribution relative to the total gravitational energy. However, studies suggest that in the early Universe, during a highly anisotropic and dynamic phase, the usual limitations on the sign and magnitude of ν may not apply. In such conditions, ν could assume much larger values (both positive and negative). This occurs because spacetime conditions are dominated by intense quantum effects and high-energy interactions that influence vacuum dynamics [52]. This behavior enables the exploration of cases where the values of ν directly impact the isotropization process and the evolution of the metric [52]. The parameter ν also regulates the temporal variation of the vacuum energy density.

Introducing the value of ρ_Λ Equation (9) in the action Equation (6), we find the following:

$$S = V_0 \int_{-\infty}^{+\infty} \left\{ -\frac{3\dot{a}^2 a}{N} + 3Na + Na^3 [\alpha \rho_\alpha - \rho_\Lambda^0 - v(H^2 - H_0^2)] \right\} dt, \quad (11)$$

From action (11), one can extract the following Lagrangian density,

$$\mathcal{L} = -\frac{3\dot{a}^2 a}{N} + 3Na + Na^3 [\alpha \rho_\alpha - \rho_\Lambda^0 - v(H^2 - H_0^2)], \quad (12)$$

for simplicity, we will consider $V_0 = 1$.

Now, we can compute the canonical momentum:

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -\frac{6\dot{a}a}{N} - 2vNa\dot{a} = -2\dot{a}a \left(\frac{3}{N} + vN \right) \quad (13)$$

Through introducing p_a Equation (13) into the Hamiltonian density definition, $\mathcal{H} = p_a \dot{a} - \mathcal{L}$ and using the Schutz formalism [65], in order to determine the Hamiltonian of the matter sector (see Appendix A for more details), one obtains

$$N\mathcal{H} = -\frac{p_a^2}{4\left(\frac{3}{N} + vN\right)a} - 3Na + Np_T + Na^3 C, \quad (14)$$

where $C = \rho_\Lambda^0 - vH_0^2$. To avoid factor-ordering ambiguities, one must use the following change in variables [66]:

$$a = \left(\frac{3x}{2} \right)^{2/3}, \quad p_a = p_x a^{1/2}. \quad (15)$$

Choosing the gauge $N = 1$, the Hamiltonian density becomes

$$\mathcal{H} = -\frac{p_x^2}{4(3+v)} - 3\left(\frac{3x}{2}\right)^{2/3} + p_T + \frac{9}{4}Cx^2. \quad (16)$$

Using the constraint equation $\mathcal{H} = 0$, one identifies the potential $V(x)$:

$$V(x) = 3\left(\frac{3x}{2}\right)^{2/3} - \frac{9x^2}{4}C, \quad (17)$$

which has the shape of a barrier. In Figure 1, we show an example of $V(x)$ Equation (17) for $\rho_\Lambda^0 = 0.22$, $v = 0.2$, and $H_0 = 1$. In this example, the maximum value of $V(x)$ (V_{max}) is 14.1421.

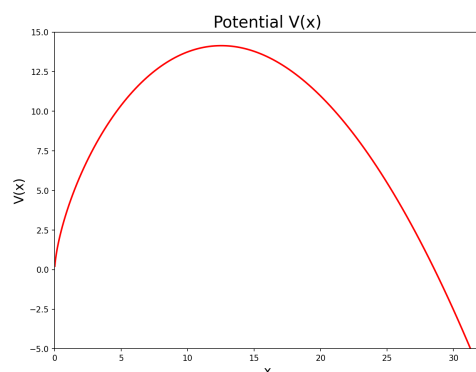


Figure 1. Potential $V(x)$ with $\rho_\Lambda^0 = 0.22$, $v = 0.2$ and $H_0 = 1$.

The classical dynamics are governed by Hamilton's equations:

$$\begin{cases} \dot{x} &= \frac{\partial \mathcal{H}}{\partial p_x} = -\frac{p_x}{2(3+\nu)}, \\ \dot{p}_x &= -\frac{\partial \mathcal{H}}{\partial x} = 3\left(\frac{3x}{2}\right)^{-1/3} - \frac{9C}{2}x, \\ \dot{T} &= \frac{\partial \mathcal{H}}{\partial p_T} = 1, \\ \dot{p}_T &= -\frac{\partial \mathcal{H}}{\partial T} = 0. \end{cases} \quad (18)$$

Therefore, one may impose the constraint equation $\mathcal{H} = 0$, leading to an equation for momentum p_x , given by

$$p_x = \pm \left[-12(3+\nu) \left(\frac{3x}{2} \right)^{\frac{2}{3}} + 4(3+\nu)p_T + 9C(3+\nu)x^2 \right]^{\frac{1}{2}}. \quad (19)$$

We began studying the dynamics produced by Hamilton's equations by constructing the phase portrait of the current model, as shown in Figure 2. The curves are labeled according to parameter p_T , which takes the values $p_T = 0, 1, 2, 5, 10, 14, 14.145, 15, 20, 25, \dots, 200, 250$, and 300. The phase portrait provides a general idea of the different types of dynamical solutions for the model. As we shall see next, all types of dynamical solutions can be associated with the four regions identified in the phase portrait Figure 2.

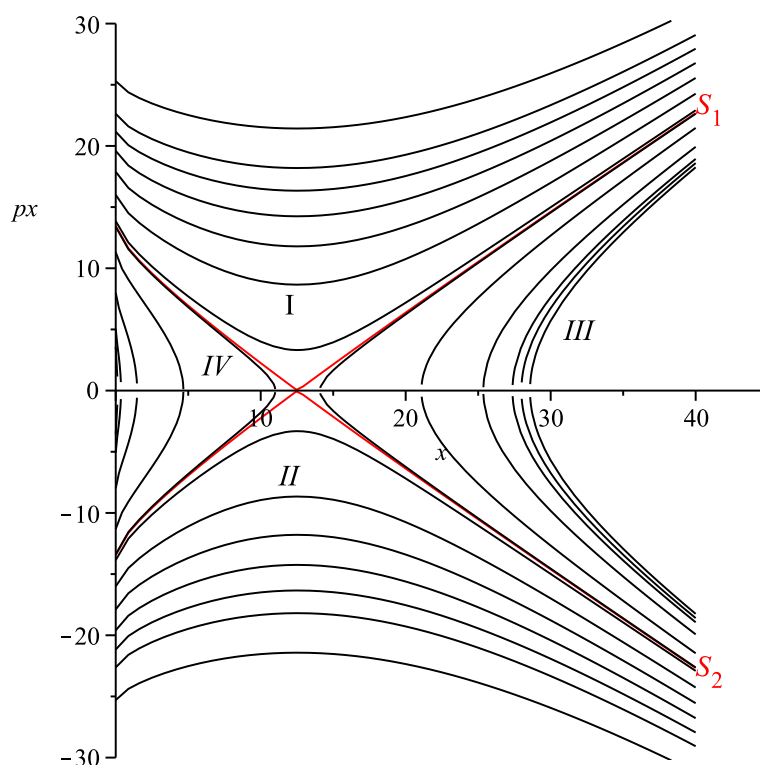


Figure 2. Phase portrait of the model made with the potential values $\rho_{\Lambda}^0 = 0.22$, $\nu = 0.2$ and $H_0 = 1$, while parameter p_T takes different values. The red lines occur when $p_T = 14.145$ and are called separatrices.

Analyzing Hamilton's equations, we observe that the phase portrait presents the cases in which $C > 0$ is a fixed point in the coordinates:

$$P \left(x = \sqrt{\frac{2}{3(\Lambda - \nu h_0^2)}}, p_x = 0 \right), \quad (20)$$

under an energy surface

$$p_T = \frac{2}{\sqrt{\Lambda - \nu h_0^2}}, \quad (21)$$

which correspond to a stationary solution (Einstein's solution). Through the linearization process, the system around fixed point P takes the form $\dot{X} = JX$, where X represents a column matrix of components and J a 2×2 matrix, called a Jacobian matrix. Thus, the system is integrable and has solutions of the following form:

$$X^{(N)}(t) = \sum_{j=1}^2 c_j A_j^{(N)} e^{\lambda_j t}, \quad (22)$$

where c_j are integration constants whose values will depend on the initial conditions; A_j are the components of the eigenvectors associated with the eigenvalues λ_j of the Jacobian matrix J , given by

$$J = \begin{pmatrix} 0 & -(12 + 4\nu) \\ -6\Lambda + 6\nu h_0^2 & 0 \end{pmatrix}. \quad (23)$$

Note that the form of the solutions (22) in the linear neighborhood of the fixed point will depend on the nature of the eigenvalues of J . These eigenvalues take the following form:

$$\lambda_1 = \frac{\sqrt{3(3+\nu)(\Lambda - \nu h_0^2)}}{3+\nu} \text{ and } \lambda_2 = -\frac{\sqrt{3(3+\nu)(\Lambda - \nu h_0^2)}}{3+\nu}. \quad (24)$$

Since the eigenvalues (24) are real numbers with opposite signs, we can see that the fixed point is a hyperbolic saddle [67]; indicating that, in general, the functions in the linear neighborhood of this point are unstable. In other words, any variation in the initial conditions will result in a departure of the system from its current configuration. The direction in which the curves approach and move away from the linear neighborhood of the hyperbolic saddle point will be shown by the eigenvectors A_j , given by

$$A_1 = \begin{pmatrix} -\frac{\sqrt{3(\Lambda - \nu h_0^2)(3+\nu)}}{6(\Lambda - \nu h_0^2)(3+\nu)} \\ 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{\sqrt{3(\Lambda - \nu h_0^2)(3+\nu)}}{6(\Lambda - \nu h_0^2)(3+\nu)} \\ 1 \end{pmatrix}. \quad (25)$$

In Figure 2, the red curves S_1 and S_2 are the separatrices, so named because they separate different types of behavior. These trajectories define the stable and unstable manifolds of the hyperbolic saddle, respectively. Here, S_1 (unstable manifold) emanates from the fixed point toward an asymptotically repulsive de Sitter solution (unstable de Sitter configuration), while S_2 (stable manifold) converges asymptotically to an attractor solution (stable de Sitter configuration).

After combining some of the Hamilton's equations from Equation (18), one can obtain a second-order differential equation for the classical evolution of the scale factor:

$$\ddot{x} + \frac{3}{2(3+\nu)} \left(\frac{3x}{2} \right)^{-1/3} - \frac{9C}{4(3+\nu)} x = 0. \quad (26)$$

When solving Equation (26) with the appropriate initial conditions, four different types of classical solutions are found, as pictured in Figure 3a,b. These solutions are as follows: (i) A contraction solution with x starting at a high value and decreasing until it reaches $x = 0$, which produces a big crunch singularity. This solution can be seen in Figure 3a. For this particular example, the initial conditions were $x_0 = 100$ and $p_T = -15$. These types of solutions are located in Region II of Figure 2; (ii) An expansion solution where x begins at low values and starts expanding at an accelerated rate to infinity. This

solution can be seen in Figure 3b. For this particular example, the initial conditions were $x_0 = 2$ and $p_T = 15$. These types of solution are located in Region I of Figure 2; (iii) An expansion followed by a contraction solution in which x begins at small values and grows until it reaches a maximum value. Then, it begins to shrink until it reaches $x = 0$, which produces a big crunch singularity. This solution is shown in Figure 3c. For this particular example, the initial conditions were $x_0 = 0.1$ and $p_T = 2$. These types of solution are located in Region IV of Figure 2; (iv) A bouncing solution, where x starts at a high value, shrinks until it reaches a minimum value, and then grows towards infinity. This solution is portrayed in Figure 3d. For this particular example, the initial conditions were $x_0 = 100$ and $p_T = 3$. These types of solution are located in Region III of Figure 2.

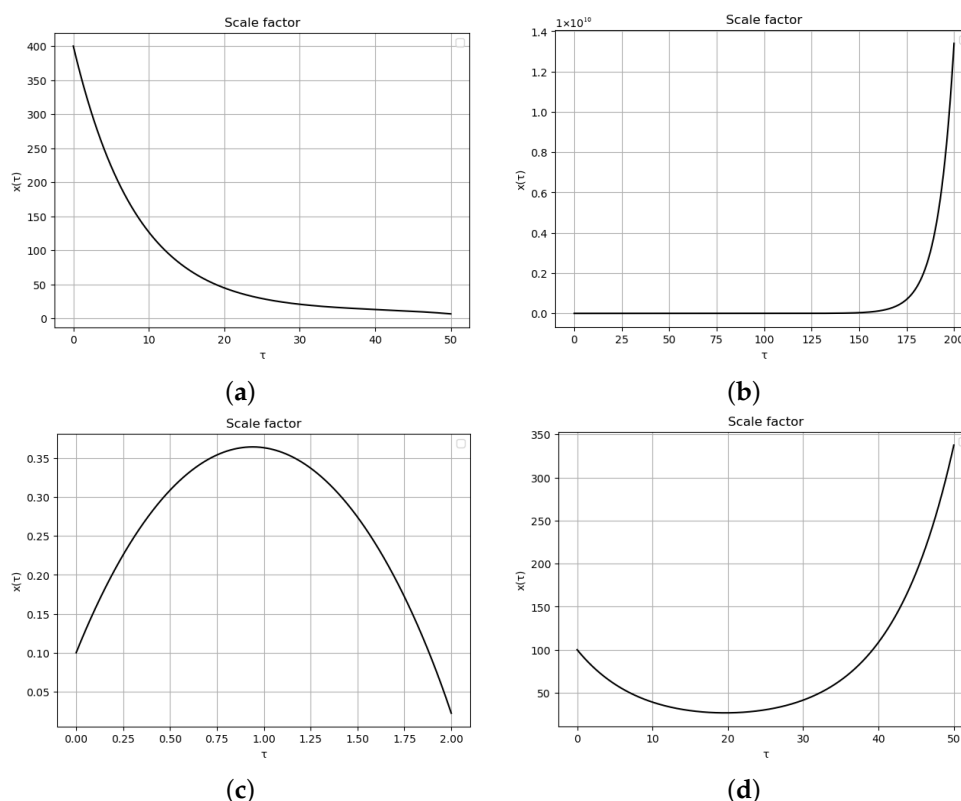


Figure 3. (a) shows a contraction solution, where x begins at a higher value and decreases to $x = 0$. (b) depicts an expanding solution, where x begins at lower value and expands towards infinity. (c) shows an expansion followed by a contraction. (d) shows a bouncing solution, where x starts at a high value, shrinks to a minimum, and begins an immediate expansion to infinity.

Since we want to investigate how the running CC with ρ_Λ given by Equation (9) influences our model at the quantum level, it is interesting to see how this quantity behaves at the classical level. In order to do that, we determined the temporal evolution of the running CC energy density ρ_Λ Equation (9) for each of the four classical solutions, obtained above, for the scale factor. Figure 4 illustrates the behavior of ρ_Λ for each scale factor solution. For direct comparison, we strategically placed Figure 4 to match the solutions of Figure 3. Figure 4a shows that ρ_Λ increases as the scale factor decreases. Figure 4b shows the variation in ρ_Λ for the expanding solution. It starts at a higher value and decreases as the scale factor increases until it reaches a constant value. In Figure 4c, one can see the behavior of ρ_Λ with respect to the big crunch case. It begins at some finite value, decreases until it reaches a minimum, and, as the scale factor becomes smaller, it grows. Finally, Figure 4d shows the behavior of ρ_Λ for the bouncing case. It starts from a finite initial value, decreases until it reaches a minimum, and then increases again. Eventually, it

asymptotically approaches a constant value. This behavior derives from the fact that, in this classical solution, the scale factor $x(t)$ grows exponentially, implying that its time derivative $\dot{x}(t)$ also increases exponentially. Consequently, the Hubble parameter $H(t) = \dot{x}(t)/x(t)$ becomes constant at later times. That explanation also works for the constant asymptotic value of ρ_Λ in the case of an expanding scale factor solution.

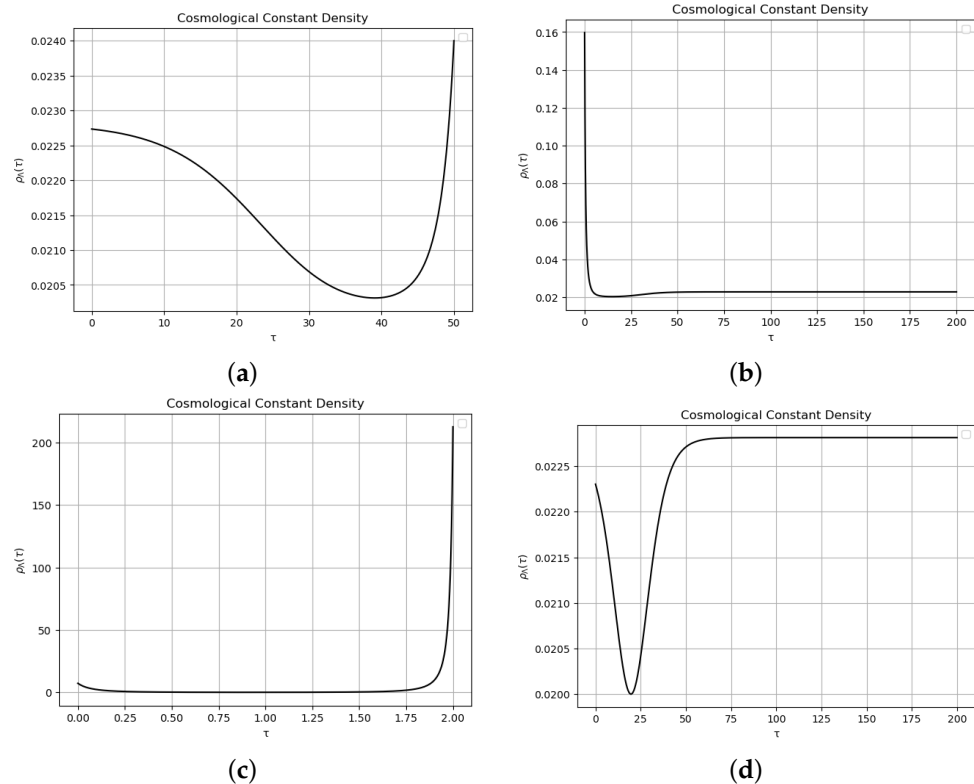


Figure 4. Here, one can see how the cosmological constant energy density behaves for different classical scale factor solutions. The solutions are (a) contraction, (b) expansion, (c) big crunch, and (d) bouncing.

3. Canonical Quantization

The next step is the quantization of the present model. We achieve this following Dirac's formalism [68,69], which consists of replacing the canonical variables x and T and their canonically conjugated momenta with their respective operators. After that, one must introduce the wavefunction of the Universe, which is a function of the operators associated with the canonical variables, written as $\Psi(x, T)$. Quantum effects are prevalent in the early Universe and the wavefunction of the Universe; it is responsible to describe those quantum effects. One can present the quantization process by replacing the canonical momenta p_x and p_T with their respective operators:

$$p_x \longrightarrow -i\frac{\partial}{\partial x}, \quad p_T \longrightarrow -i\frac{\partial}{\partial T}. \quad (27)$$

Next, we must impose the constraint equation $\hat{H}\Psi(x, T) = 0$, where $\hat{H}$ is the Hamiltonian operator. This equation gives rise to the Wheeler–DeWitt equation [1,2]. Through using the Hamiltonian from Equation (14) with gauge choice $N = 1$ and completing $\tau = -T$, one obtains the Wheeler–DeWitt equation for this model:

$$\left(\frac{1}{4(3+v)} \frac{\partial^2}{\partial x^2} - 3 \left(\frac{3x}{2} \right)^{2/3} + \frac{9x^2}{4} C \right) \Psi(x, t) = -i \frac{\partial}{\partial \tau} \Psi(x, t). \quad (28)$$

3.1. WKB Tunneling Probability

Assuming that one may write the solution to Equation (28) in the form $\Psi(x, \tau) = \psi(x)e^{-iE\tau}$, and using it in Equation (28), one finds

$$\frac{\partial^2}{\partial x^2} \psi(x) + 4(3 + \nu)[E - V(x)]\psi(x) = 0, \quad (29)$$

with $V(x)$ being obtained via Equation (17). Now, suppose that the WKB approximation is valid for the present model. Once obtained, the WKB solution will be used to calculate the tunneling probabilities through the potential barrier $V(x)$.

In order to derive the WKB solution, we must rewrite Equation (29) in the following way:

$$\frac{\partial^2}{\partial x^2} \psi(x) = -\frac{K^2(x)}{\hbar^2} \psi(x), \quad (30)$$

where,

$$K(x) = \sqrt{4(3 + \nu)(E - V(x))} \quad E > V(x), \quad (31)$$

and we are going to restore $\hbar$ only for this derivation. In this way, the nature of that approximation will become clear. Next, suppose that the solution to Equation (30) may be written in the following form:

$$\psi(x) = e^{if(x)/\hbar} \quad (32)$$

where $f(x)$ is a complex function. Now, introducing $\psi(x)$ Equation (32) in Equation (30), we obtain

$$i\hbar f''(x) - (f'(x))^2 + K^2 = 0, \quad (33)$$

where $'$ means the derivative with respect to x . Consider the following series expansion of $f(x)$ in the powers of $\hbar$:

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \dots \quad (34)$$

Through introducing the expansion presented in Equation (34) to Equation (33) and equating the different powers of $\hbar$, we can write the following equations:

$$-(f'_0)^2 + K^2 = 0, \quad if'_0 - 2f'_0 f'_1 = 0, \quad if'_1 = 2f'_0 f'_2 + (f'_1)^2, \dots \quad (35)$$

Solving Equation (35) up to the first order in $\hbar$, we obtain

$$f_0(x) = \pm \int K(x) dx, \quad e^{if_1(x)} = \frac{\bar{C}}{\sqrt{K}}, \quad (36)$$

where $\bar{C}$ is an integration constant. Combining Equations (32), (34), and (36), the final expression of $\psi(x)$ Equation (32), up to the first order in $\hbar$, is

$$\psi(x) = \frac{\bar{C}}{\sqrt{K}} e^{\pm \frac{i}{\hbar} \int K(x) dx}. \quad (37)$$

In order to guarantee the validity of the WKB approximation, it is necessary that $K^2(x)$ Equation (31) is much greater than $\hbar^2$. Let us compute the ratio $(K(x)/\hbar)^2$, by restoring the dimensions of all physical quantities present in the desired ratio. Recall that we are using the natural unit system, also known as Planck Units (PU). In Equation (31), we can observe that it is necessary to obtain the conversion factors for mass and energy from PU to SI. These conversion factors are as follows: $m = \sqrt{\hbar c/G} = 2.176434(24) \times 10^{-8}$ kg and $E = \sqrt{\hbar c^5/G} = 1.9561 \times 10^9$ J. Performing the calculations using, as an estimate for the

factor $E - V(x)$, the maximum value of $V(x)$ Equation (17) and the values of the parameters provided in the example of Figure 1, we obtain the following: $(K(x)/\hbar)^2 \approx 1.985 \times 10^{70} \text{ m}^{-2}$, which is indeed very big. Finally, adapting the WKB solution in Equation (37) for the different regions of the potential $V(x)$ Equation (17), we can write the complete WKB solution in the following way [70]:

$$\psi(x) = \begin{cases} \frac{A}{\sqrt{K(x)}} \exp(i \int_x^{x_l} K(x) dx) + \frac{B}{\sqrt{K(x)}} \exp(-i \int_x^{x_l} K(x) dx) & 0 \leq x \leq x_l, \\ \frac{C}{\sqrt{\kappa(x)}} \exp(-\int_{x_l}^x \kappa(x) dx) + \frac{D}{\sqrt{\kappa(x)}} \exp(\int_{x_l}^x \kappa(x) dx) & x_l \leq x \leq x_r, \\ \frac{F}{\sqrt{K(x)}} \exp(i \int_x^{x_r} K(x) dx) + \frac{G}{\sqrt{K(x)}} \exp(-i \int_x^{x_r} K(x) dx) & x_r \leq x < \infty, \end{cases} \quad (38)$$

where A, B, C, D, F , and G are constants, x_l is the value of x where the energy E intercepts the potential $V(a)$ from the left side, and x_r is the value of x where the energy E intercepts the potential $V(a)$ from the right side, and we return to the unit system where $c = 8\pi G = \hbar = 1$. In addition, the term $\kappa(x)$ is

$$\kappa(x) = \sqrt{4(3 + \nu)(V(x) - E)} \quad E < V(x). \quad (39)$$

Using matrix notation, the connection between coefficients A, B, C, D, F , and G is [70],

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2\theta + \frac{1}{2\theta} & i(2\theta - \frac{1}{2\theta}) \\ -i(2\theta - \frac{1}{2\theta}) & 2\theta + \frac{1}{2\theta} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}. \quad (40)$$

The parameter θ provides the height and length of the barrier. This is given by

$$\theta = \exp\left(\int_{x_l}^{x_r} \kappa(x) dx\right). \quad (41)$$

Once the WKB solution is obtained, one is able to compute the transmission coefficient, i.e., the tunneling probability. If one considers that there is no wave coming from the right ($G = 0$), the tunneling probability, being denoted by TP_{WKB} , will be [70]

$$TP_{WKB} = \frac{|F|^2}{|A|^2} = \frac{4}{\left(2\theta + \frac{1}{2\theta}\right)^2}. \quad (42)$$

For the present model, θ will be

$$\theta = \exp\left(\int_{x_l}^{x_r} \sqrt{4(3 + \nu) \left(3\left(\frac{3x}{2}\right)^{2/3} - \frac{9x^2}{4}C - E\right)} dx\right). \quad (43)$$

3.2. Integrated Tunneling Probability

The operator $\hat{H}$ in Equation (28) is self-adjoint [71] with respect to the internal product:

$$(\Psi, \Phi) = \int_0^\infty dx \Psi^*(x, \tau) \Phi(x, \tau), \quad (44)$$

if the wavefunctions satisfy one of the boundary conditions, $\Psi(0, \tau) = 0$ or $\Psi'(0, \tau) = 0$, where the prime ' means the partial derivative with respect to x . In the present paper, we shall restrict our attention to wave functions satisfying $\Psi(0, \tau) = 0$. We also demand that $\Psi(x, \tau) \rightarrow 0$ when $x \rightarrow \infty$. Along with the WKB solution, we also numerically solve the Wheeler–DeWitt Equation (28). In order to do that, we employ a finite difference procedure based on the Crank–Nicolson method [72]. We tested the validity of our numerical

solutions by computing the norm of the wavefunction for different times, and we found that it was always preserved. This test is commonly performed to evaluate the validity of numerical solutions to quantum mechanical systems [13–15,17]. The next step in solving the Wheeler-DeWitt equation Equation (28) is furnishing an initial wavefunction. That initial wavefunction fixes an energy for the dust. One may understand the relationship between the initial wavefunction energy and the dust energy because the time variable is related to the degree of freedom of the dust. Hence, the energies of the stationary states are associated with the energies of the dust fluid.

Among the several possibilities, for simplicity, we chose the following initial wavefunction:

$$\Psi(x, 0) = \frac{8\sqrt[4]{2} E^{3/4} x e^{-4x^2 E}}{\sqrt[4]{\pi}} \quad (45)$$

where E represents the mean kinetic energy associated with the dust energy and $\Psi(x, 0)$ is very concentrated in a region close to $x = 0$. The initial wavefunction Equation (45) must satisfy the following condition in order to be normalized: $\int_0^\infty |\Psi(x, 0)|^2 dx = 1$. Since we want to calculate the tunneling probability, we must compute the outgoing wavefunction. In other words, we must compute the part of the wavefunction which tunnels the potential barrier. The outgoing wavefunction propagates to infinity in the positive x direction as time goes to infinity. When performing a numerical calculation, we must specify a maximum value in the x direction. Let us call that number x_{max} . The behavior of all wavefunctions was computed, and their time evolution shows that they are well defined throughout the whole space.

Figure 5 presents an example of the probability density as a function of the scale factor x at the moment $\tau_{max} = 10$, when Ψ reaches numerical infinity, at $x_{max} = 100$. In this example, we draw the potential written in Equation (17), which is a barrier with values $\rho_\Lambda^0 = 0.22$, $\nu = 0.2$ and $H_0 = 1$. In this example, the maximum potential value is $V_{max} = 14.1421$, located at $x = 12.5353$. The fluid energy is $E = 14.1$, which is smaller than V_{max} . Then, for this example, the tunneling process will occur. For the fluid energy E , the wavefunction will reach the potential barrier at $x_l = 11.6072$ and, after tunneling, it will leave the potential barrier at $x_r = 12.6396$.

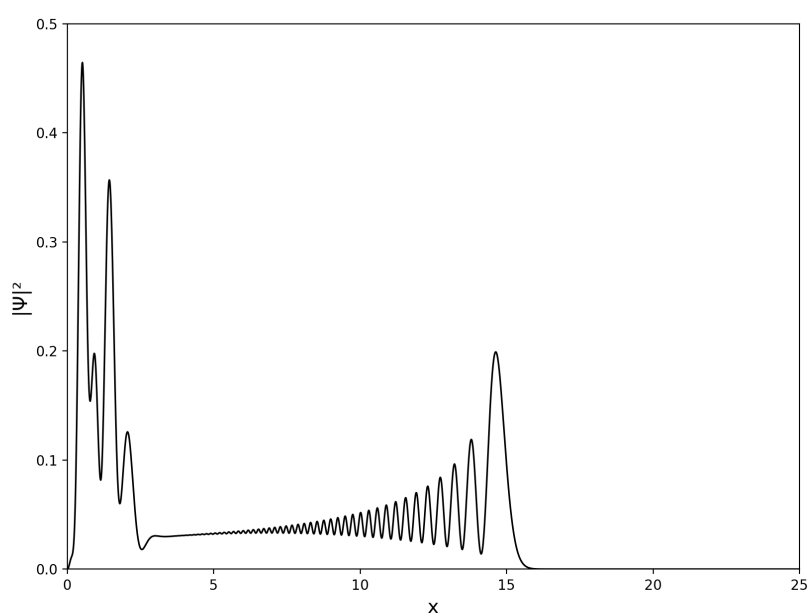


Figure 5. $|\Psi(x, t_{max})|^2$ for $\rho_\Lambda^0 = 0.22$, $\nu = 0.2$, $H_0 = 1.0$, $E = 14.09$ at the moment $\tau_{max} = 10$, when Ψ reaches numerical infinity, defined as $x_{max} = 100$.

In order to obtain the integrated tunneling probability, denoted by TP_{int} , we use the following definition [13,14]:

$$TP_{int} = \frac{\int_{x_r}^{\infty} |\Psi(x, \tau_{max})|^2 dx}{\int_0^{\infty} |\Psi(x, \tau_{max})|^2 dx}. \quad (46)$$

Here, $\Psi(x, \tau_{max})$ is the wavefunction of the universe calculated at time $\tau = \tau_{max}$, which is the moment Ψ reaches numerical infinity x_{max} . TP_{int} can be understood as the odds that the Universe will be found at the right side of the potential barrier.

4. Results

In this section, we will compute the tunneling probabilities TP_{int} and TP_{WKB} for the potential $V(x)$ presented in Equation (17) and how they change due to variations in energy E and parameters ρ_{Λ}^0 , ν and H_0 . Aiming to investigate the behavior of probabilities in the face of changes in parameters and energy, we fixed them as well as the energy, except for the quantity under study. This strategy was applied to each of the parameters and energy. The x values where energy E intercepts the potential, x_l and x_r , are also relevant. They must be determined to calculate the probabilities.

4.1. The Phenomenological Parameter ν

To study how the tunneling probabilities TP_{int} and TP_{WKB} are affected by changes in the ν parameter, we fixed the values of parameters ρ_{Λ}^0 and H_0 and the energy value E . We chose $\rho_{\Lambda}^0 = 0.22$, $H_0 = 1.0$, and $E = 4$ (this value is smaller than V_{max}). The range of values used to examine the behavior of TP_{int} and TP_{WKB} began at $\nu = -0.02$ and went all the way up to $\nu = 0.021$ in steps of 0.001. Since the Universe under study is very primitive, ν may have negative or positive values. One can see the results of the calculations in Table 1 and Figure 6, which provides a comparison between TP_{int} and TP_{WKB} . As one can see, the change in TP_{int} is much more subtle than the change in TP_{WKB} . We further analyzed each of the tunneling probabilities individually. It is possible to see, in Figure 7, that both decrease when parameter ν grows. Table 1 shows the parameter ν values, the tunneling probabilities, and the x values where the energy intercepts the potential ($E = V(x)$) from the left side (x_l) and from the right side (x_r).

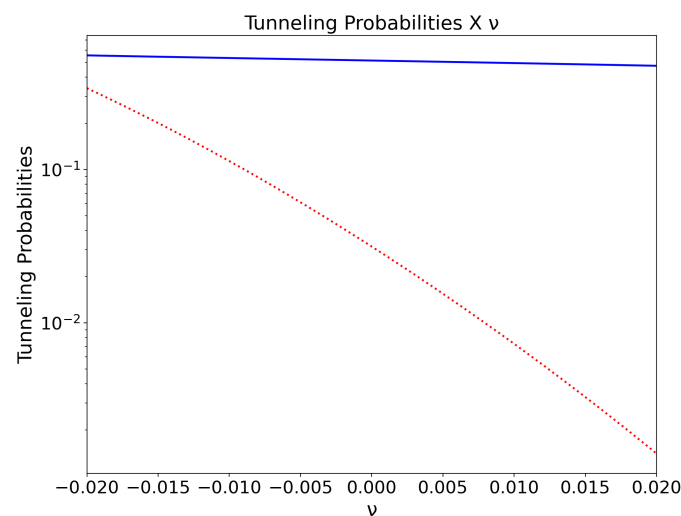


Figure 6. Comparison between TP_{int} (solid line) and TP_{WKB} (dots) as ν changes for a simulation time $\tau = 10$ and fixed energy $E = 4$. TP_{int} changes more slowly compared to TP_{WKB} , giving it the impression of being an almost constant line.

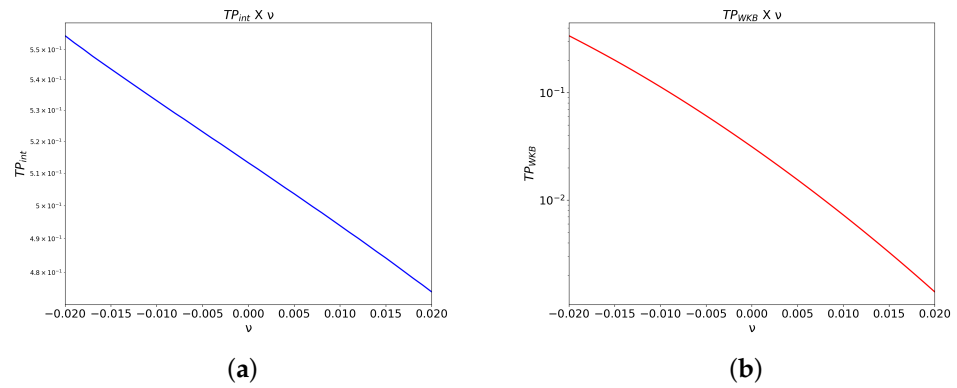


Figure 7. Individual comparison of TP_{int} (a) and TP_{WKB} (b) as ν changes for a simulation time $\tau = 10$ and fixed energy $E = 4$. Here, one can see that both decrease as ν increases.

Table 1. The variation in TP_{int} and TP_{WKB} as ν increases with simulation time $\tau = 10$ and a fixed energy $E = 4$.

ν	TP_{int}	TP_{WKB}	x_l	x_r
−0.0200	0.5547	0.3391	1.6096	2.1382
−0.0190	0.5523	0.3063	1.5982	2.1724
−0.0180	0.5502	0.2765	1.5876	2.2039
−0.0170	0.5478	0.2489	1.5773	2.2339
−0.0160	0.5457	0.2238	1.5677	2.2624
−0.0150	0.5436	0.2009	1.5587	2.2895
−0.0140	0.5414	0.1799	1.5500	2.3160
−0.0130	0.5393	0.1608	1.5417	2.3416
−0.0120	0.5373	0.1433	1.5334	2.3668
−0.0110	0.5352	0.1276	1.5259	2.3912
−0.0100	0.5331	0.1134	1.5184	2.4153
−0.0090	0.5311	0.1006	1.5112	2.4389
−0.0080	0.5291	0.0890	1.5043	2.4622
−0.0070	0.5271	0.0787	1.4975	2.4852
−0.0060	0.5251	0.0694	1.4909	2.5082
−0.0050	0.5231	0.0610	1.4846	2.5307
−0.0040	0.5211	0.0537	1.4786	2.5529
−0.0030	0.5192	0.0471	1.4726	2.5752
−0.0020	0.5172	0.0412	1.4667	2.5974
−0.0010	0.5153	0.0360	1.4609	2.6194
0.0000	0.5133	0.0314	1.4556	2.6411
0.0010	0.5114	0.0274	1.4500	2.6629
0.0020	0.5095	0.0238	1.4447	2.6847
0.0030	0.5075	0.0207	1.4397	2.7062
0.0040	0.5055	0.0179	1.4345	2.7279
0.0050	0.5037	0.0155	1.4297	2.7492
0.0060	0.5017	0.0134	1.4247	2.7709

Table 1. *Cont.*

ν	TP_{int}	TP_{WKB}	x_l	x_r
0.0070	0.4998	0.0115	1.4201	2.7923
0.0080	0.4979	0.00990	1.4154	2.8140
0.0090	0.4959	0.00850	1.4109	2.8354
0.0100	0.4940	0.00729	1.4066	2.8567
0.0110	0.4920	0.00623	1.4021	2.8784
0.0120	0.4901	0.00532	1.3978	2.8999
0.0130	0.4882	0.00454	1.3935	2.9217
0.0140	0.4862	0.00386	1.3895	2.9432
0.0150	0.4843	0.00328	1.3855	2.9649
0.0160	0.4823	0.00278	1.3816	2.9865
0.0170	0.4803	0.00235	1.3774	3.0084
0.0180	0.4783	0.00198	1.3737	3.0302
0.0190	0.4764	0.00167	1.3699	3.0521
0.0200	0.4744	0.00140	1.3660	3.0742

4.2. The Cosmological Constant Energy Density ρ_Λ^0

To examine how the tunneling probabilities TP_{int} and TP_{WKB} are affected by changes in the ρ_Λ^0 parameter, we fixed values for parameters ν and H_0 and the energy E . Here, we chose $\nu = 0.2$, $H_0 = 1.0$, and $E = 10$ (this value is smaller than V_{max}). The range of values used to examine the behavior of TP_{int} and TP_{WKB} began at $\rho_\Lambda^0 = 0.22$ and reached all the way up to $\rho_\Lambda^0 = 0.23$ in steps of 0.0005. One can see the results of the calculations in Table 2 and Figure 8, which provide a comparison between TP_{int} and TP_{WKB} . Table 2 shows the parameter ρ_Λ^0 values, the tunneling probabilities, and the x values where the energy intercepts the potential ($E = V(x)$) from the left side (x_l) and from the right side (x_r). Figure 8 shows that TP_{int} and TP_{WKB} increase as one increases parameter ρ_Λ^0 .

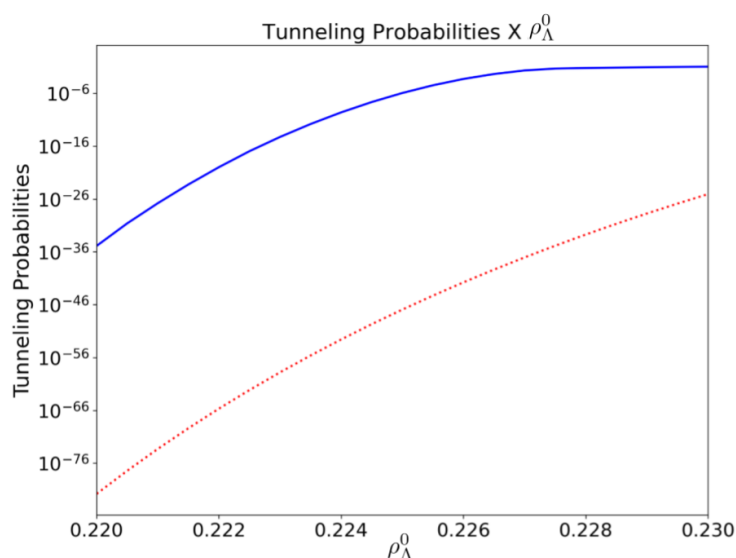


Figure 8. Comparison of TP_{int} (solid line) and TP_{WKB} (dots) as ρ_Λ^0 changes for a simulation time $\tau = 10$ and fixed energy $E = 10$.

Table 2. Variation in TP_{int} and TP_{WKB} as ρ_{Λ}^0 grows with a simulation time $\tau = 10$ and a fixed energy $E = 10$.

ρ_{Λ}^0	TP_{int}	TP_{WKB}	x_l	x_r
0.2200	1.4516×10^{-35}	1.7009×10^{-82}	4.6686	21.0203
0.2205	2.6622×10^{-31}	3.9449×10^{-78}	4.6901	20.5035
0.2210	1.7294×10^{-27}	5.3374×10^{-74}	4.7123	20.0068
0.2215	6.0803×10^{-24}	4.3775×10^{-70}	4.7348	19.5296
0.2220	1.1666×10^{-20}	2.2591×10^{-66}	4.7581	19.0698
0.2225	1.2301×10^{-17}	7.5690×10^{-63}	4.7817	18.6270
0.2230	6.128×10^{-15}	1.6990×10^{-59}	4.8062	18.1993
0.2235	1.7261×10^{-12}	2.6207×10^{-56}	4.8311	17.7867
0.2240	2.7564×10^{-10}	2.8525×10^{-53}	4.8567	17.3873
0.2245	2.4964×10^{-08}	2.2377×10^{-50}	4.8829	17.0010
0.2250	1.2787×10^{-06}	1.2935×10^{-47}	4.9101	16.6264
0.2255	3.6765×10^{-05}	5.6087×10^{-45}	4.9377	16.2636
0.2260	0.00058516	1.8597×10^{-42}	4.9665	15.9108
0.2265	0.0050401	4.7833×10^{-40}	4.9957	15.5684
0.2270	0.023677	9.7063×10^{-38}	5.0263	15.2348
0.2275	0.054146	1.5740×10^{-35}	5.0577	14.9100
0.2280	0.068165	2.0660×10^{-33}	5.0903	14.5934
0.2285	0.081331	2.2208×10^{-31}	5.1237	14.2847
0.2290	0.096297	1.9783×10^{-29}	5.1586	13.9828
0.2295	0.10885	1.4748×10^{-27}	5.1948	13.6874
0.2300	0.12086	9.2886×10^{-26}	5.2324	13.3983

4.3. Energy E

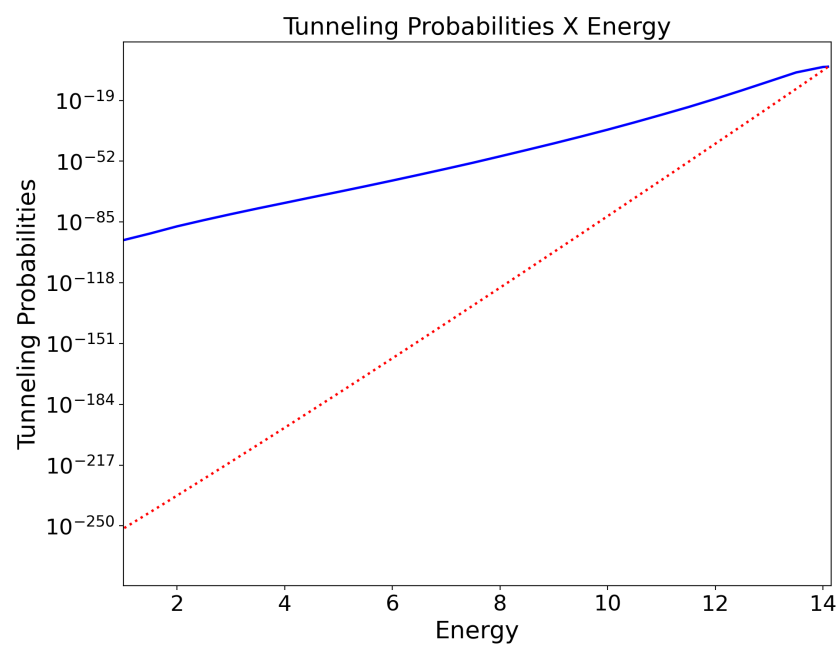
To study how the tunneling probabilities TP_{int} and TP_{WKB} are affected by changes in the fluid energy E , we fixed the values for the potential barrier parameters. Here, we chose $\rho_{\Lambda}^0 = 0.22$, $\nu = 0.2$, and $H_0 = 1.0$. The maximum value of this potential is $V_{max} = 14.1421$. The energy values used to examine the behavior of TP_{int} and TP_{WKB} began at $E = 0.5$ and reached all the way up to $E = 14$ in steps of 0.5. We also included the values $E = 14.05$, 14.07 and 14.09. One can see the results of the calculations in Table 3 and Figure 9, which present a comparison between TP_{int} and TP_{WKB} . Table 3 shows the energy values, the tunneling probabilities, and the x values where the energy intercepts the potential ($E = V(x)$) from the left side (x_l) and from the right side (x_r). Figure 9 shows that TP_{int} and TP_{WKB} increase as one increases the energy.

Table 3. The variation in TP_{int} and TP_{WKB} as E changes for a simulation time $\tau = 10$.

Energy	TP_{int}	TP_{WKB}	x_l	x_r
0.0	0.0	7.7956×10^{-270}	0.0013	28.5452
0.5	2.2991×10^{-98}	4.0960×10^{-261}	0.0454	28.2494
1.0	1.3461×10^{-95}	2.9104×10^{-252}	0.1285	27.9488
1.5	5.7261×10^{-92}	2.6130×10^{-243}	0.2365	27.6426
2.0	4.3881×10^{-88}	2.8611×10^{-234}	0.3647	27.3309
2.5	1.0625×10^{-84}	3.7323×10^{-225}	0.5109	27.0130
3.0	1.8226×10^{-81}	5.6980×10^{-216}	0.6736	26.6883
3.5	2.371×10^{-78}	1.0032×10^{-206}	0.8520	26.3566

Table 3. *Cont.*

Energy	TP_{int}	TP_{WKB}	x_l	x_r
4.0	2.3676×10^{-75}	2.0140×10^{-197}	1.0454	26.0172
4.5	2.5271×10^{-72}	4.5619×10^{-188}	1.2539	25.6698
5.0	2.4022×10^{-69}	1.1567×10^{-178}	1.4771	25.3133
5.5	2.6116×10^{-66}	3.2574×10^{-169}	1.7151	24.9474
6.0	3.2863×10^{-63}	1.0123×10^{-159}	1.9686	24.5711
6.5	4.8082×10^{-60}	3.4520×10^{-150}	2.2379	24.1833
7.0	8.1694×10^{-57}	1.2846×10^{-140}	2.5237	23.7831
7.5	1.6048×10^{-53}	5.1938×10^{-131}	2.8271	23.3690
8.0	4.514×10^{-50}	2.2714×10^{-121}	3.1494	22.9396
8.5	1.432×10^{-46}	1.0711×10^{-111}	3.4926	22.4924
9.0	5.0377×10^{-43}	5.4216×10^{-102}	3.8579	22.0259
9.5	2.3805×10^{-39}	2.9395×10^{-92}	4.2489	21.5363
10.0	1.4516×10^{-35}	1.7009×10^{-82}	4.6686	21.0203
10.5	1.3331×10^{-31}	1.0475×10^{-72}	5.1217	20.4729
11.0	1.6996×10^{-27}	6.8573×10^{-63}	5.6149	19.8868
11.5	2.7029×10^{-23}	4.7601×10^{-53}	6.1574	19.2526
12.0	7.9187×10^{-19}	3.4964×10^{-43}	6.7626	18.5558
12.5	3.6666×10^{-14}	2.7207×10^{-33}	7.4537	17.7714
13.0	2.3885×10^{-9}	2.2455×10^{-23}	8.2726	16.8539
13.5	0.00016588	1.9939×10^{-13}	9.3179	15.6916
14.0	0.15379	0.002234	11.0075	13.7286
14.05	0.17855	0.02509	11.3033	13.3118
14.07	0.19751	0.06915	11.4446	13.0716
14.09	0.21946	0.2263	11.6072	12.6396

**Figure 9.** Comparison of TP_{int} (solid line) and TP_{WKB} (dots) as energy E changes for a simulation time $\tau = 10$.

4.4. H_0

To examine how the tunneling probabilities TP_{int} and TP_{WKB} are affected by changes in the H_0 parameter, we fixed values for parameters ν and Λ and determined a fixed energy value. Here, we chose $\nu = 0.2$, $\rho_{\Lambda}^0 = 0.22$, and $E = 5$ (this value is smaller than V_{max}). The range of values used to examine the behavior of TP_{int} and TP_{WKB} began at $H_0 = 1.0$ and reached all the way up to $H_0 = 1.01$ in steps of 0.0005. One can see the results of the calculations in Table 4 and Figure 10, which provide a comparison between TP_{int} and TP_{WKB} . Table 4 shows the parameter H_0 values, the tunneling probabilities, and the x values where the energy intercepts the potential ($E = V(x)$) from the left side (x_l) and from the right side (x_r). Figure 10 shows that TP_{int} and TP_{WKB} decrease as one increases parameter H_0 .

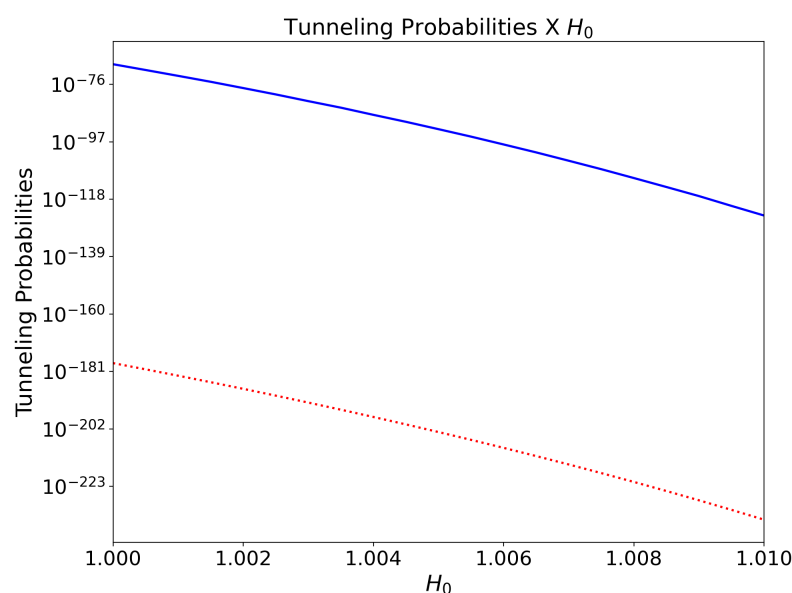


Figure 10. Comparison of TP_{int} (solid curve) and TP_{WKB} (dots) as H_0 changes with a simulation time $\tau = 10$ and fixed energy $E = 5$.

Table 4. Variation in TP_{int} and TP_{WKB} as H_0 increases with a simulation time $\tau = 10$ and a fixed energy $E = 5$.

H_0	TP_{int}	TP_{WKB}	x_l	x_r
1.0000	2.4022×10^{-69}	1.1567×10^{-178}	1.4771	25.3133
1.0005	1.8416×10^{-71}	6.1360×10^{-181}	1.4766	25.5233
1.0010	1.3295×10^{-73}	2.9014×10^{-183}	1.4762	25.7372
1.0015	9.017×10^{-76}	1.2183×10^{-185}	1.4757	25.9551
1.0020	4.5456×10^{-78}	4.5252×10^{-188}	1.4752	26.1771
1.0025	2.145×10^{-80}	1.4811×10^{-190}	1.4748	26.4032
1.0030	7.4611×10^{-83}	4.2514×10^{-193}	1.4743	26.6338
1.0035	3.0745×10^{-85}	1.0661×10^{-195}	1.4739	26.8688
1.0040	7.2884×10^{-88}	2.3235×10^{-198}	1.4734	27.1086
1.0045	2.0441×10^{-90}	4.3821×10^{-201}	1.4729	27.3529
1.0050	4.1692×10^{-93}	7.1132×10^{-204}	1.4726	27.6023
1.0055	7.8718×10^{-96}	9.8851×10^{-207}	1.4721	27.8568

Table 4. *Cont.*

H_0	TP_{int}	TP_{WKB}	x_l	x_r
1.0060	1.0714×10^{-98}	1.1699×10^{-209}	1.4717	28.1164
1.0065	1.3412×10^{-101}	1.1719×10^{-212}	1.4712	28.3816
1.0070	1.1994×10^{-104}	9.8761×10^{-216}	1.4708	28.6523
1.0075	9.8196×10^{-108}	6.9581×10^{-219}	1.4703	28.9288
1.0080	5.69×10^{-111}	4.0714×10^{-222}	1.4699	29.2112
1.0085	2.9972×10^{-114}	1.9640×10^{-225}	1.4694	29.4998
1.0090	1.4368×10^{-117}	7.7541×10^{-229}	1.4689	29.7948
1.0095	3.695×10^{-121}	2.4860×10^{-232}	1.4686	30.0964
1.0100	1.1142×10^{-124}	6.4190×10^{-236}	1.4681	30.4049

5. Conclusions

In the present work, the probability of the birth of a homogeneous and isotropic FLRW Universe with positively curved spatial sections ($k = +1$) was studied. The matter content of the model was composed of a dust perfect fluid. The potential barrier of the model originated from the geometry of space-time, the fluid energy density, and the presence of running cosmological constant, which can be interpreted as the zero-point energy of the quantum vacuum in Quantum Field Theory (QFT) [73]. The Hamiltonian was found using the ADM formalism and the model was quantized using the Dirac formalism.

We explicitly calculated the tunneling probabilities TP_{WKB} and TP_{int} for the birth of the Universe as a function of the phenomenological parameter (ν), the dust energy E , the cosmological constant energy density (ρ_Λ^0), and the Hubble parameter (H_0).

It was observed that both tunneling probabilities, TP_{WKB} and TP_{int} , decrease as one increases ν . It was also noted that TP_{WKB} and TP_{int} grow as E increases, indicating that the Universe is more likely to be born with higher values of dust energy E . The same was observed for the ρ_Λ^0 parameter, that is, TP_{WKB} and TP_{int} are bigger for higher values of ρ_Λ^0 . Finally, the tunneling probabilities decrease as one increases the value of H_0 .

So, the best conditions for the Universe to be born, in the present model, would be the highest possible values for E and ρ_Λ^0 and the lowest possible values for ν and H_0 .

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Appendix A. Schutz Formalism

According to the Schutz variational formalism for FLRW cosmological models [65], the fluid's 4-velocity U_ν is written in terms of the thermodynamical potentials μ , ϕ , θ , and S :

$$U_\nu = \frac{1}{\mu}(\phi_{,\nu} + \theta S_{,\nu}), \quad (\text{A1})$$

where μ is the specific enthalpy, S is the specific entropy, and the potentials ϕ and θ have no clear physical meaning. The normalization condition for this 4-velocity is

$$U^\nu U_\nu = -1. \quad (\text{A2})$$

Using Equation (A2) and a coordinate system comoving with the fluid, we may write μ Equation (A1) in the following way:

$$\mu = N^{-1}(\dot{\phi} + \theta \dot{S}). \quad (\text{A3})$$

The action for the fluid is [65]

$$S_m = \int d^4x \sqrt{-g} p, \quad (\text{A4})$$

where p is the fluid pressure. The perfect fluid satisfies the equation of state (3). It is possible to show, with the aid of some thermodynamic equations, that p may be written in the following form:

$$p = \alpha \left(\frac{\mu}{\alpha + 1} \right)^{1+1/\alpha} e^{-S/\alpha}. \quad (\text{A5})$$

The value of the quantity $\sqrt{-g}$, for the FLRW metric is given by $\sqrt{-g} = Na^3$. Introducing that value along with the values of p Equation (A5) and μ Equation (A3) in S_m (A4), one obtains the following value for the Lagrangian density of the fluid ($\mathcal{L}_m$):

$$\mathcal{L}_m = N^{-1/\alpha} a^3 \alpha \left(\frac{\dot{\phi} + \theta \dot{S}}{\alpha + 1} \right)^{1+1/\alpha} e^{-S/\alpha}. \quad (\text{A6})$$

From the above Lagrangian density Equation (A6), we may compute the conjugated momenta, p_ϕ and p_S :

$$p_\phi = N^{-1/\alpha} a^3 \left(\frac{\dot{\phi} + \theta \dot{S}}{\alpha + 1} \right)^{1/\alpha} e^{-S/\alpha}; \quad p_S = \theta p_\phi. \quad (\text{A7})$$

The Hamiltonian density of the matter ($\mathcal{H}_m$) is obtained with the aid of the $\mathcal{L}_m$ Equation (A6), the momenta p_ϕ and p_S . It has the following expression:

$$\mathcal{H}_m = Na^{-3\alpha} p_\phi^{1+\alpha} e^S. \quad (\text{A8})$$

In order to simplify $\mathcal{H}_m$ Equation (A8), we introduce the canonical transformations [10]:

$$T = p_S e^{-S} p_\phi^{-(1+\alpha)}; \quad p_T = p_\phi^{1+\alpha} e^S; \quad \bar{\phi} = \phi - (1+\alpha) \frac{p_S}{p_\phi}; \quad \bar{p}_\phi = p_\phi. \quad (\text{A9})$$

With these transformations, $\mathcal{H}_m$ Equation (A8) takes the following form:

$$\mathcal{H}_m = Na^{-3\alpha} p_T. \quad (\text{A10})$$

Finally, in the specific case of dust, we have $p = 0$ and $\alpha = 0$, such that the fluid's Hamiltonian density is reduced to

$$\mathcal{H}_m = Np_T. \quad (\text{A11})$$

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