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Black Hole Solution Free of Ghosts in f(R) Gravity Coupled with Two Scalar Fields

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Abstract

One extension of general relativity, known as f(R) gravity, where R denotes the Ricci scalar, is regarded as a promising candidate for addressing the anomalies observed in conventional general relativity. In this work, we apply the field equations of f(R) gravity to a spacetime with spherical symmetry with distinct metric potentials, i.e., $g_{tt} \neq g_{rr}$. By solving the resulting nonlinear differential equations, we derive a novel black hole solution without imposing constraints on the Ricci scalar or on the specific form of f(R) gravity. This solution does not reduce to the Schwarzschild solution of Einstein's general relativity. This solution is notable because it includes a gravitational mass and extra terms that make the singularities in the curvature stronger than those in black holes from Einstein's general relativity. We analyze these black holes within the framework of thermodynamics and demonstrate their consistency with standard thermodynamic quantities. Furthermore, we investigate the stability by examining odd-type perturbation modes and show that the resulting black hole is stable. Finally, we derive the coefficients of the two scalar fields and demonstrate that the black hole obtained in this study is free from ghosts.

Keywords: f(R) gravitational theory; exact spherically symmetric solution; thermodynamic quantities; stability

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1. Introduction

The Solar System serves as a laboratory for testing theories of gravitation. Einstein's general relativity (GR) has undergone extensive testing through independent observations and space fly-by experiments [1]. On the scale of the Solar system, GR has proven to be a remarkably accurate theory of gravity. The testability of multiple general relativistic effects via extremely compact star orbits close to the black hole at the center of the galaxy has been studied by Lalremruati and Kalita [2]. The Very Large Telescope has detected the Schwarzschild pericenter shift and gravitational redshift of the S2 star in the vicinity of the Galactic black hole [3,4]. To solve the primordial singularity problem [5] and replace the enigmatic dark matter and dark energy components in the standard model of cosmology, significant alternatives to GR have been put forth in recent decades [6–8]. In the context of general relativity, dark matter and dark energy are necessary to explain the formation of large-scale structures [9,10] and the accelerated expansion of the universe [11,12]. No

experiment has provided a satisfactory indication of the expected particle candidates for dark matter [13–19]. According to current theories, dark energy is a cosmological constant that accelerates the expansion of the universe by its repulsive energy density in a vacuum with negative pressure [11,12,20–22]. However, the energy density of vacuum as computed in quantum theory [23] is 10^{120} times greater than that measured from observations of the universe's accelerated expansion [20,21]. This anomaly poses a significant challenge to our understanding of gravitation in a cosmological context. Many dark energy models aim to explain cosmic acceleration, including coupled dark matter-dark energy scenarios and dynamical scalar fields with negative pressure. Those who are interested can follow some of the best literature reviews (see e.g., Sahni and Starobinsky [11]; Peebles and Ratra [12]; Copeland et al. [24]; Amendola and Tsujikawa [25]). Here, we stress that the physics of dark energy is still unknown. Because of these factors, it is thought that dark matter and dark energy are manifestations of GR modification in the large scale structure of the universe rather than exotic forms of matter-energy [7,26,27].

f(R) gravity theory is one of the most researched extensions of GR. The gravitational field equations of Einstein have been modified geometrically. Here, a general function of it, $f(R)$, takes the place of the Ricci scalar, R, in the gravitational Lagrangian. The resulting field equations in a cosmological setting alter the Friedmann-Lemaitre evolution, causing the appearance of a curvature fluid that accelerates expansion on its own without the need for additional sources of negative pressure [6,28,29]. This scenario accounts for the occurrence of inflation in the primordial universe without the need for additional scalar field [5]. It has been discovered that gravity, in the galactic scales $f(R)$, produces flat rotation curves in Low Surface Brightness galaxies without introducing exotic and previously unidentified dark matter particles [7]. The extra scalar mode of the gravitational force included in these theories is called the scalaron, and it is defined as the derivative of $\psi = df(R)/dR$. The scalaron field modifies the Schwarzschild metric, which is the weak and static gravitational field surrounding spherically symmetric bodies, according to the vacuum solution of the f(R) gravity field equations [30]. A Yukawa correction term with $e^{-(M_\psi cr/h)}/r$ variation exists in the gravitational potential of f(R) theory, where M_ψ is the mass of the scalar mode, c is the speed of light in empty space, and h is Planck's constant. Typically, this kind of adjustment is referred to as a fifth force of nature [30,31]. This is scaled by $1/r$ and added to the standard Newtonian term. The theory's testability through the observation of compact stellar orbits' pericenter shift near the supermassive black hole (Sgr A*) in the Galactic Centre has been thoroughly examined. This has involved taking into account the astrometric capabilities of currently in use large telescope facilities, like the Keck, the GRAVITY interferometer in VLT, and the upcoming Extremely Large Telescopes [32–35]. In f(R) gravity theory, a Kerr metric has been constructed recently [36]. It has been determined that the Kerr metric in f(R) gravity has the proper Schwarzschild limit. Gravity reduces to solutions in GR for infinitely large scalaron mass black hole solutions, and gravitational potential reduces to Newtonian form. Based on the discovered bright release ring of the black hole shadow in the Galactic Centre [37] and the Lense-Thirring precession of compact stellar orbits close to the black hole, one can infer that the f(R) gravity behaves like GR [36] for scalarons with mass in the range $(10^{-17}$–$10^{-16})$ eV.

Several independent investigations have placed constraints on the f(R) gravity theory in the vicinity of the black hole and in cosmological scales [38–48]. In this paper we will study f(R) gravitational theory coupled with two scalar fields trying to derive a new black hole. The paper is structured in the following way: In Section 2, we present the basis of f(R) theory coupled with two scalar fields. In Section 3 we derive the form of the coefficients of the scalar fields showing that they possess a positive behavior which ensure that the model derived in this study is free from ghosts. In Section 4, we derive a new solution in

f(R) assuming a specific form of f_R and discussing its relevant physics by calculating its invariants showing that it possesses a strong black hole compared with GR back hole when $r \to \infty$. In Section 5 we study the thermodynamical properties of the solution presented in Section 4. In Section 6 we show that the scalar fields behave in a physical pattern which ensure that our solution is free from ghosts. In Section 7, we provide the key findings of the present study.

2. Basis of f(R) Theory

We examine a 4-dimensional f(R) gravity action with two scalar fields whose action is expressed as (cf. [49–54]):

$$\mathcal{I} := \int \sqrt{-g}\left[\frac{\mathrm{f(R)}}{2\kappa^2} - \frac{1}{2}\, b(\chi,\chi_1)\partial_\mu\chi\partial^\mu\chi_1 - b_1(\chi,\chi_1)\,\partial_\mu\chi\partial^\mu\chi_1 - \frac{1}{2}\, b_2(\chi,\chi_1)\partial_\mu\chi\partial^\mu\chi_1 - \Phi(\chi,\chi_1)\right] d^4x\,. \tag{1}$$

The Ricci scalar is represented by the symbol R, the metric tensor's determinant is g, and the two scalars are χ and χ_1 and finally, $\Phi(\chi,\chi_1)$ is the potential. Moreover, there is dependence of the coefficients b, b_1, and b_2 on the scalar fields χ and χ_1.

Using the principle of variation on action (1), the field equations become [55]

$$\mathcal{I}_{\mu\nu} = \mathrm{R}_{\mu\nu}\mathrm{f_R} - \frac{1}{2}g_{\mu\nu}\mathrm{f(R)} + [g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\mathrm{f_R} = 2\kappa^2 T^{(\chi,\chi_1)}{}_{\mu\nu}, \tag{2}$$

with $\Box$ being the d'Alembertian operator and $\mathrm{f_R} = \frac{\mathrm{df}}{\mathrm{dR}}$ and $T^{(\chi,\chi_1)}{}_{\mu\nu}$ is the energy-momentum tensor of the scalar fields that has the form:

$$\begin{aligned} T^{(\chi,\chi_1)}{}_{\mu\nu} =& b(\chi,\chi_1)\partial_\mu\chi\partial_\nu\chi_1 + b_1(\chi,\chi_1)\left(\partial_\mu\chi\partial_\nu\chi_1 + \partial_\nu\chi\partial_\mu\chi_1\right) + b_2(\chi,\chi_1)\partial_\mu\chi_1\partial_\nu\chi - g_{\mu\nu}\left[\frac{1}{2}\, b(\chi,\chi_1)\partial_\rho\chi\partial^\rho\chi_1\right. \\ &\left. + b_1(\chi,\chi_1)\partial_\rho\chi\partial^\rho\chi_1 + \frac{1}{2}\, b_2(\chi,\chi_1)\partial_\rho\chi\partial^\rho\chi_1 + \Phi(\chi,\chi_1)\right], \end{aligned} \tag{3}$$

and the contracted Bianchi identities take the form

$$0 = \frac{b_\chi}{2}\,\partial_\mu\chi\partial^\mu\chi + b\nabla^\mu\partial_\mu\chi + b_\zeta\partial_\mu\chi\partial^\mu\chi_1 + \left(b_{1\chi_1} - \frac{1}{2}\, b_{2\chi}\right)\partial_\mu\chi_1\partial^\mu\chi_1 + b_1\nabla^\mu\partial_\mu\chi_1 - \Phi_\chi\,, \tag{4}$$

$$0 = \left(b_{1\chi} - \frac{1}{2}\,\chi_1\right)\partial_\mu\chi\partial^\mu\chi + b\nabla^\mu\partial_\mu\chi + \frac{1}{2}\,\alpha_{2\chi_1}\partial_\mu\chi_1\partial^\mu\chi_1 + b_2\nabla^\mu\partial_\mu\chi_1 + b_{2\chi}\partial_\mu\chi\partial^\mu\chi_1 - \Phi_{\chi_1}\,, \tag{5}$$

where $\alpha_\chi \equiv \partial\alpha(\chi,\chi_1)/\partial\chi$, and so on. Here we make the assumption

$$\chi = t\,, \qquad \chi_1 = r\,. \tag{6}$$

The assumption made in Equation (6) does not limit the generality of our analysis. In this study, we explore spherically symmetric spacetimes within the framework of theory (1), where the scalar fields χ and χ_1 typically depend on both the time coordinate t and the radial coordinate r. Once a solution is obtained, the explicit functional forms of $\chi(t,r)$ and $\chi_1(t,r)$ can be determined.

In spacetime regions where the relations are one-to-one, and provided that $\partial_\mu\chi$ is timelike and $\partial_\mu\chi_1$ is spacelike, it is possible to invert the functional dependencies and redefine the scalar fields by introducing new variables, say $\bar{\chi}$ and $\bar{\chi_1}$, such that $\chi(t,r) \to \chi(\bar{\chi},\bar{\chi_1})$ and $\chi_1(t,r) \to \chi_1(\bar{\chi},\bar{\chi_1})$. One can then identify these new scalar fields with the coordinates t and r as in Equation (6), i.e., $\bar{\chi} \to \chi = t$ and $\bar{\chi_1} \to \chi_1 = r$. This change of variables, $(\chi,\chi_1) \to (\bar{\chi},\bar{\chi_1})$, can be absorbed through redefinitions of the functions b, b_1, b_2, and Φ appearing in the action (1). Hence, the assumption made in Equation (6) does not result in any loss of generality [56].

The trace of the field Equation (2), when $T^{(\chi,\chi_1)}{}_{\mu\nu} = 0$, takes the form:

$$\mathcal{I} = 3\Box \mathrm{f}_R + \mathrm{R} f_R - 2\mathrm{f}(R) \equiv 0\,. \tag{7}$$

Using Equation (7), $f(R)$ can be written as follows:

$$\mathrm{f}(R) = \frac{1}{2}\left[3\Box \mathrm{f}_R + \mathrm{R} f_R\right]. \tag{8}$$

In the scenario where $= T^{(\chi,\chi_1)}{}_{\mu\nu} = 0$, and inserting Equation (8) in Equation (2), we obtain [57]

$$\mathcal{I}_{\mu\nu} = \mathrm{R}_{\mu\nu}\mathrm{f}_\mathrm{R} - \frac{1}{4} g_{\mu\nu}\mathrm{R}\mathrm{f}_\mathrm{R} + \frac{1}{4} g_{\mu\nu}\Box \mathrm{f}_\mathrm{R} - \nabla_\mu \nabla_\nu \mathrm{f}_\mathrm{R} = 0\,. \tag{9}$$

It is crucial to examine Equations (7) and (9) using spacetime with spherical symmetry assuming $g_{tt} \neq g_{rr}$.

3. f(R) Gravity with Two Scalar Fields

We are now ready to investigate the derivation of a spherically symmetric solution within the framework of $f(R)$ gravity coupled to two scalar fields. To this end, we begin by deriving spherically symmetric solutions of the field Equation (2). Following this, we derive the scalar fields b, b_2, and Φ in terms of Y, Y_1, and $f(R)$.

The spacetime with two distinct metric potentials is spherically symmetric and described as[1]:

$$ds^2 = -\mathrm{Y}(r)dt^2 + \mathrm{Y}_1(r)dr^2 + r^2 \sum_{i,j=1,2} \bar{g}_{ij} dx^i dx^j\,, \quad \text{with} \quad \bar{g}_{ij} = r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

where the two unknown functions, $\mathrm{Y}(r)$ and $\mathrm{Y}(r)$, depend on the radial distance r. The metric (10) can be expressed in terms of the Ricci scalar:

$$R(r) = \frac{r^2 \mathrm{Y}_1 \mathrm{Y}'^2 - r^2 \mathrm{Y}\mathrm{Y}'\mathrm{Y}'_1 - 2r^2 \mathrm{Y}\mathrm{Y}_1\mathrm{Y}''_1 - 4r\mathrm{Y}[\mathrm{Y}_1\mathrm{Y}' - \mathrm{Y}\mathrm{Y}'_1] + 4\mathrm{Y}^2(1-\mathrm{Y}_1)}{2r^2\mathrm{Y}^2}\,, \tag{11}$$

where $\mathrm{Y} \equiv \mathrm{Y}(r)$, $\mathrm{Y}_1 \equiv \mathrm{Y}_1(r)$, $\mathrm{Y}' = \frac{d\mathrm{Y}}{dr}$, $\mathrm{Y}'' = \frac{d^2\mathrm{Y}}{dr^2}$ and $\mathrm{Y}'_1 = \frac{d\mathrm{Y}_1}{dr}$.

The unavoidable recurrence of a phantom in Einstein's gravity with two scalar fields prompts us to investigate f(R) gravity in the hope to exorcise this ghost. The equation of motion for this modified gravity is in Equation (2). Assuming the energy-momentum tensor (3) of the scalar field doublet the components of the field Equation (2) become:

$$
\begin{aligned}
&t\,t-component:\\
&\frac{1}{4\mathrm{Y}^2\mathrm{Y}_1{}^2r^2}\Big[f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2-2f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2+f_R\mathrm{Y}'^2\mathrm{Y}_1r^2-4f_R\mathrm{Y}'\mathrm{Y}\mathrm{Y}_1r-4f_R\mathrm{Y}_1'\mathrm{Y}^2r\\
&-4f_R\mathrm{Y}^2\mathrm{Y}_1^2+4f_R\mathrm{Y}^2\mathrm{Y}_1-3\mathrm{Y}r^2f_R'\mathrm{Y}'\mathrm{Y}_1+2\mathrm{Y}^2r^2f_R''\mathrm{Y}_1-\mathrm{Y}^2r^2f_R'\mathrm{Y}_1'+4\mathrm{Y}^2rf_R'\mathrm{Y}_1\Big]\\
&=-\frac{\kappa^2(2\Phi\mathrm{Y}_1\mathrm{Y}+b\mathrm{Y}_1+\mathrm{Y}b_2)}{\mathrm{Y}_1\mathrm{Y}}\,,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&t\,r-component\\
&0=b_1\,,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&r\,r-component\\
&\frac{1}{4\mathrm{Y}^2\mathrm{Y}_1{}^2r^2}\Big[f_R\mathrm{Y}'^2\mathrm{Y}_1r^2-2f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2+f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2+4f_R\mathrm{Y}_1'\mathrm{Y}^2r+4f_R\mathrm{Y}'\mathrm{Y}\mathrm{Y}_1r-4f_R\mathrm{Y}^2\mathrm{Y}_1^2\\
&+4f_R\mathrm{Y}^2\mathrm{Y}_1+\mathrm{Y}r^2f_R'\mathrm{Y}'\mathrm{Y}_1-6\mathrm{Y}^2r^2f_R''\mathrm{Y}_1+3\mathrm{Y}^2r^2f_R'\mathrm{Y}_1'+4\mathrm{Y}^2rf_R'\mathrm{Y}_1\Big]=-\frac{\kappa^2(2\Phi\mathrm{Y}_1\mathrm{Y}-\mathrm{Y}b_2-b\mathrm{Y}_1)}{\mathrm{Y}_1\mathrm{Y}}\,,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\theta\,\theta=\phi\,\phi-component\\
&\frac{}{4\mathrm{Y}^2\mathrm{Y}_1^2r^2}\Big[4f_R\mathrm{Y}^2\mathrm{Y}_1^2-4f_R\mathrm{Y}^2\mathrm{Y}_1-f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2+2f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2-f_R\mathrm{Y}'^2\mathrm{Y}_1r^2+\mathrm{Y}r^2f_R'\mathrm{Y}'\mathrm{Y}_1\\
&+2\mathrm{Y}^2r^2f_R''\mathrm{Y}_1-\mathrm{Y}^2r^2f_R'\mathrm{Y}_1'-4\mathrm{Y}^2rf_R'\mathrm{Y}_1\Big]=-\frac{\kappa^2(\mathrm{Y}b_2-b\mathrm{Y}_1+2\Phi\mathrm{Y}_1\mathrm{Y})}{\mathrm{Y}_1\mathrm{Y}}\,.
\end{aligned}
\tag{15}
$$

The above system yields the form of b, b_1, b_2, and Φ in the form:

$$
\begin{aligned}
b=&\frac{1}{4\kappa^2\mathrm{Y}_1{}^2\mathrm{Y}r^2}\Big[2\mathrm{Y}r^2f_R'\mathrm{Y}'\mathrm{Y}_1+2f_R\mathrm{Y}_1'\mathrm{Y}^2r+2f_R\mathrm{Y}'\mathrm{Y}\mathrm{Y}_1r+2f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2-f_R\mathrm{Y}'^2\mathrm{Y}_1r^2\\
&-f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2-4\mathrm{Y}^2rf_R'\mathrm{Y}_1+4f_R\mathrm{Y}^2\mathrm{Y}_1^2-4f_R\mathrm{Y}^2\mathrm{Y}_1\Big]\,,\\
b_1=&0\,,\\
b_2=&\frac{1}{4\kappa^2\mathrm{Y}_1\mathrm{Y}^2r^2}\Big[2\,f_R\mathrm{Y}_1'\mathrm{Y}^2r-4\,\mathrm{Y}^2r^2f_R''\mathrm{Y}_1+2\,\mathrm{Y}^2r^2f_R'\mathrm{Y}_1'-2\,f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2+f_R\mathrm{Y}'^2\mathrm{Y}_1r^2\\
&+f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2+4\,\mathrm{Y}^2rf_R'\mathrm{Y}_1-4f_R\mathrm{Y}^2\mathrm{Y}_1^2+4\,f_R\mathrm{Y}^2\mathrm{Y}_1+2\,f_R\mathrm{Y}'\mathrm{Y}\mathrm{Y}_1r\Big]\,,\\
\Phi=&\frac{1}{8\kappa^2\mathrm{Y}_1^2\mathrm{Y}^2r^2}\Big[4f_R\mathrm{Y}^2\mathrm{Y}_1-4f_R\mathrm{Y}^2\mathrm{Y}_1^2+f_R\mathrm{Y}'\mathrm{Y}_1'\mathrm{Y}r^2-2f_R\mathrm{Y}''\mathrm{Y}\mathrm{Y}_1r^2+f_R\mathrm{Y}'^2\mathrm{Y}_1r^2-\mathrm{Y}r^2f_R'\mathrm{Y}'\mathrm{Y}_1\\
&-2\mathrm{Y}^2r^2f_R'\mathrm{Y}_1+\mathrm{Y}^2r^2f_R'\mathrm{Y}_1'+4\mathrm{Y}^2rf_R'\mathrm{Y}_1\Big]\,,
\end{aligned}
\tag{16}
$$

where $f_R'=\frac{df_R}{dr}$, $f_R''=\frac{d^2f_R}{dr^2}$ [2] From Equation (16) we can find the form of the scalar fields b, b_1, b_2, and the potential Φ as far as we find the forms of Y, Y_1, $f(R)$, and the derivatives of $f(R)$, i.e., f_R, f_R', and f_R''. In the next section we will derive the forms of Y, Y_1, $f(R)$.

4. Spherically Symmetric Solution in $f(R)$ Theory

In order to investigate Equations (7) and (9), which correspond to the vacuum case of Equation (2), i.e., $T_{\mu\nu}^{(\chi,\chi_1)}=0$, and to determine a general form of the arbitrary function $f(R)$, we analyze the equations of motion using a spherically symmetric spacetime containing two unspecified metric functions, without imposing any constraints on the Ricci scalar.

By using Equation (10) to plug in Equations (7), (9) and (11), we obtain:

$$
\begin{aligned}
&\mathcal{I}_t{}^t \equiv \frac{1}{8\mathrm{Y}^2 r^2 \mathrm{Y}_1^2}\Big[2\mathrm{Y}^2 r^2 \mathcal{F}'' \mathrm{Y}_1 - 2\mathcal{F}\mathrm{Y}'' \mathrm{Y}_1 \mathrm{Y} r^2 + \mathcal{F}\mathrm{Y}'^2 \mathrm{Y}_1 r^2 - \mathrm{Y}\{\mathrm{Y}_1(4\mathcal{F} + 3r\mathcal{F}') - r\mathcal{F}\mathrm{Y}_1'\} r\mathrm{Y}' - \mathrm{Y}^2\{r[4\,\mathcal{F} + r\mathcal{F}']\mathrm{Y}_1' \\
&-4\,\mathrm{Y}_1[r\mathcal{F}' - \mathcal{F}(\mathrm{Y}_1 - 1)]\}\Big] = 0\,, \\
&\mathcal{I}_r{}^r \equiv \frac{1}{8\mathrm{Y}^2 r^2 \mathrm{Y}_1^2}\Big[\mathcal{F}\mathrm{Y}'^2 \mathrm{Y}_1 r^2 - 2\mathcal{F}\mathrm{Y}'' \mathrm{Y}_1 \mathrm{Y} r^2 - 6\,\mathrm{Y}^2 r^2 \mathcal{F}'' \mathrm{Y}_1 + \mathrm{Y}\{r\mathcal{F}\mathrm{Y}_1' + \mathrm{Y}_1(4\,\mathcal{F} + r\mathcal{F}')\} r\mathrm{Y}' + \mathrm{Y}^2\{(4\mathcal{F} + 3r\mathcal{F}')r\mathrm{Y}_1' \\
&+4\mathrm{Y}_1[r\mathcal{F}' - \mathcal{F}(\mathrm{Y}_1 - 1)]\}\Big] = 0\,, \\
&\mathcal{I}_\theta{}^\theta \equiv \mathcal{I}_\phi{}^\phi = \frac{1}{8\mathrm{Y}^2 r^2 \mathrm{Y}_1^2}\Big[2\,\mathcal{F}\mathrm{Y}'' \mathrm{Y}_1 \mathrm{Y} r^2 + 2\,\mathrm{Y}^2 r^2 \mathcal{F}'' \mathrm{Y}_1 - \mathcal{F}\mathrm{Y}'^2 \mathrm{Y}_1 r^2 + r^2 \mathrm{Y}[\mathrm{Y}_1 \mathcal{F}' - \mathcal{F}\mathrm{Y}_1']\mathrm{Y}' - \mathrm{Y}^2\{r[4\,\mathrm{Y}_1 + \mathrm{Y}_1' r]\mathcal{F}' \\
&-4\,\mathrm{Y}_1 \mathcal{F}(\mathrm{Y}_1 - 1)\}\Big] = 0\,, \\
&\mathcal{I} \equiv \frac{1}{2\mathrm{Y}^2 r^2 \mathrm{Y}_1^2}\Big[6\,\mathrm{Y}^2 r^2 \mathcal{F}'' \mathrm{Y}_1 - 2\,\mathcal{F}\mathrm{Y}'' \mathrm{Y}_1 \mathrm{Y} r^2 + \mathcal{F}\mathrm{Y}'^2 \mathrm{Y}_1 r^2 + \mathrm{Y}\{r\mathcal{F}\mathrm{Y}_1' + 3\mathrm{Y}_1(r\mathcal{F}' - 4\mathcal{F})\} r\mathrm{Y}' - \mathrm{Y}^2\Big\{\Big(3r^2\mathcal{F}' - 4r\mathcal{F}\Big)\mathrm{Y}_1' \\
&+4\mathrm{Y}_1\Big[\Big(\mathrm{f}r^2 - \mathcal{F}\Big)\mathrm{Y}_1 + \mathcal{F} - 3\,r\mathcal{F}'\Big]\Big\}\Big] = 0\,,
\end{aligned}
\tag{17}
$$

where $\mathcal{F} \equiv \mathcal{F}(r) = \frac{d\mathrm{f}(R(r))}{dR(r)}$, $\mathcal{F}' = \frac{d\mathcal{F}(r)}{dr}$, $\mathcal{F}'' = \frac{d^2\mathcal{F}(r)}{dr^2}$, $\mathcal{F}''' = \frac{d^3\mathcal{F}(r)}{dr^3}$. We take $\mathrm{f}(R) = \mathrm{f}(r)$ since spherical symmetry is involved. It is noteworthy to mention that the differential equations given in (17) and the system of differential equations above are the same presented in [58].

Except for the trace part, Equation (17) can be rewritten as follows:

$$
0 = \frac{2\epsilon r^2[\mathrm{Y}^2\mathcal{F}'' - \mathcal{F}\mathrm{Y}''\mathrm{Y} + \mathcal{F}\mathrm{Y}'^2] + \mathrm{Y}[\mathcal{F}(r\epsilon' - 8\epsilon) - 4r\epsilon\mathcal{F}']r\mathrm{Y}' - \mathrm{Y}^2\big[r(r\epsilon' - 4\epsilon)\mathcal{F}' + 4\mathcal{F}\big(r\epsilon' + \mathrm{Y}\epsilon^2 - \epsilon\big)\big]}{8\mathrm{Y}^3 r^2 \epsilon^2}\,, \tag{18}
$$

$$
0 = \frac{2\epsilon r^2[\mathcal{F}\mathrm{Y}'^2 - \mathcal{F}\mathrm{Y}''\mathrm{Y} - 3\mathrm{Y}^2\mathcal{F}''] + \mathrm{Y}[4r\epsilon\mathcal{F}' + \mathcal{F}\{8\epsilon + r\epsilon'\}]r\mathrm{Y}' + \mathrm{Y}^2\big[r(4\epsilon + 3r\epsilon')\mathcal{F}' + 4\mathcal{F}\{\epsilon - \mathrm{Y}\epsilon^2 + r\epsilon'\}\big]}{8\mathrm{Y}^3 r^2 \epsilon^2}\,, \tag{19}
$$

$$
0 = \frac{2\mathcal{F}\mathrm{Y}''\epsilon\mathrm{Y}r^2 + 2\mathrm{Y}^2 r^2 \mathcal{F}''\epsilon - 2\mathcal{F}\mathrm{Y}'^2\epsilon r^2 - \mathcal{F}\mathrm{Y}' r^2 \epsilon'\mathrm{Y} - [r(4\epsilon + r\epsilon')\mathcal{F}' - 4\mathcal{F}\epsilon(\epsilon\mathrm{Y} - 1)]\mathrm{Y}^2}{8\mathrm{Y}^3 r^2 \epsilon^2}\,, \tag{20}
$$

where $\mathrm{Y}_1(r) = \epsilon(r)\mathrm{Y}(r)$.

4.1. Novel Solution

From Equations (18)–(20), we have two differential equations with three unknown functions, Y, ϵ, and $\mathcal{F}$[3]. Therefore to close the system, we assume that $\mathcal{F}$ takes on the following structure:

$$
\mathcal{F} = 1 + \frac{\delta}{r}\,, \tag{21}
$$

where δ has a unit of length. We assume that $\mathrm{Y}_1 = \epsilon\mathrm{Y}$. Equation (21) shows that when $\delta = 0$ we revert to the case of GR, because $f(R) = \mathrm{R}$. Using the above data then we get:

$$
\begin{aligned}
&\mathrm{Y}(r) = -\frac{2}{\delta^3(2r+\delta)^2}\Big[r^2 c_1\Big(-2\delta c_1(2r+\delta)^2\ln(2r+\delta) + \frac{3}{2}\Big(\delta c_1 - \frac{1}{3}c_2\Big)(2r+\delta)^2\ln(\delta + r) + \frac{1}{2}(2r+\delta)^2(\delta c_1 + c_2)\ln(r) \\
&+\Big\{\delta^4 c_3 + 4\delta^3 c_3 r + \Big(4c_3 r^2 + \frac{c_1}{2}\Big)\delta^2 + (c_2 - 2rc_1)\delta + 2rc_2\Big\}\delta\Big)\Big]\,, \\
&\epsilon(r) = \delta^6\Big[-2\delta c_1(2r+\delta)^2\ln(2r+\delta) + \frac{3}{2}\Big(\delta c_1 - \frac{c_2}{3}\Big)(2r+\delta)^2\ln(\delta + r) + \frac{1}{2}(2r+\delta)^2(\delta c_1 + c_2)\ln(r) + \Big(\delta^4 c_3 + 4\delta^3 c_3 r \\
&+\Big(4c_3 r^2 - \frac{c_1}{2}\Big)\delta^2 + (c_2 - 2rc_1)\delta + 2rc_2\Big)\delta\Big]^{-2}, \qquad \mathrm{Y}_1(r) = \epsilon(r)\mathrm{Y}(r)\,, \qquad \mathcal{F} = 1 + \frac{\delta}{r}\,,
\end{aligned}
\tag{22}
$$

where c_i $i = 1\cdots 3$ are constants of integration. Utilizing Equation (22) on the trace equation, more precisely on the fourth equation of Equation (17), we can derive $\mathrm{f}(r)$ as:

$$\mathrm{f}(r) = \frac{1}{(\delta+r)\delta^3 c_1 r^5}\Big[-36(2r+\delta)^3\Big(\delta+\frac{r}{3}\Big)rc_1\delta(\delta+r)\ln(2r+\delta) + 27(2r+\delta)^3\Big(\delta+\frac{r}{3}\Big)r\Big(c_1\delta-\frac{c_2}{3}\Big)(\delta+r)\ln(\delta+r)$$
$$+9(2r+\delta)^3\Big(\delta+\frac{r}{3}\Big)r(c_1\delta+c_2)(\delta+r)\ln(r) + 18\delta\Big\{\delta^7 c_3 r + \Big(\frac{23}{3}\delta_3 r^2 + \frac{c_1}{9}\Big)\delta^6 + \Big(\frac{c_2}{9}+\frac{68}{3}c_3 r^3\Big)\delta^5$$
$$+\Big(32c_3 r^4 + \frac{3r}{2}c_2 - \frac{5}{2}r^2 c_1\Big)\delta^4 + \frac{64}{3}r^2\Big(\frac{19}{64}c_2 - \frac{17}{48}rc_1 + \delta_3 r^3\Big)\delta^3 + \Big(\frac{16}{3}c_3 r^6 - 8r^4 c_1 + \frac{104}{9}r^3 c_2\Big)\delta^2 + \frac{8}{3}r^5 c_2$$
$$-\frac{8}{3}\Big(rc_1 - \frac{7}{2}c_2\Big)r^4\delta\Big\}\Big], \tag{23}$$

where c, c_1, c_2 and c_3 are considered as constants. Employing Equations (11) and (23) yields:

$$R = \frac{1}{4\delta^3 c_1 r^4(\delta+r)^2}\Big[72\Big(c_1\delta-\frac{1}{3}c_2\Big)(2r+\delta)^3 r(\delta+r)^2\ln(\delta+r) - 96\delta r c_1(\delta+r)^2(2r+\delta)^3\ln(2r+\delta) + 24r\ln(r)$$
$$(\delta+r)^2(2r+\delta)^3(c_1\delta+c_2) + 48\Big\{\delta^7 c_3 r + \Big(8c_3 r^2 + \frac{5c_1}{48}\Big)\delta^6 + \Big(25c_3 r^3 + \frac{5c_2}{48}\Big)\delta^5 + \Big(\frac{3}{2}rc_2 + 38c_3 r^4 - \frac{5}{2}r^2 c_1\Big)\delta^4 +$$
$$\Big(\frac{27}{4}r^2 c_2 - \frac{25}{3}r^3 c_1 + 28c_3 r^5\Big)\delta^3 + \Big(8c_3 r^6 + \frac{40}{3}r^3 c_2 - 10r^4 c_1\Big)\delta^2 - 4r^4(-3c_2+rc_1)\delta + 4c_2 r^5\Big\}\delta\Big]. \tag{24}$$

4.2. The Physical Characteristics of Solution (22)

In this subsection, we aim to comprehend the solution's physical characteristics mentioned in Equation (22). To achieve this goal, we describe the asymptotic behaviors of the metric potentials, $\mathrm{Y}(r)$ and $\mathrm{Y}_1(r)$, from Equation (22) as:

$$\mathrm{Y}(r) \approx \frac{4r^2 {c_1}^2\ln(2)}{\delta^2} - 2r^2 c_3 c_1 + \frac{{c_1}^2}{4} - \frac{1}{3}\frac{{c_1}^2\delta}{r} + \frac{1}{12}\frac{c_2 c_1}{r} + \frac{3}{8}\frac{{c_1}^2\delta^2}{r^2} - \frac{1}{8}\frac{c_1\delta c_2}{r^2} - \frac{31\delta^3{c_1}^2 - 11\delta^2 c_1 c_2}{80r^3}$$
$$+\frac{73\delta^4{c_1}^2 - 26\delta^3 c_1 c_2}{192r^4} + \cdots,$$
$$\mathrm{Y}_1(r) \approx \frac{{c_1}^2}{4} - \frac{1}{3}\frac{{c_1}^2\delta}{r} + \frac{1}{4}\frac{{c_1}^2\delta^2\ln(2)}{(2c_1\ln(2)-c_3\delta^2)^2 r^2} - \frac{\delta^4 c_3 c_1}{8(2c_1\ln(2)-c_3\delta^2)^2 r^2} - \frac{{c_1}^2\delta^3\ln(2)}{4r^3(2c_1\ln(2)-c_3\delta^2)^2}$$
$$+\frac{\delta^5 c_1 c_3}{4r^3(2c_1\ln(2)-c_3 c^2)^2} + \frac{5{c_1}^2 c^4\ln(2)}{8r^4(2c_1\ln(2)-c_3\delta^2)^2} - \frac{5}{16}\frac{c^6 c_1 c_3}{r^4(2c_1\ln(2)-c_3\delta^2)^2} - \frac{1}{64}\frac{{c_1}^2\delta^4}{r^4(2c_1\ln(2)-c_3\delta^2)^2}. \tag{25}$$

The above ansatz can be rewritten as

$$\mathrm{Y}(r) \approx -\Lambda_{eff}r^2 + 1 - \frac{M}{r} + \frac{\mathcal{Q}^2}{r^2} + \frac{{\mathcal{Q}_1}^2}{r^3} + \frac{{\mathcal{Q}_2}^2}{r^4}, \qquad \mathrm{Y}_1(r) \approx 1 - \frac{M}{r} - \Lambda_{1_{eff}}r^2 - {\mathcal{Q}_6}^2 r - \frac{{\mathcal{Q}_7}^2}{r} - \frac{{\mathcal{Q}_8}^2}{r^2}. \tag{26}$$

where we have put $c_1 = 2$, $c_3 = \frac{4.55}{\delta^2}$, $\Lambda_{eff} = \frac{7.09}{\delta^2}$, $M = 1.3\delta - 0.17c_2$, $\mathcal{Q}^2 = 1.5\delta^2 - 0.25\delta c_2$, ${\mathcal{Q}_1}^2 = -1.55\delta^3 + 0.275\delta^2 c_2$, ${\mathcal{Q}_2}^2 = 1.52083\delta^4 - 0.271\delta^3 c_2$, ${\mathcal{Q}_3}^2 = 0.1519079843\times10^{-2}\delta^2$, ${\mathcal{Q}_4}^2 = 0.3153079228\times10^{-3}\delta^3$, ${\mathcal{Q}_5}^2 = 0.4321192418\times10^{-4}\delta^4$, $\Lambda_{1_{eff}} = \frac{658.29}{\delta^2}$, ${\mathcal{Q}_6}^2 = -\frac{136.64}{\delta}$, ${\mathcal{Q}_7}^2 = -(0.3\delta - 0.02c_2)$, and ${\mathcal{Q}_8}^2 = 0.12\times10^{-2}\delta^2 - 0.9\times10^{-3}\delta c_2$. Using Equation (26) in (10) we get

$$ds^2 = -\Big[-\Lambda_{eff}r^2 + 1 - \frac{M}{r} + \frac{\mathcal{Q}^2}{r^2} + \frac{{\mathcal{Q}_1}^2}{r^3} + \frac{{\mathcal{Q}_2}^2}{r^4}\Big]dt^2 + \Big[1 - \Lambda_{1_{eff}}r^2 - \frac{M}{r} - {\mathcal{Q}_6}^2 r - \frac{{\mathcal{Q}_7}^2}{r} - \frac{{\mathcal{Q}_8}^2}{r^2}\Big]dr^2$$
$$+r^2(d\theta^2 + \sin^2 d\phi^2). \tag{27}$$

The line element (27) approaches Anti-de-Sitter spacetime asymptotically and differs from Schwarzschild spacetime because of the additional terms mainly originating from the dimensional parameter δ, which is caused by the higher-order curvature terms of f(R). Next, we will utilizing Equation (27) in Equation (11) to obtain:

$$R(r) = \frac{1}{4\delta^3 r^4 c_1 (\delta + r)^2}\Big[-96\delta r c_1(\delta + r)^2(2r + \delta)^3 \ln(2r + \delta) + 72(2r + \delta)^3(\delta + r)^2 r\Big(\delta c_1 - \frac{c_2}{3}\Big)\ln(\delta + r) + 24r(\delta + r)^2$$
$$(2r + \delta)^3(\delta c_1 + c_2)\ln(r) + 48\delta\Big(r\delta^7 c_3 + \Big(\frac{5}{48}c_1 + 8c_3 r^2\Big)\delta^6 + \Big(25r^3 c_3 + \frac{5}{48}c_2\Big)\delta^5 + \Big(38r^4 c_3 - 5/2r^2 c_1 + \frac{3rc_2}{2}\Big)\delta^4$$
$$+\Big(\frac{27}{4}c_2 r^2 - \frac{25}{3}r^3 c_1 + 28r^5 c_3\Big)\delta^3 + \Big(\frac{40}{3}c_2 r^3 - 10r^4 c_1 + 8r^6 c_3\Big)\delta^2 - 4r^4(rc_1 - 3c_2)\delta + 4r^5 c_2\Big)\Big],$$
$$\approx \frac{12\big(\delta^2 c_3 - 2\,c_1\,\ln(2)\big)(2\,r + \delta)^3}{r^3 \delta_1\,\delta^2},$$
$$r(R) \approx -\sqrt[3]{\delta}\Big[-24\sqrt[3]{18}\sqrt[3]{2c_1\ln(2) - \delta^2 c_3}R^{2/3}\delta^{4/3}{c_1}^{5/3}\ln(2) + 12\sqrt[3]{18}\sqrt[3]{2c_1\ln(2) - \delta^2 c_3}R^{2/3}\delta^{\frac{10}{3}}{c_1}^{2/3}c_3 + 2R\delta^2{c_1}^2 18^{\frac{2}{3}}\ln(2)$$
$$-R\delta^4 c_1 18^{\frac{2}{3}}c_3 + 288\Big(2c_1\ln(2) - \delta^2 c_3\Big)^{\frac{2}{3}}\sqrt[3]{R}\delta^{\frac{2}{3}}{c_1}^{4/3}\ln(2) - 144\Big(2c_1\ln(2) - \delta^2 c_3\Big)^{2/3}\sqrt[3]{R}\delta^{8/3}\sqrt[3]{c_1}c_3\Big]\Big[\Big(2c_1\ln(2) - \delta^2 c_3\Big)^{2/3}$$
$$\Big(R\delta^2 c_1 + 192c_1\ln(2) - 96\delta^2 c_3\Big)\sqrt[3]{R}\sqrt[3]{c_1}\Big]^{-1}.$$
$$= \frac{4\times 10^6\sqrt[3]{\delta}}{(2\times 10^7\delta^2 R - 1701685173)^2\sqrt[3]{R}}\Big[967244237\delta^{\frac{10}{3}}R^{\frac{5}{3}} - 82297258850\delta^{\frac{4}{3}}R^{\frac{2}{3}} + 2199141 01\delta^4 R^2 - 18711228250\delta^2 R$$
$$+4254212935\delta^{\frac{8}{3}}R^{\frac{4}{3}} - 3619665538 00\delta^{\frac{2}{3}}\sqrt[3]{R}\Big], \tag{28}$$

where we have used the numerical values given after Equation (26) in the last equation of Equation (28). Equation (28) shows that $\delta \neq 0$ which ensure that solution (27) can not reduce to the solution of Einstein GR. Equation (23) provides the asymptote form of $f(r)$, which takes the following shape:

$$f(r) \approx \frac{42.54212936}{\delta^2} + \frac{127.6263880}{\delta r} + \frac{127.626388}{r^2} + \frac{53.17766163\delta}{r^3} + \frac{0.5c_2\delta}{r^4} + \frac{6.97664925\delta^2}{r^4}. \tag{29}$$

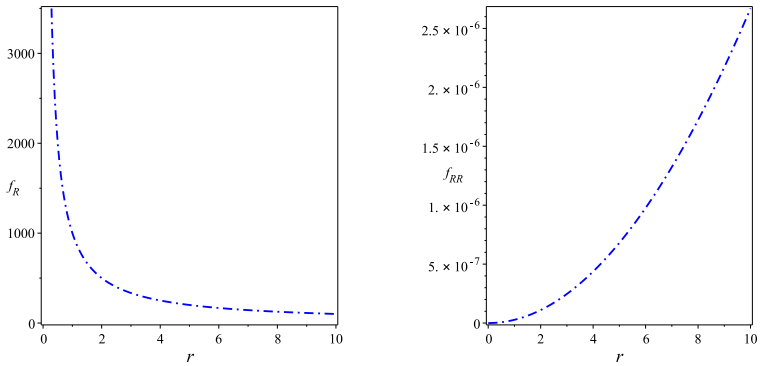
By substituting the second equation of (28) into Equation (29), we obtain:

$$f(R) \approx \beta_1 - \beta_2 R^{1/3} + \beta_3 R^{2/3} + \beta_4 R + \cdots, \tag{30}$$

with $\beta_i, i = 1\cdots 4$ are constants take the values:

$$\beta_1 = \frac{2\times 10^{-20}}{\delta^{7/3}}\Big[-2.500000044\times 10^{20}\delta^{7/3} - 1.319484608\times 10^{21}\,\delta^{5/3} - 2.127106468\times 10^{21}\sqrt[3]{\delta} - 2.901732708\times 10^{21}\delta$$
$$-1.491435932\times 10^{19}\,\delta^3 - 1068876963000000000c_2\,\delta^2\Big],$$
$$\beta_2 = 10^{-18}\,\frac{195312500.0\,\delta^{5/3}c_2 + 47859895500\delta^{4/3} + 2725253613\delta^{8/3} + 17867694300\delta^{2/3} + 33236038520\delta^2}{\delta^{4/3}},$$
$$\beta_3 = 10^{-19}\,\frac{1595329850 00\delta + 195312500\delta^{4/3}c_2 + 2725253613\delta^{7/3} + 33236038520\delta^{5/3} + 63813194000\sqrt[3]{\delta}}{\sqrt[3]{\delta}},$$
$$\beta_4 = 10^{-19}\,\delta^{2/3}\Big(195312500.0\,c_2\delta - 6381319400 + 2725253613.0\delta^2\Big). \tag{31}$$

It is important to compare our findings with those reported by Jaime et al. [58]. In their study, Jaime et al. investigated f(R) models that satisfy two key conditions: $f_R = \frac{df(R)}{dR} > 0$ and $f_{RR} = \frac{d^2 f(R)}{dR^2} > 0$. These conditions were imposed to ensure viable solutions for extended objects surrounded by matter fields, rather than black hole solutions in vacuum. Notably, the black hole solutions obtained in our study also satisfy these conditions, as demonstrated in Figure 1a,b.



(**a**) The attributes of f_R (**b**) The attributes of f_{RR}

Figure 1. (**a**) The general behavior of the function f_R is shown in (**a**), and (**b**) illustrates the corresponding behavior of f_{RR}.

Now, utilize Equation (22) to determine the invariants we get:

$$\begin{aligned}
\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)_{\text{as}_{r\to\infty}} &\approx \frac{1206.6}{\delta^4}+\frac{3610}{\delta^3 r}+\frac{4826}{\delta^2 r^2}+\frac{\left(9472\delta^2-1536c_2\delta+32{c_2}^2\right)\times 10^{-10}\ln(r)+3921\delta^2+1775\times 10^{-10}c_2\delta}{\delta^3 r^3},\\
\left(R_{\mu\nu}R^{\mu\nu}\right)_{\text{as}_{r\to\infty}} &\approx \frac{1800}{\delta^4}+\frac{5420}{\delta^3 r}+\frac{6938}{\delta^2 r^2}+\frac{\left(142\delta^3-576c_2\,\delta^2+48{c_2}^2\delta\right)\times 10^{-10}\ln(r)+4978\delta^3}{\delta^4 r^3},\\
(R)_{\text{as}_{r\to\infty}} &\approx \frac{85}{\delta^2}+\frac{128}{\delta r}+\frac{64}{r^2}+\frac{11\delta}{r^3}.
\end{aligned} \tag{32}$$

Here $\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R\right)$ are the Kretschmann scalar, the square of the Ricci tensor, and the Ricci scalar and all exhibit a physical singularity at $r = 0$. Furthermore, the equations above demonstrate that δ cannot be zero in order to guarantee that solution (22) cannot be transformed into GR in any way. Emphasizing the importance of the constant δ, which is the primary factor causing the difference of the given results from GR with specified values $\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R\right) = (16[2\Lambda^2 r^6+9M]/[3r^6], 16\Lambda^2, \mp 8\Lambda)$. Equation (32) shows that the main term of the invariants $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R)$ is $(\frac{1}{r}, \frac{1}{r}, \frac{1}{r})$ which differs from the Schwarzschild black hole where the leading term of the Kretschmann scalar is $\frac{1}{r^6}$ and the other invariants $R_{\mu\nu}R^{\mu\nu} = R = 0$ Hence, Equation (32) suggests that singularities in invariants are more powerful than the Schwarzschild black hole in GR whether as r approaches infinity or approaches zero.

5. The Thermodynamic Characteristics of the Black Hole Solution

In order to analyze the thermodynamic properties of the recently found solution defined in Equation (26), we introduced the concept of the Hawking temperature [59,60][4] as:

$$T_2 = \frac{\Upsilon'(r_2)}{4\pi\sqrt{-\Upsilon(r_2)/\Upsilon_1(r_2)}}. \tag{33}$$

In this situation, the prime symbol denotes a derivative in relation to the event horizon positioned at $r = r_2$. The value of r_2 that satisfies $\Upsilon_1(r_2) = 0$ but $\Upsilon_1'(r_2)$ is not zero, corresponds to the largest positive root. The entropy of Bekenstein-Hawking for f(R) theory is expressed as [65,66][5]:

$$S(r_2) = \frac{1}{4}Af_R(r_2). \tag{34}$$

Here, A is the measurement of the event horizon's surface area within this context. The quasi-local energy is calculated in the following way [61–66]:

$$E(r_2) = \frac{1}{4}\int \left[2f_R(r_2) + {r_2}^2\left\{f(R(r_2)) - R(r_2)f_R(r_2)\right\}\right] dr_2. \tag{35}$$

The black hole's thermodynamic stability can be determined by the heat capacity indicator H_2 as follows: if $H_2 > 0$, it is stable, if $H_2 < 0$, it is unstable. In the ensuing examination, we assess the thermal stability of these black hole solutions by monitoring their heat capacities [67,68]

$$H_2 = \frac{dE_2}{dT_2} = \frac{\partial M}{\partial r_2}\left(\frac{\partial T_2}{\partial r_2}\right)^{-1}. \tag{36}$$

In this frame, E_2 stands for the energy. In the end, the Gibbs free energy is determined in the following manner [66,69]:

$$G_2 = E_2 - T_2 S_2\,. \tag{37}$$

In the case of the form given in Equation (26), the horizons are made up of four roots that can be expressed in an algebraic equation.

$$r = \sqrt[4]{y^4\Lambda_{eff} - y^2 + My - \mathcal{Q}^2} \tag{38}$$

where y represents the variable that is influenced by other variables. Furthermore, based on Equation (22), we are able to obtain the subsequent equation for mass.

$$M = \frac{\Lambda_{eff}r^4 - r^2 - \mathcal{Q}^2}{r}. \tag{39}$$

Equation (39) shows how the black hole's overall mass is influenced by the horizon and the dimensional constant δ. The illustration in Figure 2a shows the performance of functions Y and $Y1$.

Using Equation (34), we can calculate the entropy of the specified black hole solution provided by Equation (26) in the following manner:

$$S_2 = \frac{\pi {r_2}^2}{4}\left(1 + \frac{\delta}{r_2}\right). \tag{40}$$

By looking at Equation (40), we can see that $\mathcal{Q}$ cannot be equal to zero. Given the explanation of Λ_{eff}, M, and $\mathcal{Q}$, following Equation (26), it is evident that $\delta = 0$ is not an allowable value. This implies that we are unable to regain the standard entropy of general relativity. The entropy patterns are shown in Figure 2b, demonstrating a consistent entropy pattern.

The temperature associated with the black hole solution in Equation (26) is determined to be:

$$T_2 = \frac{\delta(2\Lambda_{eff}{r_2}^4 - Mr_2 + 2\mathcal{Q}^2)}{4\pi r_2\sqrt{\mathcal{Q}^2 - \Lambda_{eff}{r_2}^4 - Mr_2 + {r_2}^2\sqrt{(r_2{\mathcal{Q}_8}^2 - {r_2}^2 + \mathcal{Q}_9)\delta^2 + \delta{r_2}^3{\mathcal{Q}_7}^2 + {r_2}^4{\mathcal{Q}_6}^2}}}. \tag{41}$$

In this context, T_2 signifies the temperature of Hawking at the event horizon. The figure labeled Figure 2c illustrates the temperature, consistently showing it as a positive value.

The expression for the quasi-local energy is given as shown in Equation (35).

$$E_2 = -16r_2 - 5\delta\ln r - \frac{4{r_2}^3}{\delta^2} - \frac{11{r_2}^2}{\delta} + \frac{\delta^2}{r_2} - \frac{\delta c_2}{4r_2}\,. \tag{42}$$

Figure 2d shows the trend in quasi-local energies, demonstrating an increase in E_2.

At last, applying Equation (36) allows us to derive the expression of heat capacity linked to black hole (26) in the following format:

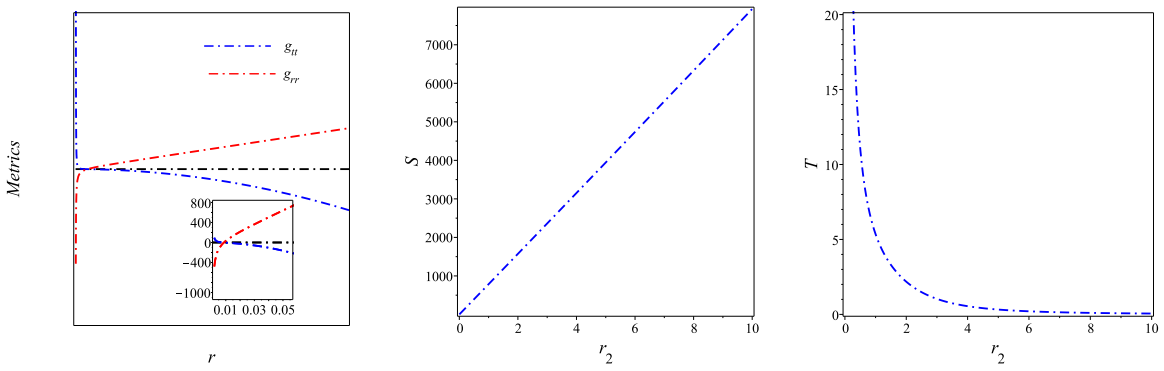
$$H_2 = -8\times10^{-29}\pi\Big[\Big\{\Big(10^8c_2-10^{12}r\Big)\delta^3-10^{12}\delta^4+\Big(-10^{12}\times r^2+17\times10^9c_2r\Big)\delta^2+6\times10^{12}r^4\Big\}^{3/2}\Big(10^9\delta^2r^2-13\times10^{12}r^3\delta$$
$$-10^4\delta^4+10^9\Lambda_{eff}\,r^4-10^9M\delta^2r+10^3\delta^3c_2\Big)^{3/2}\Big(10^6r^4-62\delta^2r^2+1\delta^4-0.2\delta^3c_2\Big)\Big]\Big[\delta^2\Big(11\times10^{12}r^{12}\Lambda_{eff}-72\times10^{12}r^{11}\delta$$
$$-10^{-12}r^{10}\Lambda_{eff}\,\delta^2+\Big(\Big(-11\times10^7+10^6\Lambda_{eff}\Big)\delta^3+\Big(5\times10^6M-72705c_2\,\Lambda_{eff}\Big)\delta^2\Big)r^9+\Big(\Big(-8\times10^6+162982\Lambda_{eff}\Big)\delta^4$$
$$+\Big(11\times10^7-10^6\Lambda_{eff}\Big)c_2\delta^3\Big)r^8+\Big(\Big(-21\times10^6-530\Lambda_{eff}\Big)\delta^5+\Big(\Big(66\Lambda_{eff}+35\times10^6\Big)c_2-25\times10^5M\Big)\delta^4\Big)r^7$$
$$+\Big[\Big(7\times10^4-226\Lambda_{eff}\Big)\delta^6+\Big[\Big(-10^6+43\Lambda_{eff}\Big)c_2-7\times10^5M\Big]\delta^5+\Big(-{c_2}^2\Lambda_{eff}+9\times10^4c_2\,M\Big)\delta^4\Big]r^6$$
$$+\Big[\Big(2\times10^5-26\Lambda_{eff}\Big)\delta^7+\Big[\Big(8\Lambda_{eff}-4\times10^4\Big)c_2-10^5M\Big]\delta^6+\Big(\Big(10^4-\Lambda_{eff}\Big){c_2}^2+10^4c_2\,M\Big)\delta^5\Big]r^5+\Big[\Big(-2\Lambda_{eff}+10^4\Big)\delta^8$$
$$+\Big[\Big(\Lambda_{eff}-10^4\Big)c_2+3\times10^3M\Big]\delta^7+\Big(\Big(6\times10^2-0.1\Lambda_{eff}\Big){c_2}^2-33c_2\,M\Big)\delta^6\Big]r^4+\Big[10^4\delta^9+\Big(10^2M-7\times10^2c_2\Big)\delta^8$$
$$+\Big(61{c_2}^2-23c_2M\Big)\delta^7+0.4{c_2}^2\delta^6M\Big]r^3+\Big[(0.4c_2+13M)\delta^9-\delta^{10}-\Big(4c_2M+0.04{c_2}^2\Big)\delta^8+\Big(0.3{c_2}^2M-10^{-4}{c_2}^3\Big)\delta^7\Big]r^2$$
$$+\Big[10^{-4}\delta^{11}+\Big(M-10^{-4}c_2\Big)\delta^{10}+\Big(10^{-4}{c_2}^2-0.3c_2M\Big)\delta^9+\Big(0.03{c_2}^2M-10^{-5}{c_2}^3\Big)\delta^8\Big]r+10^{-5}\delta^{12}-10^{-7}\delta^9{c_2}^3$$
$$-10^{-7}\delta^{11}c_2+10^{-7}\delta^{10}{c_2}^2\Big)\Big]^{-1}. \tag{43}$$

The heat capacity behavior shown in Equation (43) is displayed in Figure 2e, revealing a positive trend for the black hole described by Equation (26), indicating its thermodynamic stability. The free energy of the grand canonical ensemble, also called Gibbs free energy, is defined as follows [66,69]:

$$G(r_2) = M(r_2) - T(r_2)S(r_2)\,. \tag{44}$$

Here, $T(r_2)$, $S(r_2)$, and $E(r_2)$ denote the temperature, entropy, and quasi-local energy at the event horizon, respectively. Substituting Equations (34) and (38)–(40) into Equation (44) results in the following:

$$G(r_2) = -16r_2-5\delta\ln(r_2)-\frac{4{r_2}^3}{\delta^2}-\frac{11{r_2}^2}{\delta}+\frac{\delta^2}{r_2}-\frac{\delta c_2}{4r_2}-\frac{\Big(2\Lambda_{eff}{r_2}^4-Mr_2+2\mathcal{Q}^2\Big)\delta r_2\Big(1+\frac{\delta}{r_2}\Big)}{10\sqrt{{r_2}^2-\Lambda_{eff}\,{r_2}^4-Mr_2+\mathcal{Q}^2}}$$
$$\times\frac{1}{\sqrt{\Big(r{\mathcal{Q}_8}^2-{r_2}^2+\mathcal{Q}_9\Big)\delta^2+{\mathcal{Q}_7}^2{r_2}^3\delta+{\mathcal{Q}_6}^2{r_2}^4}}\,. \tag{45}$$



(a) **(b)** **(c)**

Figure 2. *Cont.*

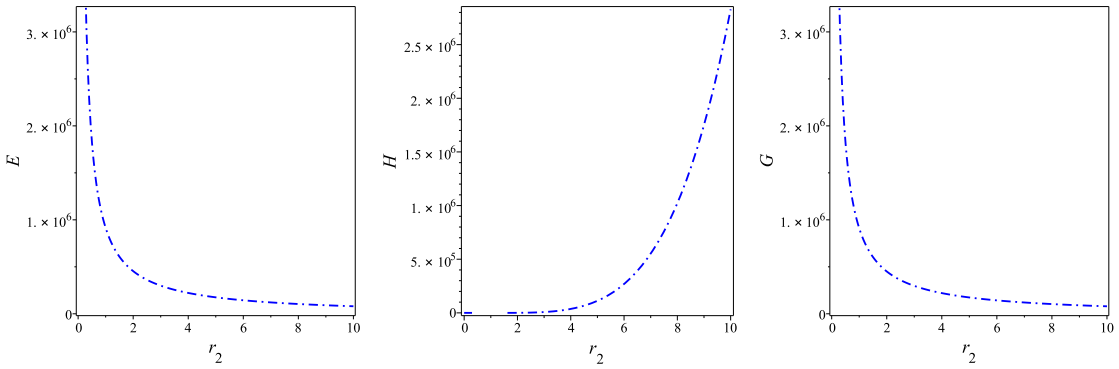


Figure 2. Figure (**a**) shows the general patterns of the metric components g_{tt} and g_{rr}; Figure (**b**) illustrates how the entropy behaves; Figure (**c**) illustrates the fluctuations in the Hawking temperature, and Figure (**f**) shows the Gibbs function's pattern. The model is characterized by all parameters that take the following specific values $\delta = 10^3$, $M = 1$, $\Lambda_{eff} = 10^2$, $c_2 = 1$, and $\mathcal{Q} = \mathcal{Q}_5 = \mathcal{Q}_6 = \mathcal{Q}_7 = \mathcal{Q}_8 = \mathcal{Q}_9 = 1$ for the calculations of the thermodynamical quantities in Planck mass units. (**a**) The properties of the metric potential g_{tt} and g_{rr}; here, we use the numerical values for $\delta = -0.1$, $M = 11$, $\Lambda_{eff} = 1$, abd $c_2 = 1$. (**b**) The entropy. (**c**) The characteristics of the Hawking temperature. (**d**) The quasi-local energy. (**e**) The heat capacity. (**f**) The characteristics of the Gibbs function.

Figure 2f illustrates the variations in the Gibbs energy trends of the black hole (26) for certain model parameter values. Figure 2f shows that the Gibbs free energy remains consistently positive, signifying higher overall stability.

6. Solution of the Two Scalar Fields

Using Equations (21) and (22) in Equation (16), we can derive the expressions of b, b_2, and Φ. Due to the lengthy form of these quantities, we show their behaviors in Figure 3, which shows that the coefficient of the doublet scalar fields and the potential are all positive that ensure the black hole presented in this study is free from ghost.

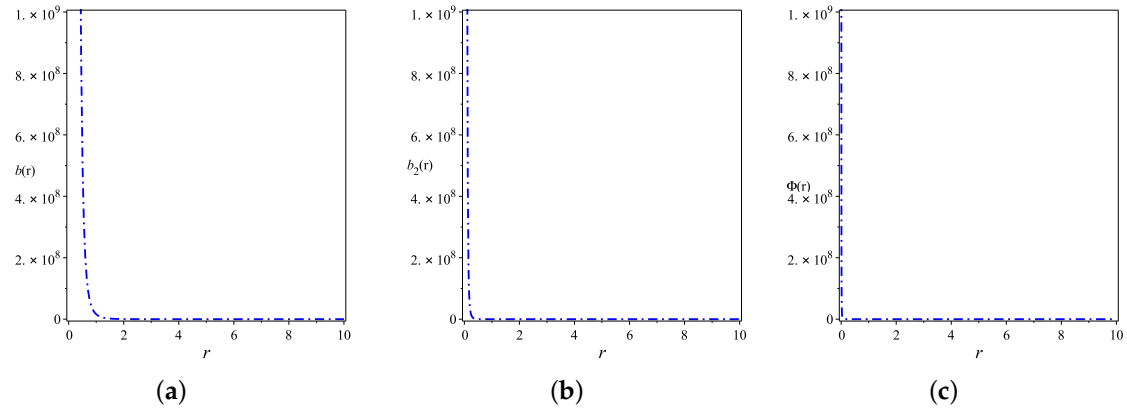


Figure 3. (**a**) The overall trends of the coefficient of the scalar fields $b(r)$ is illustrated in Figure 2a; (**b**) presents the overall trends of the coefficient of the scalar fields $b_1(r)$; and (**c**) displays how the potential behaves. (**a**) The characteristics of the coefficient of the scalar fields $b(r)$. (**b**) The characteristics of tge coefficient of the scalar fields $b_2(r)$. (**c**) The characteristics of the potential Φ.

7. Discussion and Conclusions

Spherical symmetric spacetimes play a crucial role in black hole physics because their fundamental properties are readily accessible for investigation [70]. Numerous

solutions for spherical symmetric black holes have previously been obtained within modified gravity theories involving f(R) terms and identical metric potentials [71,72]. In the present study, we consider a spherical symmetric spacetime with distinct metric potentials, without imposing any specific form on them. Before proceeding, we emphasize the following important points:

(i) In the vacuum case, analyzing the trace equation of f(R) allows us to determine a specific form of f(R).

(ii) In the field Equations (7) and (9), we employ this particular form of f(R), which yields a system of differential equations involving only the first derivative of f(R) with respect to R. By substituting this form into Equation (10), we obtain the governing differential equations of the system. We have solved these equations analytically, thereby determining both the metric potentials and the function $f(R)$.

The resulting black hole solution depends only on distinct integration constants, which distinguishes it from the Schwarzschild solution of general relativity (GR). We have shown that this solution does not coincide with the classical GR black hole. The effect of higher curvature terms in f(R) gravity is the primary reason that this solution can not be specialized the corresponding GR solution. To understand the physics of this novel solution, we analyze its asymptotic behavior and demonstrate that it approaches AdS spacetime in one limit. With respect to the second asymptote associated with the cosmological constant, the line element exhibits an asymptotic structure similar to AdS/dS spacetime.

The absence of Birkhoff's theorem in f(R) gravity has been widely discussed [73]. Several studies have explored the validity of this theorem in the conformal framework [74–81]. In our work, we did not rely on approximations or conformal transformations to obtain analytical solutions. Our findings confirm that Birkhoff's theorem does not hold in f(R) gravity. In GR, the theorem remains valid because no spin-0 modes arise in the linearized field equations; consequently, a spherical symmetric spacetime cannot couple to higher-spin excitations [73,82]. In contrast, in f(R) gravity, the differential equation governing the Ricci scalar R acts as a spin-0 mode. Therefore, any significant coupling between the metric and R can lead to the breakdown of Birkhoff's theorem, which is precisely reflected in the analytical solution of Equation (28), yielding a nontrivial value for R.

The physics of black holes is further examined by calculating curvature invariants. We find that they exhibit leading-order behavior of $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R) \sim (\frac{1}{\mathrm{r}}, \frac{1}{\mathrm{r}}, \frac{1}{\mathrm{r}})$, in contrast to the Schwarzschild black hole, where the leading Kretschmann scalar scales as $1/r^6$ and the other invariants remain constant, $R_{\mu\nu}R^{\mu\nu} = R = \text{const}$. This indicates that the singularity associated with the Kretschmann scalar is stronger in our solution than in GR. Importantly, this amplification arises from the contribution of higher-order curvature terms in the f(R) theory. In our research on this black hole, we also compute thermodynamic quantities such as the Hawking temperature, entropy, quasi-local energy, and Gibbs free energy. Notably, all these thermodynamic properties are consistent with previous studies of black holes.

In addition, we have analyzed the stability of this black hole. To this end, we reformulate the f(R) action as a scalar-tensor theory in which the scalar field interacts with the Ricci scalar. Using the odd-type perturbation method, we derive the gradient stability condition and show that the radial propagation speed of perturbations equals unity for our black holes.

Finally, we determine the explicit form of the coefficients for the scalar doublet fields. These coefficients are shown to depend on both the metric potentials and the derivatives of f(R). By employing the metric potentials and f(R) form derived in this study, one can compute the explicit coefficients and the potential Φ. Due to the length of these expressions,

we present them in Figure 3a–c. These results confirm that our model is free of ghost-like instabilities, as all coefficients remain positive.

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Notes

[1] We verified all the computations in this work with the help of Maple 16 software.

[2] Note that since we are dealing with spherical symmetry then $f(R) = f(R(r)) \equiv f(r)$. Therefore, $\mathcal{F}(r) = f_R = \frac{df(R)}{dR} \equiv \frac{df(r)}{dr}\frac{dr}{dR}$.

[3] Equations (18)–(20) are in fact two differential equations. This is can be shown by subtracting (18) and (19) and (19), (20), which show that this system yields only two independent differential equations.

[4] Because of the disparity in the metric potentials in the solution provided by Equation (22), the Hawking temperature differs from the standard temperature when the metric potentials are the same [59,61–64].

[5] Remember that the concept of entropy in linear non-metricity theory is not the same as in f(R) theory. By setting $f(R) = R$, we achieve a better understanding of the non-metricity theory.

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