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The Influence of Zero-Point Fluctuations on the Photon Wave Packet Motion in a Vacuum

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Abstract

The influence of zero-point fluctuations on photon propagation in a vacuum is investigated without using normal ordering and renormalization procedures, but in a frame of the conformally unimodular metric for a description of the fluctuating gravitational field. The complete formula for decoherence time is presented.

Keywords: zero point fluctuations; stochastic gravitational field; photon decoherence

1. Introduction

Current investigations of photon propagation in a vacuum probe fundamental physics at the intersection of quantum field theory, cosmology, and space-time symmetry tests [1]. Two phenomena remain pivotal: (i) energy-dependent velocity dispersion [2,3], potentially detectable as time delay for photons of the different energies from distant astrophysical sources, and (ii) intrinsic decoherence mechanisms linked to quantum space-time fluctuations [4–6], which may depolarize radiation over cosmological scales [7,8]. While velocity dispersion tests Lorentz invariance [1,9], decoherence bridges quantum gravity phenomenology and observational astrophysics [5,10,11].

Here, we develop a framework to quantify these effects via photon dynamics in fluctuating space-time. Formulating an optical Dirac equation for photons in curved geometry allows us to use the density matrix formalism in the first-order differential equation. The next step is to derive an equation for the photon density matrix in a random space-time, which is assumed to be Minkowski on average. In this way, one needs to calculate the correlators of a metric arising due to zero-point fluctuations of the quantum fields. Then, we introduce some quantity to describe an electromagnetic field's degree of coherence (i.e., purity of a system). Finally, an explicit formula for the decoherence time of a photon wave packet is deduced.

2. Optical Dirac Equation for the Photon Under Gravitational Background

The wave equation inherently involves a second-order time derivative, distinguishing it from the Schrödinger equation, which governs the dynamics of massive particles and is first-order in time. Decoherence effects for such particles have been analyzed within the Schrödinger framework [4]. Extending this approach to electromagnetic fields requires redefining Maxwell's equations in a matrix form. This reformulation allows for constructing an optical analogue of the Dirac equation, as discussed in [12], providing a suitable foundation for investigating decoherence in the photon sector.

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Because it is not possible to define a vacuum state which is invariant relative to general coordinate transformations, it is reasonable to define a unique class of metrics to describe the gravitational field arising due to zero-point fluctuations [4,13–15]:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2 (1 - \partial_m P^m)^2 d\eta^2 - \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta), \tag{1}$$

where $x^\mu = \{\eta, \boldsymbol{x}\}$, η is a conformal time, γ_{ij} is a spatial metric, $a = \gamma^{1/6}$ is a locally defined scale factor, and $\gamma = \det \gamma_{ij}$. This metric reflects the concept [13] that diffemorfism symmetry of general relativity has to be violated with respect to a particular class of metrics (see also [16] in this relation) to avoid the cosmological zero point energy problem. The spatial part of the interval (1) reads as

$$dl^2 \equiv \gamma_{ij} dx^i dx^j = a^2(\eta, \boldsymbol{x}) \tilde{\gamma}_{ij} dx^i dx^j, \tag{2}$$

where $\tilde{\gamma}_{ij} = \gamma_{ij}/a^2$ is a matrix with the unit determinant. According to ref. [13], 3-vector $\boldsymbol{P}$ and $\boldsymbol{N}$ and are not arbitrary, but obey the constaints $\boldsymbol{\nabla}(\nabla \cdot \boldsymbol{N}) = 0$, $\boldsymbol{\nabla}(\nabla \cdot \boldsymbol{P}) = 0$, where the ordinary, non-covariant operator $\boldsymbol{\nabla}$ is implied. Let us set $\boldsymbol{N} = 0$ and $\boldsymbol{P}$ and write Maxwell equations in this metric. The forms of electrodynamic in the curved space-time are discussed up to the present time [17,18], but here we use the Maxwell equations in the three-dimensional form suggested in [19]:

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} \left(\sqrt{\gamma} D^i \right) = 0, \tag{3}$$

$$\frac{1}{2\sqrt{\gamma}} e^{ijk} \left(\frac{\partial H_k}{\partial x^j} - \frac{\partial H_j}{\partial x^k} \right) = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \eta} (\sqrt{\gamma} D^i), \tag{4}$$

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} \left(\sqrt{\gamma} B^i \right) = 0, \tag{5}$$

$$\frac{1}{2\sqrt{\gamma}} e^{ijk} \left(\frac{\partial E_k}{\partial x^j} - \frac{\partial E_j}{\partial x^k} \right) = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \eta} (\sqrt{\gamma} B^i), \tag{6}$$

where $B^i = \gamma^{ij} H_j / \sqrt{g_{00}}$, $D^i = \gamma^{ij} E_j / \sqrt{g_{00}}$. In the conformally unimodular metric (1) $\sqrt{g_{00}} = a$, $\sqrt{\gamma} = a^3$, $\gamma^{ij} = a^{-2} \tilde{\gamma}^{ij}$. By introducing new quantities $\boldsymbol{\mathcal{D}} = a^3 \boldsymbol{D}$, and $\boldsymbol{\mathcal{B}} = a^3 \boldsymbol{B}$, the system of Equations (3)–(6) acquires the form:

$$\mathrm{div} \boldsymbol{\mathcal{D}} = 0, \tag{7}$$

$$\mathrm{rot}(\tilde{\gamma} \boldsymbol{\mathcal{B}}) = \frac{\partial}{\partial \eta} \boldsymbol{\mathcal{D}}, \tag{8}$$

$$\mathrm{div} \boldsymbol{\mathcal{B}} = 0, \tag{9}$$

$$\mathrm{rot}(\tilde{\gamma} \boldsymbol{\mathcal{D}}) = -\frac{\partial}{\partial \eta} \boldsymbol{\mathcal{B}}, \tag{10}$$

where a matrix $\tilde{\gamma}$ denotes the matrix $\tilde{\gamma}_{ij}$. These equations could be written as an optical Dirac equation [12]

$$\frac{\partial}{\partial \eta} \begin{pmatrix} \boldsymbol{\mathcal{D}} \\ i\boldsymbol{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{S}\hat{\boldsymbol{p}} \\ \boldsymbol{S}\hat{\boldsymbol{p}} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\gamma} \boldsymbol{\mathcal{D}} \\ i\tilde{\gamma} \boldsymbol{\mathcal{B}} \end{pmatrix}, \tag{11}$$

where the rotor is expressed through the matrix of spin 1

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

by the relation $(\boldsymbol{S}\hat{\boldsymbol{p}})\boldsymbol{F} = i\hat{\boldsymbol{p}} \times \boldsymbol{F}$ [12] valid for the operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$ and an arbitrary 3-vector $\boldsymbol{F}$. Denoting

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{S} \\ \boldsymbol{S} & 0 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix} \tag{13}$$

allows us to write the optical Dirac equation for a six-component wave function $\Psi \equiv \{\mathcal{D}, i\mathcal{B}\}$

$$i\frac{\partial \Psi}{\partial \eta} = \boldsymbol{\alpha}\hat{\boldsymbol{p}}\,\mathcal{G}\Psi. \tag{14}$$

Additionally, the conditions (7) and (9) on the upper and lower components of the photon wave function have to be satisfied. In a presence of a static gravitational field, the scalar product

$$< \Psi_1|\Psi_2 >= \int \Psi_1^+ \mathcal{G}\Psi_2 d^3\boldsymbol{r} \tag{15}$$

is conserved during evolution, but for the nonstationary case, it is not. We are not interested in the photon creation by the nonstationary gravitational field, so let us ad hoc modify Equation (14) to avoid the photon creation

$$i\frac{\partial \Psi}{\partial \eta} = \boldsymbol{\alpha}\hat{\boldsymbol{p}}\,\mathcal{G}\Psi - i\frac{\mathcal{G}^{-1}}{2}\frac{\partial \mathcal{G}}{\partial \eta}\Psi. \tag{16}$$

For Equation (16) the scalar product (15) is conserved even for a time-dependent gravitational field. Let us do a non-unitary transformation (see [20] in this relation)

$$\psi(\boldsymbol{r}, \eta) = \mathcal{G}^{1/2}(\boldsymbol{r}, \eta)\Psi(\boldsymbol{r}, \eta), \tag{17}$$

which leads to the photon quantum mechanics with the "flat" scalar product $< \psi_1|\psi_2 >= \int \psi_1^+\psi_2 d^3\boldsymbol{r}$. As a result, the optical Dirac Equation (16) acquires the form

$$i\frac{\partial \psi}{\partial \eta} = \mathcal{G}^{1/2}\boldsymbol{\alpha}\hat{\boldsymbol{p}}\,\mathcal{G}^{1/2}\psi, \tag{18}$$

for an arbitrary time-dependent gravitational field $\mathcal{G}$. Considering the gravitational field as a perturbation reduces (18) to the approximate equation

$$i\frac{\partial \psi}{\partial \eta} = \boldsymbol{\alpha}\hat{\boldsymbol{p}}\,\psi + \frac{1}{2}((\mathcal{G} - I)\boldsymbol{\alpha}\hat{\boldsymbol{p}} + \alpha\hat{\boldsymbol{p}}(\mathcal{G} - I))\psi, \tag{19}$$

where the first term on the right-hand side of (19) contains the Hamiltonian $H_0 \equiv \boldsymbol{\alpha}\hat{\boldsymbol{p}}$ and the remaining terms represent the interaction

$$V = \frac{1}{2}((\mathcal{G} - I)\boldsymbol{\alpha}\hat{\boldsymbol{p}} + \boldsymbol{\alpha}\hat{\boldsymbol{p}}(\mathcal{G} - I)). \tag{20}$$

Further, we will consider a fluctuating gravitational field originating due to the zero-point fluctuation of quantum fields.

3. Correlators of the Fluctuating Gravitational Field

Let us imagine empty space-time filled only by a vacuum, and take into account its quantum properties, i.e., the gravitational field created by zero-point fluctuations of quantum fields. For simplicity, only the scalar field will be considered. We assume that fluctuations of the gravitational field in a class of conformally unimodular metrics (1) are relatively small, allowing us to consider them as perturbations. Scalar perturbations of the conformally unimodular metric [21] are written as

$$ds^2 = a(\eta, \boldsymbol{x})^2 \left(d\eta^2 - \left(\left(1 + \frac{1}{3} \sum_{m=1}^{3} \partial_m^2 F(\eta, \boldsymbol{x}) \right) \delta_{ij} - \partial_i \partial_j F(\eta, \boldsymbol{x}) \right) dx^i dx^j \right), \tag{21}$$

where the perturbations of the locally defined scale factor are defined by

$$a(\eta, \boldsymbol{x}) = e^{\alpha(\eta)}(1 + \Phi(\eta, \boldsymbol{x})), \tag{22}$$

where Φ could be referred to as a gravitational potential. The stress–energy tensor in the hydrodynamic approximation [22]

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p\, g_{\mu\nu} \tag{23}$$

includes the perturbations of the energy density $\rho(\eta, \boldsymbol{x}) = \rho_v + \delta\rho(\eta, \boldsymbol{x})$ and pressure $p(\eta, \boldsymbol{x}) = p_v + \delta p(\eta, \boldsymbol{x})$ around the vacuum mean values, where the index v will denote a uniform component of the vacuum energy density and pressure.

The zero-order equations for a flat universe take the form [23]

$$M_p^{-2} e^{4\alpha} \rho_v - \frac{1}{2} e^{2\alpha} \alpha'^2 = const, \tag{24}$$

$$\alpha'' + \alpha'^2 = M_p^{-2} e^{2\alpha} (\rho_v - 3p_v), \tag{25}$$

where $\alpha(\eta) = \log a(\eta)$ and M_p denotes the reduced Planck mass $M_p = \sqrt{\frac{3}{4\pi G}}$. According to the five-vector theory of gravity [13], the first Friedmann Equation (24) is satisfied up to some constant, and the main parts of the vacuum energy density and pressure

$$\rho_v \approx N_{all} \frac{k_{max}^4}{16\pi^2 a^4}, \tag{26}$$

$$p_v = \frac{1}{3} \rho_v \tag{27}$$

do not contribute to the universe expansion. In the Formula (26), the number N_{all} of all degrees of freedom of the quantum fields in nature appears because the zero-point stress–energy tensor is an additive quantity [24]. The momentum ultraviolet cut-off [14]

$$k_{max} \approx \frac{12 M_p}{\sqrt{2 + N_{sc}}} \tag{28}$$

is proportional to the Planck mass and includes the number of minimally coupled scalar fields N_{sc} plus two, because the gravitational waves possess two additional degrees of freedom [25]. Without a real matter, and if the constant in Equation (24) compensates a vacuum energy (26) exactly, the static Minkowski space-time arises. To consider the perturbations under this background, we set $\alpha(\eta) = 0$ in (22).

Generally, a vacuum resembles some fluid, i.e., "ether" [23,26], but with some stochastic properties among the elastic ones. Let us return to the stress–energy tensor (23) and introduce other variables

$$\wp(\eta, \boldsymbol{x}) = a^4(\eta, \boldsymbol{x}) \rho(\eta, \boldsymbol{x}), \tag{29}$$

$$\Pi(\eta, \boldsymbol{x}) = a^4(\eta, \boldsymbol{x}) p(\eta, \boldsymbol{x}) \tag{30}$$

for the reasons which will be explained below. The perturbations around the uniform values can be written now as $\wp(\eta, \boldsymbol{x}) = \rho_v + \delta\wp(\eta, \boldsymbol{x})$, $\Pi(\eta, \boldsymbol{x}) = p_v + \delta\Pi(\eta, \boldsymbol{x})$. The vacuum-ether 4-velocity u is represented in the form of

$$u^\mu = \{(1 - \Phi(\eta, \boldsymbol{x})), \boldsymbol{\nabla} \frac{v(\eta, \boldsymbol{x})}{\wp(\eta, \boldsymbol{x}) + \Pi(\eta, \boldsymbol{x})}\} \approx \{(1 - \Phi(\eta, \boldsymbol{x})), \boldsymbol{\nabla} \frac{v(\eta, \boldsymbol{x})}{\rho_v + p_v}\}, \tag{31}$$

where $v(\eta, \boldsymbol{x})$ is a scalar function. Expanding all perturbations into the Fourier series $\delta\wp(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \delta\wp_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\boldsymbol{x}}$, $\Phi(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\boldsymbol{x}} \ldots$ etc., results in the equations for the perturbations:

$$-6\hat{\Phi}'_{\boldsymbol{k}} + k^2 \hat{F}'_{\boldsymbol{k}} + \frac{18}{M_p^2} \hat{v}_{\boldsymbol{k}} = 0, \tag{32}$$

$$-6k^2 \hat{\Phi}_{\boldsymbol{k}} + k^4 \hat{F}_{\boldsymbol{k}} + \frac{18}{M_p^2} \delta\hat{\wp}_{\boldsymbol{k}} = 0, \tag{33}$$

$$-12\hat{\Phi}_{\boldsymbol{k}} - 3\hat{F}''_{\boldsymbol{k}} + k^2 \hat{F}_{\boldsymbol{k}} = 0, \tag{34}$$

$$-9\hat{\Phi}''_{\boldsymbol{k}} - 9k^2 \hat{\Phi}_{\boldsymbol{k}} + k^4 \hat{F}_{\boldsymbol{k}} - \frac{9}{M_p^2} \left(3\delta\hat{\Pi}_{\boldsymbol{k}} - \delta\hat{\wp}_{\boldsymbol{k}}\right) = 0, \tag{35}$$

$$-\delta\hat{\wp}'_{\boldsymbol{k}} + k^2 \hat{v}_{\boldsymbol{k}} = 0, \tag{36}$$

$$\delta\hat{\Pi}_{\boldsymbol{k}} + \hat{v}'_{\boldsymbol{k}} = 0. \tag{37}$$

It is remarkable that the choice of the variables (29)–(31) means that the values ρ_v and p_v do not appear in the system (32)–(37). The second point is that the continuity and Newton's second law Equations (36) and (37) do not contain metric perturbation.

From now on, we will begin to consider the perturbation in Equations (32)–(37) as operators by writing a "hat" under every quantity. Here, we do not suppose the strong nonlinearity [27] and assume a smallness of the quantum fluctuations of space-time in this particular conformally unimodular metric. The system (32)–(37) for a perturbation evolution is exact in the first order on perturbations, but it is not closed. To obtain a closed system, one needs, for instance, to specify the equation of state for a perturbation of pressure. Still, as an approximation, we could calculate pressure and energy density strictly by using the quantum field theory under the unperturbed Minkowski space-time. Expressing $F_{\boldsymbol{k}}$ from Equation (33) and substituting it into Equation (35) leads to

$$\hat{\Phi}''_{\boldsymbol{k}} + \frac{1}{3} k^2 \hat{\Phi}_{\boldsymbol{k}} + \frac{1}{M_p^2} \left(3\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}}\right) = 0. \tag{38}$$

Below, we will approximately consider an operator $3\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}}$ by using the creation and annihilation operators under the Minkowski space-time background. Such an approximation allows closing the system (32)–(37).

Quantum Fields as a Source for Energy Density and Pressure Perturbation

A massless scalar field is the simplest example of quantum fields. The energy density and pressure of the scalar field in the pure Minkowski space-time (without metric perturbation) has the form [24,28]

$$\hat{p}(\eta, \boldsymbol{x}) = \frac{\hat{\varphi}'^2}{2} - \frac{(\boldsymbol{\nabla}\hat{\varphi})^2}{6}, \tag{39}$$

$$\hat{\rho}(\eta, \boldsymbol{x}) = \frac{\hat{\varphi}'^2}{2} + \frac{(\boldsymbol{\nabla}\hat{\varphi})^2}{2} \tag{40}$$

All the quantities may be expanded into the Fourier series $\hat{\varphi}(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\phi}_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\boldsymbol{x}}$, $\hat{p}(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{p}_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\boldsymbol{x}}$ etc. For $\boldsymbol{k} \neq 0$, the approximate identifying $\delta\hat{\Pi}_{\boldsymbol{k}} = \hat{p}_{\boldsymbol{k}}$ and $\delta\hat{\wp}_{\boldsymbol{k}} = \hat{\rho}_{\boldsymbol{k}}$ results in

$$\delta\hat{\Pi}_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} \frac{1}{2} \hat{\phi}_{\boldsymbol{q}}^{+\prime} \hat{\phi}'_{\boldsymbol{q}+\boldsymbol{k}} - \frac{1}{6} (\boldsymbol{q}+\boldsymbol{k}) \boldsymbol{q}\, \hat{\phi}_{\boldsymbol{q}}^{+} \hat{\phi}_{\boldsymbol{q}+\boldsymbol{k}}, \tag{41}$$

$$\delta\hat{\wp}_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} \frac{1}{2} \hat{\phi}_{\boldsymbol{q}}^{+\prime} \hat{\phi}'_{\boldsymbol{q}+\boldsymbol{k}} + \frac{1}{2} (\boldsymbol{q}+\boldsymbol{k}) \boldsymbol{q}\, \hat{\phi}_{\boldsymbol{q}}^{+} \hat{\phi}_{\boldsymbol{q}+\boldsymbol{k}}, \tag{42}$$

so that the quantity $3\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}}$ from Equation (38) is reduced to

$$3\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}} = 2 \sum_{\boldsymbol{q}} \hat{\phi}_{\boldsymbol{q}}^{+\prime} \hat{\phi}'_{\boldsymbol{q}+\boldsymbol{k}}. \tag{43}$$

Writing the quantized field explicitly with creation and annihilation operators [29]

$$\hat{\phi}_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}^{+}_{-\boldsymbol{k}} e^{i\omega_k \eta} + \hat{a}_{\boldsymbol{k}} e^{-i\omega_k \eta} \right), \tag{44}$$

allows obtaining from Equations (43) and (44)

$$3\,\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} \sqrt{\omega_q \omega_{|\boldsymbol{q}+\boldsymbol{k}|}} \Big(\hat{a}_{-\boldsymbol{q}} \hat{a}^{+}_{-\boldsymbol{q}-\boldsymbol{k}} e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} - \omega_q)\eta} + \hat{a}^{+}_{\boldsymbol{q}} \hat{a}_{\boldsymbol{q}+\boldsymbol{k}} e^{i(\omega_q - \omega_{|\boldsymbol{q}+\boldsymbol{k}|})\eta}$$
$$- \hat{a}_{-\boldsymbol{q}} \hat{a}_{\boldsymbol{q}+\boldsymbol{k}} e^{-i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} + \omega_q)\eta} - \hat{a}^{+}_{\boldsymbol{q}} \hat{a}^{+}_{-\boldsymbol{q}-\boldsymbol{k}} e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} + \omega_q)\eta} \Big), \tag{45}$$

where $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ for a massless scalar field. As is seen from Equation (45), the perturbations have the general form:

$$3\,\delta\hat{\Pi}_{\boldsymbol{k}} + \delta\hat{\wp}_{\boldsymbol{k}} = \sum_{m} \hat{\mathcal{P}}_{m\boldsymbol{k}} e^{i\Omega_{m\boldsymbol{k}}\eta}, \tag{46}$$

where the frequencies $\Omega_{m\boldsymbol{k}}$ take the values of $\omega_q - \omega_{|\boldsymbol{q}+\boldsymbol{k}|}$, $-\omega_q + \omega_{|\boldsymbol{q}+\boldsymbol{k}|}$, $\omega_q + \omega_{|\boldsymbol{q}+\boldsymbol{k}|}$ and $-\omega_q - \omega_{|\boldsymbol{q}+\boldsymbol{k}|}$ for $m = 1$ to 4. That allows finding the solution of Equation (38) as

$$\hat{\Phi}_{\boldsymbol{k}}(\eta) = -\frac{1}{M_p^2} \sum_{m} \frac{\hat{\mathcal{P}}_{m\boldsymbol{k}} e^{i\Omega_{m\boldsymbol{k}}\eta}}{\Omega^2_{m\boldsymbol{k}} - k^2/3}. \tag{47}$$

Using Equations (34) and (47), one comes to

$$\hat{F}_{\boldsymbol{k}}(\eta) = -\frac{4}{M_p^2} \sum_{m} \frac{\hat{\mathcal{P}}_{m\boldsymbol{k}} e^{i\Omega_{m\boldsymbol{k}}\eta}}{\Omega^4_{m\boldsymbol{k}} - k^4/9}. \tag{48}$$

Under Equations (45) and (47), the final expression for the metric perturbation $\hat{\Phi}_{\boldsymbol{k}}(\eta)$ acquires the form

$$\hat{\Phi}_{\boldsymbol{k}}(\eta) = \frac{1}{M_p^2} \sum_{\boldsymbol{q}} \sqrt{\omega_q \omega_{|\boldsymbol{q}+\boldsymbol{k}|}} \Bigg(\frac{1}{(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} + \omega_q)^2 - k^2/3} \Big(\hat{a}_{-\boldsymbol{q}} \hat{a}_{\boldsymbol{q}+\boldsymbol{k}} e^{-i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} + \omega_q)\eta} +$$
$$\hat{a}^{+}_{\boldsymbol{q}} \hat{a}^{+}_{-\boldsymbol{q}-\boldsymbol{k}} e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} + \omega_q)\eta} \Big) - \frac{1}{(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} - \omega_q)^2 - k^2/3} \Big(\hat{a}_{-\boldsymbol{q}} \hat{a}^{+}_{-\boldsymbol{q}-\boldsymbol{k}} e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|} - \omega_q)\eta} +$$
$$\hat{a}^{+}_{\boldsymbol{q}} \hat{a}_{\boldsymbol{q}+\boldsymbol{k}} e^{i(\omega_q - \omega_{|\boldsymbol{q}+\boldsymbol{k}|})\eta} \Big) \Bigg), \tag{49}$$

$$\hat{F}_{\boldsymbol{k}}(\eta) = \frac{4}{M_p^2}\sum_{\boldsymbol{q}}\sqrt{\omega_q\omega_{|\boldsymbol{q}+\boldsymbol{k}|}}\Bigg(\frac{1}{(\omega_{|\boldsymbol{q}+\boldsymbol{k}|}+\omega_q)^4 - k^4/9}\Big(\hat{a}_{-\boldsymbol{q}}\hat{a}_{\boldsymbol{q}+\boldsymbol{k}}e^{-i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|}+\omega_q)\eta} + \hat{a}^+_{\boldsymbol{q}}\hat{a}^+_{-\boldsymbol{q}-\boldsymbol{k}}e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|}+\omega_q)\eta}\Big) - \frac{1}{(\omega_{|\boldsymbol{q}+\boldsymbol{k}|}-\omega_q)^4 - k^4/9}\Big(\hat{a}_{-\boldsymbol{q}}\hat{a}^+_{-\boldsymbol{q}-\boldsymbol{k}}e^{i(\omega_{|\boldsymbol{q}+\boldsymbol{k}|}-\omega_q)\eta} + \hat{a}^+_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}+\boldsymbol{k}}e^{i(\omega_q-\omega_{|\boldsymbol{q}+\boldsymbol{k}|})\eta}\Big)\Bigg), \quad (50)$$

Expressions (49) and (50) allow calculating the correlators

$$S_\Phi(\tau-\eta,\boldsymbol{k}) =< 0|\hat{\Phi}^+_{\boldsymbol{k}}(\eta)\hat{\Phi}_{\boldsymbol{k}}(\tau)|0 >= \frac{18}{M_p^4}\sum_{\boldsymbol{q}}\frac{e^{i(\tau-\eta)(\omega_q+\omega_{q+k})}\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}}{\left(k^2-3(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^2\right)^2} = \frac{18}{(2\pi)^3M_p^4}\int\frac{e^{i(\tau-\eta)(\omega_q+\omega_{q+k})}\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}d^3\boldsymbol{q}}{\left(k^2-3(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^2\right)^2}, \quad (51)$$

$$S_F(\tau-\eta,\boldsymbol{k}) =< 0|\hat{F}^+_{\boldsymbol{k}}(\eta)\hat{F}_{\boldsymbol{k}}(\tau)|0 >= \frac{32*81}{M_p^4}\sum_{\boldsymbol{q}}\frac{e^{i(\tau-\eta)(\omega_q+\omega_{q+k})}\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}}{\left(k^4-9(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^4\right)^2} = \frac{32*81}{(2\pi)^3M_p^4}\int\frac{e^{i(\tau-\eta)(\omega_q+\omega_{q+k})}\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}d^3\boldsymbol{q}}{\left(k^4-9(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^4\right)^2}, \quad (52)$$

which are related to the space-time correlators

$$< 0|\hat{F}(\eta,\boldsymbol{x})\hat{F}(\tau,\boldsymbol{x}')|0 >= \sum_{\boldsymbol{k}}S_F(\tau-\eta,\boldsymbol{k})e^{ik(x-x')} = \frac{1}{(2\pi)^3}\int S_F(\tau-\eta,\boldsymbol{k})e^{ik(x-x')}d^3\boldsymbol{k},$$

$$< 0|\hat{\Phi}(\eta,\boldsymbol{x})\hat{\Phi}(\tau,\boldsymbol{x}')|0 >= \sum_{\boldsymbol{k}}S_\Phi(\tau-\eta,\boldsymbol{k})e^{ik(x-x')} = \frac{1}{(2\pi)^3}\int S_\Phi(\tau-\eta,\boldsymbol{k})e^{ik(x-x')}d^3\boldsymbol{k}.$$

We will also need the correlators at different continuous values of $\boldsymbol{k}$, $\boldsymbol{k}'$:

$$< 0|\hat{F}_{\boldsymbol{k}}(\eta)\hat{F}_{\boldsymbol{k}'}(\tau)|0 >= \int < 0|\hat{F}(\eta,\boldsymbol{x})\hat{F}(\tau,\boldsymbol{x}')|0 > e^{-ikx-ik'x'}d^3\boldsymbol{x}d^3\boldsymbol{x}' = (2\pi)^3S_F(\tau-\eta,\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}'). \quad (53)$$

Explicit calculation gives

$$\tilde{S}_\Phi(\omega,k) = \frac{1}{2\pi}\int S_\Phi(\eta,k)e^{-i\omega\eta}d\eta = \frac{18}{(2\pi)^3M_p^4}\int\frac{\delta(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{q}+\boldsymbol{k}}-\omega)\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}d^3\boldsymbol{q}}{\left(k^2-3(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^2\right)^2} = \begin{cases}\frac{1}{160\pi^2{M_p}^4}\left(5+\frac{4k^4}{(k^2-3\omega^2)^2}\right), k<\omega \\ 0,\ \text{otherwise}\end{cases}, \quad (54)$$

$$\tilde{S}_F(\omega,k) = \frac{32*81}{(2\pi)^3M_p^4}\int\frac{\delta(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{q}+\boldsymbol{k}}-\omega)\omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}d^3\boldsymbol{q}}{\left(k^4-9(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})^4\right)^2} = \begin{cases}\frac{27\left(15\omega^4-10k^2\omega^2+3k^4\right)}{10\pi^2M_p^4\left(k^4-9\omega^4\right)^2}, k<\omega \\ 0,\ \text{otherwise}\end{cases}. \quad (55)$$

A formula similar to (55) could be obtained for a fermion field, as shown in Appendix A.

4. Migdal Equation for Photon Density Matrix Evolution

The kernel of a photon density matrix is defined by $\rho(\boldsymbol{r},\boldsymbol{r}',\eta) = \psi(\boldsymbol{r},\eta)\psi^+(\boldsymbol{r}',\eta)$, but we begin with consideration of a density matrix as an operator, and write the equation for its evolution in the standard form [30] using the Hamiltonian H_0 and interaction (20):

$$i\partial_\eta\hat{\rho} = [\hat{H}_0+\hat{V},\hat{\rho}]. \quad (56)$$

A formal solution of Equation (56) could be written as

$$\hat{\rho}(\eta) = -i\int_{-\infty}^{\eta} e^{i\hat{H}_0(\tau-\eta)}[\hat{V}(\tau), \rho(\tau)]e^{-i\hat{H}_0(\tau-\eta)}d\tau. \quad (57)$$

This expression can be substituted back into Equation (56), and one comes to

$$i\partial_\eta\hat{\rho} = [\hat{H}_0, \hat{\rho}] - i\int_{-\infty}^{\eta}[\hat{V}(\eta), e^{i\hat{H}_0(\tau-\eta)}[\hat{V}(\tau), \hat{\rho}(\tau)]e^{-i\hat{H}_0(\tau-\eta)}]d\tau. \quad (58)$$

Change of the time variable $\tau \to \tau + \eta$ in the integral leads to

$$i\partial_\eta\hat{\rho} = [\hat{H}_0, \hat{\rho}] - i\int_{-\infty}^{0}[\hat{V}(\eta), e^{i\hat{H}_0\tau}[\hat{V}(\eta+\tau), \hat{\rho}(\eta+\tau)]e^{-i\hat{H}_0\tau}]d\tau. \quad (59)$$

Further approximation is to write $\hat{\rho}(\tau+\eta) \approx e^{-i\hat{H}_0\tau}\hat{\rho}(\eta)e^{i\hat{H}_0\tau}$ on the right-hand side of (59), and obtain in the second order on the interaction [31–33]:

$$\begin{aligned} i\partial_\eta\hat{\rho} = [\hat{H}_0, \hat{\rho}] - i\int_{-\infty}^{0}[\hat{V}(\eta), e^{i\hat{H}_0\tau}[\hat{V}(\tau+\eta), e^{-i\hat{H}_0\tau}\hat{\rho}(\eta)e^{i\hat{H}_0\tau}]e^{-i\hat{H}_0\tau}]d\tau = [\hat{H}_0, \hat{\rho}] - \\ i\int_{-\infty}^{0}\Big(\hat{V}(\eta)e^{i\hat{H}_0\tau}\hat{V}(\eta+\tau)e^{-i\hat{H}_0\tau}\hat{\rho}(\eta) - \hat{V}(\eta)\hat{\rho}(\eta)e^{i\hat{H}_0\tau}\hat{V}(\eta+\tau)e^{-i\hat{H}_0\tau} - \\ e^{i\hat{H}_0\tau}\hat{V}(\tau+\eta)e^{-i\hat{H}_0\tau}\hat{\rho}(\eta)\hat{V}(\eta) + \hat{\rho}(\eta)e^{i\hat{H}_0\tau}\hat{V}(\tau+\eta)e^{-i\hat{H}_0\tau}\hat{V}(\eta)\Big)d\tau. \end{aligned} \quad (60)$$

As a result, the equation for $\rho_{\boldsymbol{pp}'} = \int e^{-i\boldsymbol{pr}}\hat{\rho}\, e^{i\boldsymbol{p}'\boldsymbol{r}}d^3\boldsymbol{r}/(2\pi)^3$ acquires the form

$$\begin{aligned} i\partial_\eta\rho_{\boldsymbol{pp}'} = \boldsymbol{\alpha p}\,\rho_{\boldsymbol{pp}'} - \rho_{\boldsymbol{pp}'}\boldsymbol{\alpha p}' - \frac{i}{(2\pi)^6}\int\int_{-\infty}^{0}\Big(V_{\boldsymbol{pq}}(\eta)e^{i\boldsymbol{\alpha q}\tau}V_{\boldsymbol{qq}'}(\eta+\tau)e^{-i\boldsymbol{\alpha q}'\tau}\rho_{\boldsymbol{q}'\boldsymbol{p}'}(\eta) - \\ V_{\boldsymbol{pq}}(\eta)\rho_{\boldsymbol{qq}'}(\eta)e^{i\boldsymbol{\alpha q}'\tau}V_{\boldsymbol{q}'\boldsymbol{p}'}(\eta+\tau)e^{-i\boldsymbol{\alpha p}'\tau} - e^{i\boldsymbol{\alpha p}\tau}V_{\boldsymbol{pq}}(\eta+\tau)e^{-i\boldsymbol{\alpha q}\tau}\rho_{\boldsymbol{qq}'}(\eta)V_{\boldsymbol{q}'\boldsymbol{p}'}(\eta) + \\ \rho_{\boldsymbol{pq}}(\eta)e^{i\boldsymbol{\alpha q}\tau}V_{\boldsymbol{qq}'}(\eta+\tau)e^{-i\boldsymbol{\alpha q}'\tau}V_{\boldsymbol{q}'\boldsymbol{p}'}(\eta)\Big)d\tau d^3\boldsymbol{q}d^3\boldsymbol{q}'. \end{aligned} \quad (61)$$

Let us remember that $V_{\boldsymbol{qq}'} = \int e^{-i\boldsymbol{qx}}\hat{V}e^{i\boldsymbol{q}'\boldsymbol{x}}d\boldsymbol{x}$ is a function of $\boldsymbol{q}, \boldsymbol{q}'$ and simultaneously a 6×6 matrix. According to (13), (20), (21)

$$V_{\boldsymbol{qq}'}(\tau) = \frac{1}{2}\Big(\mathcal{D}_{\boldsymbol{q}-\boldsymbol{q}'}\boldsymbol{\alpha q}' + \boldsymbol{\alpha q}\,\mathcal{D}_{\boldsymbol{q}-\boldsymbol{q}'}\Big)F_{\boldsymbol{q}-\boldsymbol{q}'}(\tau) \equiv \mathcal{V}_{\boldsymbol{qq}'}F_{\boldsymbol{q}-\boldsymbol{q}'}(\tau), \quad (62)$$

where $\mathcal{V}_{\boldsymbol{qq}'}$ denotes the matrix part of the interaction and the scalar function $F_{\boldsymbol{q}}(\tau) = \int F(\boldsymbol{x},\tau)e^{-i\boldsymbol{qx}}d^3\boldsymbol{x}$ corresponds to $F(\boldsymbol{x},\tau) = \sum_{\boldsymbol{k}} F_{\boldsymbol{k}}(\tau)e^{i\boldsymbol{kx}} = \frac{1}{(2\pi)^3}\int F_{\boldsymbol{k}}(\tau)e^{i\boldsymbol{kx}}d^3\boldsymbol{k}$. Under Equation (21), the six-dimensional matrix $\mathcal{D}_{\boldsymbol{q}}$ is represented in the form of four three-dimensional blocks

$$\mathcal{D}_{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{q}\otimes\boldsymbol{q} - \frac{q^2}{3}I & 0 \\ 0 & \boldsymbol{q}\otimes\boldsymbol{q} - \frac{q^2}{3}I \end{pmatrix}. \quad (63)$$

The gravitational field is considered a random field, originating from the zero-point fluctuations. That means that $F_{\boldsymbol{q}-\boldsymbol{q}'}$ is a random quantity as well as the density matrix. After averaging, one comes to the approximate equation for a mean value of the density matrix $< \rho_{\boldsymbol{pp}'} >$, but to avoid introducing a new designation, we will denote the averaged density matrix by the same symbol and write:

$$\begin{aligned} i\partial_\eta \rho_{\boldsymbol{p}\boldsymbol{p}'} &= \boldsymbol{\alpha p}\, \rho_{\boldsymbol{p}\boldsymbol{p}'} - \rho_{\boldsymbol{p}\boldsymbol{p}'} \boldsymbol{\alpha p}' - \\ & \frac{i}{(2\pi)^6} \int\int_{-\infty}^{0} \Big(\mathcal{V}_{\boldsymbol{pq}} e^{i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{qq}'} e^{-i\boldsymbol{\alpha q}'\tau} \rho_{\boldsymbol{q}'\boldsymbol{p}'}(\eta) < F_{\boldsymbol{p}-\boldsymbol{q}}(\eta) F_{\boldsymbol{q}-\boldsymbol{q}'}(\eta+\tau) > - \\ & \mathcal{V}_{\boldsymbol{pq}} \rho_{\boldsymbol{qq}'} e^{i\boldsymbol{\alpha q}'\tau} \mathcal{V}_{\boldsymbol{q}'\boldsymbol{p}'} e^{-i\boldsymbol{\alpha p}'\tau} < F_{\boldsymbol{p}-\boldsymbol{q}}(\eta) F_{\boldsymbol{q}'-\boldsymbol{p}'}(\eta+\tau) > - \\ & e^{i\boldsymbol{\alpha p}\tau} \mathcal{V}_{\boldsymbol{pq}} e^{-i\boldsymbol{\alpha q}\tau} \rho_{\boldsymbol{qq}'}(\eta) \mathcal{V}_{\boldsymbol{q}'\boldsymbol{p}'} < F_{\boldsymbol{p}-\boldsymbol{q}}(\eta+\tau) F_{\boldsymbol{q}'-\boldsymbol{p}'}(\eta) > + \\ & \rho_{\boldsymbol{pq}}(\eta) e^{i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{qq}'} e^{-i\boldsymbol{\alpha q}'\tau} \mathcal{V}_{\boldsymbol{q}'\boldsymbol{p}'} < F_{\boldsymbol{q}-\boldsymbol{q}'}(\eta+\tau) F_{\boldsymbol{q}'-\boldsymbol{p}'}(\eta) > \Big) d\tau d^3\boldsymbol{q} d^3\boldsymbol{q}'. \end{aligned} \tag{64}$$

Correlators in Equation (64) are taken from (53) and contain delta functions. That allows us to perform one integration and reduce Equation (64) to

$$\begin{aligned} i\partial_\eta \rho_{\boldsymbol{p}\boldsymbol{p}'} = \boldsymbol{\alpha p}\, \rho_{\boldsymbol{p}\boldsymbol{p}'} - \rho_{\boldsymbol{p}\boldsymbol{p}'} \boldsymbol{\alpha p}' - \frac{i}{(2\pi)^3} \int\int_{-\infty}^{0} \Big(& \mathcal{V}_{\boldsymbol{pq}}\, e^{i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{qp}}\, e^{-i\boldsymbol{\alpha p}\tau} \rho_{\boldsymbol{p}\boldsymbol{p}'}(\eta) S_F(\tau, \boldsymbol{q}-\boldsymbol{p}) - \\ & \mathcal{V}_{\boldsymbol{pq}} \rho_{\boldsymbol{q},\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}'}(\eta) e^{i\boldsymbol{\alpha}(\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}')\tau} \mathcal{V}_{\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}',\boldsymbol{p}'} e^{-i\boldsymbol{\alpha p}'\tau} S_F(\tau, \boldsymbol{q}-\boldsymbol{p}) - \\ & e^{i\boldsymbol{\alpha p}\tau} \mathcal{V}_{\boldsymbol{pq}} e^{-i\boldsymbol{\alpha q}\tau} \rho_{\boldsymbol{q},\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}'}(\eta) \mathcal{V}_{\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}',\boldsymbol{p}'} S_F(-\tau, \boldsymbol{q}-\boldsymbol{p}) + \\ & \rho_{\boldsymbol{p}\boldsymbol{p}'}(\eta) e^{i\boldsymbol{\alpha p}'\tau} \mathcal{V}_{\boldsymbol{p}'\boldsymbol{q}} e^{-i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{q}\boldsymbol{p}'} S_F(-\tau, \boldsymbol{q}-\boldsymbol{p}') \Big) d\tau d^3\boldsymbol{q}. \end{aligned} \tag{65}$$

As a measure of the purity of a state, a quantity

$$\mathcal{C} = \int \mathrm{Tr} \rho_{\boldsymbol{p}\boldsymbol{p}'} \rho_{\boldsymbol{p}'\boldsymbol{p}} d^3\boldsymbol{p}' d^3\boldsymbol{p} = \int \mathrm{Tr} \rho_{\boldsymbol{p}\boldsymbol{p}} d^3\boldsymbol{p} = 1 \tag{66}$$

could be introduced, which equals unity for a completely pure state. The quantity $\mathcal{C}$ measures decoherence, and the symbol Tr denotes a track of a 6×6 matrix. A time evolution of $\mathcal{C}$ is given by

$$\begin{aligned} \frac{\partial \mathcal{C}}{\partial \eta} &= \int \mathrm{Tr} \Big(\frac{\partial \rho_{\boldsymbol{p}\boldsymbol{p}'}}{\partial \eta} \rho_{\boldsymbol{p}'\boldsymbol{p}} + \rho_{\boldsymbol{p}\boldsymbol{p}'} \frac{\partial \rho_{\boldsymbol{p}'\boldsymbol{p}}}{\partial \eta} \Big) d^3\boldsymbol{p}' d^3\boldsymbol{p} = \\ & -\frac{2}{(2\pi)^3} \int\int_{-\infty}^{0} \mathrm{Tr} \Big((\mathcal{V}_{\boldsymbol{pq}} e^{i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{qp}} e^{-i\boldsymbol{\alpha p}\tau} \rho_{\boldsymbol{p}\boldsymbol{p}'}(\eta) \rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta) \quad - \\ & \mathcal{V}_{\boldsymbol{pq}} \rho_{\boldsymbol{q},\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}'}(\eta) e^{i\boldsymbol{\alpha}(\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}')\tau} \mathcal{V}_{\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}',\boldsymbol{p}'} e^{-i\boldsymbol{\alpha p}'\tau} \rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta)) S(\tau, \boldsymbol{q}-\boldsymbol{p}) - \\ & (e^{i\boldsymbol{\alpha p}\tau} \mathcal{V}_{\boldsymbol{pq}} e^{-i\boldsymbol{\alpha q}\tau} \rho_{\boldsymbol{q},\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}'}(\eta) \mathcal{V}_{\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{p}',\boldsymbol{p}'} \rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta) - \\ & \rho_{\boldsymbol{p}\boldsymbol{p}'}(\eta) \rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta) e^{i\boldsymbol{\alpha p}\tau} \mathcal{V}_{\boldsymbol{pq}} e^{-i\boldsymbol{\alpha q}\tau} \mathcal{V}_{\boldsymbol{qp}}) S(\tau, \boldsymbol{q}-\boldsymbol{p}) \Big) d\tau d^3\boldsymbol{p}' d^3\boldsymbol{p} d^3\boldsymbol{q}. \end{aligned} \tag{67}$$

One could be usable to expand the matrix $\boldsymbol{\alpha p}$ over eagenmodes

$$\boldsymbol{\alpha p} | m, \boldsymbol{p} >= \varepsilon_m(\boldsymbol{p}) | m, \boldsymbol{p} > . \tag{68}$$

There are two longitudinal modes with zero energy, two modes with positive energy, and two modes with negative energy, as shown in Appendix B. Expansion over modes reduces Equation (69) to

$$
\begin{aligned}
\frac{\partial \mathcal{C}}{\partial \eta} = -\frac{2}{(2\pi)^3} \int \sum_{m,n=1}^{6} \Big(&< n,\boldsymbol{p}|\rho_{\boldsymbol{pp}'}(\eta)\rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta)\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}}|m,\boldsymbol{p}+\boldsymbol{q}>< m,\boldsymbol{p}+\boldsymbol{q}|\mathcal{V}_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{p}}|n,\boldsymbol{p}> \\
&\Delta(\varepsilon_m(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_n(\boldsymbol{p}) + \omega) - \\
&< n,\boldsymbol{p}|\rho_{\boldsymbol{pp}'}(\eta)\mathcal{V}_{\boldsymbol{p}',\boldsymbol{q}+\boldsymbol{p}'}\rho_{\boldsymbol{q}+\boldsymbol{p}',\boldsymbol{q}+\boldsymbol{p}}(\eta)|m,\boldsymbol{p}+\boldsymbol{q}>< m,\boldsymbol{p}+\boldsymbol{q}|\mathcal{V}_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{p}}|n,\boldsymbol{p}> \\
&\Delta(\varepsilon_m(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_n(\boldsymbol{p}) + \omega) - \\
&< n,\boldsymbol{p}|\mathcal{V}_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}}|m,\boldsymbol{p}+\boldsymbol{q}>< m,\boldsymbol{p}+\boldsymbol{q}|\rho_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}'}(\eta)\mathcal{V}_{\boldsymbol{q}+\boldsymbol{p}',\boldsymbol{p}'}\rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta)|n,\boldsymbol{p}> \\
&\Delta(-\varepsilon_m(\boldsymbol{p}+\boldsymbol{q}) + \varepsilon_n(\boldsymbol{p}) - \omega) + \\
&< m,\boldsymbol{p}+\boldsymbol{q}|\mathcal{V}_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{p}}\rho_{\boldsymbol{pp}'}(\eta)\rho_{\boldsymbol{p}'\boldsymbol{p}}(\eta)|n,\boldsymbol{p}>< n,\boldsymbol{p}|\mathcal{V}_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}}|m,\boldsymbol{p}+\boldsymbol{q}> \\
&\Delta(-\varepsilon_m(\boldsymbol{p}+\boldsymbol{q}) + \varepsilon_n(\boldsymbol{p}) - \omega)\Big) \tilde{S}_F(\omega,q)d\omega d^3\boldsymbol{p}'d^3\boldsymbol{p}d^3\boldsymbol{q},
\end{aligned} \tag{69}
$$

where $\tilde{S}(\omega,k) = \frac{1}{2\pi}\int S(\eta,k)e^{-i\omega\eta}d\eta$ and

$$
\Delta(\omega) = \int_{-\infty}^{0} e^{i\omega\tau}d\tau = \pi\delta(\omega) - i\mathcal{P}\frac{1}{\omega}. \tag{70}
$$

We could suggest the following program of calculations: one substitutes the density matrix of a wave packet of free electromagnetic waves into the right-hand side of (69) and obtains an estimate for $\frac{\partial \mathcal{C}}{\partial \eta}$, which allows extracting the typical decoherence time. Further calculations are performed for the Gaussian wave packet

$$
\begin{aligned}
\mathcal{D}_{\boldsymbol{p}}(\eta) &= \frac{2(\Delta p)^{9/2}}{\pi^{3/2}}\frac{\boldsymbol{e}\times\boldsymbol{p}}{|\boldsymbol{e}\times\boldsymbol{p}|}e^{-i\varepsilon_{\boldsymbol{p}}\eta-(\boldsymbol{p}-\boldsymbol{p}_0)^2/(\Delta p)^2}, \\
\mathcal{B}_{\boldsymbol{p}}(\eta) &= \frac{1}{\varepsilon_{\boldsymbol{p}}}\mathcal{D}(\boldsymbol{p},\eta),
\end{aligned} \tag{71}
$$

where $\varepsilon_{\boldsymbol{p}} = |\boldsymbol{p}| = p$, Δp is the width of the packet in momentum space, $\boldsymbol{e}$ is some vector characterizing the wave polarization. The 6×6 density matrix of the free electromagnetic field

$$
\rho_{\boldsymbol{pp}'}(\eta) = \psi_{\boldsymbol{p}}(\eta)\psi^{+}_{\boldsymbol{p}'}(\eta) = \begin{pmatrix} \mathcal{D}_{\boldsymbol{p}} \\ i\mathcal{B}_{\boldsymbol{p}} \end{pmatrix}\begin{pmatrix} \mathcal{D}^{*}_{\boldsymbol{p}'} & -i\mathcal{B}^{*}_{\boldsymbol{p}'} \end{pmatrix} \tag{72}
$$

can be constructed.

The first and last terms in the brackets on the right-hand side of Equation (69) do not contribute to the quantity $\frac{\partial \mathcal{C}}{\partial \eta}$. Second and third terms contain $\rho_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}'}$, $\rho_{\boldsymbol{p},\boldsymbol{p}'}$, which makes the integral over $d^3\boldsymbol{p}d^3\boldsymbol{p}'d^3\boldsymbol{q}$ convergent due to the finiteness of a momentum wave packet. The integral over ω is also convergent, because $\tilde{S}_F(q,\omega) \sim 1/\omega^4$ according to (55). It turns out that the terms with the Dirac delta function in (70) do not contribute to $\frac{\partial \mathcal{C}}{\partial \eta}$ due to energy conservation, because $\varepsilon_{\boldsymbol{p}+\boldsymbol{q}} + q - \varepsilon_{\boldsymbol{p}} > 0$. Thus, the decoherence effect is a purely off-shell effect, which is not related to the real on-shell scattering. After the calculation of tracks, the integral (69) takes the form

$$
\begin{aligned}
\frac{\partial \mathcal{C}}{\partial \eta} = \int_{\omega>q} g(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q},\eta)\exp\Big(-\frac{(\boldsymbol{p}-\boldsymbol{p}_0)^2 + (\boldsymbol{p}'-\boldsymbol{p}_0)^2 + (\boldsymbol{p}'+\boldsymbol{q}-\boldsymbol{p}_0)^2 + (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_0)^2}{\Delta p^2}\Big) \\
\frac{\tilde{S}_F(q,\omega)}{\varepsilon_{\boldsymbol{p}+\boldsymbol{q}} + \omega - \varepsilon_{\boldsymbol{p}}}d^3\boldsymbol{p}\,d^3\boldsymbol{p}'d^3\boldsymbol{q}\,d\omega,
\end{aligned} \tag{73}
$$

where $\boldsymbol{p}_0 = \{0, 0, p_0\}$, and $g(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}, \eta)$ is a some function originating from the calculation of traces. Integral (73) is complicated. Simplification consists in an expansion of the function $g(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}, \eta)$ into the Taylor series in the vicinity $\boldsymbol{p} = \boldsymbol{p}_0$, $\boldsymbol{p}' = \boldsymbol{p}_0$ and leads to the estimate

$$g(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}, \eta) \approx -\frac{(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\boldsymbol{q}_\perp q_z^2 (q^2 - 3q_z^2)^2}{9\pi^3 p_0 (\Delta p)^6}\, \eta, \tag{74}$$

where $\boldsymbol{p}_\perp \equiv \{p_x, p_y\}$, $\boldsymbol{p}'_\perp \equiv \{p'_x, p'_y\}$ and $\boldsymbol{q}_\perp \equiv \{q_x, q_y\}$. Equation (74) gives the first non-zero term in the expansion. In addition, we need the integral

$$\int_{-\infty}^{\infty} \frac{\tilde{S}_F(q,\omega)}{\varepsilon_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{p}+\boldsymbol{q}} - \omega} d\omega \approx \frac{27 \times 15}{810\, \pi^2 M_p^4} \int_q^{\infty} \frac{1}{p - |\boldsymbol{p}+\boldsymbol{q}| - \omega} \frac{d\omega}{\omega^4} =$$

$$\frac{1}{12\pi^2 M_p^4 (p - |\boldsymbol{p}+\boldsymbol{q}|^4)} \left(6\left(\frac{p - |\boldsymbol{p}+\boldsymbol{q}|}{q}\right) + 3\left(\frac{p - |\boldsymbol{p}+\boldsymbol{q}|}{q}\right)^2 + 2\left(\frac{p - |\boldsymbol{p}+\boldsymbol{q}|}{q}\right)^3 + 6\ln\left(\frac{|\boldsymbol{p}+\boldsymbol{q}| + q - p}{q}\right)\right). \tag{75}$$

After some steps of the simplification, we come to the final estimate

$$\frac{\partial \mathcal{C}}{\partial \eta} \approx -\frac{2.5 \times 10^{-6} (\Delta p)^8}{p_0^2 M_p^4} \eta, \tag{76}$$

and integration over time η leads to

$$\mathcal{C} \approx 1 - \frac{1.25 \times 10^{-6} (\Delta p)^8}{p_0^2 M_p^4} (\eta - \eta_i)^2, \tag{77}$$

where η_i is the initial conformal time at which a pure wave packet was emitted. For an expanding universe, the conformal time is related to the redshift $z = 1/a - 1$ as $dz = -\frac{1}{a^2}\frac{da}{d\eta} d\eta$, which gives $\eta(z) = \eta_i - \int_{z_i}^{z} \frac{dz}{H(z)} \approx \eta_i + \frac{z_i - z}{H_0}$, where H_0 is the Hubble constant. In terms of the redshift, expression (77) is rewritten as

$$\mathcal{C} \approx 1 - \frac{1.25 \times 10^{-6} (\Delta p)^8}{p_0^2 M_p^4 H_0^2} (z_i - z)^2, \tag{78}$$

A condition is that $\mathcal{C}$ turns to zero at present time η_0, when $z = 0$ gives an estimation of z_i:

$$z_i \sim 900 \frac{H_0 p_0 M_p^2}{(\Delta p)^4}. \tag{79}$$

For instance, at $p_0 \sim 30$ GeV and $\Delta p / p_0 \sim 0.05$ we have $z_i \sim 0.25$. That means that radiation emitted at this z_i must be fully decoherent today. Thus, the effect seems rather observable, although the decoherence depends strongly on $\Delta p / p_0$.

It seems also interesting to understand the influence of the fluctuating gravitational field on the spatial localization of the photon. This question needs calculation of a center of wave packet motion, namely, mean value of the operator $\hat{r} = i\frac{\partial}{\partial p}$ with the help of basic Equation (65). The range of the spatial localization of the photon could be estimated by calculation of the mean value $\hat{r}^2 - < r >^2$. Due to the ambiguity of the photon position operator [34,35] it is preferable to rewrite Equation (65) in the Foldy–Wouthuysen representation (see, e.g., [36] and references therein) for this aim.

5. Decoherence and Depolarization

A decrease in $\mathcal{C}$ suggests that spatial and polarization coherence is lost. Let us say spatial coherence is already lost, and only polarization decoherence remains. How is the polarization decoherence related to the polarization of radiation? The simplest way to clarify this is to consider the usual two-dimensional photon-polarization matrix, corresponding to two possible photon polarizations. Let us say this density matrix has the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta^* & 1-\alpha \end{pmatrix}. \tag{80}$$

$$Tr(\rho^2) - 1 = 2\left(\alpha^2 - \alpha + |\beta|^2\right) \tag{81}$$

The minimum of this quantity, corresponding to maximal decoherence, is reached at $\alpha = 1/2$ and $\beta = 0$ if α and β are considered independent variables. In this case, radiation will be unpolarized. It seems that decoherence also destroys the polarization of radiation. Indeed, this question needs more careful investigation because it is related to the conservation of orbital and total angular momenta [12].

6. Conclusions and Discussion

We have culculated the decoherence of a Gaussian wave packet in a vacuum. The attractive feature of the resulting formula is that it does not depend on the ultraviolet cut-off because the corresponding correlator $S_F(q,\omega)$ of the metric fluctuations decreases in a sufficient degree with a frequency increase. The Migdal equation for the density matrix evolution was applied, but it seems that these results could be reproduced in a frame of more conventional formalism with the Green functions applied for modeling decoherence in a turbulent atmosphere [37–41]. Unfortunately, a formalism will be more complicated because a fluctuating gravitational field generates both effective dielectric permittivity and magnetic permeability; hence, a necessity of a matrix 6×6 Greene function arises.

The next stage should be a discussion of the possible experiments on observing the incoherent radiation from distant astrophysical sources. One of the directions could be investigation of polarization of high energy of gamma quants [42–45] at high redshift.

A second direction could be clarifying the influence of the coherence of the photon wave packet on the development of a particle shower in a detector.

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Appendix A. Correlators of Fermion Field

In this appendix, the notation ψ is used for fermion field, and $\boldsymbol{\alpha}$, β are $4\times\ 4$ matrices [46] of the usual Dirac equation. The Lagrangian of the fermion field in the expanding uniform flat universe [47,48] with the metric $ds^2 = N(\eta)^2 a(\eta)^2 d\eta^2 - a(\eta)^2 d\boldsymbol{x}^2$ has the form

$$\mathcal{L}_{ferm} = -\frac{M_p^2 a'^2}{2N} + \sum_{\boldsymbol{k}} \frac{i\,a^3}{2}\psi^+_{\boldsymbol{k}}\partial_\eta\psi_{\boldsymbol{k}} - \frac{i\,a^3}{2}\partial_\eta\psi^+_{\boldsymbol{k}}\psi_{\boldsymbol{k}} - Na^3\psi^+_{\boldsymbol{k}}(\boldsymbol{\alpha k})\,\psi_{\boldsymbol{k}} - Na^4\,m\psi^+_{\boldsymbol{k}}\beta\psi_{\boldsymbol{k}}. \tag{A1}$$

The fist term in Equation (A1) is introduced to derive fermion energy density in a simple way, because we know that in the Friedmann Equation (24) a'^2 is proportional to ρ. Varying Lagrangian over the lapse function N and then setting it to unity gives

$$\frac{1}{2}M_p^2 a'^2 \equiv \rho a^4 = \sum_{\boldsymbol{k}} a^3 \psi_{\boldsymbol{k}}^+ (\boldsymbol{\alpha k})\, \psi_{\boldsymbol{k}} + a^4\, m \psi_{\boldsymbol{k}}^+ \beta \psi_{\boldsymbol{k}}. \tag{A2}$$

On the other hand, we could obtain the expression for the quantity $\rho - 3p$ (25) by varying action with the Lagrangian (A1) over a:

$$M_p^2 a'' = a^3(\rho - 3p) = \sum_{\boldsymbol{k}} \frac{3}{2} a^2 \big(-i\psi_{\boldsymbol{k}}^+ \psi_{\boldsymbol{k}}' + i\psi_{\boldsymbol{k}}'^+ \psi_{\boldsymbol{k}} + 2\psi_{\boldsymbol{k}}^+ (\boldsymbol{\alpha k}) \psi_{\boldsymbol{k}}\big) + 4a^3 m(\psi_{\boldsymbol{k}}^+ \beta \psi_{\boldsymbol{k}}) = a^3 m \sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}}^+ \beta \psi_{\boldsymbol{k}}, \tag{A3}$$

where the last equality in (A3) is obtained using the equation of motion

$$i\psi_k' - (\boldsymbol{\alpha k})\psi_{\boldsymbol{k}} + i\frac{3a'}{2a}\psi_{\boldsymbol{k}} - m\, a\beta\psi_{\boldsymbol{k}} = 0 \tag{A4}$$

of the fermion field. As in the case of the scalar field, after identifying $\delta\hat{\Pi}_{\boldsymbol{k}} = \hat{p}_{\boldsymbol{k}}$ and $\delta\hat{\wp}_{\boldsymbol{k}} = \hat{\rho}_{\boldsymbol{k}}$ we come to

$$\hat{\delta}\wp_{\boldsymbol{k}} + 3\delta\hat{\Pi}_{\boldsymbol{k}} = \hat{\psi}_{\boldsymbol{k}}^+ (2\boldsymbol{\alpha k} + \beta m)\hat{\psi}_{\boldsymbol{k}}^+ \tag{A5}$$

for Minkowski space-time. Certainly, another way to obtain the expression (A5) is to use the stress–energy tensor of the fermion field. The fermion field is quantized as

$$\hat{\psi}_{\boldsymbol{k}} = \hat{b}_{-\boldsymbol{k},s}^+ v_{-\boldsymbol{k},s} e^{i\omega_k \eta} + \hat{a}_{\boldsymbol{k},s} u_{\boldsymbol{k},s} e^{-i\omega_k \eta}, \tag{A6}$$

where $\omega_{\boldsymbol{k}} = \sqrt{k^2 + m^2}$ and the basic bispinors are:

$$u_{\boldsymbol{k},s}(\eta) = \sqrt{\frac{\omega_{\boldsymbol{k}} + m}{2\omega_{\boldsymbol{k}}}} \begin{pmatrix} \varphi_s \\ \frac{\sigma \boldsymbol{k}}{\omega_{\boldsymbol{k}} + m}\varphi_s \end{pmatrix},$$

$\varphi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The bispinor $v_{\boldsymbol{k},s}$ is expressed as $v_{-\boldsymbol{k},-s} = i\gamma^0\gamma^2(\bar{u}_{\boldsymbol{k},s})^T$, where the symbol T denotes the transpose vector and $\bar{u} = u^+\gamma^0$. Calculations similar to those for the scalar field give the following correlator:

$$S_F(\tau - \eta, \boldsymbol{k}) =< 0|\hat{F}_{\boldsymbol{k}}^+(\eta)\hat{F}_{\boldsymbol{k}}(\tau)|0 >= \frac{16*81}{M_p^4} \sum_{q,s,\sigma} \frac{e^{i(\tau-\eta)(\omega_q + \omega_{q+k})}(u_{q,s}^+(2\boldsymbol{\alpha} + \beta m)v_{-k-q,\sigma})(v_{-k-q,\sigma}^+(2\boldsymbol{\alpha} + \beta m)u_{q,s})}{\left(k^4 - 9(\omega_{\boldsymbol{q}} + \omega_{\boldsymbol{k}+\boldsymbol{q}})^4\right)^2}, \tag{A7}$$

which is reduced to

$$\tilde{S}_F(\omega, k) = \frac{16*81}{(2\pi)^3 M_p^4} \int \frac{4\omega_{\boldsymbol{k}+\boldsymbol{q}}^2((\boldsymbol{k}+\boldsymbol{q})\boldsymbol{q} + \omega_{\boldsymbol{q}}\omega_{\boldsymbol{k}+\boldsymbol{q}}) - m^2((\boldsymbol{k}+\boldsymbol{q})\boldsymbol{q} + 3\omega_{\boldsymbol{q}+\boldsymbol{k}}\omega_{\boldsymbol{q}})) - m^4}{\omega_{\boldsymbol{q}}} \frac{\delta(\omega_{\boldsymbol{q}} + \omega_{\boldsymbol{q}+\boldsymbol{k}} - \omega)d^3\boldsymbol{q}}{\left(k^4 - 9(\omega_{\boldsymbol{q}} + \omega_{\boldsymbol{k}+\boldsymbol{q}})^4\right)^2}. \tag{A8}$$

The integral (A8) becomes more complicated by fermion mass compared to (55), so we set $m = 0$, which is valid if the gamma quant energy is much larger than the fermion mass, and is reasonable, for example, for electrons, but not for t-quarks. Calculations at $m = 0$ give

$$\tilde{S}_F(\omega, k) = \begin{cases} \frac{2*81\left(\omega^4+\omega^2 k^2\right)}{\pi^2\left(k^4-9\omega^4\right)^2}, k < \omega \\ 0, \text{ otherwise} \end{cases}, \tag{A9}$$

which is similar to (55).

Appendix B. Solution of the Free Optical Dirac Equation

For a free photon on the Minkowski space-time background, the solution of Equation (19) without interaction is written as $\psi(\eta) = e^{-i\varepsilon_n \eta}(\boldsymbol{p})\psi_n(\boldsymbol{p})$, where ψ_n has to satisfy the eigenvalue Equation (68). The eigenfunctions of operator $\boldsymbol{\alpha p}$ are

$$\psi_1(\boldsymbol{p}) = \frac{1}{\sqrt{1+\frac{p_x^2+p_y^2}{p_z^2}}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ p_x/p_z \\ p_y/p_z \\ 1 \end{pmatrix}, \quad \psi_2(\boldsymbol{p}) = \frac{1}{\sqrt{1+\frac{p_x^2+p_y^2}{p_z^2}}}\begin{pmatrix} p_x/p_z \\ p_y/p_z \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\psi_3(\boldsymbol{p}) = \frac{1}{\sqrt{2\left(1+\frac{p_z^2}{p_x^2}\right)}}\begin{pmatrix} ip_y/p \\ \frac{p_x^2+p_z^2}{ip_x p} \\ \frac{ip_y p_z}{p_x p} \\ -p_z/p_x \\ 0 \\ 1 \end{pmatrix}, \quad \psi_4(\boldsymbol{p}) = \frac{1}{\sqrt{2}\sqrt{p_x^2+p_z^2}}\begin{pmatrix} -ip_z \\ 0 \\ ip_x \\ -p_x p_y/p \\ \frac{p_x^2+p_z^2}{p} \\ -p_y p_z/p \end{pmatrix}, \tag{A10}$$

$$\psi_5(\boldsymbol{p}) = \frac{1}{\sqrt{2\left(1+\frac{p_z^2}{p_x^2}\right)}}\begin{pmatrix} -ip_y/p \\ \frac{i(p_x^2+p_z^2)}{p_x p} \\ \frac{p_y p_z}{ip_x p} \\ -p_z/p_x \\ 0 \\ 1 \end{pmatrix}, \quad \psi_6(\boldsymbol{p}) = \frac{1}{\sqrt{2}\sqrt{p_x^2+p_y^2}}\begin{pmatrix} ip_z \\ 0 \\ -ip_x \\ -p_x p_y/p \\ -\frac{p_x^2+p_z^2}{p} \\ -p_y p_z/p \end{pmatrix}.$$

The corresponding eigenvalues $\varepsilon_n(p)$ are $\{0, 0, p, p, -p, -p\}$.

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