

Article

Power of the Radiative Friction Force for a Charged Particle Performing a Flyby Near a Rotating Black Hole

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Abstract

We analytically obtain a relativistic generalization of the classical Larmor formula for the power of the radiation friction force $P = mc^3 r_e (-w_i w^i)$ for the case where a relativistic charged particle moves in the vicinity of a rotating Kerr black hole.

Keywords: radiative friction force power; rotating black hole; charged particle; flyby

1. Introduction

As is known—see, among other works, refs. [1,2]—when a charged particle moves with acceleration, it radiates, decelerates, and interacts with the proper electromagnetic field. Hence, the processes of energy change include (1) the change (over time) in the field's energy volume density, (2) the radiation escaping through the distant surface Σ and enveloping a volume V around the particle, and (3) excitation of the electromagnetic field by the particle decelerating because of the action of the radiative friction force. Hence, three quantities—the change in the field energy per unit of time (dW_f/dt), the total energy flux of the outflowing field (P_r), and the power of the radiative friction force (P)—are connected by only one relationship. In a general case, they are far from being equal to each other in terms of the absolute magnitude. Only if we consider a stationary motion with $dW_f/dt \rightarrow 0$ or averaged over time when the value of the quantity becomes zero (for example, when a particle moves in a circular orbit and the quantities are considered averages over the period of revolution) will the equality $P = P_r$ (on average) take place.

The details of the calculations of the power of the radiative friction force (item 3) in the curved spacetime near a rotating black hole are not trivial but, to our knowledge, are not available in the published literature, although the works devoted to radiation (item 2) are plentiful. In this context (item 2), refs. [3–12] contain helpful core information. In addition to these works, we find it useful to list [13], although this is not in the focal domain of our paper. At the same time, we would like to emphasize that the characteristics of detected radio, microwave, and X radiation from the regions in the vicinity of a black hole are interpreted and described while assuming the existence of a specific magnetic field at the site. But whether the plasma that creates the magnetic field is one component or two components, collisionless or dissipative, and expansive like a halo or localized near the equatorial plane of the black hole (and whether it is a black hole in the first place) are the questions and uncertainties that are not yet fully resolved. Hence, any theoretical works on this subject are strongly magneto model-dependent.

Let us remember that the calculation of the radiation losses of a moving charge (in the context of item 2) is carried out according to the following scheme (briefly). The flux of



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energy through the surface S is proportionate to the surface integral $\int_S d\phi d\theta \sin \theta T_1^0$, where for the coordinates $q^i = (ct, r, \theta, \phi)$, c is the speed of light, t is time, and (r, θ, ϕ) are the usual spherical coordinates. The mixed energy-momentum tensor T_i^k for the electromagnetic field is also introduced, as well as the four-current j^i and the concept of a point charge moving along a geodesic trajectory. Both the energy-momentum tensor F^{ik} and the Christoffel symbols Γ_{jk}^i are fixed by the spacetime geometry. One pair of covariant Maxwell equations is given by $F_{;k}^{ik} = -4\pi j^i/c$ or $(\sqrt{-g}F^{ik})_{;k} = -4\pi\sqrt{-g}j^i/c$. (The other pair is satisfied via the introduction of the four-potential A_i .) In these equations, as is customary, the semicolon in $F_{;k}^{ik}$ denotes the covariant derivative, the comma denotes the usual derivative, and $g = \det g_{ik}$ is the determinant of the matrix constructed from the metrical tensor elements. The electromagnetic covariant tensor is obviously expressed via the four-potential A_i . Next, the Green function method or the method of multipole expansion over the vector's spherical harmonics may be used for a given current j^i . Naturally, the boundary conditions must be taken into account. For example, the waves at the “surface” of the black hole must only be incoming, or the waves at infinity must only be outflowing and outgoing.

In this paper, we focus on item 3. In other words, instead of calculating the flux of the radiated energy from *afar*, we look at the power of the radiative friction force on the charged particle's *trajectory* (in the field of a rotating black hole). All the details of the derivations are presented to offer a methodological guide for those interested in the subject. This paper is organized as follows. Section 2 formulates the model. Section 3 discusses the power of the radiation friction force in the curved spacetime near a rotating black hole. Finally, Section 4 concludes with our final remarks.

2. Model

The evolution equation for the momentum-energy tensor $T_i^k[f]$ characterizing the electromagnetic field created by a moving charge is

$$\partial_k T_i^k[f] = -\frac{1}{c} F_{ik} j^k - \mu c f_i \frac{ds}{dt} \quad (1)$$

and its various consequences were considered in [14]. This equation follows from the conservation condition for the “particle + field” system $\partial_k (T_i^k[f] + T_i^k[p]) = 0$ and the fact that the force of the radiation friction f_i is included in the equation of motion for the particle. Here and onward, we use all designations that are generally accepted; the operator is $\partial_i \equiv \partial/\partial x^i$, the electromagnetic stress-tensor F_{kl} is expressed via potentials A_l as $F_{kl} = \partial_k A_l - \partial_l A_k$, the four-vector $j^k = \rho u^k ds/dt$ is the current density, ρ and μ are the charge and mass densities, respectively, c is the speed of light, ds is the interval, etc. Let us also mention that [14] used the following definitions. A flat spacetime was characterized by the Galilean metrics with the positive signature $(+---)$; each world point was described by a vector pointing to a position or state with contra-variant components $q^i = (ct, q^\alpha)$; t was the time measured by a remote observer; the Latin indices took the values $i = 0, 1, 2, 3$, while the Greek indices took the values $\alpha = 1, 2, 3$; the spacetime metric was fixed at an interval ds ; and $ds^2 = g_{ik} dq^i dq^k$, where g_{ik} is the metric tensor. With this definition of four-coordinates, quantities s and q^i have the dimension L . Recall that for a flat spacetime, $g_{ik} = (1, -\delta_{\mu\nu})$. In this case, the interval $ds = cdt\sqrt{1-\beta^2} \equiv c\gamma^{-1}dt$, where γ was the Lorentz factor. The contra-variant components of the four-velocity were $u^0 = \gamma$, $u^\alpha = \gamma\beta^\alpha$, where $\beta^\alpha = v^\alpha/c$, and the covariant ones were $u_0 = u^0$, $u_\alpha = -u^\alpha$. Similar to the definition of the four-speed $u^i = dq^i/ds = (\gamma, \gamma\beta)$ (obviously, $u_i u^i = g_{ik} u^i u^k = u^0 u^0 - u^\alpha u^\alpha = \gamma^2 - \gamma^2 \beta^2 = 1$), the quantity $w^i = du^i/ds = d^2 q^i/ds^2$ was called the contra-variant components of the four-acceleration. (Detailed explanations can be found in [1], p. 41 or p. 318.) The dimensions of

the quantities were $[u^i] = 1$, $[w^i] = L^{-1}$. The components of the four-acceleration in a flat spacetime expressed via the three-velocity and its three-acceleration were

$$w^i = (w^0, \mathbf{w}) = \left(\frac{\gamma^4}{c^2} \boldsymbol{\beta} \cdot \mathbf{a}, \frac{\gamma^2}{c^2} \mathbf{a} + \frac{\gamma^4}{c^2} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{a}) \right). \quad (2)$$

Here, $\mathbf{a} = c(d\boldsymbol{\beta}/dt)$ is the usual three-dimensional acceleration, and the dimension of the quantity is $[a] = L^1 T^{-2}$.

Let us now focus on the physical meaning of the terms on the right side of Equation (1). It is apparent that the terms describe the “sources” of the electromagnetic field. The field density changes if there is a “collision” between a moving charge ($j^k \neq 0$) and an electromagnetic field ($F_{ik} \neq 0$). The field is also generated if there is deceleration or acceleration of the charge due to the interaction between the charge and the field it generates. The choice of signs on the right side of the equation is not arbitrary; it is determined by the fact that the equations of motion of a point charged particle have the form

$$mc \frac{du_i}{ds} = \frac{e}{c} F_{ik} u^k + m c f_i. \quad (3)$$

Obviously, the radiation friction force f_i has to satisfy the condition $u^i f_i = 0$ because of the normalization condition $u^i u_i = 1$, and $F_{ik} = -F_{ki}$.

Below, we parameterize the effect of the radiation friction in the leading approximation through the expression (see, for example, [1,2,15,16])

$$f^i = r_e (\delta_k^i - u^i u_k) \frac{d^2 u^k}{ds^2} \quad \text{see} \quad f_i = r_e (g_{ik} - u_i u_k) \frac{d^2 u^k}{ds^2}, \quad (4)$$

which certainly satisfies the mandatory requirement $u^i f_i = 0$. The quantity $r_e = 2e^2/3mc^2$ is nothing more than the so-called “classical radius of the electron” r_e (with the coefficient 2/3 included here). Variants of Equation (4), in which additional nonlocal terms are added, are also presented in the literature, but discussion of this issue is beyond the scope of our work. Equation (4) certainly can be rewritten in an alternative form, taking into account $u_k u^k = 1$. Recall that the relations used in this case are $u_k (du^k/ds) = 0$ and $u_k (d^2 u^k/ds^2) = -(du_i/ds)(du^i/ds)$.

The physical meaning of Equation (1) becomes clear after integration of the time component of the tensor over a certain three-dimensional volume V bounded by the two-dimensional surface Σ , after regrouping the terms and including some correction terms in the redefinition of the electromagnetic field energy density:

$$\frac{d}{dt} \overline{W_f} = - \oint d\Sigma \cdot \mathbf{S} + mc^3 r_e (-w_k w^k) \equiv -P_r + P. \quad (5)$$

The overhead line here signifies the volume integral.

Equation (5) explains how energy is balanced via the three processes: the radiation escaping through the distant surface Σ enveloping volume V , the excitation of the electromagnetic field by the particle decelerating under the action of the radiative friction force, and the change over time in the field’s energy volume density.

On the other hand, Equation (5) shows that the three quantities—the change in the field energy per unit time (dW_f/dt), the total energy flux of the outflowing field (P_r), and the power of the “radiative friction force” (P)—are connected by only one relationship. In a general case, they are far from being equal to each other in terms of the absolute magnitude. Only if we consider a stationary motion with $dW_f/dt \rightarrow 0$ (for example, when a particle moves along a circular orbit and the quantities are considered averages over the period of

revolution), does the equality $P = P_r$ (on average) take place. This circumstance was noted in a methodological review [17].

Let us also emphasize that P and P_r are fixed by the field characteristics in different regions of the space; the friction force is applied to the charge on its trajectory, and the field flux is calculated through a distant surface Σ . In full accord with the spirit of field theory, the energy flux through a surface Σ is determined directly by the field near this surface and not by the field on the trajectory of a charge situated inside the surface Σ . Therefore, there is no reason a priori to believe that these quantities are always equal to each other.

Thus, although the energy flow through the bounding surface does provide some information about the rate of change (P_r) in the electromagnetic field energy in the volume under consideration, this information is not complete. Obviously, the power (P) of the radiation friction forces should also be taken into account.

The meaning of the expression in Equation (1) might seem trivial, but important nuances shine through for the phenomena in the presence of an external gravity field, i.e., in the curved spacetime.

As mentioned, the first term in the right part of Equation (5) is the electromagnetic energy flow “outflowing” from the volume V through the distant surface Σ . Only the most slowly decreasing part of the field (created by the accelerating charge) makes a contribution to this term.

The second term in the right part of Equation (5) is the power of the radiation friction force. It increases the total energy of the electromagnetic field surrounding the moving charge. It is easy to verify using the definition of w^i that

$$P = mc^3 r_e (-w^i w_i) = mc^3 r_e (-g_{ik} w^k w^i) \equiv mc^3 r_e (-w^0 w^0 + w^\mu w_\mu) = mc^{-1} r_e (1 - \beta^2) a^2 \gamma^8 (1 - \beta^2 \sin^2 \psi) = \frac{2}{3} \frac{e^2}{c^3} \gamma^6 (a^2 - [\boldsymbol{\beta}, \mathbf{a}]^2), \quad (6)$$

where ψ is the angle between the directions of the spatial velocity $\mathbf{v} = c\boldsymbol{\beta}$ and the spatial acceleration $\mathbf{a}$ of the particle. It is necessary to remember the nuances of the cross-product in Cartesian and curvilinear coordinates.

When the charged particle moves with a small velocity with respect to the speed of light ($\beta \ll 1$) in an inertial frame of reference, then Equation (6) gives the well-known Larmor’s formula [18] for the bremsstrahlung:

$$dE = \frac{2}{3} \frac{e^2}{c^3} a^2 dt'. \quad (7)$$

Here, obviously, $a = a(t')$, $t' = t - R/c$, and $dt = dt'$.

To conclude this section, we point out that the expression in Equation (6) may be obtained simply from dimensional considerations and for this reason, naturally, can be generalized to an arbitrary case. Indeed, the state of a particle may be characterized by the four-vector $P^i = (E, P^\alpha)$ of energy-momentum. The position of this particle in the four-spacetime is characterized by the position vector $q^i = (ct, q^\alpha)$. If there is a change in the quantity P^i , i.e., $dP^i = (dE, dP^\alpha)$, then this change should be proportional to the change in the four-state vector dq^i , i.e., $dP^i \sim dq^i$. The power of the loss (which is comparable in magnitude with the friction force power) is determined by the magnitude of the particle charge but not by its sign, i.e., $dP^i \sim e^2 dq^i$. The proportionality coefficient in the written expression should depend on the rate of change in the state of the particle, i.e., on the invariant (scalar) constructed from four-accelerations w^i (and naturally on w_i), which for small space velocities certainly has to give the limit value $a^\alpha a_\alpha$. Here a^α is the contra-variant three-acceleration. For reasonable magnitudes of particle acceleration, in the leading approximation, this scalar is $w^i w_i$. We note here that similar to the definition of the four-

speed $u^i = dq^i/ds$, the quantity $w^i = du^i/ds = d^2q^i/ds^2$ is called the four-acceleration (see [1], § 7, p. 21 and § 87, p. 244). In addition to this, the expression for dP^i must contain the invariant, reflecting the fact that the loss of energy by the particle has the electromagnetic nature. Thus, the speed of light (in a vacuum) c must be, in principle, present in the sought-for expression. Therefore, we make the assumption that $dP^i \sim e^2 c^\kappa w^k w_k dq^i$, where κ is some numerical factor (to be determined). From the simple dimensional consideration, when letting $[dP^0] = M^1 L^2 T^{-2}$, $[e^2] = M^1 L^3 T^{-2}$, $[c] = LT^{-1}$, $[w] = L^{-1}$, $[q^0] = L$, we find that $M^1 L^2 T^{-2} = M^1 L^3 T^{-2} \times L^\kappa T^{-\kappa} \times L^{-2} \times L^1$. From here, $\kappa = 0$. Finally, we obtain

$$dP^i = \lambda e^2 w^k w_k dq^i, \quad (8)$$

where λ is a dimensionless coefficient. The value of the constant λ is found from the “correspondence principle” at the limit case of small velocities of the particle. At the limit of small velocities, β tends toward zero, the quantity $w^i w_i \rightarrow -a^2/c^4$ (see Equation (2)), and with $q^0 = ct$, this leads to the Larmor formula [18]. Thus, the parameter $\lambda = -2/3$.

3. Power of Radiation Friction Force in Curved Spacetime

The principal goal of this section is to analytically estimate the power of the radiation friction force P for a charged particle performing a flyby in the vicinity of a rotating black hole.

The basic expression for calculating the effect takes the form

$$dP^k = -\frac{2}{3} e^2 w^i w_i dq^k. \quad (9)$$

We use the sign-metric convention $+- - -$ (attention alert for readers accustomed to the alternative convention). We do not consider a magnetized black hole; we consider the simplest configuration. The square of the dimensionless interval in the vicinity of the simplest rotating black hole in the Kerr metric in the Boyer–Lindquist coordinates is written as follows:

$$ds^2 = g_{ij} dq^i dq^j = \left[\left(1 - \frac{x}{\zeta^2}\right) dt^2 - 2\omega \frac{x}{\zeta^2} \sin^2 \theta d\phi dt - \frac{\zeta^2}{\chi} dx^2 - \zeta^2 d\theta^2 - \frac{\Lambda}{\zeta^2} \sin^2 \theta d\phi^2 \right], \quad (10)$$

with

$$\begin{aligned} \zeta^2 &= x^2 + \omega^2 \cos^2 \theta, & \chi &= x^2 - x + \omega^2, \\ \Lambda &= (x^2 + \omega^2)^2 - (x^2 - x + \omega^2) \omega^2 \sin^2 \theta. \end{aligned} \quad (11)$$

Here, the units of length and time are $r_g = 2GM_h/c^2$ and r_g/c , respectively, the radial coordinate is written as $r = r_g x$, the black hole momentum is written as $J_h = M_h c r_g \omega$, M_h is the mass of the black hole, G is the gravity constant, and c is the speed of light.

The parameters q^i specify the four-coordinate location (in the considered spacetime) of some world point $q^i = (ct, r, \theta, \phi)$. These coordinates are defined from the perspective of an observer located at infinity. At large r values, the space coordinates r, θ, ϕ turn into standard spherical coordinates. Note that r here has a somewhat different meaning than the traditional “distance” from the black hole; in the considered spacetime, there exists no central point $r = 0$ in the sense of a particular spacetime point (an event on a world line for a material object).

With nonzero off-diagonal terms, the square of the interval ds can be rewritten in the form

$$\begin{aligned} ds^2 &= g_{ik}dq^i dq^k = g_{00}(dq^0)^2 + 2g_{0\alpha}dq^0 dq^\alpha + g_{\alpha\beta}dq^\alpha dq^\beta = \\ &= g_{00}\left(dq^0 + \frac{g_{0\alpha}}{g_{00}}dq^\alpha\right)^2 + \left(g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dq^\alpha dq^\beta \equiv \\ &= g_{00}(dq^0 - g_\nu dq^\nu)^2 - \gamma_{\mu\nu}dq^\mu dq^\nu = m_{ik}dQ^i dQ^k, \end{aligned} \quad (12)$$

where the three-vector $g_\mu = -g_{0\mu}/g_{00}$. This allows us to introduce and express the so-called “synchronized time” η (see, for example, [1]). Its element is $d\eta = dq^0 - g_\gamma dq^\gamma$. From here, $d\eta/ds = u^0 - g_\nu u^\nu$. Obviously, there is no difference between the times t (measured by the clock located in the frame of reference of any external observer who is at rest and observes the moving particle) and η (measured in the frame of reference in which both the initial and final coordinates of the particle can be determined simultaneously by using—as is noted in [1]—light signals), i.e., when all non-diagonal elements in the metric tensor m_{ij} are zero.

The above considerations are not a meaningless mathematical exercise. The delicacy of this issue is discussed in detail in the classic work [1].

The geometrical distance dl between two points in the 3D space is defined via elementary displacements dq^α and the 3D metric tensor $\gamma_{\alpha\beta}$: $\gamma_{\alpha\beta}dq^\alpha dq^\beta = dl^2$. The symmetric (with respect to the permutation of indices) tensor $\gamma_{\alpha\beta} = -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}g_{00}^{-1} = \gamma_{\beta\alpha}$ contains all information about the rotation of the black hole ($g_{0\beta} \neq 0$). Thus, from $ds^2 = g_{00}d\eta^2 - \gamma_{\alpha\beta}dq^\alpha dq^\beta$, it follows that $ds^2 = g_{00}d\eta^2(1 - v^2) \equiv g_{00}\gamma^{-2}d\eta^2$, where γ is the Lorentz factor. From here, by the way, $d\eta/ds = g_{00}^{-1/2}\gamma$. The square of the dimensionless “physical” 3D velocity of a particle in the usual 3D space is defined now as $v^2 = g_{00}^{-1}(dl/d\eta)^2 \equiv g_{00}^{-1}\gamma_{\alpha\beta}(dq^\alpha/d\eta)(dq^\beta/d\eta)$. The tensor $\gamma_{\alpha\beta}$ enables the process of raising and lowering the indices α, β . Then, the contra-variant components of “physical” velocities are written as $v^\alpha = g_{00}^{-1/2}(dq^\alpha/d\eta)$ (in fact, $u^\alpha = dq^\alpha/ds$), and the co-variant ones are symmetrically $v_\alpha = \gamma_{\alpha\beta}g_{00}^{-1/2}(dq^\beta/d\eta)$.

The components of the four-velocities $u^i = (u^0, u^\alpha)$ can be written in terms of 3D “physical” velocities v^i as $u^\alpha = \gamma v^\alpha$, respectively, usually for spatial components, and $u_\alpha = -\gamma_{\alpha\beta}\gamma v^\beta$. Also, $u^0 = g_{00}^{-1/2}\gamma + g_\alpha\gamma v^\alpha$, $u_0 = g_{00}^{1/2}\gamma + g_{00}g_\alpha\gamma v^\alpha$. Certainly, the condition of normalization is $u^k u_k = 1$.

The square of the interval in the spacetime $Q^i = (\eta, q^\alpha)$ is thus rewritten as $ds^2 = m_{ik}dQ^i dQ^k$, where the diagonal metric tensor $m_{ij} = (g_{00}, -\gamma_{\alpha\beta})$.

Now, we calculate the space and time components of the four-acceleration. The spacial ones are

$$\begin{aligned} w^\alpha &= \frac{du^\alpha}{ds} = g_{00}^{-1/2}\gamma \frac{d(\gamma v^\alpha)}{d\eta} = g_{00}^{-1/2}\gamma \left(\gamma \frac{dv^\alpha}{d\eta} + v^\alpha \frac{d\gamma}{d\eta} \right) = \\ &= \gamma g_{00}^{-1/2}\gamma \left(\frac{dv^\alpha}{d\eta} + \gamma^3 v^\alpha v_\beta \frac{dv^\beta}{d\eta} \right) = \left(\frac{\gamma^2}{c^2} a^\alpha + \frac{\gamma^4}{c^2} v^\alpha v_\beta a^\beta \right) = \frac{\gamma^4}{c^2} \left(\gamma^{-2} a^\alpha + v^\alpha v_\beta a^\beta \right). \end{aligned} \quad (13)$$

Here, via analogy to the three-velocity definition, $a^\nu = g_{00}^{-1/2}c^2 dv^\nu/d\eta$ is the usual three-dimensional acceleration, and the dimension of the quantity is $[a] = L^1 T^{-2}$.

For the time component w^0 of the four-acceleration, we obtain

$$\begin{aligned} w^0 &= \frac{du^0}{ds} = g_{00}^{-1/2}\gamma \left((g_{00}^{-1/2} + g_\alpha v^\alpha) \frac{d\gamma}{d\eta} + \gamma \frac{d}{d\eta} (g_{00}^{-1/2} + g_\alpha v^\alpha) \right) = \\ &= \left((g_{00}^{-1/2} + g_\alpha v^\alpha) \frac{\gamma^4}{c^2} v_\beta a^\beta + \gamma^2 g_{00}^{-1/2} \frac{d}{d\eta} g_{00}^{-1/2} + \gamma^2 v^\alpha g_{00}^{-1/2} \frac{dg_\alpha}{d\eta} + \frac{\gamma^2}{c^2} g_\alpha a^\alpha \right). \end{aligned} \quad (14)$$

Because the quantities g_{00} and g_α do not depend on the time parameter η , we omit the corresponding terms and obtain

$$w^0 = g_{00}^{-1/2} c^{-2} \gamma^4 v_\beta a^\beta + c^{-2} g_\alpha (v^\alpha \gamma^4 v_\beta a^\beta + \gamma^2 a^\alpha) = \frac{\gamma^4}{c^2} \left(g_{00}^{-1/2} v_\beta a^\beta + g_\alpha (v^\alpha v_\beta a^\beta + \gamma^{-2} a^\alpha) \right) = \frac{\gamma^4}{c^2} g_{00}^{-1/2} v_\beta a^\beta + g_\alpha w^\alpha. \quad (15)$$

We are guided by the idea that the parameters associated with a particle and the parameters characterizing the spacetime surrounding this particle are different things. Naturally, the expressions in Equations (14) and (15) transit into Equation (2) when $g_{00} \rightarrow 1$ and $g_\alpha \rightarrow 0$, i.e., we are in a flat spacetime.

If the particle motion takes place in the equatorial plane $\theta = \pi/2$ of the curved spacetime in the vicinity of a rotating black hole, then the above obtained expressions are significantly simplified as well. Indeed, in this case, the components of acceleration w^2 and velocity v^2 are equal to zero, and the components of the metric tensor m_{ik} that we need are

$$g_{00} = 1 - \frac{1}{x}, \quad g_{01} = g_{02} = 0, \quad g_{03} = \frac{\omega}{x}, \quad \gamma_{11} = \frac{x^2}{x^2 - x + \omega^2}, \\ \gamma_{33} = -g_{33} + \frac{g_{03}^2}{g_{00}} = \frac{(x^2 + \omega^2)^2 - (x^2 - x + \omega^2)\omega^2}{x^2} + \frac{\omega^2}{x^2} \frac{x}{x-1}. \quad (16)$$

Next, we calculate the convolution $w_k w^k = m_{kj} w^j w^k = g_{00} w^0 w^0 - \gamma_{\alpha\beta} w^\alpha w^\beta$, where Equations (13), (15) and (16) should be substituted.

To illustrate the details, we give here this elementary calculation. The term $-w_0 w^0$ is $-(\gamma^8/c^4) v^2 a^2 \cos^2 \psi$, where ψ is the angle between vectors $\mathbf{v}$ and $\mathbf{a}$ because of the scalar product definition. The next term $w_\alpha w^\alpha$ is equal to $(\gamma^8/c^4) (a^\beta \gamma^{-2} + v^\beta v a \cos \psi) (a_\beta \gamma^{-2} + v_\beta v a \cos \psi)$. For the product, we obtain $(\gamma^8/c^4) (a^2 \gamma^{-4} + 2\gamma^{-2} a^2 v^2 \cos^2 \psi + a^2 v^4 \cos^2 \psi) = \gamma^8 a^2 (1 - 2v^2 + v^4 + 2v^2 \cos^2 \psi - 2v^4 \cos^2 \psi + v^4 \cos^2 \psi) = \gamma^8 a^2 (1 - 2v^2 + 2v^2 \cos^2 \psi + v^4 \sin^2 \psi) = \gamma^8 a^2 (1 - 2v^2 \sin^2 \psi + v^4 \sin^2 \psi)$. Next, the expression $-w_0 w^0 + w_\alpha w^\alpha$ is transformed into $(\gamma^8/c^4) a^2 (-v^2 \cos^2 \psi + 1 - 2v^2 + 2v^2 \cos^2 \psi + v^4 \sin^2 \psi) = (\gamma^8/c^4) a^2 (1 - 2v^2 + v^2 \cos^2 \psi + v^4 \sin^2 \psi) = (\gamma^8/c^4) a^2 (1 - v^2 - v^2 \sin^2 \psi + v^4 \sin^2 \psi) = \gamma^8 a^2 (1 - v^2) (1 - v^2 + v^2 \cos^2 \psi)$, where the parameter $\cos \psi$ is given by

$$\cos \psi = \frac{\gamma_{11} a^1 v^1 + \gamma_{33} a^3 v^3}{\sqrt{\gamma_{11} (a^1)^2 + \gamma_{33} (a^3)^2} \sqrt{\gamma_{11} (v^1)^2 + \gamma_{33} (v^3)^2}}. \quad (17)$$

Finally, we find the desired expression:

$$-\frac{2}{3} e^2 c w^j w_j = \frac{2}{3} \frac{e^2}{c^3} \gamma^6 a^2 (1 - v^2 + v^2 \cos^2 \psi). \quad (18)$$

4. Conclusions

The obtained expression takes into account all the features that interest us. A particle can move with an ultra-relativistic velocity when $v \rightarrow 1$; the features of the spacetime metric are packed into the parameter $\cos \psi$; the rotation of the black hole changes only the component γ_{33} of the metric tensor; and this component does not depend on the direction of rotation of the black hole (i.e., on the sign of the parameter ω). The obtained expressions work even when the particle approaches the event horizon (i.e., when $x \rightarrow 1$). It is easy to verify that, formally, for a flat spacetime, when the parameter $x \rightarrow \infty$ in Equation (16) above tends toward infinity, Equation (18) becomes Equation (6). Thus, we obtain a relativistic generalization, i.e., Equation (18), of the classical Larmor formula, i.e., Equation (7), for

the power of the radiation friction force $P = mc^3 r_e (-w_i \dot{w}^i)$ for the case where a relativistic charged particle moves in the vicinity of a rotating Kerr black hole.

It is worth noting that the effect is maximal when the acceleration is maximal, i.e., in the vicinity of a periastron which shifts due to relativistic effects ($g_{00} \neq 1$ and $g_k \neq 0$) (see, for example, [19,20]).

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