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Nonrelativistic Superfluids in Cosmology from a Relativistic Approach: Revisiting Two Formulations of Superfluidity

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Abstract: Two formulations of superfluidity are reviewed: Landau's phenomenological two-fluid model and a relativistic effective field theory description. We demonstrate how the two-fluid formalism can be recovered from the nonrelativistic limit of the relativistic effective theory at finite temperatures. We show how self-gravitating, nonrelativistic superfluids are obtained from the Newtonian limit of the relativistic approach on curved spaces. The concepts are presented in an accessible manner for readers who may not be deeply familiar with superfluidity from a condensed matter perspective.

Keywords: dark matter; fluid dark matter; superfluids; superfluid dark matter; Bose–Einstein condensation; cosmology

1. Introduction

Despite the series of successful predictions brought forth by the Lambda-Cold Dark Matter (Λ CDM) model [1–11], the fundamental nature of dark matter (DM) remains unknown. On large scales, the standard cosmological model describes DM as a cold, pressureless fluid of non-interacting particles, and while this description suffices for explaining large-scale observations, there are several discrepancies to be found on galactic scales (for a recent overview of the small-scale problems, see [12–15] for alternative approaches).

These discrepancies can be explained by the superfluid dark matter (SFDM) model [16–18] (see also [19–32] and the reviews [33–35]). In this model, DM consists of ultralight particles with a mass on the order of 10^{-20} to a few eV [16]. On cosmological scales, DM still behaves as a cold, collisionless fluid, similar to its behaviour in the standard model; on galactic scales, however, the DM particles can thermalize and condense into a superfluid phase. This occurs in regions of high DM density, such as the core of galaxies, where the density surpasses a critical threshold, triggering a phase transition and causing the particles to form a Bose–Einstein Condensate (BEC). Additionally, the superfluid phonons couple to ordinary matter, mediating a long-range force between baryons that, for a specific superfluid equation of state, reproduces the Modified Newtonian Dynamics (MOND) law [36].

At finite temperature, the phenomenology of superfluidity is described by Landau's two-fluid model [37,38], originally developed to describe superfluid helium. Below a critical temperature T_c , the superfluid is treated as a mixture of two components: a superfluid and a normal component. The relative concentrations of these components depend on the temperature: in the $T \rightarrow 0$ limit, the system consists entirely of the superfluid component, but as the temperature rises, the proportion of the superfluid decreases while the normal



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component fraction grows. For $T \geq T_c$, the superfluid component vanishes entirely, leaving only the normal component.

While a nonrelativistic approach can be appropriate for studying low-energy systems such as liquid helium, studying high-energy superfluids requires a fully relativistic framework [39]. Relativistic effects become significant when the particles' kinetic energy is comparable to their masses, and given the vast range of energy scales over which superfluidity is observed, a relativistic treatment becomes essential. For instance, the critical temperature of different systems can range from as low as 10^{-7} K in ultra-cold atomic gases to 10^8 K for neutron matter, as indicated by astrophysical observations [40] (see [41] for a review on nuclear matter at high densities and the formation of relativistic superfluids in this context). Moreover, many UV descriptions of the origin of superfluid DM are fully relativistic, e.g., see [42–48], and the relativistic description might be more efficient when the nonrelativistic fluid is considered in a relativistic background (e.g., see [49] for a study on the behaviour of superfluid DM on a cosmic string background). In addition to the importance of relativistic effects, the relativistic approach is also more general: the nonrelativistic case can be recovered as a limiting case.

Although there are equivalent formulations of relativistic superfluids [50–57], this work considers the operational approach through the low-energy effective field theory framework proposed by Nicolis [58]. This framework provides a fully relativistic and Poincaré-invariant description of finite-temperature superfluids, in which the dynamics comes straightforwardly from a low-energy effective action via the standard variational principle. The action is derived in the traditional effective field theory approach: the infrared degrees of freedom are identified—the superfluid phase and the comoving coordinates of the volume elements of the normal fluid—and the most general action involving these degrees of freedom is constructed, ensuring that it remains consistent with the symmetries of the system. To the best of our knowledge, there is no explicit construction of the nonrelativistic limit of such a formulation, which presumably would reproduce the Galilean description of superfluids usually assumed in cosmology.

In this paper, we show how Landau's two-fluid model is recovered from the relativistic effective description. We revisit both the two-fluid formalism and the effective field theory approach to superfluidity to create a map between the field variables and the standard hydrodynamical and thermodynamical ones in the nonrelativistic limit. This paper is intended for cosmologists and astrophysicists who work with superfluid dark matter but may have little familiarity with the details of superfluidity in condensed matter physics. We aim to present these concepts in a way that is accessible without requiring deep prior knowledge of superfluidity from a condensed matter perspective. We provide a bridge between the general relativistic effective field description of superfluids and the self-gravitating, nonrelativistic description typically employed in cosmology. In doing so, we hope to close a literature gap in explicitly recovering the nonrelativistic limit from general symmetry principles, including Lorentz invariance.

This paper is structured as follows: Section 2 establishes the theoretical background, with a brief overview of Landau's two-fluid model in Section 2.1 and of the effective theory of superfluidity in Section 2.2, defining their main equations and variables; in Section 3, we recover the nonrelativistic limit from the effective theory in different regimes and discuss how to obtain self-gravitating superfluids (which are typically used in cosmology) from the Newtonian limit of the relativistic theory. In Section 4, we establish a map between the two-fluid model and the nonrelativistic limit of the field theory approach at finite temperature, and finally, in Section 5, we conclude. Throughout this paper, we use natural units in which the speed of light and Boltzmann's and Planck's constants are set to one, i.e., $c = k_B = \hbar = 1$. We adopt the metric signature $(-, +, +, +)$.

2. Preliminaries

To set the notation and the background for the rest of this paper, in this section, we briefly review the two-fluid model for nonrelativistic superfluids [59] and the effective field approach for relativistic superfluids [58].

2.1. Two-Fluid Model

The two-fluid model of superfluidity, first introduced by London [60,61] and Tisza [62] in 1938 and later reformulated by Landau [37] in 1941 (see also [63,64] for a more detailed account of the history of superfluid physics), describes a superfluid as a mixture of two different fluids: a superfluid component, composed of atoms in a BEC, and a normal component, made up of collective excitations. The former behaves as an inviscid fluid, carries zero entropy, and flows irrotationally, while the latter behaves as an ordinary viscous fluid. The normal component is formed by two different types of quasi-particles, phonons—lower-energy excitations—and rotons—higher-energy excitations. These two components are in relative motion, and we denote the velocity of the superfluid and normal components by $\vec{v}_s$ and $\vec{v}_n$, respectively.

In this framework, a superfluid with total particle number density n will satisfy

$$n = n_n + n_s, \quad (1)$$

where n_n and n_s give the contributions of the normal and superfluid components, respectively. Both n_n and n_s are temperature-dependent: at $T = 0$, $n_s = n$ and $n_n = 0$, so that all the fluid is found in the superfluid state; as the temperature increases, excitations start to appear in the form of the normal component, and at $T = T_c$, all the superfluid component is gone, so $n_s = 0$ and $n_n = n$.

The mass density of the superfluid and normal parts are defined in terms of the particle number densities as

$$\rho_{n,m} = mn_n, \quad \rho_{s,m} = mn_s, \quad (2)$$

respectively, so that the total mass density is given by

$$\rho_m = \rho_{n,m} + \rho_{s,m}. \quad (3)$$

The mass flux density of the system, $\vec{j}$, or momentum density, is given by

$$\vec{j} = \rho_{s,m}\vec{v}_s + \rho_{n,m}\vec{v}_n, \quad (4)$$

i.e., a sum of the fluxes pertaining to the normal and superfluid parts.

With these quantities in hand, we can now summarize the equations of superfluid dynamics [59]. The mass density ρ_m and the flux $\vec{j}$ obey the continuity equation

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \quad (5)$$

which expresses the law of mass conservation.

The conservation of momentum is given by the equation

$$\frac{\partial j^i}{\partial t} + \frac{\partial \Pi^{ik}}{\partial x^k} = 0, \quad (6)$$

where Π^{ik} is the momentum flux density tensor, defined by

$$\Pi^{ij} = \rho_{n,m} v_n^i v_n^j + \rho_{s,m} v_s^i v_s^j + P \delta^{ij}, \quad (7)$$

with $P = P_n + P_s$ denoting the total pressure of the system.

The law of entropy conservation is given by

$$\frac{\partial(\rho_m s)}{\partial t} + \vec{\nabla} \cdot (\rho_m s \vec{v}_n) = 0, \quad (8)$$

where the second term in parenthesis, $\rho_m s \vec{v}_n$, is the entropy flux.

Finally, the conservation of energy is given by the expression

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{Q} = 0, \quad (9)$$

where ϵ is the energy per unit volume of the fluid and $\vec{Q}$ the energy flux density vector, given by

$$\epsilon = \epsilon_n + \epsilon_s + \frac{1}{2} \rho_{n,m} v_n^2 + \frac{1}{2} \rho_{s,m} v_s^2, \quad (10)$$

$$\vec{Q} = \vec{v}_n \left(\epsilon_n + \frac{1}{2} \rho_{n,m} v_n^2 + P_n \right) + \vec{v}_s \left(\epsilon_s + \frac{1}{2} \rho_{s,m} v_s^2 + P_s \right). \quad (11)$$

Here, ϵ_n and ϵ_s are the internal energy densities of the normal and superfluid components, respectively, which are given by the usual thermodynamic relations,

$$\epsilon_n = \mu \rho_{n,m} + Ts - P_n, \quad (12)$$

$$\epsilon_s = \mu \rho_{s,m} - P_m, \quad (13)$$

where μ is the chemical potential. Equations (5), (6), (8), and (9) provide the dynamics of nonrelativistic fluids. Our goal is to recover these from the general effective field approach for relativistic superfluids, which we revisit in the next section.

2.2. Effective Theory Approach

From an effective field point of view, a BEC is described by a state that spontaneously breaks the global $U(1)$ symmetry associated with the conservation of particle number in the fluid. This is achieved by the condensation of excited particles into the ground state during the formation of the BEC. By the Goldstone theorem, there is a gapless excitation ψ , which non-linearly realizes the $U(1)$ symmetry as

$$\psi \rightarrow \psi + a, \quad \text{where } a \text{ is a constant,} \quad (14)$$

and this is the only degree of freedom present at low energies.

The superfluid phase at $T = 0$ can be described by

$$\psi = \mu t, \quad (15)$$

where μ is the chemical potential. This homogeneous and isotropic state admits gapless excitations π ,

$$\psi = \mu t + \pi, \quad (16)$$

the superfluid phonons. At finite temperature, the phonons will be excited, reach thermodynamic equilibrium, and form a thermal bath. Otherwise, if their frequencies are lower

than the phonon inverse mean free time, and their wavelengths longer than their mean free path, this thermal bath of phonons will be describable by usual hydrodynamics. We are then left with a normal fluid made up of phonons, which moves through and interacts with the background superfluid. As the temperature increases, it approaches a critical temperature T_c , above which the symmetry is restored and superfluidity is gone. Only the normal component remains, and the dynamics is once again those of an ordinary fluid.

We can thus postulate that, at temperatures between 0 and T_c , our system will be composed of two fluids, a normal one and a superfluid one, and the density of the superfluid decreases monotonically with T , going to 0 at T_c . This is the essence of Landau's two-fluid model.

For the rest of this section, we follow [58] closely. The low-energy degrees of freedom of our field theory are the superfluid phase $\psi(\mathbf{x}, t)$ —which enjoys a shift symmetry—and the comoving coordinates for the volume elements of the normal fluid component, which are parametrized by three scalar fields $\phi^i(\mathbf{x}, t)$, $i = 1, 2, 3$. These ϕ^i are the comoving (or 'Lagrangian') coordinates of the fluid element occupying physical (or 'Eulerian') position $\mathbf{x}$ at time t . The fluid's internal space is homogeneous and isotropic, such that the dynamics of ϕ^i should obey [65,66]

$$\phi^i \rightarrow \phi^i + a^i, \quad a^i = \text{const.}, \quad (17)$$

$$\phi^i \rightarrow R^i_j \phi^j, \quad R \in SO(3). \quad (18)$$

Moreover, the maps ϕ^i should be invertible, which requires $\det \partial_i \phi^j \neq 0$, where $\partial_i \equiv \partial/\partial x^i$. For an incompressible, volume-preserving fluid, the constraint becomes $\det \partial_i \phi^j = 1$.

Lastly, we note that if we try to move around a volume element of a non-dissipative fluid, we would not feel any resistance; if we try to perform the same with a solid, the motion would suffer resistance from the rest of the solid's volume. In other words, the fluid's dynamics must be invariant under volume-preserving reparametrizations [65],

$$\phi^i \rightarrow \zeta^i(\phi^j), \quad \det \frac{\partial \zeta^i}{\partial \phi^j} = 1. \quad (19)$$

Given the shift symmetries (14) and (17), ψ and ϕ^i should enter the Lagrangian with at least one derivative acting on them, so at lowest order in the derivative expansion, $\mathcal{L}$ should be made up of $\partial_\mu \psi$ and $\partial_\mu \phi^i$. In order to obey the symmetry (19), the fluid variables have to appear through the combination

$$J^\mu = \frac{1}{6} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{ijk} \partial_\alpha \phi^i \partial_\beta \phi^j \partial_\gamma \phi^k. \quad (20)$$

Now, we have the building blocks for our Lagrangian: ψ should enter as $\partial_\mu \psi$ and ϕ^i as J^μ . With these vectors, we can construct three scalar quantities,

$$X = \partial_\mu \psi \partial^\mu \psi, \quad (21)$$

$$b = \sqrt{-J^\mu J_\mu} = \sqrt{\det(\partial_\mu \phi^i \partial^\mu \phi^j)}, \quad (22)$$

$$y = -\frac{1}{b} J^\mu \partial_\mu \psi \equiv u^\mu \partial_\mu \psi, \quad (23)$$

where u^μ is the normal fluid's four-velocity, normalized such that $u_\mu u^\mu = -1$, and the fluid's comoving coordinates do not change along its integral curves, $u^\mu \partial_\mu \phi^i = 0$. The vector $\partial^\mu \psi$ can be interpreted as the four-velocity of the superfluid component (up to a normalization factor). It is also worth noting that, since $J^\mu = -bu^\mu$, it follows that J^μ is manifestly time-like, as $J^\mu J_\mu = -b^2 < 0$ for real, nonzero b . In this general construction, b

is a real and non-negative scalar; its physical interpretation (e.g., as a comoving entropy or number density) depends on the identification of the current J^μ , which we specify later.

In summary, the low-energy Lagrangian density describing superfluidity has the form [58]

$$\mathcal{L} = F(X, b, y). \quad (24)$$

To compute the associated stress tensor, we vary $\mathcal{L}$ with respect to the metric, which gives

$$T^{\mu\nu} = (yF_y - bF_b)u^\mu u^\nu + (F - bF_b)g^{\mu\nu} - 2F_X \partial^\mu \psi \partial^\nu \psi. \quad (25)$$

The current associated with the spontaneously broken $U(1)$ symmetry (14) is just

$$N^\mu = F_y u^\mu + 2F_X \partial^\mu \psi, \quad (26)$$

the particle number four-current.

Now, we can match our field theory quantities to thermodynamical ones. To do so, we need to pick a local reference frame, and since our system is composed of two fluids in relative motion, we can choose either component; we choose to move with the normal fluid component, along u^μ .

The energy–momentum tensor for a perfect fluid has the general form

$$\mathcal{T}^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}. \quad (27)$$

We denote this tensor as $\mathcal{T}^{\mu\nu}$ to distinguish it from the total stress–energy tensor $T^{\mu\nu}$ derived from variation of the action. The two coincide only when the superfluid and normal components are comoving, i.e., when $\partial_\mu \psi = -yu_\mu$. From (27), we can isolate the energy density and pressure,

$$\rho = \mathcal{T}^{\mu\nu} u_\mu u_\nu, \quad (28)$$

$$P = \frac{1}{3} \mathcal{T}^{\mu\nu} (g_{\mu\nu} + u_\mu u_\nu). \quad (29)$$

Replacing $\mathcal{T}^{\mu\nu}$ in the above expression by (25), we can easily obtain the total energy density,

$$\rho = T^{\mu\nu} u_\mu u_\nu = yF_y - F - 2y^2 F_X. \quad (30)$$

When computing the pressure, however, we have to remember that we are working with a system composed of two fluids, and that since there is no frame in which $T^{\mu\nu}$ is isotropic, Equation (29) is not valid due to the relative motion of the fluid components. Instead, we have an energy–momentum tensor, which is the sum of an isotropic tensor, $\propto g^{\mu\nu}$, and of two rank-one tensors, one $\propto u_\mu u_\nu$ and one $\propto \partial_\mu \psi \partial_\nu \psi$ —which depends on the four-velocity of the superfluid component, as earlier identified. Thus, we can identify the pressure with the coefficient of $g^{\mu\nu}$, i.e.,

$$P = F - bF_b. \quad (31)$$

From the four-current N^μ , we can extract the total particle number density,

$$n = -N^\mu u_\mu = F_y - 2yF_X. \quad (32)$$

Now that we have ρ , P , and n , we can use thermodynamic identities to find the remaining fluid variables. Starting from Euler's equation,

$$\rho + P = Ts + \mu n, \quad (33)$$

and replacing in the expression the quantities we already have, we find

$$y(F_y - 2yF_X) - bF_b = Ts + \mu(F_y - 2yF_X), \quad (34)$$

which is satisfied for

$$\mu = y, \quad Ts = -bF_b. \quad (35)$$

In order to solve the ambiguity in the second expression, we take another look at the four-vector J^μ : it follows from its definition (20) that it is an identically conserved current, i.e., $\partial_\mu J^\mu = 0$; furthermore, $J^\mu = bu^\mu$, and we see that it is aligned with u^μ . Thus, J^μ can be identified with the entropy four-current [65],

$$J^\mu = su^\mu, \quad (36)$$

which leads to the identifications

$$s = b, \quad T = -F_b. \quad (37)$$

Lastly, we want to express X in terms of fluid variables. The vector $\partial_\mu \psi$ can be decomposed into the sum of two orthogonal vectors, one parallel to u^μ and one perpendicular:

$$\begin{aligned} \partial_\mu \psi &= \left(\frac{u_\alpha \partial^\alpha \psi}{u_\nu u^\nu} \right) u_\mu + \xi_\mu \\ &= -y u_\mu + \xi_\mu. \end{aligned} \quad (38)$$

The component of $\partial_\mu \psi$ orthogonal to u^μ , ξ^μ , is just

$$\xi^\mu = \partial^\mu \psi + y u^\mu = (\eta^{\mu\nu} + u^\mu u^\nu) \partial_\nu \psi, \quad (39)$$

which is a space-like vector¹, $\xi^\mu = (0, \vec{\xi})$. Since $X = \partial_\mu \psi \partial^\mu \psi$, we can write

$$X = -y^2 + \xi^2, \quad (40)$$

where $\xi = |\vec{\xi}|$.

Computing the differential of (29), we find

$$\begin{aligned} dP &= F_X dX + F_y dy - b dF_b \\ &= s dT + n d\mu + F_X(dX + 2y dy) \\ &= s dT + n d\mu + 2F_X \xi d\xi, \end{aligned} \quad (41)$$

which defines the equation of state of the superfluid. Finally, from Equation (29), we can construct the Lagrangian

$$\begin{aligned} \mathcal{L} &= P + bF_b = P - Ts \\ &= P - T \frac{\partial P}{\partial T}. \end{aligned} \quad (42)$$

Analogously to what was performed for $\partial_\mu \psi$, we can also decompose the superfluid component's four-velocity, $\tilde{u}^\mu$, into two orthogonal vectors, u^μ and ζ^μ . The normalized four-velocity $\tilde{u}^\mu$ is given by

$$\tilde{u}^\mu = -\frac{\partial^\mu \psi}{\sqrt{-X}}, \quad (43)$$

and it can be decomposed as

$$\begin{aligned} \tilde{u}^\mu &= \left(\frac{u_\nu \tilde{u}^\nu}{u_\alpha u^\alpha} \right) u^\mu + \zeta^\mu \\ &= \frac{y}{\sqrt{-X}} u^\mu + \zeta^\mu, \end{aligned} \quad (44)$$

where ζ^μ is given by

$$\zeta^\mu = \tilde{u}^\mu - \frac{y}{\sqrt{-X}} u^\mu = -\frac{1}{\sqrt{-X}} \tilde{\zeta}^\mu. \quad (45)$$

In the normal component's frame, $u^\mu = (1, \vec{0})$ and $\tilde{\zeta}^\mu = (0, \tilde{\zeta}^i)$, so that

$$\tilde{u}^\mu = \frac{y}{\sqrt{-X}} \left(1, -\frac{\tilde{\zeta}^i}{y} \right) \equiv \gamma (1, \tilde{v}^i), \quad (46)$$

where $\tilde{v}^i$ is the velocity of the superfluid component, and $\gamma = (1 - \tilde{v}^2)^{-1/2}$ is the usual Lorentz gamma factor. When there is no relative velocity between the two components, $\tilde{\zeta}^\mu = 0$ and $y = \sqrt{-X}$.

3. Recovering the Nonrelativistic Limit for BEC DM

In this section, we recover the nonrelativistic description of superfluid DM from the relativistic approach described in the previous section. We consider the low- and high-temperature limits. We also recover the self-gravitating case that is used in cosmology by assuming a generic curved metric and then taking the Newtonian limit.

3.1. Low-Temperature Limit

Let us suppose that $T = 0$, such that the superfluid's normal component is absent and we can neglect the b and y dependence of F . In this case,

$$T_{\mu\nu} = -2F_X \partial_\mu \psi \partial_\nu \psi + F g_{\mu\nu}, \quad (47)$$

$$N^\mu = 2F_X \partial^\mu \psi, \quad (48)$$

and there is only the superfluid component frame. Note that

$$\tilde{u}^\mu = -\frac{\partial^\mu \psi}{\sqrt{-X}} \quad (49)$$

is a time-like, future-directed, and normalized four-vector that we can identify as the superfluid velocity. To check this, let us evaluate it in the homogeneous configuration $\psi = \mu t$:

$$\tilde{u}^\mu = -g^{\mu\nu} \frac{\partial_\nu \psi}{\sqrt{-X}} = -g^{\mu\nu} \delta_\nu^0 \frac{\mu}{\sqrt{\mu^2}} = -g^{\mu 0} \text{sign}(\mu), \quad (50)$$

and for Minkowski space, we have

$$\tilde{u}^\mu = \text{sign}(\mu)(1, 0, 0, 0). \quad (51)$$

So, if $\mu > 0$, the four-velocity is future-directed. And we know that, in the nonrelativistic limit, the chemical potential μ is dominated by the rest mass m of the superfluid particles, such that $\mu > 0$. This justifies the minus sign in the definition of $\tilde{u}^\mu$ above.

Then, we can write

$$T_{\mu\nu} = 2XF_X\tilde{u}_\mu\tilde{u}_\nu + Fg_{\mu\nu}, \quad (52a)$$

$$N^\mu = -2\sqrt{-X}F_X\tilde{u}^\mu, \quad (52b)$$

and the energy density, pressure, and number density of the superfluid are, respectively,

$$\rho = 2XF_X - F, \quad P = F, \quad n = -N^\mu\tilde{u}_\mu = -2\sqrt{-X}F_X. \quad (53)$$

To obtain the nonrelativistic limit, we need to isolate the rest mass contribution to ψ :

$$\psi = mt + \theta, \quad (54)$$

where $\dot{\theta} \ll m$. In this case (and for flat space),

$$\begin{aligned} X &= -(m + \dot{\theta})^2 + (\nabla\theta)^2 \\ &\approx -m^2 - 2mY, \end{aligned} \quad (55)$$

where we define

$$Y \equiv \dot{\theta} - \frac{1}{2m}(\nabla\theta)^2, \quad (56)$$

which is a typical variable for the Galilean effective approach to superfluids [35]. Note that, in the homogeneous case, $\dot{\theta}$ is the nonrelativistic chemical potential.

Moreover,

$$\begin{aligned} \tilde{u}^\mu &= \frac{(m + \dot{\theta}, -\partial^i\theta)}{\sqrt{(m + \dot{\theta})^2 - (\nabla\theta)^2}} = \frac{(1 + \dot{\theta}/m, -\partial^i\theta/m)}{\sqrt{(1 + \dot{\theta}/m)^2 - (\nabla\theta)^2/m^2}} \\ &\approx \left(1 - \frac{Y}{m}\right) \left(1 + \frac{\dot{\theta}}{m}, -\frac{\partial^i\theta}{m}\right), \end{aligned} \quad (57)$$

and in the nonrelativistic limit, the speed should be much smaller than unity, while the relativistic γ factor is approximately one. This is precisely the case when $\dot{\theta} \ll m$ and $|\partial^i\theta|/m \ll 1$, such that

$$\tilde{u}^\mu \approx \left(1, -\frac{\partial^i\theta}{m}\right) \equiv (1, \tilde{v}^i), \quad (58)$$

and Y can be written as

$$Y = \dot{\theta} - \frac{1}{2}m\tilde{v}^2. \quad (59)$$

So, $F(X)$ should be seen as a function of Y in the nonrelativistic limit:

$$F(X(Y)) \equiv \mathcal{P}(Y), \quad F_X = \frac{dY}{dX}\mathcal{P}_Y = -\frac{\mathcal{P}_Y}{2m}. \quad (60)$$

If the pressure $P = \mathcal{P}(Y)$ (see the second equation in (53)) is much smaller than the energy density, we have

$$\rho \approx 2XF_X \approx m\mathcal{P}_Y, \quad (61)$$

while the number density is

$$n \approx \mathcal{P}_Y. \quad (62)$$

Thus, we find that $\rho \approx mn$ in the nonrelativistic limit, as expected.

The equation of motion for ψ ,

$$\nabla_\mu (F_X \partial^\mu \psi) = -\nabla_\mu (\sqrt{-X} F_X \tilde{u}^\mu) = 0, \quad (63)$$

simplifies to

$$\nabla_\mu (\mathcal{P}_Y \tilde{u}^\mu) = 0, \quad (64)$$

where $\tilde{u}^\mu$ is given by (58). In flat space, this gives

$$\frac{\partial n}{\partial t} - \frac{1}{m} \nabla \cdot (n \nabla \theta) = 0, \quad (65)$$

which is just the continuity equation after multiplying by m and identifying $nm = \rho$ and $v^i = -\partial^i \theta / m$:

$$\frac{\partial \rho}{\partial t} + \nabla_i (\rho v^i) = 0. \quad (66)$$

Note that we could have also obtained this from the conservation of the energy–momentum tensor. This happens because, in the nonrelativistic limit, the conservation of mass and energy are related.

3.2. Self-Gravitating Superfluids

A self-gravitating BEC can be described after assuming a non-trivial metric in the relativistic theory. Einstein's equations may be sourced by the energy–momentum tensor (52a), which has a rather complicated dependence on the metric because it enters in the definition of X . However, for weak gravity in the nonrelativistic limit, the equations can be greatly simplified; this is the Newtonian limit of the theory. Consider the linearized metric

$$ds^2 = -(1 + 2\Phi(x^i))dt^2 + (1 - 2\Psi(x^i))\delta_{ij}dx^i dx^j, \quad (67)$$

where the Newtonian potential Φ and curvature scalar Ψ are much smaller than unity. If Einstein's equations are sourced by a perfect fluid, we find, after linearization,

$$\nabla^2 \Phi = 4\pi G(\rho + 3P), \quad (68a)$$

$$\nabla^2 \Psi = 4\pi G(\rho - P), \quad (68b)$$

and if $P \ll \rho$, we have $\Phi = \Psi$. Moreover, energy and momentum conservation gives, to leading order in v^i ,

$$\dot{\rho} + v^i \partial_i \rho + (\rho + P) \partial_i v^i = (\rho + P) v^i \partial_i (\Phi - 3\Psi), \quad (69a)$$

$$(\rho + P)(\dot{v}^i + v^j \partial_j v^i) + \partial^i P + v^i \dot{P} = -(\rho + P) \partial^i \Phi - 2\Psi \partial^i P, \quad (69b)$$

and after neglecting P , we find

$$\dot{\rho} + \partial_i (v^i \rho) = -2\rho v^i \partial_i \Phi, \quad (70a)$$

$$\rho(\dot{v}^i + v^j \partial_j v^i) + \partial^i P = -\rho \partial^i \Phi. \quad (70b)$$

For the superfluid case, we need to take into account the metric dependence of X in (52a). We have

$$\begin{aligned} X &= -(1 - 2\Phi)\dot{\psi}^2 + (1 + 2\Psi)\partial^i\psi\partial_i\psi \\ &\approx -m^2 - 2m\left(\dot{\theta} - \frac{1}{2m}(\nabla\theta)^2 - m\Phi\right) + 2\Psi(\nabla\theta)^2, \end{aligned} \quad (71)$$

while the four-velocity reads

$$\begin{aligned} \tilde{u}^\mu &= \frac{1}{\sqrt{-X}}((1 - 2\Phi)(m + \dot{\theta}), -(1 + 2\Psi)\partial^i\theta) \\ &\approx \left(1 - \Phi, -(1 + 2\Psi)\frac{\partial^i\theta}{m}\right), \end{aligned} \quad (72)$$

and to leading order in the metric perturbations these are the same as in the flat case. The first modification on X is the shift $Y \rightarrow Y - m\Phi$. Using $v^2 \approx (\nabla\theta)^2/m^2 \ll 1$, we have

$$X \approx -m^2 - 2m\left(\dot{\theta} - \frac{m}{2}v^2 - m\Phi\right). \quad (73)$$

Thus, to leading order, we might assume X and $\tilde{u}^\mu$ to be the same as in the flat case when taking the Newtonian limit of the theory.

However, since the metric is not flat, the equation of motion for θ will now give

$$\nabla_\mu(\mathcal{P}_Y\tilde{u}^\mu) = \partial_\mu(\mathcal{P}_Y\tilde{u}^\mu) + \Gamma_{\mu\rho}^\mu\mathcal{P}_Y\tilde{u}^\rho = 0, \quad (74)$$

where the non-vanishing independent Christoffel symbols for the metric (67) are

$$\begin{aligned} \Gamma_{i0}^0 &= \partial_i\Phi, & \Gamma_{00}^j &= \partial^j\Phi, \\ \Gamma_{kl}^j &= \eta_{kl}\partial^j\Psi - 2\delta^j_{(k}\partial_{l)}\Psi, \end{aligned} \quad (75)$$

such that $\Gamma_{\mu 0}^\mu = 0$ and $\Gamma_{\mu l}^\mu = \partial_l(\Phi - 3\Psi)$. So, collecting the leading order terms of $\tilde{u}^\mu$, we find

$$\begin{aligned} \partial_\mu(\mathcal{P}_Y\tilde{u}^\mu) + \mathcal{P}_Y\Gamma_{\mu\rho}^\mu\tilde{u}^\rho &\approx \partial_0\mathcal{P}_Y - \frac{1}{m}\partial_i(\mathcal{P}_Y\partial^i\theta) \\ &\quad - \mathcal{P}_Y\frac{1}{m}\partial^i\theta\partial_i(\Phi - 3\Psi) = 0, \end{aligned} \quad (76)$$

which is just the statement of energy conservation (70a), after setting $\Phi = \Psi$. Momentum conservation (70b) implies

$$m\mathcal{P}_Y\left(-\frac{\partial^i\dot{\theta}}{m} + \frac{\partial^j\theta}{m^2}\partial_j\partial^i\theta\right) \approx -\partial^i\mathcal{P} - m\mathcal{P}_Y\partial^i\Phi. \quad (77)$$

Note that, in Equations (76) and (77), Y is given by (59). To leading order, the condition for equilibrium, i.e., $\partial^i\theta = 0$ such that $Y = \dot{\theta} = \mu_{\text{n.r.}}$, is

$$\partial^i\mathcal{P} = -m\mathcal{P}_Y\partial^i\Phi \implies \partial^iY = -m\partial^i\Phi. \quad (78)$$

For a spherical configuration, we can integrate the Poisson equation to find

$$\partial_r\Phi = \frac{4\pi GM(r)}{r^2}, \quad (79)$$

$$M(r) = \int_0^r d\tilde{r} \tilde{r}^2 \rho(\tilde{r}) = \int_0^r d\tilde{r} \tilde{r}^2 m\mathcal{P}_Y(Y(\tilde{r})), \quad (80)$$

such that

$$\nabla^2 Y = -m \nabla^2 \Phi = -4\pi G m \rho \implies \partial_r Y = -\frac{4\pi G m M(r)}{r^2}, \quad (81)$$

which can be solved once an equation of state $P = \mathcal{P}(Y)$ is given.

In superfluid DM, one typically assumes that the baryons self-interact via phonon exchange. This plays a major role in explaining some correlations between the acceleration in galaxies and their baryonic mass, which can also be explained by MOND. A typical equation of state that can mimic the required MOND to fit some galactic rotation curves is [16]

$$\mathcal{P}(Y) = \frac{2}{3} \Lambda (2m)^{3/2} Y \sqrt{|Y|}, \quad (82)$$

where Λ is the energy scale associated with the baryon–photon coupling term. For a homogeneous BEC,

$$\mathcal{P} = \frac{2}{3} \Lambda (2m\mu)^{3/2} = \frac{\rho^3}{12\Lambda^2 m^6}. \quad (83)$$

Including gravity, we can estimate the size of a spherical condensed region. The Poisson equation above gives

$$\partial_r(r^2 \partial_r \mu) = -4\pi G m [m \Lambda (2m)^{3/2}] r^2 \mu^{1/2}, \quad (84)$$

which is essentially a Lane–Emden equation. The following numerical solution with the boundary conditions $\rho(0) = \rho_{\text{core}}$ and $\partial_r \rho(0) = 0$ was found in [16]:

$$\rho_{\text{core}} \approx 5.5 \frac{3M_{\text{core}}}{4\pi R_{\text{core}}^3}, \quad (85)$$

$$R_{\text{core}} \approx \left(\frac{M_{\text{core}}}{10^{11} M_{\odot}} \right)^{1/5} \left(\frac{m}{\text{eV}} \right)^{-6/5} \left(\frac{\Lambda}{\text{meV}} \right)^{-2/5} 45 \text{ kpc}. \quad (86)$$

Thus, we can have the spherical condensate extending throughout realistic galactic cores for $\Lambda \sim \text{meV}$ and $m \sim \text{eV}$.

Let us now estimate the sound speed on these spherical condensates. We have

$$c_s^2 = \frac{\partial P}{\partial \rho} \approx \frac{\rho^2}{4\Lambda^2 m^6}, \quad (87)$$

and evaluating at the core density with $\Lambda \sim \text{meV}$ and $m \sim \text{eV}$, we find

$$c_s \sim 1.6 \times 10^3 \text{ m/s} \sim 10^{-5} c. \quad (88)$$

We conclude that the typical sound speeds in DM superfluid condensates are nonrelativistic, as expected.

3.3. High-Temperature Limit

In this section, we shall consider temperatures higher or slightly higher than the critical value, $T \gtrsim T_c$, such that the superfluid component disappears and we are left with a perfect fluid. The dependence of F on X can be neglected, i.e., $F = F(b, y)$, and the energy–momentum tensor becomes²

$$T_{\mu\nu} = (yF_y - bF_b) u_{\mu} u_{\nu} + (F - bF_b) g_{\mu\nu}, \quad (89)$$

while the conserved current N^μ reduces to

$$N^\mu = \frac{\partial F}{\partial(\partial_\mu \psi)} = F_y u^\mu. \quad (90)$$

To obtain the thermodynamic variables, we first contract $T_{\mu\nu}$ with the projector $(g^{\mu\nu} + u^\mu u^\nu)$, which gives

$$P = \frac{1}{3}(g^{\mu\nu} + u^\mu u^\nu)T_{\mu\nu} = F - bF_b. \quad (91)$$

The energy density is obtained by contracting $T^{\mu\nu}$ with the four-velocity,

$$\rho = T_{\mu\nu} u^\mu u^\nu = yF_y - F, \quad (92)$$

and the particle number density comes from the current N^μ ,

$$n = -N^\mu u_\mu = F_y. \quad (93)$$

The remaining thermodynamic variables are the same as in the general case.

With these definitions, the energy–momentum tensor defined in (89) can be rewritten as

$$T^{\mu\nu} = (P + \rho)u^\mu u^\nu + g^{\mu\nu}P. \quad (94)$$

Now, we want to recover the nonrelativistic limit for $T \gtrsim T_c$. Similarly to what was performed for the low-temperature case, we write the comoving coordinates of the normal component as $\phi^i = \alpha x^i + \zeta^i$, where x^i are the physical coordinates, $\partial_i \zeta^i \ll \alpha$, and α is a constant whose physical meaning will become evident in the following discussion. In terms of x^i and ζ^i , J^μ becomes

$$\begin{aligned} J^\mu &= \frac{1}{6}\epsilon^{\mu\sigma\beta\gamma}\epsilon_{ijk}(\alpha\partial_\sigma x^i + \partial_\sigma \zeta^i)(\alpha\partial_\beta x^j + \partial_\beta \zeta^j)(\alpha\partial_\gamma x^k + \partial_\gamma \zeta^k) \\ &= -\alpha^3 g^{\mu 0} + \frac{1}{2}\alpha^2 \epsilon^{\mu i j \gamma}\epsilon_{ijk}\partial_\gamma \zeta^k + \frac{1}{2}\alpha \epsilon^{\mu i \beta \gamma}\epsilon_{ijk}\partial_\beta \zeta^j \partial_\gamma \zeta^k + \mathcal{O}(\zeta^3), \end{aligned} \quad (95)$$

and b becomes

$$\begin{aligned} b &= \sqrt{-J^\mu J_\mu} \approx \alpha^3 \left[1 + \frac{\partial_i \zeta^i}{\alpha} - \frac{1}{2} \left(\frac{\zeta^i}{\alpha} \right)^2 \right] \\ &\approx \alpha^3 + \alpha^2 \left[\partial_i \zeta^i - \frac{1}{2\alpha} \left(\zeta^i \right)^2 \right] \equiv \alpha^3 + \alpha^2 Z, \end{aligned} \quad (96)$$

where we defined the new variable

$$Z = \partial_i \zeta^i - \frac{(\zeta^i)^2}{2\alpha}. \quad (97)$$

We can rewrite the four-velocity of the normal component as $u^\mu = -\frac{J^\mu}{b}$, so that

$$u^0 \approx 1 - \frac{1}{2} \left(\frac{\zeta^i}{\alpha} \right)^2 - \frac{1}{2} \left(\frac{\zeta^i}{\alpha} \right)^2 \frac{\partial_j \zeta^j}{\alpha}, \quad (98)$$

$$u^i \approx \frac{\zeta^i}{\alpha} - \frac{\partial_j \zeta^j}{\alpha} \frac{\zeta^i}{\alpha} + \frac{1}{2} \frac{\zeta^i}{\alpha} \left(\frac{\zeta^j}{\alpha} \right)^2 + \frac{1}{2} \frac{\zeta^i}{\alpha} \left(\frac{\zeta^j}{\alpha} \right)^2 \frac{\partial_k \zeta^k}{\alpha} - \frac{1}{2} \frac{\partial_j \zeta^j}{\alpha} \frac{\zeta^i}{\alpha} \left(\frac{\zeta^k}{\alpha} \right)^2, \quad (99)$$

up to first order in $\partial_i \zeta^i$. Now, we know that in the nonrelativistic limit the four-velocity should reduce to $u^\mu \approx (1, v^i)$, with $v^i \ll 1$; this can be achieved by taking $\zeta^i \ll \alpha$, for which

$$u^\mu \approx \left(1, \frac{\zeta^i}{\alpha}\right) \equiv (1, v^i), \quad (100)$$

where $v^i = \zeta^i / \alpha$. Similarly to (59), we can rewrite Z in terms of v^i :

$$Z = \partial_i \zeta^i - \frac{1}{2} \alpha v^2. \quad (101)$$

In this limit, $b \approx -\alpha^3$, and comparing it to Equation (37), we see that α and the entropy are related by $s = \alpha^3$.

Finally, y is given by

$$\begin{aligned} y &= u^\mu \partial_\mu \psi = u^0 \partial_0 \psi + u^i \partial_i \psi \approx m + \dot{\theta} + \frac{\zeta^i}{\alpha} \partial_i \theta \\ &= m + \dot{\theta} - m v^i \tilde{v}_i \approx m, \end{aligned} \quad (102)$$

where we have once again used $\psi = mt + \theta$.

In the nonrelativistic limit, the pressure $P = F - bF_b$ is much smaller than the energy density $\rho = yF_y - F$; this can be achieved by taking $bF_b \ll yF_y$ and $F \ll yF_y$, so that

$$\rho \approx yF_y \approx mn, \quad (103)$$

as expected.

Hence, in this regime, $F(b, y)$ should be viewed as a function of Z , $F(b(Z), y) \equiv Q(Z, y)$, such that

$$F_b = \frac{dZ}{db} Q_Z = \frac{Q_Z}{\alpha^2}, \quad F_y = Q_y, \quad (104)$$

and the thermodynamic variables become

$$\rho \approx mQ_y - Q \approx mQ_y, \quad P \approx Q - \alpha Q_Z \approx 0, \quad (105)$$

$$n \approx Q_y, \quad N^\mu \approx Q_y u^\mu. \quad (106)$$

The conservation of the energy–momentum tensor in flat spacetime leads to

$$\begin{aligned} 0 &= \partial_\mu T^{\mu\nu} = \partial_\mu [(yF_y - bF_b) u^\mu u^\nu] + g^{\mu\nu} \partial_\mu (F - bF_b) \\ &= \partial_\mu [(\rho + P) u^\mu u^\nu] + \partial^\nu P. \end{aligned} \quad (107)$$

For $\nu = 0$, we have

$$\begin{aligned} 0 &= \partial_\mu [(\rho + P) u^\mu u^0] + \partial^0 P = \partial_\mu [(\rho + P) u^\mu] - \partial_0 P \\ &= \partial_0 [(\rho + P) u^0] + \partial_i [(\rho + P) v^i] - \partial_0 P = \partial_0 (\rho + P) + \partial_i [(\rho + P) v^i] - \partial_0 P \\ &= \partial_0 \rho + \partial_i [(\rho + P) v^i] = \frac{\partial \rho}{\partial t} + \partial_i [(\rho + P) v^i], \end{aligned} \quad (108)$$

and for $\nu = i$,

$$\begin{aligned} 0 &= \partial_\mu [(\rho + P) u^\mu u^i] + \partial^i P = \partial_0 [(\rho + P) u^0 v^i] + \partial_j [(\rho + P) v^i v^j] + g^{ij} \partial_j P \\ &= \partial_0 [(\rho + P) v^i] + \partial_j [(\rho + P) v^i v^j + g^{ij} P]. \end{aligned} \quad (109)$$

Considering $P \ll \rho$, we obtain

$$\frac{\partial \rho}{\partial t} + \partial_i (\rho v^i) \equiv \frac{\partial \rho}{\partial t} + \partial_i j^i = 0, \quad (110)$$

$$\frac{\partial}{\partial t} (\rho v^i) + \frac{\partial}{\partial x^k} (\rho v^i v^k + g^{ik} P) \equiv \frac{\partial j^i}{\partial t} + \frac{\partial \Pi^{ik}}{\partial x^k} = 0, \quad (111)$$

i.e., the continuity and momentum conservation equations.

To obtain the equation of entropy conservation, we replace the energy density ρ in (94) by the thermodynamic identity $\rho = Ts + \mu n - P$,

$$T^{\mu\nu} = (Ts + \mu n)u^\mu u^\nu + g^{\mu\nu} P, \quad (112)$$

and requiring its conservation in flat spacetime, we find

$$\begin{aligned} 0 &= \partial_\mu T^{\mu\nu} = \partial_\mu (Ts u^\mu + \mu n u^\mu) u^\nu + (Ts + \mu n) u^\mu \partial_\mu u^\nu + \partial^\nu P \\ &= [(s \partial_\mu T + n \partial_\mu \mu) u^\mu + T \partial_\mu (s u^\mu) + \mu \partial_\mu (n u^\mu)] u^\nu + (Ts + \mu n) u^\mu \partial_\mu u^\nu + \partial^\nu P, \end{aligned} \quad (113)$$

and using the Gibbs–Duhem relation, $s \partial_\mu T + n \partial_\mu \mu = \partial_\mu P$, together with particle number conservation, $\partial_\mu N^\mu = \partial_\mu (n u^\mu) = 0$, we find

$$0 = [u^\mu \partial_\mu P + T \partial_\mu (s u^\mu)] u^\nu + (Ts + \mu n) u^\mu \partial_\mu u^\nu + \partial^\nu P. \quad (114)$$

Contracting this expression with u_ν yields

$$\begin{aligned} 0 &= [u^\mu \partial_\mu P + T \partial_\mu (s u^\mu)] u^\nu u_\nu + (Ts + \mu n) u_\nu u^\mu \partial_\mu u^\nu + u_\nu \partial^\nu P \\ &= -u^\mu \partial_\mu P - T \partial_\mu (s u^\mu) + (Ts + \mu n) u_\nu u^\mu \partial_\mu u^\nu + u_\nu \partial^\nu P \\ &= -T \partial_\mu (s u^\mu) \\ \therefore \partial_\mu (s u^\mu) &= 0, \end{aligned} \quad (115)$$

where we have used $u^\nu u_\nu = -1 \implies u_\nu \partial_\mu u^\nu = 0$.

4. Recovering the Nonrelativistic Dynamics for $T < T_c$

In order to take the nonrelativistic limit of the theory at any temperature below T_c , we start by assuming that, in this limit, the Lagrangian F can be decomposed as

$$F(X, y, b) \rightarrow F_S(X) + F_N(y, b), \quad (116)$$

where the subscripts S and N denote the superfluid and normal parts of F , respectively. This ansatz is motivated by the fact that, to leading order, X is proportional to m^2 , which is just a parameter. As we shall see, this ansatz yields a nonrelativistic limit that matches the two-fluid model. With this assumption, the derivatives of F with respect to the variables y , X and b become, respectively,

$$F_y = (F_N)_{y'}, \quad F_X = (F_S)_X, \quad F_b = (F_N)_b, \quad (117)$$

where we make use of the fact that the normal fluid depends on the variables y and b , while the superfluid only depends on X .

Since the energy–momentum tensor is linear in the Lagrangian and its derivatives, all the hydro- and thermodynamic variables will also decompose linearly into superfluid and normal contributions. The thermodynamic variables can be written as

$$\begin{aligned} N^\mu &= -2\sqrt{-X}F_X(X, y, b)\tilde{u}^\mu + F_y(X, y, b)u^\mu \\ &= -2\sqrt{-X}(F_S)_X\tilde{u}^\mu + (F_N)_y u^\mu, \end{aligned} \quad (118)$$

$$\begin{aligned} P &= F(X, y, b) - bF_b(X, y, b) \\ &= F_N + F_S - b(F_N)_b, \end{aligned} \quad (119)$$

$$\begin{aligned} \rho &= yF_y(X, y, b) - 2y^2F_X(X, y, b) - F(X, y, b) \\ &= y(F_N)_y - 2y^2(F_S)_X - F_N - F_S, \end{aligned} \quad (120)$$

$$\begin{aligned} n &= F_y(X, y, b) - 2yF_X(X, y, b) \\ &= (F_N)_y - 2y(F_S)_X \end{aligned} \quad (121)$$

for the system between the $T = 0$ and $T = T_c$. From these equations, we can identify the superfluid and normal variables as follows:

$$N_N^\mu = (F_N)_y u^\mu, \quad N_S^\mu = -2\sqrt{-X}(F_S)_X\tilde{u}^\mu, \quad (122)$$

$$P_N = F_N - b(F_N)_b, \quad P_S = F_S, \quad (123)$$

$$\rho_N = y(F_N)_y - F_N, \quad \rho_S = -2y^2(F_S)_X - F_S, \quad (124)$$

$$n_N = (F_N)_y, \quad n_S = -2y(F_S)_X. \quad (125)$$

As expected, the normal component's thermodynamic variables are identical to those computed in Section 3.3 in the high-temperature limit, (90)–(93); the remaining variables, however, do not match completely the results from Section 3.1, due to the fact that, at $T = 0$, the dependence of F on y and b disappears. As a consistency check, we can recover the low-temperature variables (53) by replacing $y^2 \rightarrow -X$, so that

$$\rho_S = 2X(F_S)_X - F_S, \quad n_S = -2\sqrt{-X}(F_S)_X. \quad (126)$$

As was performed in the previous sections, in the nonrelativistic limit, we can write X and b in terms of Y and Z ,

$$X = -m^2 - 2mY, \quad b = \alpha^3 + \alpha^2 Z, \quad (127)$$

and since $X = -y^2 + \zeta^2$ and $\zeta^2 = y^2(v^i - \tilde{v}^i)^2 \equiv y^2(w^i)^2$, we can write y as

$$\begin{aligned} y^2 &= -\frac{X}{1 - w^2} = \frac{m^2 + 2mY}{1 - w^2} = m^2 \left(\frac{1 + 2Y/m}{1 - w^2} \right) \\ y &\approx Y + m \left(1 + \frac{w^2}{2} \right). \end{aligned} \quad (128)$$

Thus, we can write the derivatives of F as

$$\begin{aligned} F_X &= \frac{\partial F}{\partial Y} \frac{dY}{dX} = -\frac{F_Y}{2m}, & F_b &= \frac{\partial F}{\partial Z} \frac{dZ}{db} = \frac{F_Z}{\alpha^2}, \\ F_y &= \frac{\partial F}{\partial Y} \frac{dY}{dy} + \frac{\partial F}{\partial w} \frac{dw}{dy} = F_Y + \frac{F_w}{mw}, \end{aligned} \quad (129)$$

so that

$$N_N^\mu \approx \left[(F_N)_Y + \frac{(F_N)_w}{mw} \right] u^\mu, \quad N_S^\mu \approx \left(1 + \frac{Y}{m} \right) (F_S)_Y \tilde{u}^\mu, \quad (130)$$

$$P_N \approx F_N - \alpha \left(1 + \frac{Z}{\alpha} \right) (F_N)_Z, \quad P_S = F_S, \quad (131)$$

$$\rho_N \approx \left(1 + \frac{w^2}{2} + \frac{Y}{m} \right) \left[m(F_N)_Y + \frac{(F_N)_w}{w} \right] - F_N, \quad \rho_S \approx m \left(1 + w^2 + \frac{2Y}{m} \right) (F_S)_Y - F_S, \quad (132)$$

$$n_N \approx (F_N)_Y + \frac{(F_N)_w}{mw}, \quad n_S \approx \left(1 + \frac{w^2}{2} + \frac{Y}{m} \right) (F_S)_Y. \quad (133)$$

Taking $Y/m \ll 1$, $Z/\alpha \ll 1$ and $w \ll 1$, as well as $P \ll \rho$, we obtain

$$N_N^\mu \approx \left[(F_N)_Y + \frac{(F_N)_w}{mw} \right] u^\mu, \quad N_S^\mu \approx (F_S)_Y \tilde{u}^\mu, \quad (134)$$

$$P_N \approx 0, \quad P_S \approx 0, \quad (135)$$

$$\rho_S \approx m(F_S)_Y, \quad \rho_N \approx m(F_N)_Y + \frac{(F_N)_w}{w}, \quad (136)$$

$$n_S \approx (F_S)_Y, \quad n_N \approx (F_N)_Y + \frac{(F_N)_w}{mw}, \quad (137)$$

and we can see that $\rho_S \approx mn_S$ and $\rho_N \approx mn_N$.

With these expressions in hand, we can regroup the terms of $T^{\mu\nu}$ as

$$\begin{aligned} T^{\mu\nu} &= 2X(F_S)_X \tilde{u}^\mu \tilde{u}^\nu + \left[y(F_N)_y - b(F_N)_b \right] u^\mu u^\nu + [F_N + F_S - b(F_N)_b] g^{\mu\nu} \\ &= [(\rho_S + P_S) \tilde{u}^\mu \tilde{u}^\nu + P_S g^{\mu\nu}] + [(\rho_N + P_N) u^\mu u^\nu + P_N g^{\mu\nu}] \\ &\equiv T_S^{\mu\nu} + T_N^{\mu\nu}, \end{aligned} \quad (138)$$

where the linearity of the energy-momentum tensor in F and its derivatives make it so that it can be easily decomposed into normal and superfluid parts.

We now consider the nonrelativistic dynamics of our theory, in which all the particles' velocities are small compared to that of light, and try to recover the fluid Equations (5), (6), (8), and (9). These equations are written in terms of the mass density, ρ_m , the velocities of the normal and superfluid components, $\vec{v}_n$ and $\vec{v}_s$, and the energy density ϵ . The latter does not include the contribution from the rest mass energy. Before delving into the dynamical equations, let us first discuss how ϵ is recovered in the nonrelativistic limit. In our notation, ρ stands for the relativistic energy density, which includes contributions of the rest mass and kinetic energies. In order to recover the fluid equations, we need to split ρ into a mass density term, ρ_m , and an energy density ϵ .

As an example, consider a dust-like gas of massive particles, such that

$$\rho = \gamma m_0 n, \quad (139)$$

where $\gamma = (1 - v^2)^{-1/2}$ is the usual Lorentz gamma factor, m_0 denotes the rest mass of the particle, and n is the particle number density. Starting from the relativistic momentum density,

$$\vec{j} = \gamma m_0 n \vec{v} = \rho \vec{v}, \quad (140)$$

we can obtain an expression for v^2 ,

$$v^2 = \frac{j^2}{m_0^2 n^2 + j^2}, \quad (141)$$

which can then be replaced in the expression for γ , leading to

$$\gamma = \frac{1}{\sqrt{1 - \frac{j^2}{m_0^2 n^2 + j^2}}} = \sqrt{1 + \left(\frac{j}{m_0 n}\right)^2}. \quad (142)$$

Plugging γ back into the definition of ρ , we obtain

$$\rho = m_0 n \sqrt{1 + \left(\frac{j}{m_0 n}\right)^2}. \quad (143)$$

In the nonrelativistic limit, the four-velocities of the normal and superfluid become, respectively, $u^\mu \sim (1, v^i)$ and $\tilde{u}^\mu \sim (1, \tilde{v}^i)$, with $v, \tilde{v} \ll 1$, and we can expand our quantities in powers of v and $\tilde{v}$. The Lorentz factor can be expanded as

$$\gamma \sim 1 + \frac{1}{2}v^2 + \mathcal{O}(v^4), \quad (144)$$

so that, at leading order, we have

$$\rho \approx m_0 n + \frac{1}{2} \frac{j^2}{m_0 n} \approx m_0 n + \frac{m_0 n v^2}{2} = \rho_m + \epsilon. \quad (145)$$

As can be seen from this expression, the energy density ϵ is of the order v^2 . Moreover, the pressure is of the same order of magnitude as ϵ . This can be demonstrated using the case of an ideal gas: the internal energy density is given by $\epsilon = n c_v k_B T$, where $c_v \sim 1$ is the specific heat capacity at constant volume, while the pressure is $P = n k_B T$. From this example, we see that $\epsilon = \rho - \rho_m$, and this will also be true in general. For more complex fluids, ϵ will also include contributions from the internal energy of its constituents.

Now that we know how to identify ρ_m and ϵ in our expressions, Landau's two-fluid equations can be recovered.

As shown, the particle number four-current can be decomposed into a normal and a superfluid part, $N^\mu = N_N^\mu + N_S^\mu$, so that its conservation leads to

$$\begin{aligned} 0 &= \partial_\mu N^\mu = \partial_\mu (N_N^\mu + N_S^\mu) = \partial_\mu (n_N u^\mu + n_S \tilde{u}^\mu) \\ &= \partial_0 (n_N + n_S) + \partial_i (n_N v^i + n_S \tilde{v}^i) = \partial_0 n + \partial_i (N_N^i + N_S^i) \\ &= \partial_0 n + \partial_i N^i, \end{aligned} \quad (146)$$

where $n = n_N + n_S$ is the total particle number density. Multiplying the expression by m , and using $\rho_m = m n$, we recover the continuity equation,

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \quad (147)$$

To obtain the equation of momentum conservation, we start from the conservation of the energy–momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, and take $\nu = j$:

$$\begin{aligned} 0 &= \partial_\mu T^{\mu j} = \partial_\mu (T_S^{\mu j} + T_N^{\mu j}) = \partial_\mu [(\rho_S + P_S)\tilde{u}^\mu \tilde{u}^j + P_S g^{\mu j} + (\rho_N + P_N)u^\mu u^j + P_N g^{\mu j}] \\ &\equiv \partial_\mu [(\rho_S + P_S)\tilde{u}^\mu \tilde{v}^j + (\rho_N + P_N)u^\mu v^j + P g^{\mu j}] \\ &= \partial_0 [(\rho_S + P_S)\tilde{v}^j + (\rho_N + P_N)v^j] + \partial_i [(\rho_S + P_S)\tilde{v}^i \tilde{v}^j + (\rho_N + P_N)v^i v^j + P g^{ij}], \end{aligned}$$

where $P = P_S + P_N$ is the total pressure. Replacing the relativistic energy density by $\rho = \rho_m + \epsilon$, we obtain

$$\begin{aligned} 0 &= \partial_i [(\rho_{S,m} + \epsilon_S + P_S)\tilde{v}^i \tilde{v}^j + (\rho_{N,m} + \epsilon_N + P_N)v^i v^j + P g^{ij}] \\ &\quad + \partial_0 [(\rho_{S,m} + \epsilon_S + P_S)\tilde{v}^j + (\rho_{N,m} + \epsilon_N + P_N)v^j] \\ &= \partial_0 (\rho_{S,m} \tilde{v}^j + \rho_{N,m} v^j) + \partial_i (\rho_{S,m} \tilde{v}^i \tilde{v}^j + \rho_{N,m} v^i v^j + P g^{ij}) + \mathcal{O}(v^3). \end{aligned} \quad (148)$$

Keeping only terms up to second order in v , we recover the momentum conservation equation,

$$\begin{aligned} 0 &= \partial_0 (\rho_{S,m} \tilde{v}^k + \rho_{N,m} v^k) + \partial_i (\rho_{S,m} \tilde{v}^i \tilde{v}^k + \rho_{N,m} v^i v^k + P g^{ik}) \\ &= \frac{\partial j^k}{\partial t} + \frac{\partial \Pi^{ik}}{\partial x^i}, \end{aligned} \quad (149)$$

where $\vec{j}$ is the mass flux density, defined in Equation (4), and Π^{ij} is the momentum flux density tensor, defined in Equation (7).

Taking, instead, $\nu = 0$ in $\partial_\mu T^{\mu\nu} = 0$, we find

$$\begin{aligned} 0 &= \partial_\mu T^{\mu 0} = \partial_\mu (T_S^{\mu 0} + T_N^{\mu 0}) = \partial_\mu [(\rho_S + P_S)\tilde{u}^\mu + (\rho_N + P_N)u^\mu + P g^{\mu 0}] \\ &= \partial_0 (\rho_S + P_S + \rho_N + P_N - P) + \partial_i [(\rho_S + P_S)\tilde{v}^i + (\rho_N + P_N)v^i] \\ &= \partial_0 \rho + \partial_i [(\rho_S + P_S)\tilde{v}^i + (\rho_N + P_N)v^i] \\ &= \partial_0 (\rho_m + \epsilon) + \partial_i [(\rho_{S,m} + \epsilon_S + P_S)\tilde{v}^i + (\rho_{N,m} + \epsilon_N + P_N)v^i], \end{aligned} \quad (150)$$

and the leading order contribution gives

$$0 = \partial_0 \rho_m + \partial_i (\rho_{S,m} \tilde{v}^i + \rho_{N,m} v^i) = \frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot \vec{j},$$

i.e., we recover the continuity equation.

The next-order contribution gives

$$\begin{aligned} 0 &= \partial_0 \epsilon + \partial_i [(\epsilon_S + P_S)\tilde{v}^i + (\epsilon_N + P_N)v^i] \\ &= \frac{\partial \epsilon}{\partial t} + \partial_i [Q_S^i + Q_N^i] = \frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{Q}, \end{aligned} \quad (151)$$

which is the energy conservation equation, and $\vec{Q}$ the energy flux density.

Lastly, the equation of conservation of entropy is an immediate consequence of the conservation of the four-current J^μ , which we earlier identified with the entropy four-current:

$$0 = \partial_\mu J^\mu = \partial_\mu (s u^\mu) = \frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s \vec{v}). \quad (152)$$

5. Conclusions

In this paper, we revisited two distinct frameworks for describing superfluidity: Landau's phenomenological two-fluid model and a fully relativistic effective field theory. We began by reviewing the two-fluid model, outlining the fundamental equations governing superfluid dynamics, followed by an overview of the key equations and variables from the effective field theory proposed in [58]. Our objective was to recover Landau's equations governing superfluid dynamics from the nonrelativistic limit of the effective field theory, thereby linking the general relativistic description with the nonrelativistic model commonly employed in cosmology.

We examined the nonrelativistic limit of the field theory, first in the low-temperature regime, followed by the high-temperature case. For the low-temperature limit, we also analyzed the behaviour of a self-gravitating superfluid. Next, we showed how to recover the nonrelativistic limit of the full theory, starting from the assumption that the Lagrangian can be decomposed into a superfluid and normal contributions in this limit. From there, we demonstrated that, due to the energy-momentum tensor being linear in the Lagrangian and its derivatives, all thermodynamic variables can be linearly decomposed into contributions from the superfluid and normal components. This allowed us to recover both the low- and high-temperature limit variables from the full theory ones. Finally, we showed that, given our chosen ansatz, Landau's equations are straightforwardly recovered.

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Abbreviations

The following abbreviations are used in this manuscript:

DM	Dark Matter
Λ CDM	Lambda-Cold Dark Matter
BEC	Bose–Einstein Condensate
MOND	MOdified Newtonian Dynamics
UV	Ultra-Violet

Notes

1. Since we are in a frame that is comoving with the normal component, we must have $u^\mu = (1, 0, 0, 0)$. Thus, because $\xi^\mu u_\mu = 0$, we know that ξ^μ must be space-like.
2. The y -dependence cannot be neglected when the particle number is conserved. In the absence of a superfluid component, the symmetry (14) is enhanced to $\psi \rightarrow \psi + f(\phi^i)$, with f being an arbitrary function, and this precludes the X -dependence of F [65]. However, for systems in which the particle number is not conserved (e.g., a gas of photons), we would have $F(b, y) = F(b)$.

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