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The Impact of the Eccentricity on the Collapse of an Ellipsoid into a Black Hole

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Abstract: We consider the gravitational collapse of a homogeneous pressureless ellipsoid. We have shown that the minimal size r that the ellipsoid can reach during collapse depends on its initial eccentricity e_0 as $r \propto e_0^{\nu}$, where $\nu \approx 15/8$, and this dependence is very universal. We have estimated the parameters (in particular, the initial eccentricity) of a homogeneous pressureless ellipsoid, whereat it collapses directly into a black hole. Our consideration is purely Newtonian, but we present a discussion suggesting that the results obtained within the framework of general relativity are unlikely to be significantly different.

Keywords: cosmology; large-scale structure of the universe; cold dark matter; primordial black holes

1. Introduction

The task of the gravitational collapse of an ellipsoid has a long story [1–4]. In this paper we consider the collapse of a homogeneous pressureless non-rotating ellipsoid in the Newtonian approximation. It may seem strange that we consider the ellipsoid to be completely non-rotating. However, this assumption is almost always true for cosmological objects, because cosmological perturbations with non-zero angular momentum have only the damping mode [5].

The problem has already been considered by many authors. For instance, ref. [1] considered the gravitational collapse of a uniform spheroid, and [2] — the collapse of a uniform ellipsoid in the expanding matter-dominated Universe. The interest is quite natural: the problem has numerous astrophysical applications.

Let us qualitatively consider the collapse of a homogeneous dusty ellipsoid with semi-axes a_1, a_2, a_3 without initial velocities of the particles. A homogeneous ellipsoid will remain a homogeneous ellipsoid during the collapse [2]. Since the gravitational attraction is the strongest along the shortest axis (for example, the z-axis), the ellipsoid eccentricity grows with time during the collapse. Finally, the ellipsoid transforms into a flat elliptical pancake perpendicular to the z axis.

The case of a homogeneous sphere ($a_1 = a_2 = a_3$) is exceptional. Then the system collapses into a black hole (this conclusion remains true even in the general theory of relativity and even in the presence of pressure, provided that the pressure is not sufficient to stop the collapse [6]). Apparently, if an ellipsoid only slightly differs from a sphere (its eccentricity is very small), it should form a black hole as well. The aim of this work is to estimate the maximum initial eccentricity at which the ellipsoid still collapses into a black hole.

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Generally speaking, the shape of an ellipsoid depends on two ratios (for instance, a_1/a_2 and a_1/a_3), but we can avoid the necessity to work with the two-dimensional parameter space by the following trick. Generally speaking, $a_1 \neq a_2 \neq a_3$, that is, there are the shortest and longest axes, while the third has an intermediate value. Thus, we may consider only the two extreme cases: the oblate spheroid[1] ($a_1 = a_2 > a_3$, the 'pumpkin' case), which eccentricity e can be defined as

$$e \equiv \sqrt{1 - \frac{a_3^2}{a_1^2}}, \tag{1}$$

and the prolate spheroid ($a_1 = a_2 < a_3$, the 'melon' case), which eccentricity e can be defined as

$$e \equiv \sqrt{1 - \frac{a_1^2}{a_3^2}}. \tag{2}$$

The spheroid shape in both cases may be characterized by a single parameter, e.

The structure of the paper is the following: we derive the equations describing the system in Section 2, we solve them and discuss the black hole formation in Section 3, and in Section Discussion we discuss the obtained results.

2. Calculations

So we consider the gravitational collapse of homogeneous pressureless non-rotating spheroids. We use the Newtonian gravity approximation and assume that the initial velocities of all the spheroid particles are zero.

The classical mechanics shows [1,2] that the gravitational potential Φ_e inside a uniform ellipsoid is a quadratic form of the space coordinates x_i:

$$\Phi_e = -\pi G \rho_e \sum_{i=1}^{3} \alpha_i x_i^2, \tag{3}$$

where G is the gravitational constant, ρ_e is the ellipsoid density and the coefficients α_i are

$$\alpha_i = a_1 a_2 a_3 \int_0^\infty \frac{d\lambda}{(a_i^2 + \lambda)\prod_{j=1}^{3}(a_j^2 + \lambda)^{1/2}}, \tag{4}$$

where a_i are the lengths of the semi-axes.

Since we consider spheroids, which are axially symmetric about the z-axis, it is reasonable to switch to cylindrical coordinates (ω, ϕ, z). The gravitational potential of the spheroid can be written as,

$$\Phi_s = -\pi G \rho_s \Big(\alpha(e)\omega^2 + \beta(e) z^2\Big), \tag{5}$$

where ρ_s is the current density of the spheroid. Being functions of only the current eccentricity $e(t)$, $\alpha(e)$ and $\beta(e)$ can be derived from Equation (4).

For the case of a spheroid ($a_{1,2} \equiv a$, $a_3 \equiv c$ and $\alpha_{1,2} = \alpha$, $\alpha_3 = \beta$), this integral can be calculated analytically, which gives us the following result[2]:

$$\alpha(e) = \frac{\sqrt{1-e^2}}{e^3}\Big(\arcsin e - e\sqrt{1-e^2}\Big), \tag{6}$$

$$\beta(e) = \frac{2\sqrt{1-e^2}}{e^3}\left(\frac{e}{\sqrt{1-e^2}} - \arcsin e\right), \tag{7}$$

for the oblate spheroid, and

$$\alpha(e) = \frac{1}{e^2}\left(1 - \frac{1-e^2}{2e}\ln\frac{1+e}{1-e}\right), \tag{8}$$

$$\beta(e) = \frac{2(1-e^2)}{e^3}\left(\frac{1}{2}\ln\frac{1+e}{1-e} - e\right), \tag{9}$$

for the prolate case.

Consider a point of the spheroid with initial coordinates (ω_0, ϕ_0, z_0). We need to calculate the point coordinates ($\omega(t)$, $\phi(t)$, $z(t)$) as functions of time. Due to the fact that the spheroid does not rotate $\phi(t) = \phi_0$. It is convenient to introduce new variables $R(t) \equiv \omega/\omega_0$, $Z(t) \equiv z/z_0$. A homogeneous spheroid remains a homogeneous spheroid during collapse [2], and as a result, the functions $R(t)$ and $Z(t)$ are the same for any point in the spheroid. Actually, $Z(t)$ and $R(t)$ are just the compression ratios of the spheroid along and across the z axis, respectively. It is clear from the definition that $Z = R = 1$ at $t = 0$. Since we assume that the initial velocities of all the spheroid particles are zero, $\dot{Z} = \dot{R} = 0$ at $t = 0$, and we obtain the initial conditions:

$$R(0) = Z(0) = 1, \quad \dot{R}(0) = \dot{Z}(0) = 0 \tag{10}$$

The spheroid eccentricity depends on time and may be expressed[3] through $R(t)$, $Z(t)$, and the initial eccentricity e_0:

$$Z^2 = R^2\frac{(1-e^2)}{(1-e_0^2)} \tag{11}$$

in the oblate case, and

$$R^2 = Z^2\frac{(1-e^2)}{(1-e_0^2)} \tag{12}$$

in the prolate case. The equations of motion can be obtained from the Newton's second law and look like [1]

$$\frac{d^2R}{dt^2} = -\frac{3GM}{2a^3}\frac{\alpha(e)}{\sqrt{1-e^2}}\frac{1}{RZ}, \tag{13a}$$

$$\frac{d^2Z}{dt^2} = -\frac{3GM}{2a^3}\frac{\beta(e)}{\sqrt{1-e^2}}\frac{1}{R^2}, \tag{13b}$$

where M is the spheroid mass, a is the equatorial radius of the spheroid. These equations, together with initial conditions (10), fully define the evolution of the system.

3. The Black Hole Formation

Now we need a criterion of the black hole formation. At some moment t_{col}, one of the compression factors, $Z(t)$ or $R(t)$, turns to zero. It is $Z(t)$, if the spheroid is oblate (the 'pumpkin' case), and the spheroid at $t = t_{col}$ transforms into a disc of radius $\omega_0 R(t_{col})$, perpendicular to the z axis. In the opposite case, if the spheroid is prolate (the 'melon' case), $R(t)$ turns to zero at $t = t_{col}$, and the spheroid transforms into a needle of length $2z_0 Z(t_{col})$ along the z axis.

Let us denote the minimal value that $R(t)$ can reach during the collapse of the oblate spheroid by R_{fin} and the minimal value that $Z(t)$ can reach during the collapse of the prolate spheroid by Z_{fin}. We will use two reasonable estimations for R_{fin} and Z_{fin}: the simplest

$$R_{fin} = R(t_{col}), \quad Z_{fin} = Z(t_{col}), \tag{14}$$

and a more sophisticated one, which we call 'with flyby'. For example, let us consider the oblate case. After $t = t_{col}$, the disc starts to expand in the z direction but still shortens in the x and y directions. This means that the estimation $R_{fin} = R(t_{col})$ overestimates R_{fin}. A more careful method of estimating R_{fin} is to expand the solution after $t = t_{col}$. Once $Z(t_{col}) = 0$ is reached, we reverse the z component of the velocity $v_Z \to -v_Z$ and solve Equation (13) (which we may rewrite through $(\{v_R, R, v_Z, Z\})$:

$$\frac{dv_R}{dt} = -\frac{3GM}{2a^3}\frac{\alpha(t)}{\sqrt{1-e_0^2}}\frac{1}{RZ}, \tag{15a}$$

$$\frac{dR}{dt} = v_R, \tag{15b}$$

$$\frac{dv_Z}{dt} = -\frac{3GM}{2a^3}\frac{\beta(t)}{\sqrt{1-e_0^2}}\frac{1}{R^2}, \tag{15c}$$

$$\frac{dZ}{dt} = v_Z. \tag{15d}$$

Then we find the moment when $R = Z$. The spheroid occupies the smallest possible volume at this moment and therefore has the highest density inside a sphere. We have all the reasons to consider the value of $R(t)$ at this time as a better estimation of R_{fin}, which we call 'R_{fin} with flyby'.

If we consider a prolate spheroid, we should find 'Z_{fin} with flyby' instead of 'R_{fin} with flyby', of course. To do so, we act exactly as in the previous case, but we reverse the R component of the velocity $v_R \to -v_R$ instead of v_Z and substitute to (15) the equations for α and β corresponding to the prolate case (see (8) and (9)). Then we find the moment after t_{col} when $R = Z$ and name the value of $Z(t)$ at this moment 'Z_{fin} with flyby'.

We use two ways to estimate R_{fin} and Z_{fin} for the following reason. Intuitively, the black hole forms if $\omega_0 R_{fin}$ or $z_0 Z_{fin}$ become of the order of the gravitational radius of the spheroid $r_g = 2GM/c^2$, where c is the speed of light. However, determining a precise criterion is an extremely difficult task. Equation (13) are derived in the Newtonian approximation, while when a black hole is formed, the effects of general relativity inevitably become significant. At present, exact solutions of general relativity for a collapsing spheroid are unknown. We assume that a black hole forms if the minimal radius that the spheroid reaches during the collapse (aR_{fin} or aZ_{fin}) becomes smaller or equal to the radius $2r_g$ of the marginally bound orbit around the black hole [7]:

$$aR_{fin} = 2r_g = \frac{4GM}{c^2} \quad \text{or} \quad aZ_{fin} = \frac{4GM}{c^2}. \tag{16}$$

To find the dependence of R_{fin} or Z_{fin} from the initial eccentricity value, we obtain numerical solutions of (15) for $e_0 = \{0.001, 0.003, 0.01, 0.03, 0.1, 0.3\}$, and then fit the obtained dependence $R_{fin}(e_0)$ or $Z_{fin}(e_0)$ by a linear function in the double logarithmic scale. For oblate spheroids, we obtain best fits $\lg R_{fin} = -0.2191 + 1.8821 \lg e_0$ and $\lg R_{fin} = -0.6053 + 1.8833 \lg e_0$ for the simple criterion (14) and the criterion with flyby, respectively (see Figure 1). For prolate spheroids, we obtain the best fits $\lg Z_{fin} = -0.1845 + 1.8822 \lg e_0$ and $\lg Z_{fin} = -0.5914 + 1.8788 \lg e_0$ for the simple criterion (14) and the criterion with flyby, respectively (see Figure 2). In the later case, we ignore the last two deviating points at large eccentricities (red dashed line in Figure 2), since we are interested in small values of eccentricity allowing to form a black hole.

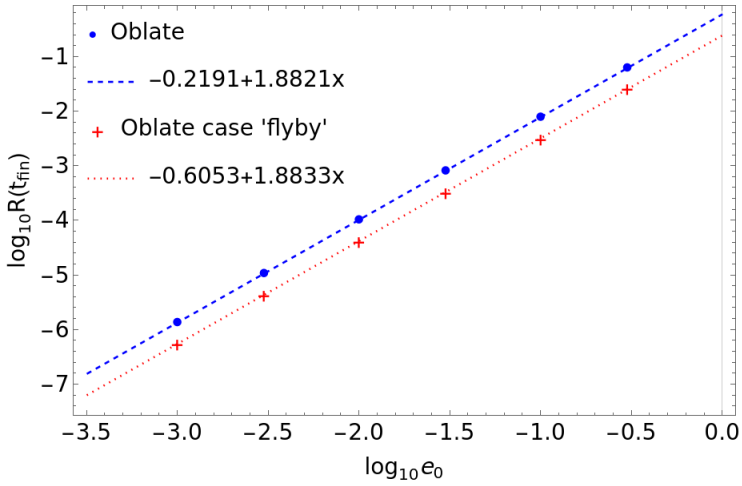


Figure 1. The maximal compression ratio (R_{fin} or Z_{fin}), as a function of the initial eccentricity e_0 of an oblate spheroid, if we use the simple criterion (14) (blue line) and the criterion with flyby (red line).

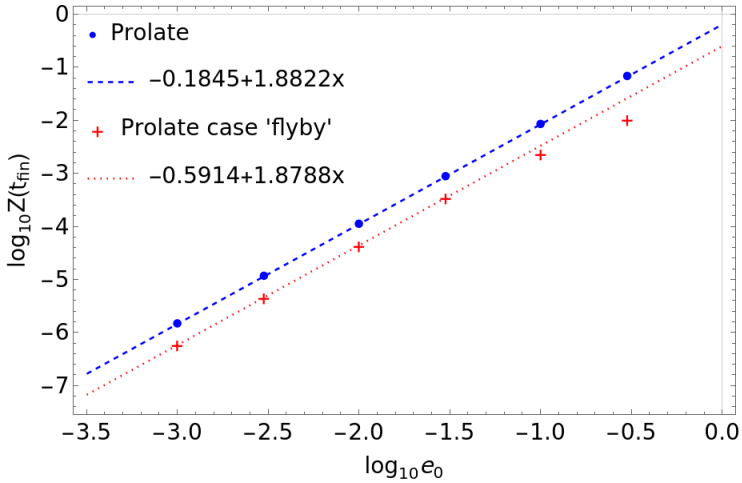


Figure 2. The maximal compression ratio (R_{fin} or Z_{fin}), as a function of the initial eccentricity e_0 of a prolate spheroid, if we use the simple criterion (14) (blue line) and the criterion with flyby (red line).

Discussion

Thus, we fit our results with expressions of the form $\mu + \nu \lg e_0$. It is notable that the value of ν in all four cases turns out to be practically the same $\nu \simeq 1.88 \simeq 15/8$. Moreover, the best fits for the oblate and prolate spheroids are also almost the same, if we use the same criterion to determine R_{fin} and Z_{fin}. Since R_{fin} and Z_{fin} have the same values if the eccentricity e_0 is the same, we may introduce the compression ratio of the spheroid χ_{fin}, which is equal to R_{fin} and Z_{fin} for an oblate and a prolate spheroid, respectively. The best fits are close to

$$\chi_{fin} = -0.2 + \frac{15}{8} \lg e_0, \tag{17}$$

if we use the simple criterion (14), and to

$$\chi_{fin} = -0.6 + \frac{15}{8} \lg e_0 \tag{18}$$

for the criterion with flyby. As we can see, μ in (17) is significantly larger than in (18), i.e., χ_{fin} is ∼2.5 larger if we apply the simple criterion (14). It is no wonder: as we discussed in the previous section, criterion (14) significantly overestimates R_{fin} and Z_{fin}. We will use the with flyby criterion as more accurate.

The fact that the fits for the oblate and prolate spheroids are almost the same is very important: it allows us to generalize our results from spheroids to an arbitrary ellipsoid. In fact, let us consider an ellipsoid with axes $a_1 \geq a_2 \geq a_3$. We may define its eccentricity e with the help of Equation (1), and so e does not depend on a_2. The value of a_2 can range from a_1 to a_3. Thus, an oblate spheroid with $a_1 = a_2 \geq a_3$ and a prolate spheroid with $a_1 \geq a_2 = a_3$ are two limiting cases of all ellipsoids with the same a_1 and a_3. However, the compression ratio (R_{fin} or Z_{fin}) behaves the same in these two cases, and therefore we have all the reasons to suppose that the compression ratio of any ellipsoid with $a_1 \geq a_2 \geq a_3$ can be well approximated by (18), where the eccentricity e_0 is given by (1).

4. Applicability of the Newtonian Approximation

All calculations in this work use the Newtonian approximation, which remains valid only when the final size of the collapsing ellipsoid greatly exceeds its gravitational radius: $a\chi_{\text{fin}} \gg r_g$. However, our analysis extends the solution to $a\chi_{\text{fin}} = 2r_g$, where relativistic effects dominate and the Newtonian framework apparently breaks down. This may raise concerns about the reliability of our results. Below, we qualitatively address the implications of general relativity (GR) and justify the use of our estimative approach.

Of course, our consideration is estimative and cannot replace precise relativistic calculations. However, such calculations are very complex and have only been carried out numerically [8]. The advantage of our calculations is their simplicity, and therefore they are easy to check. Finally, Michell in 1784 [9] found the correct equation for the gravitational radius of a black hole equating the kinetic energy of the escaping light and its gravitational energy, i.e., $c^2/2 = GM/r_g$, though it is clear now that both sides of this equation are incorrect.

Let us qualitatively discuss what may change in the solution if we perform it in the framework of general relativity. If $a\chi_{\text{fin}} \gg r_g$, the Newtonian approximation holds, and our conclusion—that black hole formation is inhibited—remains valid. If $a\chi_{\text{fin}} \leq r_g$, collapse proceeds nearly spherically even near the horizon, and the spherically symmetric collapse leads to the black hole formation [6]. Finally, in systems of astronomical interest the initial radius $a \gg r_g$, and so almost all the ellipsoid contraction, except the very last stage, can be considered in the Newtonian approximation. Therefore, one can expect that the expression (18) gives a good estimate for the initial maximum eccentricity e_0 at which the ellipsoid can still collapse into a black hole. The rigorous solution to the problem of the collapse of a homogeneous dust ellipsoid can be obtained by expanding our solution for $r \gg r_g$ to the stage when $r \sim r_g$ with the help of numerical gravity.

We could expect that our Newtonian solution gives a somewhat underestimated value of the maximum eccentricity e_0 at which the ellipsoid can still collapse into a black hole. First, the gravitational attraction of the Schwarzschild black hole (in the stationary frame of reference) is always larger than that of the Newtonian point body of the same mass [10], and as a result, the Newtonian approximation underestimates the compression ratio of the ellipsoid. Second, the ellipsoid eccentricity always grows in the Newtonian approach, while the eccentricity may even decrease near the horizon in general relativity [11,12]. Apparently, the eccentricity overestimation in the Newtonian approach impedes the black hole formation.

5. An Example of the Result Application

Let us apply the obtained results. Hereafter, we assume the standard ΛCDM cosmology with the present-day radiation fraction $\Omega_{m,0} = 5.0 \times 10^{-5}$ and $H_0 = 71$ (km/s) Mpc^{-1} [13]. Then the present-day critical density is $\rho_{c,0} \approx 0.97 \times 10^{-29}$ g/cm^3.

The result we have obtained can be applied to solve many cosmological problems. As an example, let us consider the formation of primordial black holes in the early Universe. One of the main scenarios [14] for their formation is that, for some reason that we will not discuss here, the spectrum of the primary perturbations differs significantly from the Harrison-Zeldovich one [15,16] in the short-wave region. As a result, the short perturbations become nonlinear already in the early Universe (at the radiation-dominated stage) and collapse forming black holes.

We consider the collapse of a homogeneous ellipsoid with $a_1 \geq a_2 \geq a_3$. We define the ellipsoid eccentricity e by (1). The eccentricity e should be very small for the ellipsoid to be able to collapse into a black hole. Thus, the ellipsoid is initially almost a sphere. We will use this fact. Only perturbations in dark matter can collapse in the radiation-dominated Universe. Using the method proposed in [17], it is easy to show that a matter perturbation stops expanding in the radiation-dominated Universe when its density ρ_M becomes $\sim$7.22 times higher than the radiation density ρ_γ

$$\rho_M = 7.22\rho_\gamma. \tag{19}$$

Since $\rho_\gamma \gg \rho_M$ at the radiation-dominated stage, matter perturbations should be very large to collapse. Suppose that a perturbation of mass M collapses at the redshift $\mathfrak{z} \gg 1$. Let us estimate the maximum eccentricity e_0 at the beginning of the collapse, which allows us to form a black hole. The initial radius a of the ellipsoid (we remind that it is almost a sphere) is bound with M by,

$$M = \frac{4}{3}\pi a^3 \rho_M = 7.22\frac{4}{3}\pi a^3 \rho_\gamma \approx 30 a^3 \Omega_\gamma \rho_{c0} \mathfrak{z}^4.$$

Here we use (19). So

$$a \approx \mathfrak{z}^{-4/3}\left(\frac{M}{30\Omega_\gamma \rho_{c0}}\right)^{1/3}. \tag{20}$$

Substituting this value to (16), we get

$$\chi_{fin} \simeq 1.2 \times 10^{-4} \mathfrak{z}_9^{4/3} M_{33.3}^{2/3}. \tag{21}$$

Here we have introduced the notations $M_{33.3}$, which is the ellipsoid mass divided by the solar mass $M_{33.3} = M/2 \times 10^{33}$ g, and $\mathfrak{z}_9 \equiv \mathfrak{z}/10^9$. Comparing this result with (18), we find

$$e_0 \simeq 1.7 \cdot 10^{-2} \mathfrak{z}_9^{32/45} M_{33.3}^{16/45}. \tag{22}$$

For instance, if the ellipsoids of mass $1M_\odot$ collapse in the early Universe at the temperature$\sim$240 keV, they can form black holes if their eccentricities are relatively modest ($e_0 \sim 0.02$).

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Notes

[1] We remind that a spheroid is an ellipsoid with two equal semi-axes, i.e., a spheroid with the circular symmetry.
[2] There is a misprint in the Equation (21) in [1].
[3] There is a misprint in the Equation (19a) in [1].

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