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Review

Black Hole Thermodynamics and Generalised Non-Extensive Entropy

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Abstract: The first part of this work provides a review of recent research on generalised entropies and their origin, as well as its application to black hole thermodynamics. To start, it is shown that the Hawking temperature and the Bekenstein–Hawking entropy are, respectively, the only possible thermodynamical temperature and entropy of the Schwarzschild black hole. Moreover, it is investigated if the other known generalised entropies, which include Rényi's entropy, Tsallis entropy, and the four- and five-parameter generalised entropies, could correctly yield the Hawking temperature and the ADM mass. The possibility that generalised entropies could describe hairy black hole thermodynamics is also considered, both for the Reissner–Nordström black hole and for Einstein's gravity coupled with two scalar fields. Two possibilities are investigated, namely, the case when the ADM mass does not yield the Bekenstein–Hawking entropy, and the case in which the effective mass expressing the energy inside the horizon does not yield the Hawking temperature. For the model with two scalar fields, the radii of the photon sphere and of the black hole shadow are calculated, which gives constraints on the BH parameters. These constraints are seen to be consistent, provided that the black hole is of the Schwarzschild type. Subsequently, the origin of the generalised entropies is investigated, by using their microscopic particle descriptions in the frameworks of a microcanonical ensemble and canonical ensemble, respectively. Finally, the McLaughlin expansion for the generalised entropies is used to derive, in each case, the microscopic interpretation of the generalised entropies, via the canonical and the grand canonical ensembles.

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1. Introduction

The thermodynamic properties of gravity could prove to be most important in attempting to construct a theory of quantum gravity. Every black hole (BH) can be regarded as a black body with temperature given by the Hawking temperature [1,2] and an entropy given by the Bekenstein–Hawking entropy [1,3].

In various fields of physics, statistics, and informatics, there have been proposed different forms of non-extensive entropies, with their corresponding statistics. In particular, the present authors, with some collaborators, have explicitly proposed generalised entropies, which depend on several parameters (see Refs. [4,5]). They generalise all previously known entropies, as Rényi entropy [6], the Tsallis entropy [7] (see also [8,9]), the

Sharma–Mittal entropy [10], Barrow's entropy [11], the Kaniadakis entropy [12,13], Loop Quantum Gravity's entropy [14], etc. Such entropies have been proposed to describe different kinds of physical, statistical, and information systems.

Note, however, that the Hawking temperature, $T_{\rm H}$, can be obtained from the Hawking radiation, which has a thermal distribution. This tells us that the Hawking temperature $T_{\rm H}$ is independent of the details of the gravity theory, and it is only determined by the geometry. Furthermore, if we consider the collapse of the dust shell that yields the black hole, and we assume energy conservation, the Arnowitt–Deser–Misner (ADM) mass [15] must be the thermodynamic energy of the system, at least in the case of a Schwarzschild black hole.

Recently, a number of works have appeared in which different non-extensive kinds of entropies have been applied in the study of black hole thermodynamics (see, e.g., [16–32]). Unfortunately, the Hawking temperature or black hole energy obtained in such non-extensive entropy black hole thermodynamics seems to be incorrect.

One may still conjecture that, in the early universe, non-extensive generalised entropy could be valid. With the universe's evolution, the form of the physical entropy might change to later acquire its current form. Therefore, there is good motivation for the study of different entropies, which were then applied in cosmology and BHs. In fact, various expressions of entropy lead to different holographic cosmologies [33,34] and models of holographic dark energy [35–42]. The holographic approach can be also applied to understand inflation in the early universe [43]. This makes it possible to describe dark energy and inflation via holographic cosmology in a unified way. A microscopic description of the generalised entropy has been also proposed. It might be helpful in clarifying the structure of a quantum gravity theory, which is still to be constructed.

In this review paper, we confirm once more that the Hawking temperature and the ADM mass may correspond to the thermodynamic temperature and energy uniquely, at least in the case of a Schwarzschild black hole. This shows that the Bekenstein–Hawking entropy is also a unique BH entropy. As a follow-up, we review several approaches in which the generalised entropy could be applied for the consideration of several kinds of black holes with hair(s).

In the next section, we show that the temperature and the entropy of a Schwarzschild black hole are given by the Hawking temperature (Section 2.1) and the Bekenstein–Hawking entropy, respectively, by identifying the ADM mass with the thermodynamic energy (Section 2.2). In Section 3, we discuss in more detail the question of whether the Hawking temperature and the Bekenstein–Hawking entropy are unique. To this purpose, we show that the ADM mass should be thermodynamic energy, by using the geometry of the black hole and the energy conservation via Birkhoff's theorem. In Section 4, for non-extensive entropy, we explicitly consider if it could give the Hawking temperature and the ADM mass correctly. In particular, the Rényi entropy is discussed in Section 4.1, Tsallis entropy in Section 4.2, and further generalised entropies, as the four- and five-parameter generalised entropies, in Section 4.3. In Section 5, we study if it might be possible that hairy BH thermodynamics could be described by generalised entropies, for the Reissner–Nordström black hole in Section 5.1, and for Einstein's gravity coupled to two scalar fields, in Section 5.2. In the latter case, after showing the general formulation, in Section 5.2.1, and some examples, in Section 5.2.2, we consider two kinds of possibilities. Namely, the case that the ADM mass does not give the Bekenstein–Hawking entropy, in Section 5.2.3, and the case that the effective mass expressing the energy inside the horizon does not give the Hawking temperature, in Section 5.2.4. In Section 6, for the models obtained in Section 5.2, we obtain the radii of the photon sphere and of the black hole shadow. Then, observations give constraints on the BH parameters. These turn out to be consistent, if the black

holes are of the Schwarzschild type, although future observations may also give some information about BH thermodynamics. In Section 7, for more general expressions of the generalised entropies, we propose microscopic particle descriptions of the corresponding thermodynamic system. We investigate this problem by using a microcanonical ensemble, in Section 7.1, and a canonical ensemble, in Section 7.2. In Section 8, by using the expression of the McLaughlin expansion for the generalised entropies, we consider the microscopic interpretation of the generalised entropies in the frame of a canonical ensemble, in Section 8.1, and of a grand canonical ensemble, in Section 8.2. The last section of the paper contains a summary and final discussion.

2. Entropy Consistent with Hawking Radiation

The Hawking radiation has a thermal distribution, from which we can find the Hawking temperature $T_{\rm H}$. The geometry with the horizon generates Hawking's radiation. Therefore, the Hawking temperature, $T_{\rm H}$, is only determined by the geometry and is independent of the details of the gravity theory, which realises the geometry.

Let us consider a system whose size is R and the energy and the entropy inside the system are E and $\mathcal{S}$, respectively. Then, the Bekenstein bound is given by [44]

$$2\pi RE > \mathcal{S}\,. \tag{1}$$

In the case of a black hole, R can be identified with the diameter of the horizon, that is, twice the horizon radius. We also need to check if the bound (1) is satisfied for general entropy because this bound ensures that the generalised second law of the thermodynamics is not violated.

2.1. Hawking Temperature from Geometry

First, we find the Hawking temperature. When the metric can be regarded as static, that is, the time-dependence of the metric can be neglected, we consider the line element with a horizon at $r = r_{\rm H}$,

$$ds^2 = -P(r)(r-r_{\rm H})dt^2 + \frac{dr^2}{P(r)(r-r_{\rm H})} + r^2 d\Omega_{(2)}^2\,, \quad d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2\,. \tag{2}$$

We assume that $P(r)$ is positive everywhere and sufficiently smooth in the region near the horizon $r = r_{\rm H}$. Therefore, we may approximate $P(r)$ by a constant, $P(r) \sim P(r_{\rm H})$. We now introduce a new coordinate ρ defined by

$$d\rho = \frac{dr}{\sqrt{P(r_{\rm H})(r-r_{\rm H})}}\,, \tag{3}$$

that is,

$$\rho = 2\sqrt{\frac{r-r_{\rm H}}{P(r_{\rm H})}}\,. \tag{4}$$

By Wick-rotating the time coordinate t as $t \to i\tau$, we obtain the following Euclidean metric

$$ds^2 = \frac{P(r_{\rm H})^2}{4}\rho^2 d\tau^2 + d\rho^2 + r(\rho)^2 d\Omega_{(2)}^2\,. \tag{5}$$

We avoid the conical singularity at $\rho = 0$ by imposing the periodicity on τ,

$$\frac{P(r_{\rm H})}{2}\tau \sim \frac{P(r_{\rm H})}{2}\tau + 2\pi\,. \tag{6}$$

In the finite temperature formalism of the path-integral, the periodicity $\frac{4\pi}{P(r_{\rm H})}$ corresponds to the inverse of the temperature

$$T_{\rm H} = \frac{P(r_{\rm H})}{4\pi}\,, \tag{7}$$

which we call the Hawking temperature. In the case of Schwarzschild spacetime,

$$P(r) = P_{\rm Schw}(r) \equiv \frac{1}{r}\,, \quad r_{\rm H} = 2GM\,. \tag{8}$$

Here, G is Newton's gravitational constant and M is the ADM BH mass.

2.2. Bekenstein–Hawking Entropy from Thermodynamics

As is well-known, the area law for Bekenstein–Hawking entropy [3] can always be obtained if we identify the thermodynamic energy E with the black hole mass M, $E = M$, and the temperature of the system with the Hawking temperature (7) [2], $T = T_{\rm H} = \frac{1}{8\pi GM}$. In fact, the thermodynamic relation $dE = Td\mathcal{S}$ yields

$$d\mathcal{S} = \frac{dE}{T} = 8\pi GM dM = d\left(4\pi GM^2\right)\,, \tag{9}$$

which can be integrated to be

$$\mathcal{S} = 4\pi GM^2 + \mathcal{S}_0\,, \tag{10}$$

where $\mathcal{S}_0$ is a constant of the integration. If we assume $\mathcal{S} = 0$ when $M = 0$, that is, when there is no black hole, we find $\mathcal{S}_0 = 0$ and we obtain

$$\mathcal{S} = \frac{\pi {r_{\rm H}}^2}{G} = \frac{A}{4G}\,. \tag{11}$$

Here $A \equiv 4\pi {r_{\rm H}}^2$ is the area of the horizon. Therefore, the Bekenstein–Hawking entropy, that is, the area law for BH entropy, can be obtained by assuming $E = M$ and $T = T_{\rm H}$ by using the thermodynamic relation $d\mathcal{S} = dE/T$. Note that the Bekenstein–Hawking entropy $\mathcal{S}$, of course, satisfies the Bekenstein bound in (1) because $2\pi RE = 4\pi r_{\rm H}\frac{r_{\rm H}}{2G} = 2\mathcal{S} > \mathcal{S}$.

3. Uniqueness of Hawking Temperature and Bekenstein–Hawking Entropy

We now consider whether the Hawking temperature and the Bekenstein–Hawking entropy could be unique. For this purpose, we need to consider the following two points,

1. Can the thermodynamic energy E be identified with the black hole mass M (i.e., $E = M$)?
2. Is the temperature of the black hole given by the Hawking temperature, $T = T_{\rm H}$?

For the first point, we should be careful in the following situation, that is if BH is not a Schwarzschild BH nor an isolated one, there is no Arnowitt–Deser–Misner mass. Then, the mass M may be the quasilocal mass contained in the horizon sphere or given by the "black hole part" of the spacetime. For several quasilocal mass prescriptions, see Ref. [45] for a review.

3.1. $E = M$?

To consider the first point, the following 'thought experiments' could be useful.

1. We assume an infalling spherically symmetric shell of dust with mass M and the initial radius sufficiently large. The Birkhoff theorem [46] tells that the spacetime outside the shell is Schwarzschild spacetime (8). The mass M is only the mass of the shell. Inside the shell, the spacetime is empty and flat.
2. By the collapse of the shell, the radius becomes smaller and smaller. A black hole is formed when the shell crosses the Schwarzschild radius $r_{\rm H} = 2M$ in (8).
3. The resulting geometry is always asymptotically flat and the shell mass M appearing in the horizon radius is surely the energy E of the system, $E = M$ because the energy should be conserved during the collapse of the shell due to the Birkhoff theorem. That is, the geometry outside of the shell does not change during the collapse. Therefore the energy of the final black hole must be the mass of the shell. We should note that due to spherical symmetry, the quadrupole does not appear and the gravitational waves, which might carry the energy, are not emitted during the collapse.

One may consider other definitions of the mass or the energy of the black hole like the Misner–Sharp–Hernandez quasilocal mass $M_{\rm MSH}$ defined in any spherically symmetric spacetime by [47,48] and the Brown–York quasilocal energy [49]. The obtained results are consistent with the above arguments or totally unphysical (for more detailed arguments, see [50]).

3.2. $T = T_{\rm H}$?

The second point is discussed in the previous section. As mentioned there, Hawking radiation is obtained if the geometry with the horizon is prescribed and the standard Hawking temperature is the parameter appearing as the temperature in the thermal distribution of the emitted Hawking radiation. We may imagine that we put the black hole in a heat bath at temperature T. Then, the thermal equilibrium between the black hole radiation and the heat bath occurs when the radiation temperature equals the temperature of the heat bath, $T = T_{\rm H}$. Therefore, the heat bath can be used as a thermometer and the temperature measured by the heat bath must be the standard Hawking temperature of the Hawking radiation and, therefore, we find $T = T_{\rm H}$.

4. Consistency of General Entropies

Due to various motivations, different kinds of entropy other than the Bekenstein–Hawking one [2,3] have been proposed like Tsallis [7], Rényi [6], Barrow [11], Sharma–Mittal [10], Kaniadakis [12] and loop quantum gravity entropies [14]). Furthermore, generalised entropy with three, four, five and six parameters has been proposed in [4,5,51]. These generalised entropies give all the aforementioned known entropies within a certain choice of entropic parameters.

4.1. Rényi Entropy

First, we consider Rényi entropy [52–55]

$$\mathcal{S}_{\rm R} = \frac{1}{\alpha} \ln(1 + \alpha \mathcal{S}) \,. \tag{12}$$

Here, $\mathcal{S}$ is Bekenstein–Hawking entropy (11) and α is a parameter specifying the deformation from Bekenstein–Hawking entropy. In the limit of $\alpha \to 0$, the expression (12) reduces to Bekenstein–Hawking entropy. Using Equation (10) with $\mathcal{S}_0 = 0$, we find,

$$\mathcal{S}_{\mathrm{R}} = \frac{1}{\alpha}\ln\left(1+4\pi\alpha GM^2\right). \tag{13}$$

Note that the Rényi entropy satisfies the Bekenstein bound (1) because $\mathcal{S}_{\mathrm{R}} < \mathcal{S} < 2\pi RE$ as long as $\mathcal{S}_{\mathrm{R}} > 0$.

4.1.1. Assumption $M = E$

If mass M coincides with energy E of the system due to energy conservation [52–55], the consistency of the system with the thermodynamic equation $d\mathcal{S} = dE/T$ requires defining the "Rényi temperature" T_{R} by

$$\frac{1}{T_{\mathrm{R}}} \equiv \frac{d\mathcal{S}_{\mathrm{R}}}{dM} = \frac{8\pi GM}{1+4\pi\alpha GM^2}, \tag{14}$$

that is,

$$T_{\mathrm{R}} = \frac{1}{8\pi GM} + \frac{\alpha M}{2} = T_{\mathrm{H}} + \frac{\alpha}{16\pi G T_{\mathrm{H}}}, \tag{15}$$

which is different from the Hawking temperature T_{H} and, therefore, the "Rényi temperature" T_{R} is not the temperature perceived by any observer detecting Hawking radiation, as we stressed. Hence, the "Rényi temperature" T_{R} could be physically irrelevant for black hole thermodynamics.

4.1.2. $T = T_{\mathrm{H}}$?

Instead of assuming that the thermodynamic energy E is identical with BH mass M, we now assume that the thermodynamic temperature T coincides with the Hawking temperature T_{H}.

By using the thermodynamic relation $dE = Td\mathcal{S}$, the assumptions $T = T_{\mathrm{H}}$ and $\mathcal{S} = \mathcal{S}_{\mathrm{R}}$ show that the corresponding thermodynamic energy E_{R} is given by

$$dE_{\mathrm{R}} = T_{\mathrm{H}} d\mathcal{S}_{\mathrm{R}} = \frac{1}{8\pi GM}\frac{8\pi GM dM}{1+4\pi\alpha GM^2} = \frac{dM}{1+4\pi\alpha GM^2}, \tag{16}$$

which can be integrated to give,

$$E_{\mathrm{R}} = \frac{\arctan\left(\sqrt{4\pi\alpha G}M\right)}{\sqrt{4\pi\alpha G}} = M - \frac{4\pi\alpha GM^2}{3} + \mathcal{O}\left(\alpha^2\right). \tag{17}$$

Here the integration constant is fixed so that $E_{\mathrm{R}} = 0$ when $M = 0$. The correction $-\frac{4\pi\alpha GM^2}{3} + \mathcal{O}(\alpha^2)$ shows that the expression (17) of the thermodynamic energy E_{R} is different from BH mass M, $E_{\mathrm{R}} \neq E$, which looks unphysical. The more important thing is that it seems to conflict with energy conservation when we consider the spherically symmetric dust shell collapses to a Schwarzschild black hole.

4.2. Tsallis Entropy

Let us consider Tsallis entropy [7] in BH thermodynamics as is discussed in [50].

Tsallis entropy may be considered as an alternative to Bekenstein–Hawking entropy [8] (see also [9]),

$$\mathcal{S}_{\mathrm{T}} = \frac{A_0}{4G}\left(\frac{A}{A_0}\right)^{\delta}. \tag{18}$$

Here, A_0 is a constant with the dimension of the area and δ specifying the non-extensivity. In the limit of $\delta \to 1$, the expression in (18) reduces to the standard Bekenstein–Hawking entropy (11). Note, however, that the Bekenstein bound (1) is violated for the large black hole because $\frac{\mathcal{S}_\mathrm{T}}{\mathcal{S}} \to \infty$ when $\mathcal{S} \to \infty$ if $\delta > 1$.

4.2.1. $M = E$?

Again, by assuming that the thermodynamic energy E is given by BH mass M, we obtain $A = 4\pi(2GM)^2 = 16\pi G^2 E^2$ and the expression in (18) has the following form,

$$\mathcal{S}_\mathrm{T} = \frac{A_0}{4G}\left(\frac{16\pi G^2 E^2}{A_0}\right)^\delta , \tag{19}$$

which may allow us to define "Tsallis temperature" as follows,

$$T_\mathrm{T} \equiv \frac{dE}{d\mathcal{S}_\mathrm{T}} = \frac{2G}{\delta A_0 E^{2\delta-1}}\left(\frac{A_0}{16\pi G^2}\right)^\delta = \frac{2G}{\delta A_0 M^{2\delta-1}}\left(\frac{A_0}{16\pi G^2}\right)^\delta . \tag{20}$$

The Tsallis temperature is, of course, different from the Hawking temperature (7), $T = T_\mathrm{H} = \frac{1}{8\pi GM}$ unless $\delta = 1$.

4.2.2. $T = T_\mathrm{H}$?

Instead of identifying the black hole mass M with the thermodynamic energy E, we now assume that the BH temperature is the Hawking temperature. Because we have $A = 4\pi(4\pi T_\mathrm{H})^{-2} = \frac{1}{4\pi {T_\mathrm{H}}^2}$, we find

$$\mathcal{S}_\mathrm{T} = \frac{{A_0}^{1-\delta}}{4G\left(4\pi {T_\mathrm{H}}^2\right)^\delta} , \tag{21}$$

which may lead to the "Tsallis energy" E_T given by

$$dE_\mathrm{T} = T_\mathrm{H} d\mathcal{S}_\mathrm{T} = -\frac{\delta {A_0}^{1-\delta} dT_\mathrm{H}}{2G(4\pi)^\delta {T_\mathrm{H}}^{2\delta}} . \tag{22}$$

By integrating (22), we obtain

$$E_\mathrm{T} = \frac{\delta {A_0}^{1-\delta}}{2(2\delta-1)G(4\pi)^\delta {T_\mathrm{H}}^{2\delta-1}} = \frac{\delta {A_0}^{1-\delta}(8\pi GM)^{2\delta-1}}{2(2\delta-1)G(4\pi)^\delta} . \tag{23}$$

Here, we have fixed the integration constant by imposing the condition that $E_\mathrm{T} = 0$ when $M = 0$. The standard relation $E_\mathrm{T} = M$ is reproduced for $\delta = 1$ when the Tsallis entropy reduces to the Bekenstein–Hawking entropy.

4.2.3. Bekenstein–Hawking Entropy as Tsallis Entropy

The standard thermodynamics is related to the extensive system. In the system, if we separate the system with thermodynamic energy E into two systems with E_1 and E_2 with $E = E_1 + E_2$, the standard entropy $\mathcal{S}_\mathrm{standard}(E)$ is extensive, that is,

$$\mathcal{S}_\mathrm{standard}(E_1 + E_2) = \mathcal{S}_\mathrm{standard}(E_1) + \mathcal{S}_\mathrm{standard}(E_2) . \tag{24}$$

Conversely, the original Tsallis entropy $\tilde{\mathcal{S}}_\mathrm{T}$ has the following properties,

$$\left(\tilde{\mathcal{S}}_\mathrm{T}(E_1 + E_2)\right)^{\frac{1}{\delta}} = \left(\tilde{\mathcal{S}}_\mathrm{T}(E_1)\right)^{\frac{1}{\delta}} + \left(\tilde{\mathcal{S}}_\mathrm{T}(E_2)\right)^{\frac{1}{\delta}} . \tag{25}$$

As pointed in [56], the standard Bekenstein–Hawking entropy is recovered with $\delta = 2$

$$(\mathcal{S}(E_1 + E_2))^{\frac{1}{2}} = (\mathcal{S}(E_1))^{\frac{1}{2}} + (\mathcal{S}_{\rm T}(E_2))^{\frac{1}{2}}\,. \tag{26}$$

As claimed in [56], this property could be explained by the quantum process where a black hole could split into smaller black holes.

We should note that the black hole is not in equilibrium with the heat bath or environment. It is like indoor bright red charcoal. The black hole is hotter than the vacuum. The non-extensivity of the Bekenstein–Hawking entropy in (26) could indicate that the internal energy could not be extensive, either. In Tsallis entropy, long-range force is supposed to generate non-extensivity because the long-range force makes the internal energy non-extensive.

4.3. Generalised Entropies

The generalised four- and six-parameter generalised entropies have the following forms [4,5],

$$\mathcal{S}_4(\alpha_\pm, \delta, \gamma) = \frac{1}{\gamma}\left[\left(1 + \frac{\alpha_+}{\delta}\mathcal{S}\right)^{\delta} - \left(1 + \frac{\alpha_-}{\delta}\mathcal{S}\right)^{-\delta}\right], \tag{27}$$

and

$$\mathcal{S}_6(\alpha_\pm, \delta_\pm, \gamma_\pm) = \frac{1}{\alpha_+ + \alpha_-}\left[\left(1 + \frac{\alpha_+}{\delta_+}\mathcal{S}^{\gamma_+}\right)^{\delta_+} - \left(1 + \frac{\alpha_-}{\delta_-}\mathcal{S}^{\gamma_-}\right)^{-\delta_-}\right], \tag{28}$$

respectively. Here, $\mathcal{S} = \frac{A}{4G}$ represents the Bekenstein–Hawking entropy (11). Both of these entropies reduce to all the aforementioned known entropies for a suitable limit of the respective parameters, that is, Tsallis, Rényi, Barrow, Sharma–Mittal, Kaniadakis, and loop quantum gravity entropies. For instance, we find

- $\mathcal{S}_4$ reduces to Tsallis entropy in the limit of $\alpha_+ \to \infty$, $\alpha_- = 0$ and $\gamma = (\alpha_+/\beta)^{\beta}$.
- The six-parameter entropy, $\mathcal{S}_6$ reduces to Tsallis entropy for $\alpha_+ = \alpha_- \to 0$ and $\gamma_+ = \gamma_-$.

In addition to the four- and six-parameter generalised entropies, a three-parameter entropy was also proposed in [4] in the following form

$$\mathcal{S}_3(\alpha, \delta, \gamma) = \frac{1}{\gamma}\left[\left(1 + \frac{\alpha}{\delta}\mathcal{S}\right)^{\delta} - 1\right]. \tag{29}$$

$\mathcal{S}_3$ cannot, however, be reduced to Kaniadakis entropy in any parameter limit. Therefore the four-parameter entropy is the minimal generalisation because the minimum number of parameters required in an entropy function for generalising all the known entropies is four. In $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_6$, the Bekenstein bound (1) can be violated in some parameter regions because they have limits where these entropies reduce to Tsallis entropy, where the Bekenstein bound is violated for a large black hole if $\delta > 1$.

We should also note that $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_6$ share the following properties:

1. They obey the third law of thermodynamics, i.e., they vanish in the limit of $\mathcal{S} \to 0$.
2. They are monotonically increasing functions of the variable $\mathcal{S}$.
3. They diverge in the limit $\mathcal{S} \to \infty$.

For the last point, when we consider the cosmology, A is given by the area of the apparent horizon, $A = \frac{4\pi}{H^2}$. Here, H is the Hubble rate. Therefore, $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_6$ diverge when the Hubble rate vanishes, $H = 0$ because the Bekenstein–Hawking entropy $\mathcal{S}$ itself diverges at $H = 0$. This singular behaviour is common to all the known entropies like

Tsallis, the Rényi, the Barrow, the Kaniadakis, the Sharma–Mittal and the loop quantum gravity entropies.

In order to solve the problem of a singularity when $H \to 0$, a five-parameter entropy was proposed in [51], which has the following form,

$$\mathcal{S}_5(\alpha_\pm, \delta, \gamma, \epsilon) = \frac{1}{\gamma}\left[\left\{1 + \frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_+}{\delta}\mathcal{S}\right)\right\}^{\delta} - \left\{1 + \frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_-}{\delta}\mathcal{S}\right)\right\}^{-\delta}\right]. \tag{30}$$

Due to the tanh function, the entropy (30) does not show a singularity even if $\mathcal{S}$ diverges or $H \to 0$. This entropy, therefore, permits a bouncing scenario, where H vanishes at the bouncing time.

In the following, for the generalised entropies $\mathcal{S}_4$ in (27) and $\mathcal{S}_5$ in (30), we investigate if the mass M coincides with the thermal energy E by assuming that the temperature T is given by the Hawking temperature $T_{\rm H}$, $T = T_{\rm H}$, and also inversely, if the temperature T is given by the Hawking temperature $T_{\rm H}$ by assuming the mass M coincides with the thermal energy E, $E = M$.

4.3.1. $E = M$?

The case of four-parameter generalised entropy $\mathcal{S}_4$

By substituting $\mathcal{S} = 4\pi GM^2$ in (10) with $\mathcal{S}_0 = 0$ to the four-parameter generalised entropy in (27), we find

$$dE_4 = T_{\rm H} d\mathcal{S}_4 = \frac{1}{\gamma}\left[\left(1 + \frac{4\pi\alpha_+}{\delta}GM^2\right)^{\delta-1}\alpha_+ + \left(1 + \frac{4\pi\alpha_-}{\delta}GM^2\right)^{-\delta-1}\alpha_-\right]dM\,. \tag{31}$$

Here, E_4 is the energy defined by the first relation $dE_4 = T_{\mathcal{H}} d\mathcal{S}_4$. The above expression does not give $dE_4 = dM$ nor $E_4 = M$ in general, of course.

When M is small, Equation (31) gives,

$$dE_4 \sim \frac{\alpha_+ + \alpha_-}{\gamma}dM\,, \tag{32}$$

which can be integrated to be

$$E_4 \sim E_{4(0)} + \frac{\alpha_+ + \alpha_-}{\gamma}M\,, \tag{33}$$

Here $E_{4(0)}$ is a constant of the integration. Equation (33) indicates $E_4 \neq M$ in general but if we choose

$$\frac{\alpha_+ + \alpha_-}{\gamma} = 1\,, \quad E_{4(0)} = 0\,, \tag{34}$$

we obtain $E_4 = M$.

Conversely, when M is large, if we choose $\delta > 0$, we obtain

$$E_4 \sim E_{4(1)} + \frac{\alpha_+ M}{\gamma(2\delta - 1)}\left(\frac{4\pi\alpha_+}{\delta}GM^2\right)^{\delta-1}. \tag{35}$$

Here, $E_{4(1)}$ is a constant of the integration. Regardless, Equation (35) generally gives $E_4 \neq M$ but if we choose

$$\delta = 1\,, \quad \frac{\alpha_+}{\gamma} = 1\,, \quad E_{4(1)} = 0\,, \tag{36}$$

we obtain $E_4 = M$.

Note that the condition (36) is compatible with the condition (33) if

$$\alpha_+ = \gamma\,, \quad \alpha_- = 0\,, \quad \delta = 1\,, \tag{37}$$

and we obtain an expression of the entropy that realises $E_4 = M$ in both of the limits $M \to 0$ and $M \to +\infty$. The condition (37), however, shows that the four-parameter generalised entropy $\mathcal{S}_4$ in (27) reduces to the standard Bekenstein–Hawking entropy, $\mathcal{S}_4 \to \mathcal{S}$.

The case of five-parameter generalised entropy $\mathcal{S}_5$

In the case of the five-parameter generalised entropy in (27), we find

$$\begin{aligned} dE_5 &= T_{\rm H} d\mathcal{S}_5(\alpha_\pm, \delta, \gamma, \epsilon) \\ &= \frac{1}{\gamma}\left[\left\{1 + \frac{1}{\epsilon}\tanh\left(\frac{4\pi\epsilon\alpha_+}{\delta}GM^2\right)\right\}^{\delta-1} \frac{\alpha_+}{\cosh^2\left(\frac{4\pi\epsilon\alpha_+}{\delta}GM^2\right)}\right. \\ &\quad \left. + \left\{1 + \frac{1}{\epsilon}\tanh\left(\frac{4\pi\epsilon\alpha_-}{\delta}GM^2\right)\right\}^{-\delta-1} \frac{\alpha_-}{\cosh^2\left(\frac{4\pi\epsilon\alpha_-}{\delta}GM^2\right)}\right] dM\,. \end{aligned} \tag{38}$$

Here, E_5 is the energy defined by $dE_5 = T_{\mathcal{H}} d\mathcal{S}_5$ and the above expression tells $dE_5 \neq dM$ nor $E_5 \neq M$ in general.

When M is small, one again obtains (32) and (33). The obtained result tells $E_4 \neq M$ again in general but if we choose the parameters as in (34), we obtain $E_4 = M$.

When M is large, by assuming $\alpha_+ > \alpha_- > 0$, we find

$$dE_5 \sim \frac{4\alpha_-}{\gamma}\left(1 + \frac{1}{\epsilon}\right)^{-\delta-1} \exp\left(-\frac{8\pi\epsilon\alpha_-}{\delta}GM^2\right) dM\,. \tag{39}$$

The integration of the above equation is given by using Gauss' error function erf, which is defined by

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2} dt\,, \tag{40}$$

as follows

$$E_5 \sim E_{5(0)} + \frac{1}{\gamma}\left(1 + \frac{1}{\epsilon}\right)^{-\delta-1} \sqrt{\frac{\alpha_-\delta}{2\epsilon G}}\,\mathrm{erf}\left(M\sqrt{\frac{8\pi\epsilon\alpha_- G}{\delta}}\right). \tag{41}$$

Therefore, there is no choice of the parameter that reproduces $E_4 = M$ except the limit that $\mathcal{S}_5$ in (30) reduces to the standard Bekenstein–Hawking entropy, $\mathcal{S}_5 \to \mathcal{S}$.

4.3.2. $T = T_{\rm H}$?

In the case of the four-parameter generalised entropy in (27), if we identify the thermodynamic energy E with the mass M, the corresponding temperature T_4 is given by

$$T_4 \equiv \frac{d\mathcal{S}_4}{dM} = \frac{8\pi GM}{\gamma}\left[\left(1 + \frac{4\pi\alpha_+}{\delta}GM^2\right)^{\delta-1}\alpha_+ + \left(1 + \frac{4\pi\alpha_-}{\delta}GM^2\right)^{-\delta-1}\alpha_-\right]. \tag{42}$$

Equation (42) does not give the Hawking temperature $T_{\rm H}$, $T_4 \neq T_{\rm H} = \frac{1}{8\pi GM}$ besides the limit that $\mathcal{S}_4$ in (27) reduces to the standard Bekenstein–Hawking entropy, $\mathcal{S}_4 \to \mathcal{S}$.

Conversely, in the case of the five-parameter generalised entropy in (30), we obtain

$$T_5 \equiv \frac{d\mathcal{S}_5}{dM} = \frac{8\pi GM}{\gamma}\left[\left\{1+\frac{1}{\epsilon}\tanh\left(\frac{4\pi\epsilon\alpha_+}{\delta}GM^2\right)\right\}^{\delta-1}\frac{\alpha_+}{\cosh^2\left(\frac{4\pi\epsilon\alpha_+}{\delta}GM^2\right)}\right.$$
$$\left.+\left\{1+\frac{1}{\epsilon}\tanh\left(\frac{4\pi\epsilon\alpha_-}{\delta}GM^2\right)\right\}^{-\delta-1}\frac{\alpha_-}{\cosh^2\left(\frac{4\pi\epsilon\alpha_-}{\delta}GM^2\right)}\right]. \quad (43)$$

Equation (43) does not give the Hawking temperature $T_{\rm H}$, $T_5 \neq T_{\rm H} = \frac{1}{8\pi GM}$, either.

5. A More General Black Hole

The thermodynamic relation $dE = Td\mathcal{S}$ does not generally hold, for example, if there is a chemical potential. The first law of thermodynamics is

$$dE = dQ + dW\,. \quad (44)$$

$dQ = Td\mathcal{S}$ is the heat which flows into the system and dW is the work that the system received. The variation in the work dW can be expressed as

$$dW = -PdV + \sum_i \mu_i dN_i\,. \quad (45)$$

Here, P and V are the pressure and the volume of the system and dN_i is the number of the i-th kind of particles that flow into the system and μ_i is the corresponding chemical potential.

When we discussed if the thermodynamic energy should be the ADM mass in Section 3.1 by using the falling dust shell, we have assumed that the region outside the dust shell is a vacuum. In a realistic situation, all the matter does not fall into the black hole and instead, the matter outside the horizon contributes to the ADM mass. In the case of a Reissner–Nordström black hole, the ADM mass includes the contributions from the electromagnetic field outside the horizon. More in general, if BH has any hair, the energy density of the hair contributes to the ADM mass and changes the thermodynamic relation $dE = Td\mathcal{S}$ as in (44). In this section, we discuss the possibility that the generalised entropies could be given by a hairy black hole. First, we review the thermodynamics of a Reissner–Nordström black hole, and after that, we consider a black hole with scalar hair(s). For the construction of a black hole with scalar hair(s), we use the model in which the Einstein gravity couples with two scalar fields.

5.1. Reissner–Nordström Black Hole

The metric of a Reissner–Nordström BH is given by the following line element,

$$ds^2 = -\left(1-\frac{2GM}{r}+\frac{GQ^2}{r^2}\right)dt^2 + \frac{dr^2}{1-\frac{2GM}{r}+\frac{GQ^2}{r^2}} + r^2 d\Omega^2_{(2)}\,. \quad (46)$$

Here, Q is the electric charge of the black hole and the ADM mass is given by M as in a Schwarzschild black hole. As is well-known, a Reissner–Nordström black hole has two horizons. The radii $r_\pm$ of the horizons are given by

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}\,, \quad (47)$$

Here r_+ is the radius of the outer horizon and r_- is that of the inner one. Equation (47) shows that the Bekenstein–Hawking entropy $\mathcal{S}$ is given by,

$$\mathcal{S}_\pm = \frac{\pi {r_\pm}^2}{G} = \frac{\pi\left(GM \pm \sqrt{G^2M^2 - GQ^2}\right)^2}{G}. \tag{48}$$

Here $\mathcal{S}_+$ is the entropy corresponding to the outer horizon and $\mathcal{S}_-$ to the inner one. The Hawking temperature T_H corresponding to the outer horizon is given by

$$T_\mathrm{H} = \frac{\sqrt{G^2M^2 - GQ^2}}{2\pi\left(GM + \sqrt{G^2M^2 - GQ^2}\right)^2}, \tag{49}$$

Then we find

$$T_\mathrm{H} d\mathcal{S}_+ = dM - \frac{Q}{GM + \sqrt{G^2M^2 - GQ^2}} dQ \tag{50}$$

Then, there is a correction by the last term.

One may consider the possibility of defining a generalised entropy $\mathcal{S}_\mathrm{g}$ instead of (50),

$$T_\mathrm{H} d\mathcal{S}_\mathrm{g} = dM. \tag{51}$$

In the case of a Reissner–Nordström black hole, this is generally impossible because the system depends on two variables M and Q. Let us first assume $\mathcal{S}_\mathrm{g}$, $\mathcal{S}_\mathrm{g} = \mathcal{S}_\mathrm{g}(M, Q)$. Then, Equation (51) can be rewritten as,

$$T_\mathrm{H}\left(\frac{\partial \mathcal{S}_\mathrm{g}}{\partial M} dM + \frac{\partial \mathcal{S}_\mathrm{g}}{\partial Q} dQ\right) = dM. \tag{52}$$

Then, we find $\frac{\partial \mathcal{S}_\mathrm{g}}{\partial Q} = 0$ and, therefore, the integrablity condition requires $\frac{\partial T_\mathrm{H}}{\partial Q}$ because $\frac{\partial \mathcal{S}_\mathrm{g}}{\partial M}$ should not depend on Q. This conflicts with the expression of the Hawking temperature in (49), which explicitly depends on Q. A possibility is to consider a one-dimensional line in the two-dimensional M-Q as $Q = Q(M)$. Then, Equation (52) tells,

$$\mathcal{S}_\mathrm{g}(M) \int \frac{dM}{T_\mathrm{H}(M, Q(M))}. \tag{53}$$

As an example, we consider the case $Q = q_0 M$ with a constant satisfying a condition ${q_0}^2 < G$. In this case, Equation (49) gives

$$T_\mathrm{H} = \frac{\sqrt{G^2 - G{q_0}^2}}{2\pi\left(G + \sqrt{G^2 - G{q_0}^2}\right)^2 M}, \tag{54}$$

and therefore Equation (53) can be integrated to give,

$$\mathcal{S}_\mathrm{g} = \frac{\pi\left(G + \sqrt{G^2 - G{q_0}^2}\right)^2 M^2}{\sqrt{G^2 - G{q_0}^2}}. \tag{55}$$

Here, we choose the constant of the integration so that $\mathcal{S}_\mathrm{g}$ vanishes when M vanishes. The obtained expression (55) is proportional to M^2, which is similar to the Bekenstein–Hawking entropy in (10) with $\mathcal{S}_0 = 0$ although the coefficient is different. Another function $Q = Q(M)$ gives more general expressions but it depends on the physical process of the

black hole creation. The case $Q = q_0 M$ could correspond to the process that a BH is created only by the accretion of the charged particle whose ratio of charge to mass is q_0.

5.2. Gravity Coupled with Two Scalar Fields

In [57], it was shown that arbitrarily given spherically symmetric spacetimes can be realised within Einstein's gravity coupled with two scalar fields even if the spacetime is time-dependent. The original model of Ref. [57], however, includes ghosts, which make the model inconsistent. After that, it was found that the ghosts could be excluded by imposing constraints using Lagrange multiplier fields [58].

The action in the model of Ref. [57] includes two scalar fields ϕ and χ, which couple with Einstein's gravity,

$$S_{\mathrm{GR}\phi\chi} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}A(\phi,\chi)\partial_\mu\phi\partial^\mu\phi - B(\phi,\chi)\partial_\mu\phi\partial^\mu\chi \right. \\ \left. -\frac{1}{2}C(\phi,\chi)\partial_\mu\chi\partial^\mu\chi - V(\phi,\chi) + \mathcal{L}_{\mathrm{matter}}\right]. \tag{56}$$

Here, $A(\phi,\chi)$, $B(\phi,\chi)$, and $C(\phi,\chi)$ are called kinetic functions and $V(\phi,\chi)$ is the potential, which are functions of the two scalar fields ϕ and χ. Furthermore, $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian density of matter. The gravitational coupling constant κ is related to Newton's gravitational constant G as $\kappa^2 = 8\pi G$. In this section, we mainly use the geometrised units $c = G = 1$.

General spherically symmetric and time-dependent spacetime is described by the metric given by the following line element,

$$ds^2 = -\mathrm{e}^{2\nu(t,r)}dt^2 + \mathrm{e}^{2\lambda(t,r)}dr^2 + r^2 d\Omega^2_{(2)}\,. \tag{57}$$

We also assume,

$$\phi = t\,, \quad \chi = r\,, \tag{58}$$

which does not lead to any loss of generality [57].

We should note, however, that the functions A and/or C are often negative, which makes ϕ and/or χ to be ghosts. The ghosts can be eliminated by imposing constraints by using the Lagrange multiplier fields λ_ϕ and λ_χ and modifying the action (56) $S_{\mathrm{GR}\phi\chi} \to S_{\mathrm{GR}\phi\chi} + S_\lambda$, where the additional term S_λ is given by

$$S_\lambda = \int d^4x\sqrt{-g}\left[\lambda_\phi\left(\mathrm{e}^{-2\nu(t=\phi,r=\chi)}\partial_\mu\phi\partial^\mu\phi + 1\right) + \lambda_\chi\left(\mathrm{e}^{-2\lambda(t=\phi,r=\chi)}\partial_\mu\chi\partial^\mu\chi - 1\right)\right]. \tag{59}$$

By varying S_λ with respect to λ_ϕ and λ_χ, we obtain the following constraints:

$$0 = \mathrm{e}^{-2\nu(t=\phi,r=\chi)}\partial_\mu\phi\partial^\mu\phi + 1\,, \quad 0 = \mathrm{e}^{-2\lambda(t=\phi,r=\chi)}\partial_\mu\chi\partial^\mu\chi - 1\,, \tag{60}$$

which is consistent with the assumption (58). The constraints from Equation (60) make the scalar fields ϕ and χ non-dynamical, and the fluctuations in ϕ and χ around the background (58) do not propagate (see [58] for details).

We now construct a model that has a solution realising the functions $\mathrm{e}^{2\nu(t,r)}$ and $\mathrm{e}^{2\lambda(t,r)}$ in Equation (57). Matter is assumed to be a perfect fluid with the energy density ρ and the pressure p,

$$T_{\mathrm{matter}\,tt} = -g_{tt}\rho\,, \quad T_{\mathrm{matter}\,ij} = p\,g_{ij}\,. \tag{61}$$

Here $i, j = r, \vartheta, \varphi$. For the spacetime given by Equation (57), the Einstein equations can re rewritten as follows,

$$A = \frac{e^{2\nu}}{\kappa^2}\left\{-e^{-2\nu}\left[\ddot{\lambda} + (\dot{\lambda} - \dot{\nu})\dot{\lambda}\right] + e^{-2\lambda}\left[\frac{\nu' + \lambda'}{r} + \nu'' + (\nu' - \lambda')\nu' + \frac{e^{2\lambda} - 1}{r^2}\right]\right\} \quad (62)$$
$$-e^{2\nu}(\rho + p)\,,$$
$$B = \frac{2\dot{\lambda}}{\kappa^2 r}\,,$$
$$C = \frac{e^{2\lambda}}{\kappa^2}\left\{e^{-2\nu}\left[\ddot{\lambda} + (\dot{\lambda} - \dot{\nu})\dot{\lambda}\right] - e^{-2\lambda}\left[-\frac{\nu' + \lambda'}{r} + \nu'' + (\nu' - \lambda')\nu' + \frac{e^{2\lambda} - 1}{r^2}\right]\right\},$$
$$V = \frac{e^{-2\lambda}}{\kappa^2}\left(\frac{\lambda' - \nu'}{r} + \frac{e^{2\lambda} - 1}{r^2}\right) - \frac{1}{2}(\rho - p)\,. \quad (63)$$

This tells that we obtain a model that realises the spacetime described by the metric (57) by finding the (t, r)-dependence of ρ and p and by replacing (t, r) in Equation (62) with (ϕ, χ).

5.2.1. Black Hole with Scalar Hair

We now consider the time-independent geometry, that is, static, spherical, and asymptotically flat spacetimes,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2_{(2)}\,. \quad (64)$$

Asymptotic flatness corresponds to $\lim_{r\to+\infty}\lambda(r) = 0$ and we normalise the time coordinate t, to $\lim_{r\to+\infty}\nu(r) = 0$.

Let us now investigate the effects of scalar hair and write the energy density of the scalar fields as ρ. Then, as in the standard Tolman–Oppenheimer–Volkov (TOV) equation, the time-time component of the Einstein equations gives

$$-\kappa^2\rho = \frac{1}{r^2}\left(re^{-2\lambda} - r\right)'. \quad (65)$$

Here, a prime "′" denotes differentiation with respect to r. The mass function $m(r)$ is defined by

$$e^{-2\lambda} \equiv 1 - \frac{2Gm(r)}{r}\,, \quad (66)$$

which gives $4\pi r^2\rho = m'(r)$ and by integrating the expression, we obtain

$$m(r) = 4\pi\int^r r'^2\rho(r')dr'\,. \quad (67)$$

In the case of a compact star like a neutron star, the lower limit of the integration is chosen to be $r = 0$. In the case of a black hole, the boundary condition is given at the horizon $r = r_{\rm H}$ so that

$$2Gm(r_{\rm H}) = r_{\rm H}\,. \quad (68)$$

If the geometry is asymptotically Schwarzschild spacetime, the ADM mass is given by

$$M = m(r\to\infty) = 4\pi\int^\infty dr\, r^2\rho(r)\,. \quad (69)$$

Note that $m(r=\infty)$ is not the total mass, which should be defined by

$$\bar{M} = \int d^3x\,\sqrt{\gamma}\,\rho(r) = 4\pi\int_0^\infty \rho(r) r^2 \mathrm{e}^{\lambda(r)} dr = 4\pi\int_0^\infty \rho(r) r^2 \left[1-\frac{2Gm(r)}{r}\right]^{-1/2} dr$$
$$= 4\pi\int_0^\infty dr\,\rho(r) r^2\left[1+\frac{Gm(r)}{r}-\frac{3G^2m^2(r)}{r^2}+\mathcal{O}\left(G^3\right)\right]. \tag{70}$$

Here, γ is the determinant of the three-dimensional spatial metric,

$$\gamma_{\ell m}\,dx^\ell dx^m = \mathrm{e}^{2\lambda}dr^2 + r^2 d\Omega^2_{(2)}\,. \tag{71}$$

The second term in the last line of Equation (70) can be interpreted as the Newtonian gravitational potential energy

$$-4\pi G\int_0^\infty dr\,\rho(r)\,r^2\frac{m(r)}{r} = -\frac{G}{2}\int dV\int dV'\,\frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,. \tag{72}$$

Here, dV and dV' are three-dimensional volume elements and the general-relativistic nonlinear corrections are identified by the G^2 term and higher power terms of G.

The above arguments show that the contribution to the mass from the scalar hair can be given by

$$M_{\rm hair} = m(r=\infty) - m(r=r_{\rm H})\,. \tag{73}$$

This term gives a correction as in the second term of Equation (44)

$$dM = T_H d\mathcal{S} + dM_{\rm hair}\,. \tag{74}$$

Then, the correction of the general entropy from the Bekenstein–Hawking entropy might be interpreted as the contribution from $M_{\rm hair}$. As we can identify $M = m(r=\infty)$, however, Equation (74) can be rewritten as

$$dm(r=r_{\rm H}) = T_H d\mathcal{S}\,. \tag{75}$$

Because $m(r=r_{\rm H}) = \frac{r_{\rm H}}{2G}$ and the Hawking temperature is given by $T_{\rm H} = \frac{1}{4\pi r_{\rm H}}$, Equation (75) is approved only if we choose $\mathcal{S}$ to be the Bekenstein–Hawking entropy, $\mathcal{S} = \frac{\pi {r_{\rm H}}^2}{G}$ as in the standard black hole thermodynamics.

We should note, however, that there might be a possibility to define a generalised entropy $\mathcal{S}_{\rm g}$ as in (51) by using the first law in (44) as follows

$$T_{\rm H} d\mathcal{S}_{\rm g} \equiv dQ + dW = dE\,. \tag{76}$$

We investigate the possibility in the following.

Even for more general gravity theories including the modified gravities, as an analogue of (65), we may define the effective energy density $\rho_{\rm eff}$ by using only geometry,

$$-\kappa^2\rho_{\rm eff} = \frac{1}{r^2}\left(r\mathrm{e}^{-2\lambda}-r\right)', \tag{77}$$

and also define the effective mass function as in (66)

$$\mathrm{e}^{-2\lambda} \equiv 1-\frac{2Gm_{\rm eff}(r)}{r}\,. \tag{78}$$

Then, by integrating (77), we obtain the counterpart of (67),

$$m_{\rm eff}(r) = 4\pi \int_0^r dr' r'^2 \rho_{\rm eff}(r') \,. \tag{79}$$

We may interpret $m_{\rm eff}(r)$ as the mass acted upon by the attractive force at radius r. We use this definition later.

5.2.2. Examples

In order to consider examples, we now assume [59],

$$\mathrm{e}^{2\nu} = \mathrm{e}^{-2\lambda} = \frac{1}{h_2(r)}\left(1 - \frac{r_{\rm H}}{r}\right), \tag{80}$$

with a constant radius of the horizon $r_{\rm H}$. We do not include matter besides the two scalar fields ϕ and χ. Then, the expressions (62) give,

$$\begin{aligned}
A(\phi) =& \frac{1}{\kappa^2 h_2(\phi)}\left(1 - \frac{r_{\rm H}}{\phi}\right)\left\{-\frac{h_2(\phi)h_2''(\phi) - 2h_2'(\phi)^2}{2h_2(\phi)^3}\left(1 - \frac{r_{\rm H}}{\phi}\right) - \frac{r_{\rm H}h_2'(\phi)}{\phi^2 h_2(\phi)^2} + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi)}\right)\right\},\\
B(\phi) =& 0\,,\\
C(\phi) =& -\frac{h_2(\phi)}{\kappa^2\left(1 - \frac{r_{\rm H}}{\phi}\right)}\left\{-\frac{h_2(\phi)h_2''(\phi) - 2h_2'(\phi)^2}{2h_2(\phi)^3}\left(1 - \frac{r_{\rm H}}{\phi}\right) - \frac{r_{\rm H}h_2'(\phi)}{\phi^2 h_2(\phi)^2} + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi)}\right)\right\},\\
V(\phi) =& \frac{1}{\kappa^2}\left\{\frac{h_2'(\phi)}{\phi h_2(\phi)^2}\left(1 - \frac{r_{\rm H}}{\phi}\right) + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi)}\right)\right\}.
\end{aligned} \tag{81}$$

We should note that A, C, and V in (81) depend explicitly on the horizon radius $r_{\rm H}$, that is, the horizon radius is fixed in this model. There could be other solutions besides Equation (80), but it is not be easy to find them. This problem can be bypassed by using the trick of Ref. [60]. We add a new term in the Lagrangian density including new fields σ and ρ_μ as $\mathcal{L}_{\rho\sigma} = \rho^\mu \partial_\mu \sigma$. The variation of $\mathcal{L}_{\rho\sigma}$ with respect to ρ^μ yields a constant σ,

$$\partial_\mu \sigma = 0\,, \tag{82}$$

We now identify σ with the horizon radius $r_{\rm H}$. By replacing $r_{\rm H}$ with σ in Equations in (81), $r_{\rm H}$ is given as an integration constant appearing from Equation (82),

$$\begin{aligned}
A(\phi,\sigma) =& \frac{1}{\kappa^2 h_2(\phi,\sigma)}\left(1 - \frac{\sigma}{\phi}\right)\left\{-\frac{h_2(\phi,\sigma)h_{2,\phi\phi}(\phi,\sigma) - 2h_{2,\phi}(\phi,\sigma)^2}{2h_2(\phi,\sigma)^3}\left(1 - \frac{\sigma}{\phi}\right)\right.\\
&\left. - \frac{\sigma h_{2,\phi}(\phi,\sigma)}{\phi^2 h_2(\phi,\sigma)^2} + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi,\sigma)}\right)\right\},\\
B(\phi,\sigma) =& 0\,,\\
C(\phi,\sigma) =& -\frac{h_2(\phi,\sigma)}{\kappa^2\left(1 - \frac{\sigma}{\phi}\right)}\left\{-\frac{h_2(\phi,\sigma)h_{2,\phi\phi}(\phi,\sigma) - 2h_{2,\phi}(\phi,\sigma)^2}{2h_2(\phi,\sigma)^3}\left(1 - \frac{\sigma}{\phi}\right)\right.\\
&\left. - \frac{\sigma h_{2,\phi}(\phi,\sigma)}{\phi^2 h_2(\phi,\sigma)^2} + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi,\sigma)}\right)\right\},\\
V(\phi,\sigma) =& \frac{1}{\kappa^2}\left\{\frac{h_{2,\phi}(\phi,\sigma)}{\phi h_2(\phi,\sigma)^2}\left(1 - \frac{\sigma}{\phi}\right) + \frac{1}{\phi^2}\left(1 - \frac{1}{h_2(\phi,\sigma)}\right)\right\}.
\end{aligned} \tag{83}$$

Here, $h_2(\phi,\sigma)_{,\phi} \equiv \partial h_2(\phi,\sigma)/\partial\phi$, $h_2(\phi,\sigma)_{,\phi\phi} \equiv \partial^2 h_2(\phi,\sigma)/\partial\phi^2$. By the choice of h_2, we obtain several examples.

5.2.3. Thermodynamics

As an example, we consider the case

$$h_2 = 1 + \frac{2GM(r_{\rm H}) - r_{\rm H}}{r} = 1 + \frac{2GM(\sigma) - \sigma}{\phi}\,. \tag{84}$$

Then when r is large, Equation (80) indicates

$$\mathrm{e}^{2\nu} = \mathrm{e}^{-2\lambda} \sim 1 - \frac{2GM(r_{\rm H})}{r}\,. \tag{85}$$

Therefore, $M(r_{\rm H})$ is the ADM mass.

In order to consider the possibility of (76), as an example, we consider the Rényi entropy in (12), which has now the following form

$$\mathcal{S}_{\rm R} = \frac{1}{\alpha}\ln\left(1 + \alpha\frac{\pi {r_{\rm H}}^2}{G}\right). \tag{86}$$

Because the Hawking temperature is given by $T_{\rm H} = \frac{1}{4\pi r_{\rm H}}$, if we assume (76), we find

$$M'(r_{\rm H})dr_{\rm H} = \frac{1}{2G\left(1+\alpha\frac{\pi {r_{\rm H}}^2}{G}\right)}dr_{\rm H} = \frac{1}{2\sqrt{\pi\alpha G}}d\left(\mathrm{Arctan}\left(r_{\rm H}\sqrt{\frac{\pi\alpha}{G}}\right)\right). \tag{87}$$

Therefore, in (84) if we choose

$$M(\sigma) = \frac{1}{2\sqrt{\pi\alpha G}}d\mathrm{Arctan}\left(r_{\rm H}\sqrt{\frac{\pi\alpha}{G}}\right)\Big|_{r_{\rm H}=\sigma}, \tag{88}$$

we obtain a model whose entropy is described by the Rényi entropy $\mathcal{S}_{\rm R}$.

Similarly, for the generalised entropy $\mathcal{S}_{\rm g} = \mathcal{S}_{\rm g}(r_{\rm H})$, if we choose $M(\sigma)$ in (84) by

$$M(\sigma) = \int^{\sigma} dr_{\rm H}\frac{\mathcal{S}'_{\rm g}(r_{\rm H})}{4\pi r_{\rm H}}\,, \tag{89}$$

a model whose entropy is $\mathcal{S}_{\rm g}$ can be constructed.

5.2.4. Thermodynamics Based on $m_{\rm eff}$

Here, based on [59], we consider the thermodynamics by using $m_{\rm eff}$ in (79). Instead of (80), we assume,

$$\mathrm{e}^{-2\lambda(r)} = \mathrm{e}^{2\nu(r)}h_3(r)\,. \tag{90}$$

Here $h_3(r)$ is a positive function of r. As in (83), the geometry (90) is realised by using (62) with the Lagrangian density $\mathcal{L}_{\rho\sigma} = \rho^\mu\partial_\mu\sigma$,

$$
\begin{aligned}
A(\phi,\sigma) &= \frac{1}{\kappa^2 h_3(\phi,\sigma)^2}\left(\frac{\sigma}{\phi}-1\right)^2\left[-\frac{h_{3,\phi}(\phi,\sigma)}{4\phi h_3(\phi,\sigma)}+\frac{3h_{3,\phi}(\phi,\sigma)}{4(\phi-\sigma)h_3(\phi,\sigma)}\right.\\
&\qquad\left.+\frac{h_{3,\phi\phi}(\phi,\sigma)}{2\,h_3(\phi,\sigma)}-\frac{1}{4}\left(\frac{h_{3,\phi}(\phi,\sigma)}{h_3(\phi,\sigma)}\right)^2\right],\\
B(\phi,\sigma) &= 0\,,\\
C(\phi,\sigma) &= \frac{1}{\kappa^2}\left[\frac{5h_{3,\phi}(\phi,\sigma)}{4\phi h_3(\phi,\sigma)}-\frac{3h_{3,\phi}(\phi,\sigma)}{4(\phi-\sigma)h_3(\phi,\sigma)}-\frac{h_{3,\phi\phi}(\phi,\sigma)}{2h_3(\phi,\sigma)}+\left(\frac{h_{3,\phi}(\phi,\sigma)}{2h_3(\phi,\sigma)}\right)^2\right],\\
V(\phi,\sigma) &= \frac{1}{2\kappa^2\phi}\left(1-\frac{\sigma}{\phi}\right)\frac{h_{3,\phi}(\phi,\sigma)}{h_3(\phi,\sigma)}\,,
\end{aligned}
\tag{91}
$$

Here, σ is identified with the radius of the horizon, $\sigma = r_{\rm H}$.

One should note that $\mathrm{e}^{-2\lambda(r)}$ must vanish when $\mathrm{e}^{2\nu(r)}$ vanishes in order to avoid the curvature singularity. Both $\mathrm{e}^{-2\lambda(r)}$ and $\mathrm{e}^{2\nu(r)}$ vanish at the horizon, so one can write the horizon radius by $m_{\rm eff}(r)$,

$$
r_{\rm H} = 2Gm_{\rm eff}(r_{\rm H})\,. \tag{92}
$$

As we find the Hawking temperature (7), we now consider the temperature of the black hole. Near the horizon, we write the radial coordinate as $r \equiv r_{\rm H} + \delta r$. Then, we obtain,

$$
\mathrm{e}^{-2\lambda} \sim \frac{C(r_{\rm H})(r-r_{\rm H})}{r_{\rm H}}\,,\quad \mathrm{e}^{2\nu} \sim \frac{C(r_{\rm H})(r-r_{\rm H})}{h_3(r_{\rm H})r_{\rm H}}\,. \tag{93}
$$

Here, $C(r_{\rm H}) \equiv 1 - m'(r_{\rm H})$. By a Wick rotation, $t \to i\tau$, the line element (2) near the horizon behaves as

$$
ds^2 \sim \frac{C(r_{\rm H})\delta r}{h_3(r_{\rm H})r_{\rm H}}d\tau^2 + \frac{r_{\rm H}}{C(r_{\rm H})\delta r}d(\delta r)^2 + r_{\rm H}^2\,d\Omega_{(2)}^2\,. \tag{94}
$$

Using a new radial coordinate ρ defined by $d\rho = d(\delta r)\sqrt{\frac{r_{\rm H}}{C(r_{\rm H})\delta r}}$, which gives,

$$
\rho = 2\sqrt{\frac{r_{\rm H}\delta r}{C(r_{\rm H})}} \quad \text{or} \quad \delta r = \frac{C(r_{\rm H})\rho^2}{4r_{\rm H}}\,, \tag{95}
$$

we rewrite line element (94) as

$$
ds^2 \simeq \frac{C(r_{\rm H})^2}{4h_3(r_{\rm H})r_{\rm H}^2}\rho^2 d\tau^2 + d\rho^2 + r_{\rm H}^2 d\Omega_{(2)}^2\,. \tag{96}
$$

In order to avoid conical singularities near $\rho = 0$ in the Euclidean space, we need to impose the periodicity of the Euclidean time coordinate τ,

$$
\frac{C(r_{\rm H})\tau}{2r_{\rm H}\sqrt{h_3(r_{\rm H})}} \simeq \frac{C(r_{\rm H})\tau}{2\,r_{\rm H}\sqrt{h_3(r_{\rm H})}} + 2\pi\,. \tag{97}
$$

Because the period of the Euclidean time corresponds to the temperature T, we find

$$
T = \frac{C(r_{\rm H})}{4\pi r_{\rm H}\sqrt{h_3(r_{\rm H})}} = \frac{C(r_{\rm H})}{8\pi G m_{\rm eff}(r_{\rm H})\sqrt{h_3(r_{\rm H})}} = \frac{C(r_{\rm H})T_{\rm H}}{\sqrt{h_3(r_{\rm H})}}\,. \tag{98}
$$

Here, the Hawking temperature $T_{\rm H}$ is now given by,

$$T_{\rm H} \equiv \frac{1}{8\pi G m_{\rm eff}(r_{\rm H})}\,. \tag{99}$$

Therefore, we find the temperature T deviates from the Hawking temperature by the factor $\frac{C(r_{\rm H})}{\sqrt{h_3(r_{\rm H})}}$, which cannot be absorbed by rescaling time.

By the analogy of the thermodynamic relation $dE = TdS$, we define the entropy proper to the black hole.

$$d\mathcal{S}_{\rm bh} = \frac{dm_{\rm eff}(r_{\rm H})}{T_{\rm H}}\,. \tag{100}$$

By integrating (100), we obtain,

$$\mathcal{S}_{\rm bh} = \int \frac{dm_{\rm eff}(r_{\rm H})}{T}\,. \tag{101}$$

We now consider the possibility that $\mathcal{S}_{\rm bh}$ could be different from Bekenstein–Hawking entropy.

By solving the field equations of a certain gravitational theory, there appear several constants of the integration, c_i $(i = 1, \cdots, N)$. For example, in general relativity, the mass M of a Schwarzschild black hole (8) appears as an integration constant. Both the mass M and charge Q in a Reissner–Nordström black hole (46) are also constants of the integration. The horizon radius $r_{\rm H}$ could be given by a function of c_i as in the usual Schwarzschild black hole, where we find $r_{\rm H} = 2GM$ as a function of the integration constant M. Other quantities could be also obtained as functions of c_i, such as $h_3(r = r_{\rm H}(c_i); c_i)$, etc. We may also assume that the constants c_i's are parameterised using a single parameter ξ, $c_i = c_i(\xi)$ as mentioned before Equation (53) in the case of a Reissner–Nordström black hole.

Equation (98) can be used to rewrite Equation (101) in the following form

$$\mathcal{S}_{\rm bh} = \frac{1}{2G}\int d\xi \frac{\left[4\pi r_{\rm H}(c_i(\xi))\sqrt{h_3(r = r_{\rm H}(c_i(\xi)); c_i(\xi))}\right]}{1 - \frac{\partial m(r; c_i(\xi))}{\partial r}\Big|_{r=r_{\rm H}(c_i(\xi))}} \sum_{i=1}^{N} \frac{\partial r_{\rm H}(c_i)}{\partial c_i}\frac{\partial c_i}{\partial \xi}\,. \tag{102}$$

Choosing $\xi = r_{\rm H}$, Equation (102) is simplified to be,

$$\mathcal{S}_{\rm bh} = \frac{1}{2G}\int_0^{r_{\rm H}} d\xi \frac{\left(4\pi\xi\sqrt{h_3(r = \xi; c_i(\xi))}\right)}{1 - \frac{\partial m(r; c_i(\xi))}{\partial r}\Big|_{r=\xi}}\,. \tag{103}$$

Here, the constant of the integration is fixed using the condition $\mathcal{S}_{\rm bh} = 0$ at $r_{\rm H} = 0$. In the case of a Schwarzschild black hole, where $h_3(x) = 1$, $m = M =$ const., Bekenstein–Hawking entropy (11) is reproduced. In general, however, if $h_3(r \to r_{\rm H})$ non-trivially contribution to the entropy, $\mathcal{S}_{\rm bh}$ may be different from the Bekenstein–Hawking entropy $\mathcal{S}_{\rm BH}$.

In fact, Equation (103) gives,

$$\frac{h_3(r = r_{\rm H}; c_i(r_{\rm H}))}{\left(1 - \frac{\partial m(r; c_i(r_{\rm H}))}{\partial r}\Big|_{r=r_{\rm H}}\right)^2} = 16G^2\left[\mathcal{S}'_{\rm bh}(A)\right]^2\,. \tag{104}$$

Therefore, for certain expressions of the general entropies, we find the corresponding form of

$$\Theta \equiv \frac{h_3(r = r_{\rm H}; c_i(r_{\rm H}))}{\left(1 - \frac{\partial m(r; c_i(r_{\rm H}))}{\partial r}\Big|_{r=r_{\rm H}}\right)^2}. \tag{105}$$

For example, in the case of Rényi entropy (12), we obtain

$$\Theta = \frac{1}{\left(1 + \frac{\pi\alpha {r_{\rm H}}^2}{G}\right)^2}, \tag{106}$$

and for Tsallis entropy (18), Equation (104) becomes

$$\Theta = \delta^2 \left(\frac{4\pi {r_{\rm H}}^2}{A_0}\right)^{2(\delta-1)}. \tag{107}$$

Furthermore for the three-parameter generalised entropy $\mathcal{S}_3$ in (29), we find

$$\Theta = \frac{\alpha^2}{\gamma^2}\left[1 + \left(\frac{\pi\alpha {r_{\rm H}}^2}{\beta G}\right)\right]^{2\beta-2}. \tag{108}$$

and the six-parameter entropy $\mathcal{S}_6$ in (28) yields

$$\Theta = \frac{1}{(\alpha_+ + \alpha_-)^2}\left[\alpha_+\gamma_+\left(\frac{\pi {r_{\rm H}}^2}{G}\right)^{\gamma_+-1}\left(1 + \frac{\alpha_+}{\beta_+}\left(\frac{\pi {r_{\rm H}}^2}{G}\right)^{\gamma_+}\right)^{\beta_+-1} + \alpha_-\gamma_-\left(\frac{\pi {r_{\rm H}}^2}{G}\right)^{\gamma_--1}\left(1 + \frac{\alpha_-}{\beta_-}\left(\frac{\pi {r_{\rm H}}^2}{G}\right)^{\gamma_-}\right)^{-\beta_--1}\right]^2. \tag{109}$$

Even for the four-parameter generalised entropy $\mathcal{S}_4$ in (27) and the five-parameter generalised entropy $\mathcal{S}_5$ in (30), we can find the corresponding quantity Θ.

Application of alternative entropies to Bekenstein–Hawking entropy to black holes lead to inconsistencies in the thermodynamics as we discussed but the inconsistencies might be avoided for non-Schwarzschild black holes in modified gravity if the horizon radius and, therefore, the area appearing in Bekenstein's area law are modified as we have shown. Hence, the consistency of new entropy proposals with Hawking temperature and area law could be possible for the above black holes as is shown in this section.

6. Photon Spheres and Black Hole Shadows

Recently, there has been much interest in BH shadows. Let us briefly discuss this topic here in relation to different BH thermodynamics. A photon sphere is the set of the circular orbit of the photon. The radius $r_{\rm ph}$ of the photon sphere gives the radius $r_{\rm sh}$ of the black hole shadow as follows,

$$r_{\rm sh} = r{\rm e}^{-\nu(r)}\Big|_{r=r_{\rm ph}}. \tag{110}$$

The orbit of the photon is governed by the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = \frac{1}{2}\left(-{\rm e}^{2\nu}\dot{t}^2 + {\rm e}^{2\lambda}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right). \tag{111}$$

Here, the "dot" or "$\dot{\cdot}$" expresses the derivative with respect to the affine parameter. The fact that the geodesic of the photon is null tells $\mathcal{L} = 0$. We find the conserved quantities corresponding to energy E and angular momentum L because there are no the explicit dependencies on t and ϕ in the Lagrangian $\mathcal{L}$,

$$E \equiv \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\mathrm{e}^{2\nu}\dot{t}\,, \quad L \equiv \frac{\partial V}{\partial \dot{\phi}} = r^2 \sin^2\theta\dot{\phi}\,. \tag{112}$$

The total energy $\mathcal{E}$ of the system should be also conserved and given by,

$$\mathcal{E} \equiv \mathcal{L} - \dot{t}\frac{\partial \mathcal{L}}{\partial \dot{t}} - \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mathcal{L}\,, \tag{113}$$

We should note that $\mathcal{E} = \mathcal{L}$ vanishes identically $\mathcal{E} = \mathcal{L} = 0$ for the null geodesic.

Without any loss of generality, we can choose the coordinate system in which the orbit of the photon is on the equatorial plane with $\theta = \frac{\pi}{2}$. For the coordinate choice, the condition $\mathcal{E} = \mathcal{L} = 0$ can be written as

$$0 = -\frac{E^2}{2}\mathrm{e}^{-2(\nu+\lambda)} + \frac{1}{2}\dot{r}^2 + \frac{L^2\mathrm{e}^{-2\lambda}}{2r^2}\,, \tag{114}$$

We write this system in an analogous way to the classical dynamical system with potential $W(r)$,

$$0 = \frac{1}{2}\dot{r}^2 + W(r)\,, \quad W(r) \equiv \frac{L^2\mathrm{e}^{-2\lambda}}{2r^2} - \frac{E^2}{2}\mathrm{e}^{-2(\nu+\lambda)}\,. \tag{115}$$

Because the radius of the circular orbit is defined by $\dot{r} = 0$, the radius is given by solving $W(r) = W'(r) = 0$ by using the analogy with classical mechanics. For Schwarzschild spacetime, we find $r_{\rm ph} = 3M$ and $r_{\rm sh} = 3\sqrt{3}M$.

In the model (80) with (84), we find

$$W(r) = \frac{L^2(r - r_H)}{2r^2(r + 2GM(r_{\rm H}) - r_{\rm H})} - \frac{E^2}{2}\,, \tag{116}$$

which gives

$$W'(r) = -\frac{2L^2\left(r^2 + GM(r_{\rm H})r - 2r_{\rm H}r - 2GM(r_{\rm H})r_{\rm H} + {r_{\rm H}}^2\right)}{2r^3(r + 2GM(r_{\rm H}) - r_{\rm H})^2}\,, \tag{117}$$

which gives

$$r = r_{\rm ph} \equiv \frac{1}{2}\left[-GM(r_{\rm H}) + 2r_{\rm H} \pm \sqrt{G^2M(r_{\rm H})^2 + 4GM(r_{\rm H})r_{\rm H}}\right]. \tag{118}$$

In the Schwarzschild black hole case, $2GM(r_{\rm H}) = r_{\rm H}$, the above expression gives $r = 0, 3GM(r_{\rm H})$. The case of $r = 0$ is unphysical because the origin is inside the horizon. The second case $r = 3GM(r_{\rm H})$ gives the standard result. In general, a minus sign in the front of the square root in (118) gives a radius smaller than the horizon radius and, therefore, we choose the plus sign in (118).

Then, Equation (110) gives the radius $r_{\rm sh}$ of the black hole shadow,

$$r_{\rm sh} = \frac{2GM(r_{\rm H})}{8}\left(3 + \sqrt{1+8\eta}\right)^{\frac{3}{2}}\left(-1 + \sqrt{1+8\eta}\right)^{\frac{1}{2}}. \tag{119}$$

Here

$$\eta \equiv \frac{r_{\rm H}}{2GM(r_{\rm H})}\,, \tag{120}$$

which is a unity for a Schwarzschild black hole.

We now compare the obtained result with observation. For M87*, the constraint for the radius is given by $\frac{2r_{\rm sh}}{GM} \sim 11.0 \pm 1.5$ [61] or $\frac{r_{\rm sh}}{GM} \sim 5.5 \pm 0.8$ and For Sgr A*, we have $4.21 \lesssim \frac{r_{\rm sh}}{GM} \lesssim 5.56$ [62]. By using the parameter η (120), the constraint from M87* is rewritten as $0.86 < \eta < 1.33$ and Sgr A* as $0.73 < \eta < 1.11$. Therefore, the results are consistent with aSchwarzschild black hole, where $\eta = 1$.

If by future observations, we find that η cannot be unity, then the black hole is not a Schwarzschild black hole and, therefore, the thermodynamics could be different from that of a Schwarzschild black hole. If we also obtain more information like the ADM mass of the black hole, we may obtain some clues as to what kind of thermodynamics the black hole obeys. In particular, if we obtain information from several black holes, we may find a more universal thermodynamics that governs the black holes.

7. Microcanonical and Canonical Descriptions for Generalised Entropy

From the viewpoint of quantum gravity, the microscopic understanding of generalised entropy could be important and suggestive. In this section, based on [63], we consider the origins of various entropies in microscopic particle descriptions of the thermodynamic system. Note that basically, a microscopic description describes some particle system that obeys the corresponding statistics (entropy). As we have entropy which depends on several parameters, we can eventually propose novel, not yet discovered information and statistical systems that obey these entropies.

7.1. Microcanonical Description

In this subsection, by using the microcanonical ensemble in thermodynamics, we consider how various generalised entropies appear in the isolated system with fixed energy E.

The standard Gibbs entropy is expressed as,

$$\mathcal{S}(E) = -\sum_{i=1}^{W(E)} P_i(E) \ln P_i(E)\,. \tag{121}$$

We choose the Boltzmann constant to be unity. Under the assumption that the number of states with a fixed energy E is $W(E)$ and the probability of realising the i-th state with energy E is denoted by $P_i(E)$, we obtain,

$$\sum_{i=1}^{W(E)} P_i(E) = 1\,. \tag{122}$$

A generalization of the entropy with a parameter δ is proposed in [7] by Tsallis,

$$\mathcal{S}_\delta(E) \equiv \frac{1-\sum_{i=1}^{W(E)} (P_i(E))^\delta}{\delta - 1} = \sum_{i=1}^{W(E)} \frac{P_i(E)\left(1-(P_i(E))^{\delta-1}\right)}{\delta-1}\,. \tag{123}$$

In the limit of $\delta \to 1$, $\mathcal{S}_\delta(E)$ reduces to the standard expression in (121).

A further generalisation is given by the following expression,

$$\mathcal{S}(E) = \sum_{i=1}^{W(E)} s_i(P_i(E)) \,. \tag{124}$$

We may regard $\mathcal{S}(E)$ as a function of P_i and consider the maximum of $\mathcal{S}(E)$ in (124) under the constraint (122), which is just the thermal equilibrium. Then, we obtain an expression of the generalised entropy in the thermal equilibrium in the following form (see [63] for more detailed calculations),

$$\mathcal{S}(E) = W(E) s\left(\frac{1}{W(E)}\right) . \tag{125}$$

In the microcanonical approach, we define the temperature T by

$$\frac{1}{T} \equiv \frac{dS(E)}{dE} \,. \tag{126}$$

This expression corresponds to (121).

We may consider the continuous phase space of N particles $\left(q^i, p_i\right)$ $(i = 1, 2, \cdots, N)$ instead of considering the discrete states, which may be regarded with the limit of $W(E) \to \infty$. In this limit, Equation (124) has the following forms,

$$S = \int_E \prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar}\right) s\left(q^k, p_k, P\left(q^k, p_k, E\right)\right) . \tag{127}$$

We should note that s may generally depend on q^i and p_i explicitly. In (127), $\int_E \prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar}\right) \cdots$ expresses the integration of the phase space for fixed energy E.

In general, the function s includes a finite or infinite number of parameters, $\{\alpha_n\}$, $n = 1, 2, \cdots$, $s = s\left(\{\alpha_n\}; q^k, p_k, P\left(q^k, p_k, E\right)\right)$, In a limit of the parameters, $\{\alpha_n\}$, s may reduce to that in the Gibbs entropy (121). As mentioned in Section 4.3, we impose the following conditions for s with the parameters $\{\alpha_n\}$,

1. Generalised third law: The generalised entropy vanishes when temperature T vanishes as in the case of the Gibbs entropy (121). Note, however, Bekenstein–Hawking entropy $\mathcal{S}$ (11) for black hole diverges when Hawking temperature $T_{\rm H}$ vanishes and $\mathcal{S}$ vanishes when $T_{\rm H} \to \infty$.
2. Monotonicity: The generalised entropy is a monotonically increasing function of the Gibbs entropy (121).
3. Positivity: The generalised entropy should be positive, as the number of states is greater than unity.
4. Gibbs entropy limit: The generalised entropy reduces to the Gibbs entropy (121) in an appropriate limit of the parameters $\{\alpha_n\}$.

In standard thermodynamics, the following zeroth law must be also imposed,

- When two systems denoted by A and B are in thermal equilibrium with a third system denoted by C, the system A is also in equilibrium with the system B.

The zeroth law does not hold in the case of non-extensive entropies like the Tsallis entropy [64]. This tells the generalised entropies do not always satisfy the zeroth law.

As we obtain (125) (see [63] for more detailed calculations), we find

$$S = V_{\rm phase} s\left(\frac{1}{V_{\rm phase}}\right) . \tag{128}$$

Here, V_{phase} is the volume of the phase space, which can be finite because the energy E is fixed. By the choice of s, we obtain several kinds of entropy.

Just for a simple example, we may consider one non-relativistic particle with mass m moving on the two-dimensional space with the area A. Because the energy E is fixed and given by

$$E = \frac{{p_x}^2 + {p_y}^2}{2m}, \tag{129}$$

the volume of the momentum space is equal to the area of a two-dimensional sphere with the radius $\sqrt{2mE}$, $4\pi\left(\sqrt{2mE}\right)^2 = 8\pi mE$. Therefore, we obtain

$$V_{\text{phase}} = \frac{8\pi mEA}{\hbar^3}. \tag{130}$$

By the choice of $s(\xi) = -\xi \ln \xi$, the standard expression of the Gibbs entropy, denoted by $\mathcal{S}_0$ is obtained, which we now denote S_0,

$$\mathcal{S}_0 = \ln V_{\text{phase}} = k \ln\left(\frac{8\pi mEA}{\hbar^3}\right). \tag{131}$$

Conversely, if $s(\xi)$ is given by

$$s(\xi) = \frac{\xi}{\gamma}\left[\left(1 - \frac{\alpha}{\delta}\ln\xi\right)^{\delta} - 1\right], \tag{132}$$

with positive dimensionless parameters (α, γ, δ), we obtain an expression similar to the three-parameter entropy (29) in [4]. Conversely, if $s(\xi)$ is given by

$$s(\xi) = \frac{\xi}{\gamma}\left[\left(1 - \frac{\alpha_+}{\delta}\ln\xi\right)^{\delta} - \left(1 - \frac{\alpha_-}{\delta}\ln\xi\right)^{-\delta}\right], \tag{133}$$

we obtain an expression corresponding to a four-parameters generalised entropy (27) proposed in [5]. It is straightforward to find $s(\xi)$ corresponding to other versions of generalised entropy.

In the case of non-extensive systems, such as gravitational or electromagnetic ones, the standard Gibbs additive entropy (121) should be replaced by the non-extensive Tsallis entropy [7]. The non-extensive entropy indicates that the numbers of states show the running behaviour by the change in the energy scale, as in the renormalisation group of quantum field theory. Because the entropy corresponds to the physical degrees of freedom of a system, the renormalisation group of a quantum theory implies that the degrees of freedom depend on the energy scale. In the low-energy regime, massive modes decouple, and, therefore, the degrees of freedom decrease. In the case of gravity, if the space-time fluctuations become large in the ultraviolet regime, the degrees of freedom might increase. Conversely, if gravity becomes topological, the degrees of freedom decrease. The latter situation is consistent with holography. This could suggest that the generalised entropy might also appear by reflecting the quantum structure of gravity.

7.2. Canonical Description

We now consider the canonical ensemble in thermodynamics, where the system is in equilibrium with the heat bath with temperature T. Even for the canonical ensemble, various versions of entropy appear to originate from the integration measure in the phase space.

The partition function of N particles is defined by

$$Z(\tilde{\beta}) = \int \prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar} \right) \mathrm{e}^{-\tilde{\beta} H\left(q^i, p_i\right)} . \tag{134}$$

Here, q^i and p_i are the coordinates of the position and the momenta for the i-th particle, respectively, as in the last subsection. We define $\tilde{\beta}$ as usual, $\tilde{\beta} \equiv \frac{1}{T}$. The reason why we use the measure $\prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar} \right)$ is because it is invariant under the canonical transformation in classical mechanics. We should note, however, that in quantum mechanics, only the Cartesian coordinates have a special meaning.

More in general, instead of $\prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar} \right)$, we may consider measure given by $\mathrm{e}^{-M\left(q^i, p_i\right)} \prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar} \right)$ and we may define the partition function as follows,

$$Z(\tilde{\beta}) = \int \prod_{i=1}^{N} \left(\frac{dq^i dp_i}{\hbar} \right) \mathrm{e}^{-\tilde{\beta} H\left(q^i, p_i\right) - M\left(q^i, p_i\right)} . \tag{135}$$

In the situation that we confine the particles in the box with edge length L, $M\left(q^i, p_i\right)$ is given by

$$\mathrm{e}^{-M\left(q^i, p_i\right)} = \prod_{i=1}^{N} \theta\left(q^i\right) \theta\left(L - q^i\right), \quad \text{or} \quad M = -\sum_{i=1}^{N} \left(\ln \theta\left(q^i\right) + \ln \theta\left(L - q^i\right) \right). \tag{136}$$

Here, $\theta(\xi)$ is the usual Heaviside step function,

$$\theta(\xi) = \begin{cases} 1 & \text{when } \xi \geq 0 \\ 0 & \text{when } \xi < 0 \end{cases} . \tag{137}$$

We may consider the following model as an example,

$$\begin{aligned} &\left(q^i\right) = (x, y, z), \quad H = \frac{1}{2m} \left(p_x^2 + p_y^2 + p_z^2 \right), \\ &\mathrm{e}^{-M} = 4\pi R^2 \delta\left(R^2 - x^2 - y^2 - z^2 \right) \mathrm{e}^{-X\left(4\pi\left(x^2 + y^2 + z^2\right)\right)} . \end{aligned} \tag{138}$$

Here, X is an adequate function. After the integration in the phase space, we obtain,

$$Z(\tilde{\beta}) = \frac{8\pi^2 R^3}{\hbar^3} \left(\frac{2m\pi}{\tilde{\beta}} \right)^{\frac{3}{2}} \mathrm{e}^{-X\left(4\pi R^2\right)} , \tag{139}$$

which gives the following free energy $F(\tilde{\beta})$,

$$F(\tilde{\beta}) = -\frac{1}{\tilde{\beta}} \ln Z(\tilde{\beta}) = -\frac{1}{\tilde{\beta}} \left(\ln \left(\frac{8\pi^2 R^3}{\hbar^3} \left(\frac{2m\pi}{\tilde{\beta}} \right)^{\frac{3}{2}} \right) - X\left(4\pi R^2\right) \right). \tag{140}$$

This expression give the following thermodynamic energy $E(\tilde{\beta})$,

$$E(\tilde{\beta}) = F(\tilde{\beta}) + \tilde{\beta} \frac{\partial F(\tilde{\beta})}{\partial \tilde{\beta}} = \frac{3}{2\tilde{\beta}} , \tag{141}$$

and the entropy $\mathcal{S}$

$$\mathcal{S} = \tilde{\beta}(E-F) = \left\{ \frac{3}{2} + \ln\left(\frac{8\pi^2 R^3}{\hbar^3} \left(\frac{2m\pi}{\tilde{\beta}} \right)^{\frac{3}{2}} \right) - X\left(4\pi R^2\right) \right\}. \tag{142}$$

If we assume $X(\xi)$ is given by $X(\xi) = -\frac{\xi}{4G}$ with Newton's gravitational constant G, the last term in (142) may dominate for large R, which results in Bekenstein–Hawking entropy,

$$\mathcal{S} = \frac{A}{4G}, \quad A \equiv 4\pi R^2. \tag{143}$$

Conversely, if we choose $X(\xi)$ by $X(\xi) = -\frac{A_0}{4G}\left(\frac{\xi}{A_0}\right)^{\delta}$, the Tsallis entropy in (18) can be obtained,

$$\mathcal{S} \to \frac{A_0}{4G}\left(\frac{A}{A_0}\right)^{\delta}. \tag{144}$$

The function $M(q^i, p_i)$ appearing in the measure should be given by the properties of the corresponding physical system but we can find the measure which gives the corresponding kind of (generalised) entropy.

In the case of Rényi entropy in (12), we find $X(\xi) = -\frac{1}{\alpha}\ln\left(1 + \frac{\alpha\xi}{4G}\right)$. For the three-parameter entropy (29), we obtain

$$X(\xi) = -\frac{1}{\gamma}\left[\left(1 + \frac{\alpha\xi}{4G\delta}\right)^{\delta} - 1\right], \tag{145}$$

Furthermore, a four-parameters generalised entropy (27) is given by

$$X(\xi) = -\frac{1}{\gamma}\left[\left(1 + \frac{\alpha_+\xi}{4G\delta}\right)^{\delta} - \left(1 + \frac{\alpha_-\xi}{4G\delta}\right)^{-\delta}\right]. \tag{146}$$

Thus, we have shown that the function $X(x)$ corresponding to the generalised entropy can be always found.

The general measure may originate from the modification of the commutation relation $\left[q^i, p_j\right] = i\hbar\delta^i{}_j$. We consider the following commutation relation (here, we write the reduced Planck constant or Dirac's constant $\hbar$ explicitly),

$$\left[q^i, p_i\right] = i\hbar \mathrm{e}^{M^i{}_j(q^k, p_k)}, \tag{147}$$

which induces the metric in the phase space as follows,

$$ds^2 = \sum_{i,j=1}^{N} g^j{}_i dq^i dp_j, \quad g^j{}_i \equiv \left(L^{-1}\right)^j{}_i, L^i{}_j \equiv \mathrm{e}^{M^i{}_j(q^k, p_k)}. \tag{148}$$

Here, $\left(L^{-1}\right)^j{}_i$ is the inverse matrix of $L^i{}_j$ when $L^i{}_j$ is regarded as an $N \times N$ matrix, $\sum_{k=1}^{N} L^i{}_k \left(L^{-1}\right)^k{}_j = \delta^i{}_j$. The metric $g^j{}_i$ gives the following volume form,

$$dV = \det\left(g^j{}_i\right) \prod_{i=1}^{N} \left(dq^i dp_i\right). \tag{149}$$

Due to the symplectic structure of the phase space, $\det\left(g^j{}_i\right)$ is Pfaffian. In the case that $\mathrm{e}^{M^i{}_j(q^k,p_k)}$ is proportional to the unit matrix, $\mathrm{e}^{M^i{}_j(q^k,p_k)} = \mathrm{e}^{\frac{1}{N}M(q^i,p_i)}\delta^i{}_j$, dV reduces to the previous expression of the general measure,

$$dV = \hbar^N \mathrm{e}^{-M(q^i,p_i)} \prod_{i=1}^{N}\left(\frac{dq^i dp_i}{\hbar}\right). \tag{150}$$

We should note that we cannot rewrite the metric in (148) and the commutation relations in (147) in a diagonal form like $\left[Q^i, P_i\right] = i\hbar\delta^i{}_j$ by any redefinition of the variables $Q^i = Q^i\left(q^j, p_j\right), P_i = P_i\left(q^j, p_j\right)$ if there is a non-trivial curvature given by the metric in (148).

For the three-parameter entropy (29), using (138) with (145), we find Equation (147) has the following form,

$$\left[q^i, p_j\right] = \frac{i\hbar \mathrm{e}^{\frac{1}{\gamma}\left[\left(1+\frac{\alpha c^3\pi\left(x^2+y^2+z^2\right)}{\hbar G\beta}\right)^{\beta}-1\right]}}{4\pi R^2\delta\left(R^2-x^2-y^2-z^2\right)}\delta^i{}_j\,. \tag{151}$$

Here, the inverse power of the delta function does not have a physical meaning but the delta function $\delta(x)$ can be defined by,

$$\delta(x) \equiv \lim_{\lambda\to\infty}\sqrt{\frac{\lambda}{\pi}}\mathrm{e}^{-\lambda x^2}\,. \tag{152}$$

This suggests that instead of (151), by choosing the parameter λ sufficiently large, we replace the commutation relation in (151) by,

$$\begin{aligned}\left[q^i, p_j\right] &= \frac{i\hbar}{4\pi R^2}\sqrt{\frac{\pi}{\lambda}}\mathrm{e}^{\frac{1}{\gamma}\left[\left(1+\frac{\alpha c^3\pi\left(x^2+y^2+z^2\right)}{\hbar G\beta}\right)^{\beta}-1\right]+\lambda\left(R^2-x^2-y^2-z^2\right)^2}\delta^i{}_j\,,\\ & i,j = x,y,z,\quad \left(q^x,q^y,q^z\right) = (x,y,z)\,,\end{aligned} \tag{153}$$

which might be the origin of the three-parameter entropy (29).

It is known that due to the generalised uncertainty principle based on the introduction of the minimal length [65], the modification of the canonical commutation relations can be generated. The motivation of the minimal length comes from string theory where the minimum size of the fundamental string is finite.

8. Microscopic Interpretation of Generalised Entropy

Except the analogy of the Tsallis entropy in (18) and Barrow entropy [11], the generalised entropies $\mathcal{S}_\mathrm{g}$, which are functions of the Bekenstein–Hawking entropy $\mathcal{S}$, coincide with $\mathcal{S}$ in the limit of $\mathcal{S}\to 0$ and they have the McLaughlin expansion with respect to $\mathcal{S}$,

$$\mathcal{S}_\mathrm{g}\sum_{n=0}^{\infty}\frac{f_\mathrm{g}^{(n)}}{n!}\mathcal{S}^n\,. \tag{154}$$

Here, $f_\mathrm{g}^{(n)}$ is defined by $f_\mathrm{g}^{(n)} \equiv \left.\frac{\partial^n\mathcal{S}_\mathrm{g}}{\partial\mathcal{S}^n}\right|_{\mathcal{S}=0}$ and the functions of the parameters specifying the generalised entropy $\mathcal{S}_\mathrm{g}$. The explicit forms of $f_\mathrm{g}^{(n)}$ for $\mathcal{S}_3$ in (29), the four-parameter one $\mathcal{S}_4$ in (27), the five-parameter one $\mathcal{S}_5$ in (30), and the six-parameter entropy $\mathcal{S}_6$ in (28), see [66]. In this section, based on [66], we consider the generalised entropies in the microscopic viewpoint of the canonical and grand canonical ensembles.

8.1. Canonical Description

In the canonical prescription, the phase space density of a thermodynamic system composed of N particles is expressed as

$$\rho_{\mathrm{c}}\left(q^j, p_j\right) = \frac{\exp\left(-\tilde{\beta}H\right)}{Z(T,V,N)}\,, \tag{155}$$

where $\tilde{\beta} = \frac{1}{T}$. Here, T is the temperature as before and we choose the Boltzmann constant as unity. The index j runs from $j = 1$ to $j = 3N$ and $\left\{q^j, p_j\right\}$ are generalised coordinates and generalised momenta of the system, respectively. We denote the Hamiltonian of the system by $H(q^j, p_j)$ and

$$Z(T,V,N) = \int \frac{d^{3N}qd^{3N}p}{h^{3N}}\mathrm{e}^{-\tilde{\beta}H}\,, \tag{156}$$

is the partition function, which depends on the temperature (T), volume (V), and number of particles (N) of the system. In (156), h is the Planck constant. The expression of the partition function $Z(T,V,N)$ (156) indicates that the total probability is surely unity,

$$\int \frac{d^{3N}qd^{3N}p}{h^{3N}}\rho_{\mathrm{c}}(q^j, p_j) = 1\,, \tag{157}$$

which allows us to define the ensemble average of a general microscopic quantity $v(q^j, p_j)$ by,

$$\langle v(q,p)\rangle = \int \frac{d^{3N}qd^{3N}p}{h^{3N}}v(q,p)\rho_{\mathrm{c}}(q,p)\,. \tag{158}$$

The Gibbs entropy, which we denote by $\mathcal{S}_0$, corresponding to (9) is defined by

$$\mathcal{S}_0 = \langle -\ln\rho_{\mathrm{c}}\rangle = -\int \frac{d^{3N}qd^{3N}p}{h^{3N}}\rho_{\mathrm{c}}\ln\rho_{\mathrm{c}} = \tilde{\beta}\langle H\rangle + \ln Z\,. \tag{159}$$

Here, we have used Equations (156)–(158). Similarly, we obtain the ensemble average of $(-\ln\rho_{\mathrm{c}})^2$ as follows,

$$\left\langle(-\ln\rho_{\mathrm{c}})^2\right\rangle = \int \frac{d^{3N}qd^{3N}p}{h^{3N}}\rho_{\mathrm{c}}(\ln\rho_{\mathrm{c}})^2 = \tilde{\beta}^2\left\langle H^2\right\rangle + 2\tilde{\beta}\langle H\rangle\ln Z + (\ln Z)^2\,, \tag{160}$$

which is rewritten as,

$$\left\langle(-\ln\rho_{\mathrm{c}})^2\right\rangle = {\mathcal{S}_0}^2 + \tilde{\beta}^2\sigma_2(H)\,. \tag{161}$$

Here, we define $\sigma_2(H) = \left\langle H^2\right\rangle - \langle H\rangle^2$. Therefore, we obtain,

$${\mathcal{S}_0}^2 = \left\langle(-\ln\rho_{\mathrm{c}})^2\right\rangle - \tilde{\beta}^2\sigma_2(H)\,, \tag{162}$$

that is, ${\mathcal{S}_0}^2$ is the sum of the ensemble average of $(-\ln\rho_{\mathrm{c}})^2$ and a term including $\sigma_2(H)$. In the standard extensive thermodynamic system, $\frac{\sigma_2(H)}{\langle H\rangle}$ is proportional to $\frac{1}{\sqrt{N}}$, which goes to vanish in the thermodynamic limit, $N \to \infty$, and Equation (162) reduces to the form ${\mathcal{S}_0}^2 = \left\langle(-\ln\rho_{\mathrm{c}})^2\right\rangle$. However, for non-extensive systems, $\frac{\sigma_2(H)}{\langle H\rangle}$ does not vanish even in the thermodynamic limit and ${\mathcal{S}_0}^2$ includes the extra term as in the second term of Equation (162).

By a similar procedure, we obtain ${\mathcal{S}_0}^n$ for a general positive integer n as follows,

$$ {\mathcal{S}_0}^n = \left\langle (-\ln \rho_c)^n \right\rangle - \sum_{l=2}^{n} \frac{n!}{l!(n-l)!} \left(\tilde{\beta}\right)^l \sigma_l(H) (\ln Z)^{n-l} . \tag{163} $$

Here, $\sigma_l(H) = \left\langle H^l \right\rangle - \langle H \rangle^l$. Because $\sigma_1(H) = 0$, we can take the sum of the second term of (163) from $l = 1$, $\sum_{l=2}^{n} \to \sum_{l=1}^{n}$.

Using

$$ \left\langle H^l \right\rangle = \frac{1}{Z} \int \frac{d^{3N}q \, d^{3N}p}{h^{3N}} \mathrm{e}^{-\tilde{\beta}H} H^l , \tag{164} $$

we can express $\sigma_l(H)$ in terms of the partition function Z, as follows,

$$ \sigma_l(H) = (-1)^l \left\{ \frac{1}{Z} \frac{\partial^l Z}{\partial \tilde{\beta}^l} - \left(\frac{1}{Z} \frac{\partial Z}{\partial \tilde{\beta}} \right)^l \right\} . \tag{165} $$

Using the expression of ${\mathcal{S}_0}^n$ in (163) with (165), we define an entropy similar to the form of generalised entropy, as follows,

$$ \begin{aligned} \mathrm{S}_{\mathrm{can}} &= \sum_{n=0}^{\infty} \frac{f_{\mathrm{g}}^{(n)}}{n!} {\mathcal{S}_0}^n \\ &= \sum_{n=0}^{\infty} \frac{f_{\mathrm{g}}^{(n)}}{n!} \left\{ \left\langle (-\ln \rho_c)^n \right\rangle - \sum_{l=2}^{n} \frac{n!}{l!(n-l)!} \left(\tilde{\beta}\right)^l \sigma_l(H) (\ln Z)^{n-l} \right\} . \end{aligned} \tag{166} $$

Especially in the cases of the three-parameter entropy $\mathcal{S}_3$ in (29) and the four-parameter entropy $\mathcal{S}_4$ in (27), we obtain

$$ \begin{aligned} \mathcal{S}_{\mathrm{can}\,3} &= \sum_{n=0}^{\infty} \frac{f_3^{(n)}(\alpha, \delta, \gamma)}{n!} \left\{ \left\langle (-\ln \rho_c)^n \right\rangle - \sum_{l=2}^{n} \frac{n!}{l!(n-l)!} \left(\tilde{\beta}\right)^l \sigma_l(H) (\ln Z)^{n-l} \right\} \\ &= \frac{1}{\gamma} \left[\left(1 + \frac{\alpha}{\delta} \mathcal{S}_0 \right)^{\delta} - 1 \right] , \end{aligned} \tag{167} $$

$$ \begin{aligned} S_{\mathrm{can}\,4} &= \sum_{n=0}^{\infty} \frac{f_4^{(n)}(\alpha_{\pm}, \delta, \gamma)}{n!} \left\{ \left\langle (-\ln \rho_c)^n \right\rangle - \sum_{l=2}^{n} \frac{n!}{l!(n-l)!} \left(\tilde{\beta}\right)^l \sigma_l(H) (\ln Z)^{n-l} \right\} \\ &= \frac{1}{\gamma} \left[\left(1 + \frac{\alpha_+}{\delta} \mathcal{S}_0 \right)^{\delta} - \left(1 + \frac{\alpha_-}{\delta} \mathcal{S}_0 \right)^{-\delta} \right] . \end{aligned} \tag{168} $$

In this way, we may consider the cases of the five-parameter entropy $\mathcal{S}_5$ in (30), and the six-parameter entropy $\mathcal{S}_6$ in (28).

8.2. Grand Canonical Description

The grand canonical phase space density with chemical potential μ in addition to Hamiltonian H is defined by,

$$ \rho_{\mathrm{gc}}\left(q^j, p_j, N\right) = \frac{\exp\{-\tilde{\beta}(H - \mu N)\}}{\mathcal{Z}(T, V, \mu)} . \tag{169} $$

Again, j runs from $j = 1$ to $j = 3N$. Because the particle number N in a grand canonical ensemble fluctuates, a single microstate is characterised by $\{q^j, p_j, N\}$. In (169), $\mathcal{Z}$ is a grand canonical partition function given by,

$$\mathcal{Z}(T,V,\mu)=\sum_N\int\frac{d^{3N}qd^{3N}p}{h^{3N}}\mathrm{e}^{-\tilde{\beta}(H-\mu N)}\,. \tag{170}$$

Then, the ensemble average of a microscopic quantity $v(q^j,p_j,N)$ in grand canonical description is given by,

$$\langle v(q,p,N)\rangle=\sum_N\int\frac{d^{3N}qd^{3N}p}{h^{3N}}v(q,p,N)\rho_{\mathrm{gc}}(q,p,N)\,. \tag{171}$$

For the grand canonical ensemble, the Gibbs entropy $\mathcal{S}_0$ is defined by,

$$\mathcal{S}_0=\langle-\ln\rho_{\mathrm{gc}}\rangle=-\sum_N\int\frac{d^{3N}qd^{3N}p}{h^{3N}}\rho_{\mathrm{gc}}\ln\rho_{\mathrm{gc}}=\tilde{\beta}\langle H\rangle-\mu\langle N\rangle+\ln\mathcal{Z}\,. \tag{172}$$

In general, we obtain,

$$\mathcal{S}_0{}^n=\left\langle\left(-\ln\rho_{\mathrm{gc}}\right)^n\right\rangle-\sum_{l=2}^{n}\frac{n!}{l!(n-l)!}(\tilde{\beta})^l\sigma_l(H-\mu N)(\ln\mathcal{Z})^{n-l}\,. \tag{173}$$

Here, $\sigma_l(H-\mu N)=\left\langle(H-\mu N)^l\right\rangle-\langle H-\mu N\rangle^l$. Because

$$\left\langle(H-\mu N)^l\right\rangle=\frac{1}{\mathcal{Z}}\sum_N\int\frac{d^{3N}q\,d^{3N}p}{h^{3N}}\mathrm{e}^{-\tilde{\beta}(H-\mu N)}(H-\mu N)^l\,, \tag{174}$$

the following expression can be obtained,

$$\sigma_l(H-\mu N)=(-1)^l\left\{\frac{1}{\mathcal{Z}}\frac{\partial^l\mathcal{Z}}{\partial\tilde{\beta}^l}-\left(\frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\tilde{\beta}}\right)^l\right\}. \tag{175}$$

We now define the following entropy in the grand canonical ensemble,

$$\mathcal{S}_{\text{gr-can}}=\sum_{n=0}^{\infty}\frac{f_{\mathrm{g}}^{(n)}}{n!}\left\{\left\langle\left(-\ln\rho_{\mathrm{gc}}\right)^n\right\rangle-\sum_{l=2}^{n}\frac{n!}{l!(n-l)!}(\tilde{\beta})^l\sigma_l(H-\mu N)(\ln\mathcal{Z})^{n-l}\right\}. \tag{176}$$

Then, we also obtain the expressions corresponding to (167).

In the grand canonical description, chemical potential corresponds to the work necessary to add a particle to the system by maintaining the equilibrium of the system. For the system to maintain the equilibrium, the particle must have a certain energy that is comparable to the mean energy of all the other particles.

9. Summary and Discussion

In this review paper, we have first discussed if the Hawking temperature [1,2] in (7) (in the case of Schwarzschild spacetime, we use (8)) and the ADM mass [15] could actually provide the thermodynamic temperature and energy uniquely.

We have considered these problems in Section 2. The Hawking temperature is given by the thermal distribution of the Hawking radiation, which is generated only by the geometry of the object but does not depend on the details of the gravity theory. In this sense, the Hawking temperature is a unique possibility of the thermal temperature. In regards to the ADM mass, if we consider the fall of the dust shell as a "thought experiment", as described in Section 3, by using energy conservation and Birkhoff's theorem [46], the thermodynamic energy must be given by the ADM mass. Then, the thermodynamic

relation $dE = Td\mathcal{S}$ tells us that the entropy of the system should be Bekenstein–Hawking entropy [1,3].

After that, in Section 4, we have explicitly checked if the generalised entropies could yield both the Hawking temperature and the ADM mass correctly. In particular, we have considered Rényi entropy (12) [52–55], in Section 4.1, and Tsallis entropy (18) [7], in Section 4.2. We have further investigated generalised entropies, like the four- and five-parameter generalised entropies, in (27) and (30) [4,5,51] in Section 4.3.

Despite the uniqueness of Bekenstein–Hawking entropy, we consider the possibility that the generalised entropies could become true thermodynamical entropies. One possibility, which we discussed in Section 5, is given by hairy black holes because the energy density of the hair contributes non-trivially to the ADM mass. We have considered the case of a Reissner–Nordström black hole with the hair of the electric field, in Section 5.1, and the case of Einstein's gravity coupled with two scalar fields, in Section 5.2. By using the case of two scalar fields, we can realise an arbitrarily given spherically symmetric spacetime, which can be time-dependent in general [57]. The ghosts in the original model [57] can be eliminated via some constraints [58]. After providing some examples, in Section 5.2.2, in the framework of the model with the two scalar fields, we have proposed two mechanisms to produce the generalised entropies in BH thermodynamics. In one case, Section 5.2.3, we have investigated the possibility that, as in a Reissner–Nordström black hole, the horizon radius is not given only by the ADM mass and, therefore, the entropy becomes a non-trivial function of the Bekenstein–Hawking entropy, as shown for the Rényi entropy in (88) and for arbitrary generalised entropies in (89). We have also considered the case where the effective mass expressing the energy inside the horizon does not give the naive Hawking temperature, as in (98) of Section 5.2.4. We have shown how Rényi entropy (12), Tsallis entropy (18), the three-parameter generalised entropy $\mathcal{S}_3$ (29), and the six-parameter entropy $\mathcal{S}_6$ (28) are generated in Equations (106), (107), (108), and (109), respectively. Therefore, the inconsistency of new entropy proposals, with a Hawking temperature between the area law, could be avoided for the above black holes with one or more hair types. In [67], the thermodynamic relations in the regular black holes were investigated and shown that the naive first law of thermodynamics using Bekenstein–Hawking entropy is broken and the thermodynamic energy should be corrected by a factor. The reasons for the breakdown of the first law should be also due to hair. Hairs coming from the electromagnetic fields, scalar fields, the Gauss–Bonnet terms, etc., outside the black hole horizon, contribute to the ADM mass as in the gravity theories coupled with the two scalar fields as investigated in this paper. Even in general modified gravity theories, there are hairs outside the horizon in general. These hairs contribute to the ADM mass and a breakdown of the naive first law occurs.

The radii of the photon sphere and the black hole shadow have been calculated, for the models found in Section 5.2, in (118) and (119) and we obtained observational constraints on the parameters of the models in Section 6. The parameters are consistent provided the BH is of the Schwarzschild kind. However, there is no direct relation between the shadow and BH thermodynamics.

After that, we reviewed the generalised entropy description in the microcanonical, canonical, and grand canonical ensembles. The origins of the generalised entropies were discussed in Section 7 in the formulations of a microcanonical ensemble, in Section 7.1, and of a canonical ensemble, in Section 7.2. After that, we used the McLaughlin expansion for the generalised entropies in Section 8 and possible interpretations were given.

To date, no observations exist to indicate the possibility that BH entropy should be given by any of the non-extensive entropies. However, eventually, future observations of black hole shadows and primordial gravitational waves from primordial black holes,

as well as cosmological ones, might reveal significant discrepancies with Einsteinian gravity predictions. That would open the window for modified gravity theories and generalised entropies, which could correspond to the ones considered here. Until such observational results are obtained, it is important to be ready and to consider what kind of novel physical effects could appear thanks to the generalised entropies. In parallel, we need to consider how a generalised entropy may follow from a more fundamental, possibly quantum, theory of gravity, like superstring theory. Finally, generalised statistics/entropy may provide new connections between BH thermodynamics, cosmology, and information theory, for instance, via the Landauer principle [68,69].

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