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Gauge-Invariant Perturbation Theory on the Schwarzschild Background Spacetime: Part III—Realization of Exact Solutions

Kouji Nakamura

Gravitational-Wave Science Project, National Astronomical Observatory of Japan, 2-21-1, Osawa, Mitaka 181-8588, Tokyo, Japan; dr.kouji.nakamura@gmail.com

Abstract: This is the Part III paper of our series of papers on a gauge-invariant perturbation theory on the Schwarzschild background spacetime. After reviewing our general framework for the gauge-invariant perturbation theory and the proposal for gauge-invariant treatments of $l = 0, 1$ mode perturbations on the Schwarzschild background spacetime in the Part I paper, we examine the problem of whether the $l = 0, 1$ even-mode solutions derived in the Part II paper are physically reasonable. We consider the linearized versions of the Lemaître–Tolman–Bondi solution and the non-rotating C-metric. As a result, we show that our derived even-mode solutions to the linearized Einstein equations realize these two linearized solutions. This supports the conclusion that our derived solutions are physically reasonable, which implies that our proposal for gauge-invariant treatments of $l = 0, 1$ mode perturbations is also physically reasonable. We also briefly summarize the conclusions of our series of papers.

Keywords: black hole; Schwarzschild spacetime; perturbation theory; gauge invariance

1. Introduction

Gravitational-wave observations are now at the stage where we can directly measure many events through ground-based gravitational-wave detectors [1–4]. One of the future directions of gravitational-wave astronomy will be to establish "a precise science" through the statistics of many events. Toward further development, the projects for future ground-based gravitational-wave detectors [5,6] are progressing to achieve more sensitive detectors, along with some projects for space gravitational-wave antennas [7–10]. Although there are many targets for these detectors, the extreme mass ratio inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar-mass object around a supermassive black hole, is a promising target for the Laser Interferometer Space Antenna [7]. As the mass ratio of this EMRI is very small, we can describe the gravitational waves from EMRIs through black-hole perturbations [11]. Furthermore, the development of higher-order black-hole perturbation theories is required to support gravitational-wave physics as a precise science. The motivation for our series of papers [12–15], including this paper, lies in the theoretical development of black-hole perturbation theories toward higher-order perturbations.

In black-hole perturbation theories, further developments are possible, even in perturbation theories on the Schwarzschild background spacetime. Based on the works by Regge and Wheeler [16] and Zerilli [17,18], many studies on the perturbations in the Schwarzschild background spacetime have been carried out [19–31]. In these works, perturbations are decomposed through the spherical harmonics Y_{lm} because of the spherical symmetry of the background spacetime, and $l = 0, 1$ modes should be treated separately. Due to this separate treatment, "gauge-invariant" treatments of $l = 0$ and $l = 1$ modes are unclear.

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Owing to this situation, in our previous papers [12,14], we proposed the strategy of gauge-invariant treatments for these $l = 0, 1$ mode perturbations, which is referred to as Proposal 1 in the current paper. We have been developing the general formulation of a higher-order gauge-invariant perturbation theory in a generic background spacetime to establish unambiguous, sophisticated non-linear general-relativistic perturbation theories [32–35]. Although we have applied this general framework to cosmological perturbations [36], we apply it to black-hole perturbations in our series of papers, i.e., [12–15], including this paper.

In the Part I paper [14], we derived the linearized Einstein equations in a gauge-invariant manner following Proposal 1. Generally, spherically symmetric background spacetime perturbations are classified into even- and odd-mode perturbations. In the Part I paper [14], we also gave a strategy for solving odd-mode perturbations, including $l = 0, 1$ modes. Furthermore, we derived the explicit solutions for $l = 0, 1$ odd-mode perturbations to the linearized Einstein equations following Proposal 1. Moreover, in the Part II paper [15], we gave a strategy for solving even-mode perturbations, including $l = 0, 1$ modes, and we also derived the explicit solutions for $l = 0, 1$ even-mode perturbations following Proposal 1.

In this paper, we investigate whether the solutions for even-mode perturbations derived in the Part II paper [15] are physically reasonable. We consider the correspondence between our linearized solutions and two exact solutions. One of the exact solutions discussed in this paper is the Lemaître–Tolman–Bondi (LTB) solution, and the other is the non-rotating C-metric [37,38]. This series of papers is the full version of our short paper [12].

The organization of this paper is as follows. In Section 2, after briefly reviewing the framework of the gauge-invariant perturbation theory, we summarize our proposal in ref. [12,14]. Then, we summarize the linearized even-mode Einstein equations in the Schwarzschild background spacetime derived in ref. [14]. These are based on Proposal 1. In Section 3, we discuss the realization of the LTB solution using our derived solutions for $l = 0, 1$-mode even-mode perturbations to the linearized Einstein equations following Proposal 1. In Section 4, we discuss the realization of the C-metric from our solutions derived in the Part II paper [15]. The final section (Section 5) is devoted to a summary and discussion. This final section also includes conclusions from this series of papers on a gauge-invariant perturbation theory in the Schwarzschild background spacetime.

This paper uses the same notation used in our previous papers [12–14], as well as the unit $G = c = 1$, where G is Newton's constant of gravitation and c is the velocity of light.

2. Brief Review of the General-Relativistic Gauge-Invariant Perturbation Theory

In this section, we review the premise of our series of papers [12,14,15], which is necessary to understand the contents of this paper. In Section 2.1, we briefly review our framework for the gauge-invariant perturbation theory [32,33]. This is an important premise of our series of papers [12,14,15], including this paper. Section 2.2 reviews the gauge-invariant perturbation theory on spherically symmetric spacetimes, including our proposal in refs. [12,14]. In Section 2.3, we summarize the even-mode linearized Einstein equations in the Schwarzschild background spacetime and their explicit solutions for $l = 0, 1$ modes, which are necessary for the arguments in this paper.

2.1. General Framework of Gauge-Invariant Perturbation Theory

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$, which is identified with nature itself, and we want to

describe this spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$ through perturbations. The other is the background spacetime $(\mathscr{M}, g_{ab})$, introduced as a reference based on our convenience. Note that these two spacetimes are distinct. Furthermore, in any perturbation theory, we always write equations for the perturbation of the variable Q as follows:

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{1}$$

Equation (1) gives a relation between variables on different manifolds. In actuality, $Q(\text{“}p\text{”})$ on the left-hand side of Equation (1) is a variable on $\mathscr{M}_\epsilon = \mathscr{M}_{\rm ph}$, whereas $Q_0(p)$ and $\delta Q(p)$ on the right-hand side of Equation (1) are variables on $\mathscr{M}$. Because we regard Equation (1) as a field equation, Equation (1) includes an implicit assumption of the existence of a point identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon : p \in \mathscr{M} \mapsto \text{“}p\text{”} \in \mathscr{M}_\epsilon$. This identification map is a “gauge choice” in general-relativistic perturbation theories. This is the notion of the “second-kind gauge” pointed out by Sachs [39]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on a single manifold, called the “first-kind gauge” [14]. This distinction between the first and the second kind of gauges is extensively explained in the Part I paper [14] and is also important for understanding the results in Sections 3 and 4 in this paper.

To compare the variable Q on $\mathscr{M}_\epsilon$ with its background value Q_0 on $\mathscr{M}$, we use the pull-back $\mathscr{X}_\epsilon^*$ of the identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon$ and we evaluate the pulled-back variable $\mathscr{X}_\epsilon^* Q$ on the background spacetime $\mathscr{M}$. Furthermore, in perturbation theories, we expand the pull-back operation $\mathscr{X}_\epsilon^*$ to the variable Q with respect to the infinitesimal parameter ϵ for the perturbation as follows:

$$\mathscr{X}_\epsilon^* Q = Q_0 + \epsilon \overset{(1)}{{}_{\mathscr{X}}Q} + O(\epsilon^2). \tag{2}$$

Equation (2) is evaluated on the background spacetime $\mathscr{M}$. When we have two different gauge choices $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$, we can consider the “gauge transformation”, which is the change of the point identification $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$. This gauge transformation is given by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon : \mathscr{M} \to \mathscr{M}$. In actuality, the diffeomorphism Φ_ϵ induces a pull-back from the representation $\mathscr{X}_\epsilon^* Q_\epsilon$ to the representation $\mathscr{Y}_\epsilon^* Q_\epsilon$ as $\mathscr{Y}_\epsilon^* Q_\epsilon = \Phi_\epsilon^* \mathscr{X}_\epsilon^* Q_\epsilon$. From general arguments of the Taylor expansion [40], the pull-back Φ_ϵ^* is expanded as

$$\mathscr{Y}_\epsilon^* Q_\epsilon \quad = \quad \mathscr{X}_\epsilon^* Q_\epsilon + \epsilon \pounds_{\xi_{(1)}} \mathscr{X}_\epsilon^* Q_\epsilon + O(\epsilon^2), \tag{3}$$

where $\xi^a_{(1)}$ is the generator of Φ_ϵ. From Equations (2) and (3), the gauge transformation for the first-order perturbation $^{(1)}Q$ is given by

$$\overset{(1)}{{}_{\mathscr{Y}}Q} - \overset{(1)}{{}_{\mathscr{X}}Q} \quad = \quad \pounds_{\xi_{(1)}} Q_0. \tag{4}$$

We also employ the “order by order gauge invariance” as a concept of gauge invariance [41]. We call the kth-order perturbation $\overset{(k)}{{}_{\mathscr{X}}Q}$ gauge invariant if and only if

$$\overset{(k)}{{}_{\mathscr{X}}Q} = \overset{(k)}{{}_{\mathscr{Y}}Q} \tag{5}$$

for any gauge choice $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$.

Based on the above setup, we proposed a formulation to construct gauge-invariant variables of higher-order perturbations [32,33]. First, we expand the metric on the physical spacetime $\mathscr{M}_\epsilon$, which was pulled back to the background spacetime $\mathscr{M}$ through a gauge choice $\mathscr{X}_\epsilon$ as follows:

$$\mathscr{X}_\epsilon^* \bar{g}_{ab} \quad = \quad g_{ab} + \epsilon \, {}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \tag{6}$$

Although the expression (6) depends entirely on the gauge choice $\mathscr{X}_\epsilon$, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\epsilon$ in the expression if there is no possibility of confusion. The important premise of our formulation of higher-order gauge-invariant perturbation theory was the following conjecture [32,33] for the linear metric perturbation h_{ab}:

Conjecture 1. *If the gauge transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab}, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

We call $\mathscr{F}_{ab}$ and Y^a the "gauge-invariant" and "gauge-dependent" parts of h_{ab}, respectively.

The proof of Conjecture 1 is highly non-trivial [34], and it was found that the gauge-invariant variables are essentially non-local. Despite this non-triviality, once we accept Conjecture 1, we can decompose the linear perturbation of an arbitrary tensor field ${}^{(1)}_{\mathscr{X}}Q$, whose gauge transformation is given by Equation (4), through the gauge-dependent part Y_a of the metric perturbation in Conjecture 1 as

$$ {}^{(1)}_{\mathscr{X}}Q = {}^{(1)}\mathscr{Q} + \pounds_{\mathscr{X}Y} Q_0, \tag{7} $$

where ${}^{(1)}\mathscr{Q}$ is the gauge-invariant part of the perturbation ${}^{(1)}_{\mathscr{X}}Q$. For example, the linearized Einstein tensor ${}^{(1)}_{\mathscr{X}}G_a{}^b$ or the linear perturbation of the energy-momentum tensor ${}^{(1)}_{\mathscr{X}}T_a{}^b$ are also decomposed as

$$ {}^{(1)}_{\mathscr{X}}G_a{}^b = {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] + \pounds_{\mathscr{X}Y} G_a{}^b, \quad {}^{(1)}_{\mathscr{X}}T_a{}^b = {}^{(1)}\mathscr{T}_a{}^b + \pounds_{\mathscr{X}Y} T_a{}^b, \tag{8} $$

where G_{ab} and T_{ab} are the background values of the Einstein tensor and the energy-momentum tensor, respectively. The explicit form of the gauge-invariant part ${}^{(1)}\mathscr{G}_a{}^b$ of the linear-order perturbation of the Einstein tensor is not important within this paper. Using the background Einstein equation $G_a{}^b = 8\pi T_a{}^b$, the linearized Einstein equation ${}^{(1)}_{\mathscr{X}}G_{ab} = 8\pi {}^{(1)}_{\mathscr{X}}T_{ab}$ is automatically given in the gauge-invariant form

$$ {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] = 8\pi {}^{(1)}\mathscr{T}_a{}^b[\mathscr{F}, \phi], \tag{9} $$

even if the background Einstein equation is non-trivial. Here, "ϕ" in Equation (9) symbolically represents the matter degree of freedom.

For the componentsof this paper, it is important to note that the decomposition of the metric perturbation h_{ab} into its gauge-invariant part $\mathscr{F}_{ab}$ and into its gauge-dependent part Y^a is not unique, as noted in refs. [14,41]. In actuality, the decomposition of the metric perturbation h_{ab} is also given by

$$ h_{ab} = \mathscr{F}_{ab} - \pounds_Z g_{ab} + \pounds_{Z+Y} g_{ab} =: \mathscr{H}_{ab} + \pounds_X g_{ab}, \tag{10} $$

where Z^a is gauge-invariant in the second kind, i.e., ${}_{\mathscr{Y}}Z^a - {}_{\mathscr{X}}Z^a = 0$. The tensor field $\mathscr{H}_{ab} := \mathscr{F}_{ab} - \pounds_Z g_{ab}$ is also regarded as the gauge-invariant part (in the sense of the second kind) of the perturbation h_{ab} because ${}_{\mathscr{Y}}\mathscr{H}_{ab} - {}_{\mathscr{X}}\mathscr{H}_{ab} = 0$. Similarly, the vector field $X^a := Z^a + Y^a$ is also regarded as the gauge-dependent part of the perturbation h_{ab} because ${}_{\mathscr{Y}}X^a - {}_{\mathscr{X}}X^a = \xi^a_{(1)}$. The difference between the variables $\mathscr{H}_{ab}$ and $\mathscr{F}_{ab}$ is given by $\pounds_{-Z} g_{ab}$. Here, we note that if the gauge-invariant variable $\mathscr{F}_{ab}$ is a solution to the linearized Einstein

Equation (9), $\mathscr{H}_{ab} = \mathscr{F}_{ab} - \pounds_Z g_{ab}$ is also a solution to the linearized Einstein equation (9) due to a symmetry of the linearized Einstein equation (9), as explained in the Part I paper [14]. This implies that the terms in the form $\pounds_Z g_{ab}$ may always appear in the solutions to the linearized Einstein equation due to the above symmetry of the linearized Einstein equation. As our formulation already excludes the second-kind gauge completely, we should regard the gauge-invariant term $\pounds_{-Z} g_{ab}$ as the first-kind gauge of the background spacetime induced by the infinitesimal coordinate transformations on the physical spacetime $\mathscr{M}_\epsilon$, as discussed in the Part I paper [14].

2.2. Linear Perturbations on Spherically Symmetric Background

Here, we consider the 2 + 2 formulation of the perturbation of a spherically symmetric background spacetime, which was originally proposed by Gerlach and Sengupta [23–26]. Spherically symmetric spacetimes are characterized by the direct product $\mathscr{M} = \mathscr{M}_1 \times S^2$, and their metric is

$$g_{ab} = y_{ab} + r^2 \gamma_{ab}, \tag{11}$$

$$y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \tag{12}$$

where $x^A = (t, r)$, $x^p = (\theta, \phi)$, and γ_{pq} is the metric on the unit sphere. In the Schwarzschild spacetime, the metric (11) is given by

$$y_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \quad f = 1 - \frac{2M}{r}, \tag{13}$$

$$\gamma_{ab} = (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b = \theta_a\theta_b + \phi_a\phi_b, \tag{14}$$

$$\theta_a = (d\theta)_a, \quad \phi_a = \sin\theta(d\phi)_a. \tag{15}$$

On this background spacetime $(\mathscr{M}, g_{ab})$, the components of the metric perturbation are given by

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} + h_{pq}(dx^p)_a(dx^q)_b. \tag{16}$$

Here, we note that the components h_{AB}, h_{Ap}, and h_{pq} are regarded as components of scalar, vector, and tensor on S^2, respectively. In the Part I paper [14], we showed the linear independence of the set of harmonic functions

$$\left\{ S_\delta, \hat{D}_p S_\delta, \epsilon_{pq}\hat{D}^q S_\delta, \frac{1}{2}\gamma_{pq} S_\delta, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\right) S_\delta, 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta \right\}, \tag{17}$$

where $\hat{D}_p$ is the covariant derivative associated with the metric γ_{pq} on S^2, $\hat{D}^p = \gamma^{pq}\hat{D}_q$, and $\epsilon_{pq} = \epsilon_{[pq]} = 2\theta_{[p}\phi_{q]}$ is the totally antisymmetric tensor on S^2. In the set of harmonic function (17), the scalar harmonic function S_δ is given by

$$S_\delta = \begin{cases} Y_{lm} & \text{for} \quad l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} \quad l = 1; \\ k_{(\hat{\Delta})} & \text{for} \quad l = 0. \end{cases} \tag{18}$$

Here, functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$ are the kernel modes of the derivative operator $\hat{\Delta}$ and $[\hat{\Delta}+2]$, respectively, and we employ the explicit form of these functions as follows:

$$k_{(\hat{\Delta})} = 1+\delta\ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}, \quad \delta\in\mathbb{R}, \tag{19}$$

$$k_{(\hat{\Delta}+2,m=0)} = \cos\theta+\delta\left(\frac{1}{2}\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta}-1\right), \quad \delta\in\mathbb{R}, \tag{20}$$

$$k_{(\hat{\Delta}+2,m=\pm1)} = \left[\sin\theta+\delta\left(+\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta}+\cot\theta\right)\right]e^{\pm i\phi}. \tag{21}$$

Then, we consider the mode decomposition of the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ as follows:

$$h_{AB} = \sum_{l,m}\tilde{h}_{AB}S_\delta, \tag{22}$$

$$h_{Ap} = r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_pS_\delta+\tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^qS_\delta\right], \tag{23}$$

$$h_{pq} = r^2\sum_{l,m}\left[\frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)}S_\delta+\tilde{h}_{(e2)}\left(\hat{D}_p\hat{D}_q-\frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_\delta+2\tilde{h}_{(o2)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^rS_\delta\right]. \tag{24}$$

As the linear independence of each element of the set of harmonic functions (17) is guaranteed, the one-to-one correspondence between the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ and the mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ with the decomposition formulae (22)–(24) is guaranteed, including $l=0,1$ mode if $\delta\neq0$. Then, the mode-by-mode analysis including $l=0,1$ is possible when $\delta\neq0$. However, the mode functions (19)–(21) are singular if $\delta\neq0$. When $\delta=0$, we have $k_{(\hat{\Delta})}\propto Y_{00}$ and $k_{(\hat{\Delta}+2)m}\propto Y_{1m}$. Because of this situation, we proposed the following strategy:

Proposal 1. *We decompose the metric perturbation h_{ab} on the background spacetime with the metric (11)–(14) through Equations (22)–(24) with the harmonic function S_δ given by Equation (18). Then, Equations (22)–(24) become invertible, including $l=0,1$ modes. After deriving the mode-by-mode field equations such as linearized Einstein equations by using the harmonic functions S_δ, we choose $\delta=0$ as regular boundary condition for solutions when we solve these field equations.*

As shown in the Part I paper [14], once we accept Proposal 1, then Conjecture 1 becomes the following statement:

Theorem 1. *If the gauge transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab}-{}_{\mathscr{X}}h_{ab}=\pounds_{\xi_{(1)}}g_{ab}$ with the background metric g_{ab} with spherically symmetry, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab}=:\mathscr{F}_{ab}+\pounds_Yg_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed into ${}_{\mathscr{Y}}\mathscr{F}_{ab}-{}_{\mathscr{X}}\mathscr{F}_{ab}=0$ and ${}_{\mathscr{Y}}Y^a-{}_{\mathscr{X}}Y^a=\xi^a_{(1)}$ under the gauge transformation, respectively.*

In actuality, the gauge-dependent variable Y_a is given by

$$Y_a := \sum_{l,m}\tilde{Y}_AS_\delta(dx^A)_a+\sum_{l,m}\left(\tilde{Y}_{(e1)}\hat{D}_pS_\delta+\tilde{Y}_{(o1)}\epsilon_{pq}\hat{D}^qS_\delta\right)(dx^p)_a, \tag{25}$$

where

$$\tilde{Y}_A := r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)}, \tag{26}$$

$$\tilde{Y}_{(e1)} := \frac{r^2}{2}\tilde{h}_{(e2)}, \tag{27}$$

$$\tilde{Y}_{(o1)} := -r^2\tilde{h}_{(o2)}. \tag{28}$$

Furthermore, including $l = 0, 1$ modes, the components of the gauge-invariant part $\mathscr{F}_{ab}$ of the metric perturbation h_{ab} is given by

$$\mathscr{F}_{AB} = \sum_{l,m}\tilde{F}_{AB}S_\delta, \tag{29}$$

$$\mathscr{F}_{Ap} = r\sum_{l,m}\tilde{F}_A\epsilon_{pq}\hat{D}^qS_\delta, \quad \hat{D}^p\mathscr{F}_{Ap} = 0, \tag{30}$$

$$\mathscr{F}_{pq} = \frac{1}{2}\gamma_{pq}r^2\sum_{l,m}\tilde{F}S_\delta, \tag{31}$$

where $\tilde{F}_{AB}$, $\tilde{F}_A$, and $\tilde{F}$ are given by

$$\tilde{F}_{AB} := \tilde{h}_{AB} - 2\bar{D}_{(A}\tilde{Y}_{B)}, \tag{32}$$

$$\tilde{F}_A := \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)}, \tag{33}$$

$$\tilde{F} := \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^Ar + \tilde{h}_{(e2)}l(l+1). \tag{34}$$

Thus, we have constructed gauge-invariant metric perturbations on the Schwarzschild background spacetime including $l = 0, 1$ modes.

To discuss the linearized Einstein equation (9) and the linear perturbation of the continuity equation

$$\nabla_a{}^{(1)}\mathscr{T}_b{}^a = 0 \tag{35}$$

of the gauge-invariant energy-momentum tensor ${}^{(1)}\mathscr{T}_b{}^a := g^{ac}{}^{(1)}\mathscr{T}_{bc}$ on a vacuum background spacetime, we consider the mode decomposition of the gauge-invariant part ${}^{(1)}\mathscr{T}_{bc}$ of the linear perturbation of the energy-momentum tensor through the set (17) of the harmonics as follows:

$$\begin{aligned}{}^{(1)}\mathscr{T}_{ab} = {} & \sum_{l,m}\tilde{T}_{AB}S_\delta(dx^A)_a(dx^B)_b + r\sum_{l,m}\left\{\tilde{T}_{(e1)A}\hat{D}_pS_\delta + \tilde{T}_{(o1)A}\epsilon_{pr}\hat{D}^rS_\delta\right\}2(dx^A)_{(a}(dx^p)_{b)} \\ & + r^2\sum_{l,m}\left\{\tilde{T}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_qS_\delta - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^rS_\delta\right)\right. \\ & \left. + \tilde{T}_{(o2)}2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^rS_\delta\right\}(dx^p)_a(dx^q)_b. \end{aligned} \tag{36}$$

In terms of these mode coefficients, the components of the continuity equation (35) for the gauge-invariant part of the linearized energy-momentum tensor are summarized as follows:

$$\bar{D}^C\tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^Dr)\tilde{T}_D{}^B - \frac{1}{r}l(l+1)\tilde{T}^B_{(e1)} - \frac{1}{r}(\bar{D}^Br)\tilde{T}_{(e0)} = 0, \tag{37}$$

$$\bar{D}^C\tilde{T}_{(e1)C} + \frac{3}{r}(\bar{D}^Cr)\tilde{T}_{(e1)C} + \frac{1}{2r}\tilde{T}_{(e0)} - \frac{1}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0, \tag{38}$$

$$\bar{D}^C\tilde{T}_{(o1)C} + \frac{3}{r}(\bar{D}^Dr)\tilde{T}_{(o1)D} + \frac{1}{r}(l-1)(l+2)\tilde{T}_{(o2)} = 0. \tag{39}$$

In the Part I paper [14], we derived the linearized Einstein equations, discussed the odd-mode perturbation $\tilde{F}_{Ap}$ in Equation (30), and derived the $l = 1$ odd-mode solutions to these equations. The Einstein equation for even mode $\tilde{F}_{AB}$ and $\tilde{F}$ in Equations (29) and (31), respectively, also derived in the Part I paper [14], and $l = 0, 1$ even-mode solutions are derived in the Part II paper [15]. As these solutions include the Kerr parameter perturbation and the Schwarzschild mass parameter perturbation of the linear order in the vacuum case, these are physically reasonable. Then, we conclude that our proposal is also physically reasonable. This paper aims to check that our derived solutions include the linearized LTB solution and non-rotating C-metric with the Schwarzschild background. For this purpose, even-mode solutions are necessary. Therefore, we review the strategy to derive the even-mode solutions below.

2.3. Even-Mode Linearized Einstein Equations

The even-mode part of the linearized Einstein Equation (9) is summarized as follows:

$$\tilde{F}_D{}^D = -16\pi r^2 \tilde{T}_{(e2)}, \tag{40}$$

$$\bar{D}^D \tilde{\mathbb{F}}_{AD} - \frac{1}{2}\bar{D}_A \tilde{F} = 16\pi \left[r\tilde{T}_{(e1)A} - \frac{1}{2} r^2 \bar{D}_A \tilde{T}_{(e2)} \right] =: 16\pi S_{(ec)A}, \tag{41}$$

where the variable $\tilde{\mathbb{F}}_{AB}$ is the traceless part of the variable $\tilde{F}_{AB}$ defined by

$$\tilde{\mathbb{F}}_{AB} := \tilde{F}_{AB} - \frac{1}{2} y_{AB} \tilde{F}_C{}^C. \tag{42}$$

We also have the evolution equations

$$\left(\bar{D}_D \bar{D}^D + \frac{2}{r}(\bar{D}^D r)\bar{D}_D - \frac{(l-1)(l+2)}{r^2} \right) \tilde{F} - \frac{4}{r^2}(\bar{D}_C r)(\bar{D}_D r)\tilde{\mathbb{F}}^{CD} = 16\pi S_{(F)}, \tag{43}$$

$$S_{(F)} := \tilde{T}_C{}^C + 4(\bar{D}_D r)\tilde{T}^D_{(e1)} - 2r(\bar{D}_D r)\bar{D}^D \tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)}. \tag{44}$$

$$\begin{aligned} &\left[-\bar{D}_D \bar{D}^D - \frac{2}{r}(\bar{D}_D r)\bar{D}^D + \frac{4}{r}(\bar{D}^D \bar{D}_D r) + \frac{l(l+1)}{r^2} \right] \tilde{\mathbb{F}}_{AB} \\ &\quad + \frac{4}{r}(\bar{D}^D r)\bar{D}_{(A}\tilde{\mathbb{F}}_{B)D} - \frac{2}{r}(\bar{D}_{(A} r)\bar{D}_{B)}\tilde{F} \\ &= 16\pi S_{(\mathbb{F})AB}, \end{aligned} \tag{45}$$

$$\begin{aligned} S_{(\mathbb{F})AB} := {} & \tilde{T}_{AB} - \frac{1}{2} y_{AB} \tilde{T}_C{}^C - 2\left(\bar{D}_{(A}(r\tilde{T}_{(e1)B)}) - \frac{1}{2} y_{AB} \bar{D}^D (r\tilde{T}_{(e1)D}) \right) \\ & + 2\left((\bar{D}_{(A} r)\bar{D}_{B)} - \frac{1}{2} y_{AB} (\bar{D}^D r)\bar{D}_D \right)(r\tilde{T}_{(e2)}) \\ & + r\left(\bar{D}_A \bar{D}_B - \frac{1}{2} y_{AB} \bar{D}^D \bar{D}_D \right)(r\tilde{T}_{(e2)}) \\ & + 2\left((\bar{D}_A r)(\bar{D}_B r) - \frac{1}{2} y_{AB} (\bar{D}^C r)(\bar{D}_C r) \right)\tilde{T}_{(e2)} \\ & + 2y_{AB}(\bar{D}^C r)\tilde{T}_{(e1)C} - r y_{AB}(\bar{D}^C r)\bar{D}_C \tilde{T}_{(e2)}, \end{aligned} \tag{46}$$

for the variable $\tilde{F}$ and the traceless variable $\tilde{\mathbb{F}}_{AB}$. Of course, we have to take into account the even-mode part of the continuity equations (37) and (38) of the linearized energy-momentum tensor. We note that these equations are valid not only for $l \geq 2$ modes but also $l = 0, 1$ modes in our formulation.

To evaluate Equations (43)–(46), it is convenient to introduce the component $X_{(e)}$ and $Y_{(e)}$ of the traceless variable $\tilde{\mathbb{F}}_{AB}$ by

$$\tilde{\mathbb{F}}_{AB} =: X_{(e)}\Big\{-f(dt)_A(dt)_B - f^{-1}(dr)_A(dr)_B\Big\} + 2Y_{(e)}(dt)_{(A}(dr)_{B)}, \tag{47}$$

and the Moncrief variable $\Phi_{(e)}$ defined by

$$\Phi_{(e)} := \frac{r}{\Lambda}\left[fX_{(e)} - \frac{1}{4}\Lambda\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}\right], \tag{48}$$

where

$$\Lambda = \mu + 3(1-f), \quad \mu := (l-1)(l+2). \tag{49}$$

From Equations (41) and (45), we obtain the initial value constraints for the variable $\tilde{F}$ and $Y_{(e)}$ as follows:

$$\begin{aligned} l(l+1)\Lambda\tilde{F} &= -8f\Lambda\partial_r\Phi_{(e)} + \frac{4}{r}[6f(1-f) - l(l+1)\Lambda]\Phi_{(e)} - 64\pi r^2 S_{(\Lambda\tilde{F})}, & (50)\\ l(l+1)Y_{(e)} &= r\partial_t\Big(2X_{(e)} + r\partial_r\tilde{F}\Big) + \frac{3f-1}{2f}r\partial_t\tilde{F} + 16\pi r^2 S_{(Y_{(e)})}, & (51) \end{aligned}$$

where the source term $S_{(\Lambda\tilde{F})}$ and $S_{(Y_{(e)})}$ are given by

$$\begin{aligned} S_{(\Lambda\tilde{F})} &:= \tilde{T}_{tt} + rf^2\partial_r\tilde{T}_{(e2)} + 2f(f+1)\tilde{T}_{(e2)} + \frac{1}{2}f(l-1)(l+2)\tilde{T}_{(e2)}, & (52)\\ S_{(Y_{(e)})} &:= \tilde{T}_{tr} + r\partial_t\tilde{T}_{(e2)}. & (53) \end{aligned}$$

Furthermore, we obtain the evolution equations for the variables $\Phi_{(e)}$ and $\tilde{F}$ as follows:

$$\begin{aligned} -\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\Big[f\partial_r\Phi_{(e)}\Big] - V_{even}\Phi_{(e)} &= 16\pi\frac{r}{\Lambda}S_{(\Phi_{(e)})}, & (54)\\ -\frac{1}{f}\partial_t^2\tilde{F} + \partial_r(f\partial_r\tilde{F}) + \frac{1}{r^2}3(1-f)\tilde{F} + \frac{4\Lambda}{r^3}\Phi_{(e)} &= 16\pi S_{(F)}, & (55) \end{aligned}$$

where the potential function V_{even} in Equation (54) is defined by

$$V_{even} := \frac{1}{r^2\Lambda^2}\Big[\Lambda^3 - 2(2-3f)\Lambda^2 + 6(1-3f)(1-f)\Lambda + 18f(1-f)^2\Big], \tag{56}$$

and the source terms in Equations (54) and (55) are given by

$$\begin{aligned} S_{(\Phi_{(e)})} :=\ & \frac{1}{2}\left(\frac{\Lambda}{2f} - 1\right)\tilde{T}_{tt} + \frac{1}{2}\left((2-f) - \frac{1}{2}\Lambda\right)f\tilde{T}_{rr} - \frac{1}{2}r\partial_r\tilde{T}_{tt} + \frac{1}{2}f^2r\partial_r\tilde{T}_{rr}\\ & -\frac{f}{2}\tilde{T}_{(e0)} - l(l+1)f\tilde{T}_{(e1)r}\\ & +\frac{1}{2}r^2\partial_t^2\tilde{T}_{(e2)} - \frac{1}{2}f^2r^2\partial_r^2\tilde{T}_{(e2)} - \frac{1}{2}3(1+f)rf\partial_r\tilde{T}_{(e2)}\\ & -\frac{1}{2}(7-3f)f\tilde{T}_{(e2)} + \frac{1}{4}(l(l+1) - 1 - f)(l(l+1)+2)\tilde{T}_{(e2)}\\ & -\frac{3(1-f)}{\Lambda}\left[\tilde{T}_{tt} + rf^2\partial_r\tilde{T}_{(e2)} + \frac{1}{2}(1+7f)f\tilde{T}_{(e2)}\right], \end{aligned} \tag{57}$$

$$S_{(F)} := -\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr} + 4f\tilde{T}_{(e1)r} - 2rf\partial_r\tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)}, \tag{58}$$

respectively. The consistency of evolution equations (54) and (55) with the initial value constraint (50) leads the identity

$$\begin{aligned} 0 &= r^2\Lambda\partial_t^2 S_{(\Lambda\tilde{F})} - \left[(5-3f)\Lambda + 3(1-f)(1+f) + 18\frac{1}{\Lambda}f(1-f)^2\right] f S_{(\Lambda\tilde{F})} \\ &\quad -2[3(1-f)+2\Lambda]f^2 r\partial_r S_{(\Lambda\tilde{F})} - \Lambda r^2 f\partial_r\left[f\partial_r S_{(\Lambda\tilde{F})}\right] \\ &\quad +\frac{1}{4}[(1-3f)-\Lambda]\Lambda^2 f S_{(F)} \\ &\quad -2rf^2\Lambda\partial_r S_{(\Phi_{(e)})} - [\Lambda+(1+3f)]\Lambda f S_{(\Phi_{(e)})}. \end{aligned} \tag{59}$$

In actuality, we can confirm Equation (59) from the definitions (52), (57), and (58), and the continuity equations (37) and (38), i.e.,

$$-\partial_t\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rt} + \frac{(1+f)f}{r}\tilde{T}_{rt} - \frac{f}{r}l(l+1)\tilde{T}_{(e1)t} = 0, \tag{60}$$

$$-\partial_t\tilde{T}_{tr} + \frac{1-f}{2rf}\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rr} + \frac{(3+f)f}{2r}\tilde{T}_{rr} - \frac{f}{r}l(l+1)\tilde{T}_{(e1)r} - \frac{f}{r}\tilde{T}_{(e0)} = 0, \tag{61}$$

$$-\partial_t\tilde{T}_{(e1)t} + f^2\partial_r\tilde{T}_{(e1)r} + \frac{(1+2f)f}{r}\tilde{T}_{(e1)r} + \frac{f}{2r}\tilde{T}_{(e0)} - \frac{f}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0. \tag{62}$$

For the mode with $l \neq 0$, the master equation (54) is solved through appropriate boundary conditions for the Cauchy problem and we obtain the Moncrief variable $\Phi_{(e)}$. Then, we obtain the variable $\tilde{F}$ through Equation (50). From the solution $(\Phi_{(e)}, \tilde{F})$, we obtain the component $X_{(e)}$ through the definition (48) of the Moncrief variable. Through the solution $(\Phi_{(e)}, \tilde{F}, X_{(e)})$, we obtain the component $Y_{(e)}$ through Equation (51). We can check the evolution equation (55) as a consistency check of solutions. Together with Equation (40), we obtain the solution $(\tilde{\mathbb{F}}_{AB}, \tilde{F})$ as a solution to the linearized Einstein equations when $l \neq 0$.

In actuality, from the above strategy, for the $l = 1$-mode perturbation, we can derive the solution to the linearized Einstein equation through the strategy for $l \neq 0$ mode perturbation described above. For $m = 0$ mode, in the Part II paper [15], we derived the following solution to the linearized Einstein equation:

$$\begin{aligned} \mathscr{F}_{ab} &= \pounds_V g_{ab} - \frac{16\pi r^2}{3(1-f)}\Bigg[f^2\left\{\frac{1+f}{2}\tilde{T}_{rr} + rf\partial_r\tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r}\right\}(dt)_a(dt)_b \\ &\quad +\frac{2r}{f}\left\{\partial_t\tilde{T}_{tt} - \frac{3f(1-f)}{2r}\tilde{T}_{tr}\right\}(dt)_{(a}(dr)_{b)} \\ &\quad +\frac{r}{f}\left\{\partial_r\tilde{T}_{tt} - \frac{3(1-3f)}{2rf}\tilde{T}_{tt}\right\}(dr)_a(dr)_b \\ &\quad +r^2\tilde{T}_{tt}\gamma_{ab}\Bigg]\cos\theta, \end{aligned} \tag{63}$$

where the vector field V_a is given by

$$V_a := -r\partial_t\Phi_{(e)}\cos\theta(dt)_a + \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a - r\Phi_{(e)}\sin\theta(d\theta)_a. \tag{64}$$

On the other hand, for the $l = 0$-mode, we may choose $\tilde{T}_{(e1)A} = 0$ and $\tilde{T}_{(e2)} = 0$ and we may observe that the tensor $\tilde{F}_{AB}$ is traceless. Furthermore, Equations (50) and (51) yield the r- and t-derivative of the Moncrief variable $\Phi_{(e)}$, respectively. The integrability of these equations is guaranteed by the continuity equation (37). Then, we obtain the Moncrief variable $\Phi_{(e)}$. In this case, the master equation (54) is trivial and the evolution equation (55) gives the variable $\tilde{F}$. Then, we obtain the variable $(\Phi_{(e)}, \tilde{F})$. Through the definition (48) of the Moncrief variable $\Phi_{(e)}$, we obtain the component $X_{(e)}$. To obtain the component $Y_{(e)}$,

we regard the constraints (41) as the equation for the component $Y_{(e)}$. Through this strategy, in the Part II paper [15], we derived the $l = 0$ mode solution

$$\begin{aligned}\mathscr{F}_{ab} &= \frac{2}{r}\left(M_1 + 4\pi\int dr\frac{r^2}{f}T_{tt}\right)\left((dt)_a(dt)_a + \frac{1}{f^2}(dr)_a(dr)_a\right)\\ &\quad +2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} + \pounds_V g_{ab},\end{aligned} \tag{65}$$

where

$$V_a = \left(\frac{f}{4}\mathrm{Y} + \frac{rf}{4}\partial_r\mathrm{Y} - \frac{r\Xi(r)}{(1-3f)} + f\int dr\frac{2\Xi(r)}{f(1-3f)^2}\right)(dt)_a + \frac{1}{4f}r\partial_t\mathrm{Y}(dr)_a. \tag{66}$$

Here, the variable $\tilde{F} =: \partial_t\mathrm{Y}$ must satisfy Equation (55), and $\Xi(r)$ is an arbitrary function of r.

3. Realization of LTB Solution as a Perturbation on the Schwarzschild Spacetime

3.1. Perturbative Expression of the LTB Solution on Schwarzschild Background Spacetime

Here, we consider the Lemaître–Tolman–Bondi (LTB) solution [42], which is an exact solution to the Einstein equation with the matter field

$$T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a, \tag{67}$$

and the metric

$$g_{ab} = -(d\tau)_a(d\tau)_b + \frac{(\partial_R r)^2}{1+f(R)}(dR)_a(dR)_b + r^2\gamma_{ab}, \quad r = r(\tau, R). \tag{68}$$

This solution is a spherically symmetric solution to the Einstein equation. The function $r = r(\tau, R)$ satisfies the differential equation

$$(\partial_\tau r)^2 = \frac{F(R)}{r} + f(R). \tag{69}$$

Here, we note that $F(R)$ is an arbitrary function of R, which represents initial distribution of the dust matter. $f(R)$ is also an arbitrary function of R that represents initial distribution of the energy of dust field in the Newtonian sense. The solution to Equation (69) is given in the three following cases:

(i) $f(R) > 0$ **:**

$$r = \frac{F(R)}{2f(R)}(\cosh\eta - 1), \quad \tau_0(R) - \tau = \frac{F(R)}{2f(R)^{3/2}}(\sinh\eta - \eta), \tag{70}$$

(ii) $f(R) < 0$ **:**

$$r = \frac{F(R)}{-2f(R)}(1 - \cos\eta), \quad \tau_0(R) - \tau = \frac{F(R)}{2(-f(R))^{3/2}}(\eta - \sin\eta), \tag{71}$$

(iii) $f(R) = 0$ **:**

$$r = \left(\frac{9F(R)}{4}\right)^{1/3}[\tau_0(R) - \tau]^{2/3}. \tag{72}$$

The energy density ρ is given by

$$8\pi\rho = \frac{\partial_R F}{(\partial_R r) r^2}. \tag{73}$$

The LTB solution includes the three arbitrary functions $f(R)$, $F(R)$, and $\tau_0(R)$.

Here, we consider the vacuum case $\rho = 0$. In this case, from Equation (73), we have

$$\partial_R F = 0. \tag{74}$$

Furthermore, we consider the case $f = 0$. Here, we choose $\tau_0 = R$, i.e., $\partial_R \tau_0 = 1$. In this case, Equation (72) yields

$$\begin{aligned} (dr)_a &= \left(\frac{9F}{4}\right)^{1/3} \frac{2}{3} [R-\tau]^{-1/3} [(dR)_a - (d\tau)_a] \\ &= \left(\frac{F}{r}\right)^{1/2} [(dR)_a - (d\tau)_a], \end{aligned} \tag{75}$$

$$(\partial_R r) = \left(\frac{F}{r}\right)^{1/2}, \tag{76}$$

and

$$(dR)_a = (d\tau)_a + \left(\frac{F}{r}\right)^{-1/2} (dr)_a. \tag{77}$$

Then, the metric (68) is given by

$$\begin{aligned} g_{ab} &= -(d\tau)_a (d\tau)_b + (\partial_R r)^2 (dR)_a (dR)_b + r^2 \gamma_{ab} \\ &= -\left(1 - \frac{F}{r}\right) \left[(d\tau)_a - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_a \right] \\ &\qquad \times \left[(d\tau)_b - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_b \right] \\ &\quad + \left(1 - \frac{F}{r}\right)^{-1} (dr)_a (dr)_b + r^2 \gamma_{ab}. \end{aligned} \tag{78}$$

Here, we define the time function t by

$$(dt)_a := (d\tau)_a - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_a. \tag{79}$$

Then, we obtain

$$g_{ab} = -f (dt)_a (dt)_b + f^{-1} (dr)_a (dr)_b + r^2 \gamma_{ab}, \quad f = 1 - \frac{2M}{r}, \tag{80}$$

with the identification $F = 2M$. This is the Schwarzschild metric with the mass parameter M.

Now, we consider the perturbation on the Schwarzschild spacetime which is derived by the exact LTB solution (68) so that

$$F(R) = 2[M + \epsilon m_1(R)] + O(\epsilon^2), \tag{81}$$

$$f(R) = 0 + \epsilon f_1(R) + O(\epsilon^2), \tag{82}$$

$$\tau_0(R) = R + \epsilon \tau_1(R) + O(\epsilon^2). \tag{83}$$

Through these perturbations (81)–(83), we consider the perturbative expansion of the function r which is determined by Equation (69):

$$r(\tau, R) = r_s(\tau, R) + \epsilon r_1(\tau, R) + O(\epsilon^2). \tag{84}$$

Here, the function $r_s(\tau, R)$ is given by Equation (72), i.e.,

$$r_s(\tau, R) = r(\tau, R) = \left(\frac{9M}{2}\right)^{1/3} [R - \tau]^{2/3}. \tag{85}$$

In Equations (83) and (85), we choose the background value of the function $\tau_0(R)$ to be R.

Through this perturbative expansion, we evaluate Equation (69) and we obtain

$$O(\epsilon^0) \quad : \quad (\partial_\tau r_s)^2 - \frac{2M}{r_s} = 0 \tag{86}$$

$$O(\epsilon^1) \quad : \quad (\partial_\tau r_s)(\partial_\tau r_1) - \frac{m_1(R)}{r_s} + \frac{M}{r_s^2} r_1 - \frac{1}{2} f_1(R) = 0. \tag{87}$$

Using Equation (86), the linear perturbation (87) yields

$$(1 - f)^{1/2} (\partial_\tau r_1) + \frac{m_1(R)}{r} - \frac{M}{r^2} r_1 + \frac{1}{2} f_1(R) = 0, \tag{88}$$

where we use

$$\partial_\tau r_s = -(1 - f)^{1/2} \tag{89}$$

and the replacement $r_s \to r$. The solution to Equation (88) is given by

$$\begin{aligned} r_1 \quad = \quad & \left(\frac{M}{6}\right)^{1/3} \frac{m_1(R)}{M} [R - \tau]^{2/3} - \frac{3}{20} \left(\frac{6}{M}\right)^{1/3} f_1(R) [R - \tau]^{+4/3} \\ & + B(R) [R - \tau]^{-1/3}. \end{aligned} \tag{90}$$

From the comparison with Equation (85), $B(R)$ is the perturbation of the $\tau_1(R)$ as $\tau_0(R) = R + \tau_1(R)$ in the exact solution (70)–(72). Furthermore, the solution (90) can be also derived from the exact solutions (70)–(72) for each case. From Equation (73), the perturbative dust energy density is given by

$$8\pi\rho = \frac{2\partial_R m_1(R)}{(\partial_R r) r^2}. \tag{91}$$

Through the perturbative solution (90), the metric (68) is given by

$$\begin{aligned} g_{ab} \quad = \quad & -(d\tau)_a (d\tau)_b + (\partial_R r)^2 (dR)_a (dR)_b + r^2 \gamma_{ab} \\ & + \epsilon [(2(\partial_R r_1) - f_1(\partial_R r))(\partial_R r)(dR)_a (dR)_b + 2 r r_1 \gamma_{ab}] + O(\epsilon^2) \\ =: \quad & g^{(0)}_{ab} + \epsilon \, {}_{\mathscr{Z}}h_{ab} + O(\epsilon^2). \end{aligned} \tag{92}$$

As shown in Equation (80), the background metric $g^{(0)}_{ab}$ is given by the Schwarzschild metric in the static chart. On the other hand, the linear order perturbation ${}_{\mathscr{Z}}h_{ab}$ (in the gauge $\mathscr{X}_\epsilon$) is given by

$${}_{\mathscr{Z}}h_{ab} := (2(\partial_R r_1) - f_1(R)(\partial_R r))(\partial_R r)(dR)_a (dR)_b + 2 r r_1 \gamma_{ab}. \tag{93}$$

Here, we fix the second-kind gauge so that

$$\mathscr{X}_\epsilon : (\tau, R, \theta, \phi) \in \mathscr{M} \mapsto (\tau, R, \theta, \phi) \in \mathscr{M}_{ph}. \tag{94}$$

Of course, if we employ the different gauge choice $\mathscr{Y}_\epsilon$ from the above gauge-choice $\mathscr{X}_\epsilon$, we obtain the different expression of the metric perturbation ${}_{\mathscr{Y}}h_{ab} = {}_{\mathscr{X}}h_{ab} + \pounds_\xi g_{ab}$, where ξ^a is the generator of second-kind gauge transformation $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$.

3.2. Expression of the Perturbative LTB Solution in Static Chart

Here, we consider the expression of the linear perturbation ${}_{\mathscr{X}}h_{ab}$ given by Equation (93). Here, the radial coordinate r is related to the coordinates τ and R through Equation (85) as

$$R - \tau = \left(\frac{2}{9M}\right)^{1/2} r^{3/2}. \tag{95}$$

Then, we obtain

$$R = \tau + \frac{4M}{3}\left(\frac{r}{2M}\right)^{3/2}. \tag{96}$$

Furthermore, the relation of the time function t and coordinates τ and r is given by Equation (79) as

$$t = \tau + 4M\left[\left(\frac{r}{2M}\right)^{1/2} + \frac{1}{2}\ln\left\{\frac{\left(\frac{r}{2M}\right)^{1/2} - 1}{\left(\frac{r}{2M}\right)^{1/2} + 1}\right\}\right]. \tag{97}$$

Then, we have obtained

$$\tau = t - 4M\left[\left(\frac{r}{2M}\right)^{1/2} + \frac{1}{2}\ln\left\{\frac{\left(\frac{r}{2M}\right)^{1/2} - 1}{\left(\frac{r}{2M}\right)^{1/2} + 1}\right\}\right], \tag{98}$$

$$R = t - 4M\left[\frac{1}{3}\left(\frac{r}{2M}\right)^{3/2} + \left(\frac{r}{2M}\right)^{1/2} + \frac{1}{2}\ln\left\{\frac{\left(\frac{r}{2M}\right)^{1/2} - 1}{\left(\frac{r}{2M}\right)^{1/2} + 1}\right\}\right]. \tag{99}$$

Through the coordinates (t, r), we express the linear perturbation ${}_{\mathscr{X}}h_{ab}$ in Equation (93). From Equations (77) and (79) with $F = 2M$, we obtain

$$(dR)_a = (dt)_a + f^{-1}(1-f)^{-1/2}(dr)_a, \quad f = 1 - \frac{2M}{r}, \tag{100}$$

$$(d\tau)_a = (dt)_a + f^{-1}(1-f)^{1/2}(dr)_a. \tag{101}$$

Before evaluating the metric perturbation ${}_{\mathscr{X}}h_{ab}$, we consider the perturbation of the energy-momentum tensor of the matter field. In the case of the LTB solution, the matter field is characterized by the dust field, whose energy-momentum tensor (67) is given by

$$T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a, \quad u^a = (\partial_\tau)^a. \tag{102}$$

In our case, the linearized Einstein equation gives Equation (91), i.e.,

$$8\pi\rho = \frac{2\partial_R m_1(R)}{(\partial_R r)r^2}. \tag{103}$$

As we have

$$(\partial_R r) = (1-f)^{1/2} \tag{104}$$

from Equation (95), we obtain

$$\rho = \frac{\partial_R m_1(R)}{4\pi r^2}(1-f)^{-1/2}. \tag{105}$$

On the other hand, substituting Equation (101) into Equation (102), we obtain

$$\begin{aligned} T_{ab} &= \rho(d\tau)_a(d\tau)_b \\ &= \rho\Big((dt)_a + f^{-1}(1-f)^{1/2}(dr)_a\Big)\Big((dt)_b + f^{-1}(1-f)^{1/2}(dr)_b\Big) \\ &= \rho(dt)_a(dt)_b + \rho\frac{(1-f)^{1/2}}{f}2(dt)_{(a}(dr)_{b)} + \rho\frac{1-f}{f^2}(dr)_a(dr)_b. \end{aligned} \tag{106}$$

Then, we obtain the components of the energy-momentum tensor for the static coordinate (t, r) as

$$\tilde{T}_{tt} = \rho, \quad \tilde{T}_{tr} = \frac{(1-f)^{1/2}}{f}\rho \quad \tilde{T}_{rr} = \frac{1-f}{f^2}\rho. \tag{107}$$

Here, we note that the function ρ is given by the Einstein equation (105) and R is given by Equation (99).

Here, we check the components (60) and (61) with $l = 0$ of the divergence of the energy-momentum tensor in the LTB case. Using Equations (100) and (107), we obtain

$$\begin{aligned} \partial_t \tilde{T}_{tt} = \partial_t\rho = \frac{\partial_R^2 m_1(R)}{4\pi r^2}(1-f)^{-1/2}\left(\frac{\partial R}{\partial t}\right) &= \frac{\partial_R^2 m_1(R)}{4\pi r^2}(1-f)^{-1/2}, \\ -f^2\partial_r\tilde{T}_{rt} = -f^2\partial_r\left(\frac{\partial_R m_1(R)}{4\pi r^2 f}\right) &= -\frac{\partial_R^2 m_1(R)}{4\pi r^2}(1-f)^{-1/2} + \frac{\partial_R m_1(R)}{4\pi r^3}(1+f), \\ -\frac{(1+f)f}{r}\tilde{T}_{rt} = -\frac{(1+f)f}{r}\frac{(1-f)^{1/2}}{f}\rho &= -\frac{\partial_R m_1(R)}{4\pi r^3}(1+f). \end{aligned}$$

Then, we can confirm Equation (60) with $l = 0$. Next, we check Equation (61) with $l = 0$. As we may choose $\tilde{T}_{(e1)A} = 0$ and $\tilde{T}_{(e2)} = 0$, Equation (62) yields $\tilde{T}_{(e0)} = 0$. Furthermore, using

$$\begin{aligned} \partial_t\tilde{T}_{tr} &= \frac{(1-f)^{1/2}}{f}\partial_t\rho = \frac{\partial_R^2 m_1(R)}{4\pi r^2 f}, \\ -\frac{1-f}{2rf}\tilde{T}_{tt} &= -\frac{1-f}{2rf}\rho = -(1-f)^{1/2}\frac{\partial_R m_1(R)}{8\pi r^3 f}, \\ -f^2\partial_r\tilde{T}_{rr} &= \frac{(2-f)(1-f)}{rf}\rho - (1-f)\partial_r\rho = -\frac{\partial_R^2 m_1(R)}{4\pi r^2 f} + (4+f)(1-f)^{1/2}\frac{\partial_R m_1(R)}{8\pi r^3 f}, \\ -\frac{(3+f)f}{2r}\tilde{T}_{rr} &= -\frac{(1-f)(3+f)}{2rf}\rho = -(3+f)(1-f)^{1/2}\frac{\partial_R m_1(R)}{8\pi r^3 f}, \end{aligned}$$

we can confirm Equation (61) with $l = 0$. Thus, the definitions (107) of the components $\tilde{T}_{tt}$, $\tilde{T}_{tr}$, $\tilde{T}_{rr}$ and the result of the Einstein equation (105) are justified. We also note that the continuity equations (61) and (60) with $l = 0$ and $\tilde{T}_{(e0)} = 0$ are important premises of the solution (65) for $l = 0$ mode perturbations.

Now, we consider the problem of whether the form the perturbation ${}^{\mathscr{Z}}h_{ab}$ given by Equation (93) is described by the solution (65) or not. Substituting Equation (100) into Equation (93), we obtain

$$\begin{aligned} {}^{\mathscr{Z}}h_{ab} &= \Big(2(\partial_R r_1)(1-f)^{1/2} - f_1(R)(1-f)\Big)(dt)_a(dt)_b \\ &+\frac{1}{f}\Big(2(\partial_R r_1) - f_1(R)(1-f)^{1/2}\Big)2(dt)_{(a}(dr)_{b)} \\ &+\Big(2(\partial_R r_1)(1-f)^{-1/2} - f_1(R)\Big)f^{-2}(dr)_a(dr)_b \\ &+2rr_1\gamma_{ab}. \end{aligned} \tag{108}$$

Here, we use Equation (104). Comparing Equation (108) with Equation (65), we easily see that the last term $2rr_1\gamma_{ab}$ should be included in the term of the Lie derivative of the background metric g_{ab}. Then, we consider the components of $\pounds_{V_{(1)}}g_{ab}$ with the generator $V_{(1)a} = V_{(1)r}(dr)_a$. The components of $\pounds_{V_{(1)}}g_{ab}$ are summarized as follows:

$$\pounds_{V_{(1)}}g_{tt} = -ff'V_{(1)r}, \quad \pounds_{V_{(1)}}g_{tr} = \partial_t V_{(1)r}, \quad \pounds_{V_{(1)}}g_{rr} = 2\partial_r V_{(1)r} + \frac{f'}{f}V_{(1)r}, \tag{109}$$

$$\pounds_{V_{(1)}}g_{\theta\theta} = 2rfV_{(1)r}, \quad \pounds_{V_{(1)}}g_{\phi\phi} = 2rf\sin^2\theta V_{r_{(1)}}. \tag{110}$$

If the last term $2rr_1\gamma_{ab}$ in Equation (108) is included in the term $\pounds_{V_{(1)}}g_{ab}$, we should choose the component $V_{(1)r}$ as

$$V_{(1)r} = \frac{r_1}{f}. \tag{111}$$

Substituting Equation (111) into Equation (109), we obtain

$$\pounds_{V_{(1)}}g_{tt} = -\frac{1-f}{r}r_1, \quad \pounds_{V_{(1)}}g_{tr} = \frac{1}{f}\partial_t r_1, \quad \pounds_{V_{(1)}}g_{rr} = -\frac{1-f}{rf^2}r_1 + \frac{2}{f}(\partial_r r_1). \tag{112}$$

Then, we have

$$\begin{aligned} {}^{\mathscr{Z}}h_{ab} &= \left(2(\partial_R r_1)(1-f)^{1/2} - f_1(R)(1-f) + \frac{1-f}{r}r_1\right)(dt)_a(dt)_b \\ &+\frac{1}{f}\Big(2(\partial_R r_1) - f_1(R)(1-f)^{1/2} - \partial_t r_1\Big)2(dt)_{(a}(dr)_{b)} \\ &+\left(2(\partial_R r_1)(1-f)^{-1/2} - f_1(R) + \frac{1-f}{r}r_1 - 2f(\partial_r r_1)\right)f^{-2}(dr)_a(dr)_b \\ &+\pounds_{V_{(1)}}g_{ab}. \end{aligned} \tag{113}$$

Here, we note the inverse relation of Equations (100) and (101) as follows:

$$(dt)_a = \frac{1}{f}(d\tau)_a - \frac{1-f}{f}(dR)_a, \tag{114}$$

$$(dr)_a = -(1-f)^{1/2}(d\tau)_a + (1-f)^{1/2}(dR)_a. \tag{115}$$

From Equations (114) and (115), we obtain

$$\begin{aligned} (\partial_R r_1) &= \frac{\partial t}{\partial R}(\partial_t r_1) + \frac{\partial r}{\partial R}(\partial_r r_1) \\ &= -\frac{1-f}{f}(\partial_t r_1) + (1-f)^{1/2}(\partial_r r_1). \end{aligned} \tag{116}$$

Then, we obtain

$$(\partial_t r_1) = -\frac{f}{1-f}(\partial_R r_1) + \frac{f}{1-f}(1-f)^{1/2}(\partial_r r_1). \tag{117}$$

Substituting Equation (117) into Equation (113), we obtain

$$\begin{aligned}
{}^{\mathscr{Z}}h_{ab} = & \left(2(\partial_R r_1)(1-f)^{1/2} - f_1(R)(1-f) + \frac{1-f}{r}r_1\right)(dt)_a(dt)_b \\
& + \frac{1}{f}(1-f)^{-1/2}\Big(+(2-f)(1-f)^{-1/2}(\partial_R r_1) - f_1(R)(1-f) - f(\partial_r r_1)\Big)2(dt)_{(a}(dr)_{b)} \\
& + \left(2(\partial_R r_1)(1-f)^{-1/2} - f_1(R) + \frac{1-f}{r}r_1 - 2f(\partial_r r_1)\right)f^{-2}(dr)_a(dr)_b \\
& + \pounds_{V_{(1)}} g_{ab}.
\end{aligned} \tag{118}$$

Here, we also note that, apart from the term $\pounds_V g_{ab}$, the solution (65) is traceless. Therefore, the trace part of (t,r) components in Equation (118) should be included in the term of the Lie derivative of the background metric g_{ab}. To see this, we consider the components $\pounds_{V_{(2)}} g_{ab}$ with the generator $V_{(2)a} = V_{(2)t}(dt)_a$ as follows:

$$\pounds_{V_{(2)}} g_{tt} = 2\partial_t V_{(2)t}, \quad \pounds_{V_{(2)}} g_{tr} = \partial_r V_{(2)t} - \frac{f'}{f}V_{(2)t}. \tag{119}$$

Substituting Equation (119) into Equation (118), we obtain

$$\begin{aligned}
{}^{\mathscr{Z}}h_{ab} = & \left(2(\partial_R r_1)(1-f)^{1/2} - f_1(R)(1-f) + \frac{1-f}{r}r_1 - 2\partial_t V_{(2)t}\right)(dt)_a(dt)_b \\
& + \frac{(1-f)^{-1/2}}{f}\Big((2-f)(1-f)^{-1/2}(\partial_R r_1) - f_1(R)(1-f) - f(\partial_r r_1) \\
& \qquad - f(1-f)^{1/2}\partial_r V_{(2)t} + \frac{1-f}{r}(1-f)^{1/2}V_{(2)t}\Big)2(dt)_{(a}(dr)_{b)} \\
& + \left(2(\partial_R r_1)(1-f)^{-1/2} - f_1(R) + \frac{1-f}{r}r_1 - 2f(\partial_r r_1)\right)f^{-2}(dr)_a(dr)_b \\
& + \pounds_{V_{(1)}+V_{(2)}} g_{ab}.
\end{aligned} \tag{120}$$

Apart from the term $\pounds_{V_{(1)}+V_{(2)}} g_{ab}$, the remaining term in ${}^{\mathscr{Z}}h_{ab}$ should be traceless. Then, we obtain

$$\begin{aligned}
0 &= g^{ab}\left[{}^{\mathscr{Z}}h_{ab} - \pounds_{V_{(1)}+V_{(2)}} g_{ab}\right] \\
&= \frac{1}{f}\Big(+2f(\partial_R r_1)(1-f)^{-1/2} - 2f(\partial_r r_1) - f f_1(R) + 2\partial_t V_{(2)t}\Big).
\end{aligned} \tag{121}$$

Here, we choose $V_{(2)t}$ so that

$$\partial_t V_{(2)t} = -f(\partial_R r_1)(1-f)^{-1/2} + f(\partial_r r_1) + \frac{1}{2}f f_1(R). \tag{122}$$

Through this expression of $(\partial_R r_1)$ given by Equation (116), Equation (122) is given by

$$\partial_t V_{(2)t} = \partial_t((1-f)^{1/2} r_1) + \frac{1}{2}f f_1(R) \tag{123}$$

and

$$V_{(2)t} = (1-f)^{1/2} r_1 + \frac{1}{2}f\int dt f_1(R), \tag{124}$$

where we choose the arbitrary function r to be zero. From Equation (124), we obtain

$$\begin{aligned} & -f(1-f)^{1/2}\partial_r V_{(2)t} + \frac{1-f}{r}(1-f)^{1/2}V_{(2)t} \\ = \; & \frac{1}{2r}(1-f)(2-f)r_1 - f(1-f)\partial_r r_1 - \frac{1}{2}f^2(1-f)^{1/2}\int dt\partial_r f_1(R). \end{aligned} \tag{125}$$

Here, we note that $f_1 = f_1(R)$ and its derivative with respect to r is given by

$$\partial_r f_1(R) = \frac{\partial R}{\partial r}\frac{d}{dR}f_1(R) = \frac{1}{f(1-f)^{1/2}}\frac{d}{dR}f_1(R). \tag{126}$$

On the other hand, the derivative of $f_1(R)$ with respect to t is given by

$$\partial_t f_1(R) = \frac{\partial R}{\partial t}\frac{d}{dR}f_1(R) = \frac{d}{dR}f_1(R). \tag{127}$$

Then, we obtain

$$\partial_r f_1(R) = \frac{1}{f(1-f)^{1/2}}\partial_t f_1(R). \tag{128}$$

Substituting Equation (128) into Equation (125), we obtain

$$\begin{aligned} & -f(1-f)^{1/2}\partial_r V_{(2)t} + \frac{1-f}{r}(1-f)^{1/2}V_{(2)t} \\ = \; & \frac{1}{2r}(1-f)(2-f)r_1 - f(1-f)\partial_r r_1 - \frac{1}{2}ff_1. \end{aligned} \tag{129}$$

Furthermore, by the substitution of Equations (122) and (129) into Equation (120), we obtain

$$\begin{aligned} {}^{\mathscr{Z}}h_{ab} = \; & 2\left((\partial_R r_1)(1-f)^{-\frac{1}{2}} + \frac{1-f}{2r}r_1 - f(\partial_r r_1) - \frac{1}{2}f_1(R)\right) \\ & \times\left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right) \\ & + \frac{(2-f)(1-f)^{-\frac{1}{2}}}{f}\left((\partial_R r_1)(1-f)^{-1/2} + \frac{1-f}{2r}r_1 - f(\partial_r r_1) - \frac{1}{2}f_1(R)\right) \\ & \times 2(dt)_{(a}(dr)_{b)} \\ & + \pounds_{V_{(1)}+V_{(2)}}g_{ab}. \end{aligned} \tag{130}$$

Here, we consider the information from the solution (90) of the linearized LTB solution. To consider the necessary information from the solution (90), we consider the derivative $\partial_r r_1$ in Equation (130). Using Equations (100) and (101), we obtain

$$\begin{aligned} \partial_r r_1 &= \frac{\partial R}{\partial r}(\partial_R r_1) + \frac{\partial \tau}{\partial r}(\partial_\tau r_1) \\ &= \frac{1}{f}(1-f)^{-1/2}(\partial_R r_1) + \frac{1}{f}(1-f)^{1/2}(\partial_\tau r_1). \end{aligned} \tag{131}$$

From Equations (131) and (88), we obtain

$$2(\partial_R r_1)(1-f)^{-1/2} - 2f\partial_r r_1 + \frac{1-f}{r}r_1 - f_1(R) = \frac{2m_1(R)}{r}. \tag{132}$$

Through Equation (132), we obtain

$$\begin{aligned} {}^{\mathscr{Z}}h_{ab} &= \frac{2m_1(R)}{r}\left[(dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right] + \frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}2(dt)_{(a}(dr)_{b)} \\ &\quad + \pounds_{V_{(LTB)}}g_{ab}, \end{aligned} \tag{133}$$

where $V_{(LTB)a}$ is given by

$$V_{(LTB)a} := V_{(1)a} + V_{(2)a} = \left[(1-f)^{1/2}r_1 + \frac{1}{2}f\int dt f_1(R)\right](dt)_a + \frac{r_1}{f}(dr)_a. \tag{134}$$

Now, we check whether the linear-order perturbative solution (133) has the form of the general solution (65) for the $l = 0$ mode perturbations or not. Here, we only consider the case $M_1 = 0$ in Equation (65), because we can add the term M_1 with an appropriate term of the Lie derivative of the background metric. First, we consider the first term in Equation (133). From Equation (65), the expression

$$m_1(t,r) = 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right] \tag{135}$$

should appear in Equation (133). From Equations (107) and (105), we obtain

$$4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right] = 4\pi\int dr\left[\frac{r^2}{f}\rho\right] = \int dr\Big[f^{-1}(1-f)^{-1/2}\partial_R m_1(R)\Big]. \tag{136}$$

Here, we note that

$$\partial_r m_1(R) = \frac{\partial R}{\partial r}\partial_R m_1(R) + \frac{\partial \tau}{\partial r}\partial_\tau m_1(R) = f^{-1}(1-f)^{-1/2}\partial_R m_1(R). \tag{137}$$

Substituting Equation (137) into Equation (136), we obtain

$$4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right] = \int dr\partial_r m_1(R) = m_1(R). \tag{138}$$

Thus, we may regard that the first term in Equation (65) realizes the first term in Equation (133) of the linearized LTB solution.

Next, we consider the second term in Equation (65). In this case, we evaluate the integration

$$4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right). \tag{139}$$

From Equation (107), we obtain

$$\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr} = \frac{1}{f}\rho + f\frac{1-f}{f^2}\rho = \frac{2-f}{f}\rho = \frac{1}{4\pi r^2 f}(2-f)(1-f)^{-1/2}\partial_R m_1(R). \tag{140}$$

Here, we consider $\partial_t m_1(R)$ as

$$\partial_t m_1(R) = \frac{\partial R}{\partial t}\partial_R m_1(R) + \frac{\partial \tau}{\partial t}\partial_\tau m_1(R) = \partial_R m_1(R). \tag{141}$$

Then, we obtain

$$4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt}+f\tilde{T}_{rr}\right) = 4\pi r\int dt\left(\frac{1}{4\pi r^2 f}(2-f)(1-f)^{-1/2}\partial_t m_1(R)\right)$$
$$= \frac{2-f}{rf(1-f)^{1/2}}\int dt\partial_t m_1(R)=\frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}. \quad (142)$$

Thus, we confirmed that the second term in Equation (65) realizes the second term in Equation (133) in the linearized LTB solution.

The remaining term in Equation (133) is the Lie derivative of the background spacetime. Here, we note that there is always ambiguity of the gauge-choice $\pounds_V g_{ab}$ with an arbitrary vector field V^a in the linear perturbation ${}_{\mathscr{Z}}h_{ab}$. Through this degree of freedom, we can always adjust the solution ${}_{\mathscr{Z}}h_{ab}$ so that the last term in Equation (133) is identical to the last term in Equation (65).

Thus, the linearized version (133) of the LTB exact solution with Schwarzschild background spacetime is realized from the solution (65) of the $l=0$ mode perturbations. In this sense, the solutions (65) of the $l=0$ mode perturbations are justified by the LTB solutions. It is important to note that the arbitrary functions $f_1(R)$ and $\tau_1(R)$ of the perturbative LTB solution are included only in the vector field $V_{(LTB)a}$ in Equation (133). Therefore, we may say that the term $\pounds_{V_{(LTB)}}g_{ab}$ includes physical information of the LTB solution, i.e., this term has its physical meaning.

4. Realization of the Linearized Non-Rotating C-Metric

4.1. The Linearized Non-Rotating C-Metric

Here, we consider the non-rotating vacuum C-metric [38], in which conical singularities may occur both in the axis $\theta=0$ and $\theta=\pi$. The C-metric is well known as the solution describing uniformly accelerating black holes that are pulled or pushed by the straight string at $\theta=0$ or $\theta=\pi$. The C-metric is described by the metric

$$g_{ab} = \frac{1}{(1+\alpha r\cos\theta)^2}\left(-Q(dt)_a(dt)_b+\frac{1}{Q}(dr)_a(dr)_b\right.$$
$$\left.+\frac{r^2}{P}(d\theta)_a(d\theta)_b+Pr^2\sin^2\theta(d\varphi)_a(d\varphi_b)\right), \quad (143)$$

where

$$P=1+2\alpha m\cos\theta,\quad Q=(1-\alpha^2r^2)\left(1-\frac{2m}{r}\right),\quad \varphi\in(-C\pi,+C\pi) \quad (144)$$

includes the singularities both in the axis $\theta=0$ and $\theta=\pi$. To see this, we note that the metric given by Equations (143) and (144) includes three positive real parameters: m, α (satisfying $2\alpha m<1$), and C (which is hidden in the range of the rotational coordinate $\varphi\in(-C\pi,+C\pi)$).

Here, we consider the two-dimensional section of the spacetime with the metric (143) as

$$(1+\alpha r\cos\theta)^2\,g_{ab}|_{r=const.,t=const.} =: \frac{r^2}{P}\bar{\gamma}_{ab}=\frac{r^2}{P}\Big((d\theta)_a(d\theta)_b+P^2\sin^2\theta(d\varphi)_a(d\varphi)_b\Big). \quad (145)$$

In addition to the conformal factor r^2/P, the "radius", which is the proper distance along the $(\partial/\partial\theta)^a$, is given by

$$\int_0^\theta \sqrt{\bar{\gamma}_{ab}(\partial/\partial\theta)^a(\partial/\partial\theta)^a}d\theta = \theta,$$
$$\int_\theta^\pi \sqrt{\bar{\gamma}_{ab}(\partial/\partial\theta)^a(\partial/\partial\theta)^a}d\theta = \pi - \theta. \tag{146}$$

On the other hand, the "circumference", which is the proper distance along $(\partial/\partial\varphi)^a$ from $\varphi = -C\pi$ to $\varphi = C\pi$, for any θ is given by

$$\int_{-C\pi}^{+C\pi} \sqrt{\bar{\gamma}_{ab}(\partial/\partial\varphi)^a(\partial/\partial\varphi)^a}d\varphi$$
$$= \int_{-C\pi}^{+C\pi} (1 + 2\alpha m\cos\theta)\sin\theta d\varphi = 2\pi C(1 + 2\alpha m\cos\theta)\sin\theta. \tag{147}$$

From Equations (146) and (147), we obtain the following results: in the neighborhood of $\theta = 0$,

$$\frac{\text{circumference at } \theta}{\text{radius from } \theta = 0} = \frac{2\pi C(1 + 2\alpha m\cos\theta)\sin\theta}{\theta}; \tag{148}$$

and in the neighborhood of $\theta = \pi$,

$$\frac{\text{circumference at } \theta}{\text{radius from } \theta = \pi} = \frac{2\pi C(1 + 2\alpha m\cos\theta)\sin\theta}{\pi - \theta}$$
$$= \frac{2\pi C(1 - 2\alpha m\cos(\pi - \theta))\sin(\pi - \theta)}{\pi - \theta}. \tag{149}$$

Then, we obtain

$$\lim_{\theta\to 0}\frac{\text{circumference at } \theta}{\text{radius from } \theta = 0} = 2\pi C(1 + 2\alpha m),$$
$$\lim_{\theta\to \pi - 0}\frac{\text{circumference at } \theta}{\text{radius from } \theta = \pi} = 2\pi C(1 - 2\alpha m). \tag{150}$$

These imply the existence of a conical singularity with a different conicity (unless $\alpha m = 0$). The deficit or excess angle of either of these two conical singularities can be removed in an appropriate choice of the constant C, but not both simultaneously. In general, the constant C can, thus, be seen to determine the balance between the deficit/excess angles on the two halves of the axis. In particular, one natural choice is to remove the conical singularity at $\theta = 0$ by setting $C = (1 + 2\alpha m)^{-1}$. In this choice, the deficit angle at the poles $\theta = 0, \pi$ are, respectively,

$$\delta_0 = 0, \quad \delta_\pi = 2\pi - \frac{2\pi(1 - 2\alpha m)}{1 + 2\alpha m} = 2\pi\frac{4\alpha m}{1 + 2\alpha m}. \tag{151}$$

To compare the Schwarzschild spacetime, it is convenient to rescale the range of the rotational coordinate to 2π. This can be achieved by the simple rescaling

$$\varphi = C\phi, \tag{152}$$

where $\phi \in (-\pi, \pi)$. For this choice, the metric (143) is given by

$$g_{ab} = \frac{1}{(1+\alpha r\cos\theta)^2}\left(-Q(dt)_a(dt)_b + \frac{1}{Q}(dr)_a(dr)_b + \frac{r^2}{P}(d\theta)_a(d\theta)_b + PC^2r^2\sin^2\theta(d\phi)_a(d\phi_b)\right), \tag{153}$$

where P and Q are still given by Equation (144).

Now, we consider the situation where the black hole mass m is finite, and the acceleration α is infinitesimally small. In this case, the Rindler horizon $r = 1/\alpha$ is larger than the black hole horizon $r = 2m$. Therefore, this situation is naturally given by the inequality

$$1/\alpha > 2m, \quad \text{i.e.,} \quad 2m\alpha < 1. \tag{154}$$

Furthermore, we consider the situation

$$2m\alpha \ll 1. \tag{155}$$

This situation is appropriate for the consideration of the linearized C-metric spacetime around the Schwarzschild spacetime. Moreover, we consider the situation where the constant C is finite. We observe that the metric on the physical spacetime is given by

$$\bar{g}_{ab}(\bar{M}, \bar{\alpha}, \bar{C}; \bar{x}) = \bar{g}_{\mu\nu}(\bar{M}, \bar{\alpha}, \bar{C}; \bar{x})(d\bar{x}^\mu)_a(d\bar{x}^\nu)_b. \tag{156}$$

This metric is given by the replacements $m \to \bar{M}$, $\alpha \to \bar{\alpha}$, $C \to \bar{C}$, and $\{t, r, \theta, \phi\} \to \{\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}\}$ in Equations (143) and (144).

As a gauge choice of the second kind, we consider the point identification between the background spacetime with the metric (11)–(14) in terms of the coordinates $\{x^\mu\} = \{t, r, \theta, \phi\}$ and the physical spacetime with the metric (156) by

$$\bar{x}^\mu \overset{\leftarrow}{=} x^\mu. \tag{157}$$

We call this gauge choice $\mathcal{X}_\epsilon$. We also consider the situation of perturbation

$$\bar{M} := M + \epsilon M_1, \tag{158}$$

$$\bar{\alpha} := \alpha + \epsilon\alpha_1, \tag{159}$$

$$\bar{C} := C + \epsilon C_1. \tag{160}$$

Then, the pull-back $\mathcal{X}_\epsilon^*$ of the metric $\bar{g}_{ab}$ on the physical spacetime with the metric (156) to the background spacetime with the metric (11)–(14) is given by

$$\begin{aligned}\mathcal{X}_\epsilon^*\bar{g}_{ab} &=: {}_{\mathcal{X}}\bar{g}_{ab} \\ &= g_{ab}(M, \alpha, C : x) + \epsilon M_1 \partial_{M\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x) \\ &\quad + \epsilon\alpha_1\partial_{\alpha\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x) + \epsilon C_1\partial_{C\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x) \\ &\quad + O(\epsilon^2)\end{aligned} \tag{161}$$

in the second-kind gauge choice $\mathcal{X}_\epsilon$. As the linear-order perturbations ${}_{\mathcal{X}}h_{ab}$ under the gauge-choice $\mathcal{X}_\epsilon$ are defined by Equation (6), we obtain the representation of the linear perturbation ${}_{\mathcal{X}}h_{ab}$ under the gauge choice $\mathcal{X}_\epsilon$ as follows:

$$\begin{aligned}{}_{\mathcal{X}}h_{ab} &= M_1\partial_{M\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x) + \alpha_1\partial_{\alpha\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x) \\ &\quad + C_1\partial_{C\,{}_{\mathcal{X}}}\bar{g}_{ab}(M, \alpha, C; x).\end{aligned} \tag{162}$$

On the other hand, if we apply the other gauge choice $\mathscr{Y}_\epsilon$, we have other representation of the linear-order perturbation

$$\mathscr{Y}_\epsilon^* \bar{g}_{ab} = g_{ab} + \epsilon {}_{\mathscr{Y}}h_{ab} + O(\epsilon^2). \tag{163}$$

As the gauge choice $\mathscr{Y}_\epsilon$, we consider the point identification

$$\bar{x}^\mu \overset{\leftarrow}{=} x'^\mu \tag{164}$$

from the background spacetime with the metric (153) to the physical spacetime with the metric (156). We assume that the coordinates $\{x'^\mu\}$ in the gauge choice $\mathscr{Y}_\epsilon$ are related to the coordinate $\{x^\mu\}$ as

$$x'^\mu = x^\mu - \epsilon \xi^\mu + O(\epsilon^2). \tag{165}$$

This is the coordinate transformation induced by the second-kind gauge transformation $\Phi_\epsilon = \mathscr{X}_\epsilon \circ \mathscr{Y}_\epsilon^{-1} : \mathscr{Y}_\epsilon \to \mathscr{X}_\epsilon$. The metric on the physical spacetime pulled-back by the second-kind gauge choice $\mathscr{Y}_\epsilon$ is given by

$$\begin{aligned} \mathscr{Y}_\epsilon^* \bar{g}_{ab}(\bar{M}, \bar{\alpha}, \bar{C}; x'^\mu) &= g_{ab} \\ &\quad + \epsilon \big(M_1 \partial_M g_{\mu\nu}(M, \alpha, C; x) + \alpha_1 \partial_\alpha \bar{g}_{\mu\nu}(M, \alpha, C; x) \\ &\qquad + C_1 \partial_C \bar{g}_{\mu\nu}(M, \alpha, C; x) \big) (dx^\mu)_a (dx^\nu)_b \\ &\quad + \epsilon \pounds_\xi g_{ab} \\ &\quad + O(\epsilon^2). \end{aligned} \tag{166}$$

Comparing Equations (11)–(14) and (166), the perturbation ${}_{\mathscr{Y}}h_{ab}$ in the gauge choice $\mathscr{Y}_\epsilon$ is given by

$$\begin{aligned} {}_{\mathscr{Y}}h_{ab} &= \big(M_1 \partial_M \bar{g}_{\mu\nu}(M, \alpha, C; x) + \alpha_1 \partial_\alpha \bar{g}_{\mu\nu}(M, \alpha, C; x) \\ &\quad + C_1 \partial_C \bar{g}_{\mu\nu}(M, \alpha, C; x) \big) (dx^\mu)_a (dx^\nu)_b + \pounds_\xi g_{ab}. \end{aligned} \tag{167}$$

Together with Equation (162), we obtain the gauge transformation

$${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_\xi g_{ab}. \tag{168}$$

Now, we consider the explicit expression of the perturbation h_{ab}. From the definitions (144) of the functions P and Q, we obtain

$$\partial_M P = 2\alpha \cos\theta, \quad \partial_\alpha P = 2M \cos\theta, \tag{169}$$

$$\partial_M Q = (1 - \alpha^2 r^2) \left(-\frac{2}{r} \right), \quad \partial_\alpha P = -2\alpha r^2 \left(1 - \frac{2m}{r} \right). \tag{170}$$

Through these formulae and Equation (153), we obtain

$$\begin{aligned}
\partial_M \bar{g}_{ab} &= \frac{1}{(1+\alpha r\cos\theta)^2} \\
&\quad\times\left[(1-\alpha^2 r^2)\left(\frac{2}{r}\right)\left(+(dt)_a(dt)_b+\frac{1}{Q^2}(dr)_a(dr)_b\right)\right. \\
&\quad\left.+2\alpha\cos\theta\frac{r^2}{P^2}\left(-(d\theta)_a(d\theta)_b+C^2P^2\sin^2\theta(d\phi)_a(d\phi_b)\right)\right], \qquad (171)\\
\partial_\alpha \bar{g}_{ab} &= -2\frac{r\cos\theta}{(1+\alpha r\cos\theta)}g_{ab} \\
&\quad+\frac{2r^2}{(1+\alpha r\cos\theta)^2} \\
&\quad\times\left[\alpha\left(1-\frac{2M}{r}\right)\left(+(dt)_a(dt)_b+\frac{1}{Q^2}(dr)_a(dr)_b\right)\right. \\
&\quad\left.+M\cos\theta\left(-\frac{1}{P^2}(d\theta)_a(d\theta)_b+C^2\sin^2\theta(d\phi)_a(d\phi_b)\right)\right], \qquad (172)\\
\partial_C \bar{g}_{ab} &= \frac{1}{(1+\alpha r\cos\theta)^2}2PCr^2\sin^2\theta(d\phi)_a(d\phi_b). \qquad (173)
\end{aligned}$$

In the case of $\alpha = 0$ and $C = 1$, which are background values of these parameters for the Schwarzschild spacetime, we obtain

$$\partial_M \bar{g}_{ab}|_{\alpha=0,C=1} = \left(\frac{2}{r}\right)\left((dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right), \quad f=1-\frac{2M}{r}, \qquad (174)$$

$$\partial_\alpha \bar{g}_{ab}|_{\alpha=0,C=1} = -2r\cos\theta g_{ab}+2Mr^2\cos\theta\left(-(d\theta)_a(d\theta)_b+\sin^2\theta(d\phi)_a(d\phi_b)\right), \qquad (175)$$

and

$$\partial_C \bar{g}_{ab}|_{\alpha=0,C=1} = 2r^2\sin^2\theta(d\phi)_a(d\phi)_b. \qquad (176)$$

Thus, the linear-order perturbation ${}_{\mathscr{X}}h_{ab}$ defined by Equation (174) in the second-kind gauge choice $\mathscr{X}_e$ is given by

$$\begin{aligned}
{}_{\mathscr{X}}h_{ab} &= \frac{2M_1}{r}\left((dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right) \\
&\quad+2\alpha_1\left[-r\cos\theta g_{ab}+Mr^2\cos\theta\left(-(d\theta)_a(d\theta)_b+\sin^2\theta(d\phi)_a(d\phi)_b\right)\right] \\
&\quad+2C_1r^2\sin^2\theta(d\phi)_a(d\phi)_b \qquad (177)
\end{aligned}$$

and that in the gauge $\mathscr{Y}$ is given by

$$\begin{aligned}
{}_{\mathscr{Y}}h_{ab} &= \frac{2M_1}{r}\left((dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right) \\
&\quad+2\alpha_1\left[-r\cos\theta g_{ab}+Mr^2\cos\theta\left(-(d\theta)_a(d\theta)_b+\sin^2\theta(d\phi)_a(d\phi)_b\right)\right] \\
&\quad+2C_1r^2\sin^2\theta(d\phi)_a(d\phi)_b \\
&\quad+\pounds_\xi g_{ab}. \qquad (178)
\end{aligned}$$

Here, we note that the first lines in Equations (177) and (178), which correspond to the mass perturbations, are included in the perturbation $\mathscr{F}_{AB}$ in Equations (29). In the previous papers [12,15], we already saw this term in the analyses of the $l = 0$ mode vacuum perturbations. On the other hand, the second and third lines in Equations (177) and (178), which correspond to the acceleration perturbations and perturbations of the deficit/excess

angle, are not so simple. We also note that the deficit/excess angle perturbation of the third line in Equations (177) and (178) may depend on the mass perturbation and acceleration perturbations.

4.2. Components of Metric Perturbation of the Linearized Non-Rotating C-Metric

Here, we consider the component ${}_{\mathscr{X}}h_{ab}$, which is given by Equation (177). We omit the gauge-index $\mathscr{X}$ in the notation of ${}_{\mathscr{X}}h_{ab}$. The first term is the additional mass parameter perturbation of the Schwarzschild spacetime shown in the papers [12,15], which is also described in Equation (65) for $l = 0$ mode perturbation. The $(t-r)$-part h_{ab} is given by

$$h_{AB} = \frac{2M_1}{r}\Big((dt)_A(dt)_B + f^{-2}(dr)_A(dr)_B\Big) - 2\alpha_1 r\cos\theta y_{AB}. \tag{179}$$

In the expression of h_{ab} in Equation (177), there is no component of h_{Ap}. Furthermore, the angular part of h_{ab} is given by

$$\begin{aligned} h_{pq} &= 2\alpha_1\Big[-r\cos\theta r^2\gamma_{pq} + Mr^2\cos\theta\Big(-(d\theta)_p(d\theta)_q + \sin^2\theta(d\phi)_p(d\phi)_q\Big)\Big] \\ &\quad +2C_1 r^2\sin^2\theta(d\phi)_a(d\phi)_b. \end{aligned} \tag{180}$$

This component h_{pq} is decomposed as Equation (24). The trace of h_{pq} is given by

$$\sum_{l,m}\tilde{h}_{(e0)}S = \frac{1}{r^2}\gamma^{pq}h_{pq} = -4\alpha_1 r\cos\theta + 2C_1. \tag{181}$$

On the other hand, the traceless part of h_{pq} is given by

$$\frac{1}{r^2}h_{pq} - \frac{1}{2}\gamma_{pq}\frac{1}{r^2}\gamma^{rs}h_{rs} = (2\alpha_1 M\cos\theta + C_1)\Big(-(d\theta)_p(d\theta)_q + \sin^2\theta(d\phi)_p(d\phi)_q\Big). \tag{182}$$

4.3. Harmonic Decomposition of the Perturbative Non-Rotating C-Metric

Here, we choose the mode function S as the Legendre function $P_l(\cos\theta)$ as

$$S_l = P_l(\cos\theta). \tag{183}$$

This choice corresponds to the fact that we concentrate only on the $m = 0$ modes for arbitrary l. Then, the vector harmonics are defined by

$$\hat{D}_p S_l = -\sqrt{1-z^2}\frac{d}{dz}P_l(z)(d\theta)_p, \quad z := \cos\theta, \tag{184}$$

$$\epsilon_{pq}\hat{D}^q S_l = (1-z^2)\frac{d}{dz}P_l(z)(d\phi)_p. \tag{185}$$

Furthermore, the tensor harmonics consist of the trace part

$$\frac{1}{2}\gamma_{pq}S_l = \frac{1}{2}\gamma_{pq}P_l(z), \tag{186}$$

the traceless even part

$$\Big(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\Big)S_l = P_l^2(z)\frac{1}{2}(\theta_p\theta_q - \phi_p\phi_q), \quad (l \geq 2), \tag{187}$$

and the traceless odd part

$$2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_l = P_l^2(z)2\theta_{(p}\phi_{q)}, \quad (l \geq 2). \tag{188}$$

These are derived from the formula

$$\hat{D}_q\hat{D}_r S_l = \left[(1-z^2)\frac{d^2}{dz^2}P_l(z) - z\frac{d}{dz}P_l(z)\right]\theta_q\theta_r + \left[-z\frac{d}{dz}P_l(z)\right]\phi_r\phi_q. \tag{189}$$

For $l = 0, 1$ modes, tensor harmonics that correspond to Equations (187) and (188) vanish. These correspond to the fact that we have already imposed the regularity $\delta = 0$ in Proposal 1.

Here, we check the formulae for the orthogonality of the harmonics, which are necessary later. First, we point out the orthogonality of the scalar harmonics (183) for $l, l' \geq 0$ modes:

$$\int d^2\Omega S_{l'}S_l = 2\pi\int_0^\pi \sin\theta d\theta P_{l'}(\cos\theta)P_l(\cos\theta) = 2\pi\int_{-1}^{-1} dz P_{l'}(z)P_l(z) = \frac{4\pi}{2l+1}\delta_{ll'}. \tag{190}$$

Next, we consider the orthogonality of the even tensor harmonics (187) for $l, l' \geq 2$ modes:

$$\begin{aligned}&\int d^2\Omega\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_l\left(\hat{D}^p\hat{D}^q - \frac{1}{2}\gamma^{pq}\hat{D}^r\hat{D}_r\right)S_{l'}\\ =\ & \frac{2\pi(l-1)l(l+1)(l+2)}{(2l+1)}\delta_{ll'}.\end{aligned} \tag{191}$$

Now, we consider the perturbative C-metric given by Equations (179) and (180), and $h_{Ap} = 0$. The angular part h_{pq} given by Equation (180) is also decomposed into the trace and the traceless part as Equations (181) and (182). As there is no h_{Ap} component, the perturbative C-metric does not have any vector part. Furthermore, the traceless part of the angular components (182) does not have $(\theta - \phi)$ component. Therefore, the perturbative C-metric does not have any tensor odd mode. Furthermore, we do not have any vector and tensor perturbations of odd modes in $l = 0, 1$ modes.

For our convention, we introduce the notation $|K^l_{pq}\rangle$ to consider the orthogonality Equation (191) of the even-mode tensor harmonics by

$$|K^l_{pq}\rangle := \left[\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right]S_l. \tag{192}$$

The orthogonality condition (191) is denoted as

$$\begin{aligned}\langle K^{pq}_{l'}|K^l_{pq}\rangle \ &:=\ \int d^2\Omega\left[\hat{D}^p\hat{D}^q - \frac{1}{2}\gamma^{pq}\hat{D}_r\hat{D}^r\right]P_{l'}\left[\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_s\hat{D}^s\right]P_l\\ &=\ \frac{2\pi(l-1)l(l+1)(l+2)}{2l+1}\delta_{ll'}.\end{aligned} \tag{193}$$

When an arbitrary traceless tensor f_{pq}, which represents the vector $|f_{pq}\rangle$ of the function space, is given by

$$|f_{pq}\rangle = \sum_l g_l|K^l_{pq}\rangle, \tag{194}$$

applying $\langle K^{pq}_{l'}|$ from the left, we obtain

$$\begin{aligned}\langle K^{pq}_{l'}|f_{pq}\rangle &= \sum_{l\geq 2}^{\infty} g_l \langle K^{pq}_{(e)l'}|K^{(e)l}_{pq}\rangle \\ &= \sum_{l\geq 2}^{\infty} g_l \frac{2\pi(l-1)l(l+1)(l+2)}{2l+1}\delta_{ll'} \\ &= g_l \frac{2\pi(l-1)l(l+1)(l+2)}{2l+1} \quad (l\geq 2). \end{aligned} \tag{195}$$

Then, we have

$$g_l = \frac{2l+1}{2\pi(l-1)l(l+1)(l+2)}\langle K^{pq}_{l}|f_{pq}\rangle \quad (l\geq 2). \tag{196}$$

As we note that the traceless part (182) in h_{pq} does not have the odd-mode part, we obtain

$$\begin{aligned}\frac{1}{r^2}\mathscr{Z}h_{pq} - \frac{1}{2}\gamma_{pq}\frac{1}{r^2}\gamma^{rs}\mathscr{Z}h_{rs} &= \sum_{l\geq 2}\tilde{h}_{(e2)}|K^{l}_{pq}\rangle \\ &= (2\alpha_1 M\cos\theta + C_1)\left(-\theta_p\theta_q + \phi_p\phi_q\right). \end{aligned} \tag{197}$$

Then, we obtain

$$\begin{aligned}\tilde{h}_{(e2)} &= -\frac{2l+1}{2\pi(l-1)l(l+1)(l+2)}2\pi\int_{-1}^{1} dx(2\alpha_1 Mx + C_1)P_l^2(x) \\ &= -\frac{2l+1}{2\pi(l-1)l(l+1)(l+2)}2\pi\int_{-1}^{1} dx(2\alpha_1 Mx + C_1)(1-x^2)\frac{d^2}{dx^2}P_l(x) \\ &= -\frac{2(2l+1)}{(l-1)l(l+1)(l+2)}\left[2\alpha_1 M + C_1 + (-2\alpha_1 M + C_1)(-1)^l\right]. \end{aligned} \tag{198}$$

In summary, the perturbative C-metric h_{ab} given by Equations (179) and (180) is decomposed based on the scalar harmonic function $S_l = P_l(\cos\theta)$. Together with corresponding gauge-invariant and gauge-dependent variables, these are summarized as follows: For $l = 0$ mode, the mode coefficients of the harmonic decomposition are given by

$$\tilde{h}_{AB} = \frac{2M_1}{r}\Big((dt)_A(dt)_B + f^{-2}(dr)_A(dr)_B\Big), \tag{199}$$

$$\tilde{h}_{(e1)A} = 0, \quad \tilde{h}_{(o1)A} = 0, \tag{200}$$

$$\tilde{h}_{(e0)} = 2C_1, \quad \tilde{h}_{(e2)} = 0, \quad \tilde{h}_{(o2)} = 0. \tag{201}$$

The components of the gauge-dependent part $\tilde{Y}_A$, $\tilde{Y}_{(o)}$, and $\tilde{Y_{(e)}}$ of the metric perturbation for $l = 0$ mode are given by Equations (26)–(28). Substituting Equations (199)–(201) into these equations, we obtain

$$\tilde{Y}_A := r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)} = 0, \quad \tilde{Y}_{(o1)} := -r^2\tilde{h}_{(o2)} = 0, \quad \tilde{Y}_{(e1)} := \frac{r^2}{2}\tilde{h}_{(e2)} = 0. \tag{202}$$

The components of the gauge-invariant variables $\tilde{F}_{AB}$, $\tilde{F}_A$, and $\tilde{F}$ defined by Equations (32)–(34) are given by

$$\begin{aligned}
\tilde{F}_{AB} &:= \tilde{h}_{(k_{\hat{\Delta}})AB} - 2\bar{D}_{(A}\tilde{Y}_{(k_{\hat{\Delta}})B)} = \frac{2M_1}{r}\Big((dt)_A(dt)_B + f^{-2}(dr)_A(dr)_B\Big), && (203)\\
\tilde{F}_A &:= \tilde{h}_{(k_{\hat{\Delta}},o)A} + r\bar{D}_A\tilde{h}_{(k_{\hat{\Delta}},o1)} = 0, && (204)\\
\tilde{F} &:= \tilde{h}_{(k_{\hat{\Delta}},e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r = 2C_1. && (205)
\end{aligned}$$

For $l = 1$ mode, the mode coefficients of harmonic decomposition are given by

$$\begin{aligned}
h_{AB} &= -2\alpha_1 r(-f(dt)_A(dt)_B + f^{-1}(dr)_A(dr)_B), && (206)\\
\tilde{h}_{(e1)A} &= 0, \quad \tilde{h}_{(o1)A} = 0, && (207)\\
\tilde{h}_{(e0)} &= -4\alpha_1 r, \quad \tilde{h}_{(e2)} = 0, \quad \tilde{h}_{(o2)} = 0. && (208)
\end{aligned}$$

The components of the gauge-dependent part $\tilde{Y}_A$, $\tilde{Y}_{(e1)}$, and $\tilde{Y}_{(o1)}$ of the metric perturbation for $l = 1$ modes are given by Equations (26)–(28). Substituting Equations (206)–(208) into these equations, we obtain

$$\tilde{Y}_A := r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)} = 0, \quad \tilde{Y}_{(e1)} := \frac{r^2}{2}\tilde{h}_{(e2)} = 0, \quad \tilde{Y}_{(o1)} := -r^2\tilde{h}_{(o2)} = 0. \tag{209}$$

The components of the gauge-invariant variables $\tilde{F}_{AB}$, $\tilde{F}_A$, and $\tilde{F}$ defined by Equations (32)–(34) are given by

$$\begin{aligned}
\tilde{F}_{AB} &:= -2\alpha_1 r(-f(dt)_A(dt)_B + f^{-1}(dr)_A(dr)_B), && (210)\\
\tilde{F}_A &:= \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)} = 0, && (211)\\
\tilde{F} &:= -4\alpha_1 r. && (212)
\end{aligned}$$

For $l \geq 2$ modes, the mode coefficients of harmonic decomposition are given by

$$\begin{aligned}
h_{AB} &= 0, \quad \tilde{h}_{(e1)A} = 0, \quad \tilde{h}_{(o1)A} = 0, && (213)\\
\tilde{h}_{(e0)} &= 0, \quad \tilde{h}_{(o2)} = 0, && (214)\\
\tilde{h}_{(e2)} &= -\frac{2(2l+1)}{(l-1)l(l+1)(l+2)}\Big[(2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\Big]. && (215)
\end{aligned}$$

The components of the gauge-dependent part $\tilde{Y}_A$, $\tilde{Y}_{(e1)}$, and $\tilde{Y}_{(o1)}$ of the metric perturbation for $l \geq 2$ mode are given by Equations (26)–(28). Substituting Equations (213)–(215) into these equations, we obtain

$$\begin{aligned}
\tilde{Y}_A &:= r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)} = 0, && (216)\\
\tilde{Y}_{(e1)} &:= \frac{r^2}{2}\tilde{h}_{(e2)} = -\frac{r^2(2l+1)}{(l-1)l(l+1)(l+2)}\Big[(2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\Big], && (217)\\
\tilde{Y}_{(o1)} &:= -r^2\tilde{h}_{(o2)} = 0. && (218)
\end{aligned}$$

The components of the gauge-invariant variables $\tilde{F}_A$, $\tilde{F}$, and $\tilde{F}_{AB}$, defined by Equations (32)–(34) are given by

$$\begin{aligned}
\tilde{F}_{AB} &:= \tilde{h}_{AB} - 2\bar{D}_{(A}\tilde{Y}_{B)} = 0, && (219)\\
\tilde{F}_A &:= \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)} = 0, && (220)\\
\tilde{F} &:= \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r + \tilde{h}_{(e2)}l(l+1)\\
&= -\frac{2(2l+1)}{(l-1)(l+2)}\Big[(2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\Big]. && (221)
\end{aligned}$$

4.4. Realization of $l \geq 2$ Mode Perturbations

As shown above, the perturbative expression of the C-metric does not include odd-mode perturbation as in Equation (220). Furthermore, we also obtain $\tilde{F}_{AB} = 0$ from Equation (219), and the gauge-invariant variable $\tilde{F}$ is constant, given by Equation (221). Therefore, we obtain

$$\tilde{F}_{AB} = 0, \quad \partial_r\tilde{F} = \partial_t\tilde{F} = 0, \tag{222}$$

for $l \geq 2$ mode perturbations.

For $l \geq 2$ mode, the linearized Einstein equations for even-mode perturbations are given in Section 2.3. From Equation (40) and the first condition (222), we obtain

$$\tilde{T}_{(e2)} = 0. \tag{223}$$

As Equation (222) implies $X_{(e)} = Y_{(e)} = 0$, and $\tilde{F}$ is constant, we obtain

$$\tilde{T}_{(e1)A} = 0 \tag{224}$$

from Equations (41) and (223). Furthermore, from Equation (45) and (222), we obtain

$$S_{(\mathbb{F})AB} = 0. \tag{225}$$

Together with Equations (223) and (224), Equation (225) yields

$$\begin{aligned}
\tilde{T}_{tr} &= 0, && (226)\\
\tilde{T}_{tt} + f^2\tilde{T}_{rr} &= 0. && (227)
\end{aligned}$$

Moreover, from Equations (48) and (222), Equation (55) with the source term (58) is given by

$$\tilde{F} = 16\pi\frac{r^2}{(l-1)(l+2)}\left(\frac{1}{f}\tilde{T}_{tt} - f\tilde{T}_{rr}\right) = \frac{32\pi r^2}{(l-1)(l+2)f}\tilde{T}_{tt} = \text{constant}. \tag{228}$$

Finally, from Equations (223) and (224), the component (62) yields

$$\tilde{T}_{(e0)} = 0 \tag{229}$$

for $l \geq 2$ modes. Thus, from the definition of these components (36), for $l \geq 2$ mode, we obtain

$$\begin{aligned}
{}^{(1)}\mathcal{T}_{ac} &= \sum_{l=2}^{\infty} P_l(\cos\theta)\left(\tilde{T}_{tt}(dt)_a(dt)_c + \tilde{T}_{rr}(dr)_a(dr)_c\right)\\
&=: -\frac{1}{r^2}y_{ab}\sum_{l=2}^{\infty}\lambda_l P_l(\cos\theta), && (230)
\end{aligned}$$

where we defined

$$16\pi\lambda_l := 16\pi\frac{r^2}{f}\tilde{T}_{tt} = -(2l+1)\Big[(2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\Big] \tag{231}$$

from Equations (221) and (228).

We check the other components of the linearized Einstein equation. To carry this out, we see that the Moncrief variable $\Phi_{(e)}$ in the case of Equation (222) is given by

$$\Phi_{(e)} = -\frac{r}{4}\tilde{F}, \tag{232}$$

through the definition (48). Through Equations (223) and (226), and from Equations (52) and (53), the source terms $S_{(\Lambda\tilde{F})}$ and $S_{(Y_2)}$ are given by

$$S_{(\Lambda\tilde{F})} = \tilde{T}_{tt}, \quad S_{(Y_{(e)})} = 0. \tag{233}$$

From $S_{(Y_{(e)})} = 0$ and the vanishing components $(X_{(e)}, Y_{(e)})$ of $\tilde{\mathbb{F}}_{AB}$ and the derivative of $\tilde{F}$ yields that Equation (51) is trivial. On the other hand, by substituting Equation (48) and $S_{(\Lambda\tilde{F})} = \tilde{T}_{tt}$ into Equation (50), we obtain the same result as Equation (228).

Finally, we check the master equation (54) with the source term $S_{(\Phi_{(e)})}$ given by Equation (57). Substituting Equations (223), (224), (227)–(229) into Equation (57),

$$S_{(\Phi_{(e)})} = \frac{1}{2f}\Lambda\tilde{T}_{tt} + \frac{2f-1}{f}\tilde{T}_{tt} - \frac{3(1-f)}{\Lambda}\tilde{T}_{tt}. \tag{234}$$

By substituting Equations (56), (228), (232), and (234) into Equation (54), and using Equation (49), we can confirm that Equation (54) is trivial.

4.5. Realization of $l = 0$ Mode Perturbations

The solutions to the Einstein equation for $l = 0$ mode with a generic matter field are extensively discussed in the Part II paper [15]. Following Proposal 1, we impose the regularity to the harmonic S_δ, and the components $\tilde{T}_{(e1)A}$ and $\tilde{T}_{(e2)}$ do not appear in the expression of the linear perturbations of the energy-momentum tensor. Therefore, we may safely choose

$$\tilde{T}_{(e1)A} = 0, \quad \tilde{T}_{(e2)} = 0. \tag{235}$$

From Equations (62) and (235), we have

$$\tilde{T}_{(e0)} = 0, \tag{236}$$

and Equations (60) and (61) are given by

$$-\partial_t\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rt} + \frac{(1+f)f}{r}\tilde{T}_{rt} = 0, \tag{237}$$

$$-\partial_t\tilde{T}_{tr} + \frac{1-f}{2rf}\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rr} + \frac{(3+f)f}{2r}\tilde{T}_{rr} = 0. \tag{238}$$

On the other hand, the metric perturbation of $l = 0$ mode summarized in Equations (199)–(201) is given by

$$h_{ab}(l=0) = \frac{2M_1}{r}\Big((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\Big) + r^2 C_1\gamma_{ab}. \tag{239}$$

The mass perturbation M_1 is discussed in the Part II paper [15]. To include this mass perturbation parameter M_1 as the solution to the Einstein equation in our formulation, we have to introduce the term

$$\mathscr{Z}h_{ab}(l=0) = \mathscr{X}h_{ab}(l=0) + \pounds_V g_{ab}, \tag{240}$$

where the generator V_a is given by

$$V_a = \left(\frac{f}{4}\mathrm{Y} + \frac{rf}{4}\partial_r \mathrm{Y} - \frac{r}{1-3f}\Xi(r) + f\int dr \frac{2}{f(1-3f)^2}\Xi(r)\right)(dt)_a + \frac{r}{4f}\partial_t \mathrm{Y}(dr)_a. \tag{241}$$

Here, the function $\mathrm{Y}(t,r)$ is the solution to the equation

$$-\frac{1}{f}\partial_t^2 \mathrm{Y} + \partial_r(f\partial_r \mathrm{Y}) + \frac{1}{r^2}3(1-f)\mathrm{Y} - \frac{4}{r^3}2M_1 t = 0, \tag{242}$$

and $\Xi(r)$ in Equation (241) is an arbitrary function of r.

As we always introduce the mass parameter perturbation M_1 if we introduce the last term in Equation (240), we ignore this mass perturbation at this moment. Then, we may concentrate on the perturbation C_1 in the metric perturbation

$$h_{ab}(l=0) \quad = \quad r^2 C_1 \gamma_{ab}. \tag{243}$$

As the $\theta\theta$- and $\phi\phi$-components in the solution (65) are described by the term $\pounds_V g_{ab}$, we consider the components $\pounds_{V_{(1)}} g_{ab}$ with the generator $V_{(1)a} = V_{(1)r}(r)(dr)_a$. In this case, the non-vanishing components of $\pounds_{V_{(1)}} g_{ab}$ are summarized as

$$\begin{aligned} &\pounds_{V_{(1)}} g_{tt} = -ff' V_{(1)r}, \quad \pounds_{V_{(1)}} g_{rr} = 2\partial_r V_{(1)r} + \frac{f'}{f}V_{(1)r}, \\ &\pounds_{V_{(1)}} g_{\theta\theta} = 2rf V_{(1)r}, \quad \pounds_{V_{(1)}} g_{\phi\phi} = 2rf\sin^2\theta V_{(1)r}. \end{aligned} \tag{244}$$

Here, we choose

$$V_{(1)r} = \frac{r^2}{2rf}C_1 \tag{245}$$

so that the $\theta\theta$- and $\phi\phi$-components are described by the term $\pounds_{V_{(1)}} g_{ab}$. Then, we have

$$h_{ab}(l=0) \quad = \quad \frac{1-f}{2}C_1(dt)_a(dt)_b + \frac{1-3f}{2f^2}C_1(dr)_a(dr)_b + \pounds_{V_{(1)}} g_{ab}. \tag{246}$$

As in the case of the LTB solution, we make the (t,r) part in Equation (246) traceless through the introduction of the term $\pounds_{V_{(2)}} g_{ab}$ with $V_{(2)a} = V_{(2)t}(dt)_a$ with the condition $\partial_\theta V_{(2)t} = \partial_\phi V_{(2)t} = 0$. In this choice of $V_{(2)a}$, the non-vanishing components of $\pounds_{V_{(2)}} g_{ab}$ are given by

$$\pounds_{V_{(2)}} g_{tt} = 2\partial_t V_{(2)t}, \quad \pounds_V g_{tr} = \partial_r V_{(2)t} - \frac{f'}{f}V_{(2)t}. \tag{247}$$

Through this expression (247), $h_{ab}(l=0)$ is given by

$$
\begin{aligned}
h_{ab}(l=0) &= \left(\frac{1-f}{2}C_1 - 2\partial_t V_{(2)t}\right)(dt)_a(dt)_b + \frac{1-3f}{2}C_1 f^{-2}(dr)_a(dr)_b \\
&\quad -\left(\partial_r V_{(2)t} - \frac{f'}{f}V_{(2)t}\right)2(dt)_{(a}(dr)_{b)} + \pounds_{V_{(1)}+V_{(2)}}g_{ab} \\
&= \left(fC_1 - 2\partial_t V_{(2)t}\right)(dt)_a(dt)_b + \frac{1-3f}{2}C_1\left((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right) \\
&\quad -\left(\partial_r V_{(2)t} - \frac{f'}{f}V_{(2)t}\right)2(dt)_{(a}(dr)_{b)} + \pounds_{V_{(1)}+V_{(2)}}g_{ab}. \qquad (248)
\end{aligned}
$$

Here, we choose $V_{(2)t}$ so that

$$V_{(2)t} = \frac{1}{2}ftC_1. \qquad (249)$$

The choice (249) yields

$$\partial_r V_{(2)t} - \frac{f'}{f}V_{(2)t} = 0, \qquad (250)$$

and

$$\mathscr{L}h_{ab}(l=0) = \frac{1-3f}{2}C_1\left((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right) + \pounds_{V_{(C_1)}}g_{ab}, \qquad (251)$$

where

$$V_{(C_1)a} = \frac{1}{2}ftC_1(dt)_a + \frac{r}{2f}C_1(dr)_a. \qquad (252)$$

Now, we compare the metric perturbation (251) with the derived $l=0$ solution (65) with the generator (66). As we ignore the mass parameter perturbation M_1, we obtain the relations

$$4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right] = \frac{1-3f}{4}rC_1, \qquad (253)$$

$$4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right) = 0. \qquad (254)$$

If the condition (254) is satisfied for an arbitrary time t, we obtain

$$\tilde{T}_{tt} + f^2\tilde{T}_{rr} = 0. \qquad (255)$$

As we ignore the integration constant M_1, the condition (253) gives

$$\frac{r^2}{f}\tilde{T}_{tt} = \partial_r\left(\frac{1-3f}{16\pi}r\right)C_1 = -\frac{1}{8\pi}C_1. \qquad (256)$$

Furthermore, we may choose $\tilde{T}_{tr} = 0$ without contradiction to the linear perturbation of the continuity equations (60) and (61) with $l=0$. From the definition (36) of the components $\tilde{T}_{tt}$ and $\tilde{T}_{rr}$, for $l=0$ mode, we obtain

$${}^{(1)}\mathscr{T}_{ac} = -\frac{1}{r^2}y_{ab}\lambda_{l=0} \qquad (257)$$

from Equations (255) and (256). Here, we define

$$\lambda_{l=0} := \frac{r^2}{f}\tilde{T}_{tt} = -\frac{C_1}{8\pi}. \tag{258}$$

Next, we compare the generator $V_{(C_1)a}$ defined by Equation (252) and the generator V_a given by Equation (66) in the $l = 0$ mode solution (65). Comparing the r-component of Equation (252) with Equation (66), we choose

$$\frac{1}{4f} r\partial_t \mathrm{Y} = \frac{r}{2f} C_1, \tag{259}$$

and obtain

$$\mathrm{Y} = 2C_1 t. \tag{260}$$

Here, we ignore the integration constant in the integration of Equation (259). Substituting this result (260) into Equation (66),

$$V_a = \left(\frac{f}{4} 2C_1 t - \frac{r\Xi(r)}{(1-3f)} + f \int dr \frac{2\Xi(r)}{f(1-3f)^2} \right) (dt)_a + \frac{r}{2f} C_1 (dr)_a. \tag{261}$$

When choosing $\Xi(r) = 0$, the generator (261) coincides with the generator $V_{(C_1)a}$ given by Equation (252). Thus, the $l = 0$ mode solution (65) realizes the $l = 0$ mode part (251) of the C-metric.

In summary, we have obtained the $l = 0$ metric perturbation

$$\begin{aligned} {}^{(2)}h_{ab}(l=0) &= {}^{(2)}h_{ab}(l=0) + \pounds_{V_{M_1}} g_{ab} \\ &= \left(\frac{2M_1}{r} + \frac{1-3f}{2} C_1 \right) \left((dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b \right) \\ &\quad + \pounds_{V_{(M_1)} + V_{(C_1)}} g_{ab}, \end{aligned} \tag{262}$$

where

$$V_{(M_1)a} = \frac{1}{4} f \left(\mathrm{Y}_{M_1} + r\partial_r \mathrm{Y}_{M_1} \right) (dt)_a + \frac{1}{4f} r\partial_t \mathrm{Y}_{M_1} (dr)_a, \tag{263}$$

$$V_{(C_1)a} = \frac{1}{2} f t C_1 (dt)_a + \frac{r}{2f} C_1 (dr)_a. \tag{264}$$

Here, the function $\mathrm{Y}_{M_1}(t, r)$ is the solution to the Equation (55) with $\tilde{F} =: \partial_t \mathrm{Y}_{M_1}$, i.e.,

$$-\frac{1}{f}\partial_t^2 \mathrm{Y}_{M_1} + \partial_r (f\partial_r \mathrm{Y}_{M_1}) + \frac{1}{r^2} 3(1-f)\mathrm{Y}_{M_1} - \frac{4}{r^3} 2M_1 t = 0. \tag{265}$$

The $l = 0$ mode energy-momentum tensor for the C-metric is given by

$${}^{(1)}\mathcal{T}_{ac} = -\frac{1}{r^2} y_{ab} \lambda_{l=0} \tag{266}$$

with

$$\lambda_{l=0} = -\frac{C_1}{8\pi}. \tag{267}$$

Here, we note that the result (267) is also realized by the substitution $l = 0$ into Equation (231), although Equation (231) is derived only in the case of $l \geq 2$ modes. This indicates that the formula (231) is also valid even for $l = 0$ mode perturbations.

4.6. Realization of $l = 1$ Mode Perturbations

The $l = 1$ mode of the C-metric is summarized as Equations (206)–(210). We consider the continuity equation of the energy-momentum tensor (60)–(62) with $l = 1$. As in the $l \geq 2$ and $l = 0$ cases, we choose $\tilde{T}_{(e1)A} = 0 = \tilde{T}_{(e2)}$. These and Equation (62) with $l = 1$ yield

$$\tilde{T}_{(e0)} = 0. \tag{268}$$

Furthermore, we also assume that

$$\tilde{T}_{rt} = 0, \tag{269}$$

inspecting the $l \geq 2$ and $l = 0$ cases. Then, Equations (60) and (61) with $l = 1$ are given by

$$\partial_t \tilde{T}_{tt} = 0, \tag{270}$$

$$\partial_r(\tilde{T}_{tt} + f^2\tilde{T}_{rr}) - \frac{f}{r^2}\partial_r(\frac{r^2}{f}\tilde{T}_{tt}) + \frac{5f-1}{2rf}\left(\tilde{T}_{tt} + f^2\tilde{T}_{rr}\right) = 0. \tag{271}$$

Equation (270) indicates that we have the static energy density T_{tt}. As in the case of $l \geq 2$ and $l = 0$ modes, we define $\lambda_{l=1}$ by

$$\lambda_{l=1} := \frac{r^2}{f}\tilde{T}_{tt} \tag{272}$$

and we assume that $\lambda_{l=1}$ is constant and

$$\tilde{T}_{tt} + f^2\tilde{T}_{rr} = 0. \tag{273}$$

Due to these assumptions, Equation (271) is trivial. Through the above components of the energy-momentum tensor, Equation (63) is given by

$$\begin{aligned} \mathscr{F}_{ab} &= \pounds_{V_{(vac)}} g_{ab} \\ &\quad -\frac{16\pi\lambda_{l=1}f}{3(1-f)}\left[\frac{1+f}{2}(dt)_a(dt)_b - \frac{1-3f}{2f^2}(dr)_a(dr)_b + r^2\gamma_{ab}\right]\cos\theta. \end{aligned} \tag{274}$$

Here, we note that the $l = 1$ mode solution (211)–(210) is summarized as

$$\begin{aligned} \mathscr{F}_{ab} &= \tilde{F}_{AB}\cos\theta(dx^A)_a(dx^B)_b + \frac{1}{2}\gamma_{pq}\tilde{F}\cos\theta(dx^p)_a(dx^q)_b \\ &= -2\alpha_1 r\cos\theta g_{ab}. \end{aligned} \tag{275}$$

Comparing (274) and (275), we rewrite (274) as

$$\begin{aligned} \mathscr{F}_{ab} &= \pounds_{V_{(vac)}} g_{ab} \\ &\quad -\frac{16\pi\lambda_{l=1}f}{3(1-f)}\left[-\frac{1-f}{2}(dt)_a(dt)_b + \frac{1+3f}{2f^2}(dr)_a(dr)_b + \frac{1+f}{f}r^2\gamma_{ab}\right]\cos\theta \\ &\quad +\frac{16\pi\lambda_{l=1}}{3(1-f)}\cos\theta g_{ab}. \end{aligned} \tag{276}$$

Here, we explain the choice of the coefficients of the last term in Equation (277). If the expression (231) of the λ_l for $l \geq 2$ is also valid even for $l = 1$, Equation (231) is given by

$$16\pi\lambda_{l=1} = -12\alpha_1 M \tag{277}$$

and the last term in Equation (277) is given by

$$\frac{16\pi\lambda_{l=1}}{3(1-f)}\cos\theta g_{ab} = -2\alpha_1 r\cos\theta g_{ab}. \tag{278}$$

This is the $l = 1$ mode solution described by Equations (206)–(210). As the remaining problem, we have to consider the problem of whether the middle term in Equation (277) has the form $\pounds_W g_{ab}$ or not. If the middle term in Equation (277) does have the form $\pounds_W g_{ab}$, we may say that our $l = 1$ mode solution (63) does describe the linearized C-metric apart from the term of the Lie derivative of the background metric g_{ab}.

Now, we concentrate on the problem of whether the middle term in Equation (277) has the form $\pounds_W g_{ab}$ or not. To show this, we consider the components of $\pounds_W g_{ab}$ for an appropriate vector field W_a. We consider the generator W_a that satisfies $W_\phi = 0$, $\partial_\phi W_\theta = \partial_\phi W_r = \partial_\phi W_t = 0$. Furthermore, we assume that $W_t =: w_t\cos\theta$, $W_r =: w_r\cos\theta$, and $W_\theta =: w_\theta\sin\theta$ using $\pounds_W g_{t\theta} = 0$. Then, the non-trivial components of $\pounds_W g_{ab}$ are summarized as follows:

$$\pounds_W g_{tt} = \left(2\partial_t w_t - f f' w_r\right)\cos\theta, \tag{279}$$

$$\pounds_W g_{tr} = \left(\partial_t w_r + \partial_r w_t - \frac{f'}{f}w_t\right)\cos\theta, \tag{280}$$

$$\pounds_W g_{t\theta} = \left(\partial_t w_\theta - w_t\right)\sin\theta, \tag{281}$$

$$\pounds_W g_{rr} = \left(2\partial_r w_r + \frac{f'}{f}w_r\right)\cos\theta, \tag{282}$$

$$\pounds_W g_{r\theta} = \left(\partial_r w_\theta - w_r - \frac{2}{r}w_\theta\right)\sin\theta, \tag{283}$$

$$\pounds_W g_{\theta\theta} = 2(w_\theta + r f w_r)\cos\theta, \tag{284}$$

$$\pounds_W g_{\phi\phi} = 2(w_\theta + r f w_r)\sin^2\theta\cos\theta. \tag{285}$$

The middle term in Equation (277) has only its diagonal components, and we may choose $\pounds_W g_{t\theta} = \pounds_W g_{r\theta} = 0$. From these equations, Equations (281) and (283) yield

$$w_t = \partial_t w_\theta, \quad w_r = \partial_r w_\theta - \frac{2}{r}w_\theta. \tag{286}$$

Furthermore, from Equations (284) and (285), the second equation in Equation (286), and the term proportional to γ_{ab} in the second line of Equation (277), we have

$$r f\partial_r w_\theta + (1-2f)w_\theta = -\frac{r^2(1+f)}{6(1-f)}16\pi\lambda_{l=1}. \tag{287}$$

A solution to Equation (287) is given by

$$w_\theta = -\frac{r^2}{6(1-f)}16\pi\lambda_{l=1}. \tag{288}$$

Then, from Equations (286), we obtain

$$w_t = 0, \quad w_r = -\frac{r}{6(1-f)}16\pi\lambda_{l=1}. \tag{289}$$

Substituting Equations (288) and (289) into Equations (279)–(285), the non-vanishing components of $\pounds_W g_{ab}$ are given by

$$\pounds_W g_{ab} = \frac{16\pi\lambda_{l=1}f}{3(1-f)}\left[\frac{1-f}{2}(dt)_a(dt)_b - \frac{1+3f}{2f^2}(dr)_a(dr)_b - \frac{1+f}{f}r^2\gamma_{ab}\right]\cos\theta. \tag{290}$$

Through Equation (290), Equation (277) is given by

$$\mathscr{T}_{ab} = \pounds_{V_{(l=1)}} g_{ab} + \frac{16\pi\lambda_{l=1}}{3(1-f)} \cos\theta g_{ab}, \tag{291}$$

where $V_{(l=1)a} := V_{(vac)a} + W_a$. Comparing with Equations (210) and (212), we obtain Equations (277) and (278) as expected. This also indicates that the coefficient (231) for $l \geq 2$ is also valid not only for $l = 0$ mode but also $l = 1$ mode, i.e., the coefficient (231) is valid for any $l \geq 0$. Furthermore, we have seen above that the equation of state (273) with (272) yields

$${}^{(1)}\mathscr{T}_{ac}(l=1) = -\frac{1}{r^2} y_{ab} \lambda_{l=1} P_1(\cos\theta), \tag{292}$$

where $\lambda_{l=1}$ is a constant, which is given by Equation (277). At the same time, we may say that our $l = 1$ mode solution (63) does describe the linearized C-metric apart from the term of the Lie derivative of the background metric g_{ab}.

4.7. Source Term of the Linearized C-Metric

Here, we summarize the energy-momentum tensor $\mathscr{T}_{ab}$ for the linearized C-metric as follows:

$${}^{(1)}\mathscr{T}_{ac} = -\frac{1}{r^2} y_{ab} \sum_{l=0}^{\infty} \lambda_l P_l(\cos\theta), \tag{293}$$

where the constant λ_l is given by

$$16\pi\lambda_l = -(2l+1)\left[(2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right] \tag{294}$$

for $l \geq 0$. Substituting Equation (294) into Equation (293), we obtain

$$\begin{aligned} {}^{(1)}\mathscr{T}_{ac} = \frac{1}{4r^2} y_{ab} \Bigg[& (+2\alpha_1 M + C_1)\left(\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\right) \\ & +(-2\alpha_1 M + C_1)\left(\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)(-1)^l P_l(\cos\theta)\right)\Bigg]. \end{aligned} \tag{295}$$

Inspecting the work by Kodama [43], we consider the mode decomposition of the δ-function on S^2 [44,45]

$$\delta^{(2)}(\mathbf{n} - \mathbf{n}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}'), \tag{296}$$

where $\mathbf{n}$ and $\mathbf{n}'$ are the position vectors that point to the points on S^2 in embedded in $\mathbb{R}^3$, respectively. The summation over m is given by [44,45]

$$\sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}') = \frac{2l+1}{4\pi} C_l^{1/2}(\mathbf{n}\cdot\mathbf{n}'), \tag{297}$$

where $C_l^{1/2}(x)$ is the Gegenbauer polynomial. In $\mathbb{R}^3$, any point of the unit sphere S^2 is specified by the orthogonal coordinates (x, y, z) with the center of S^2 in $\mathbb{R}^3$

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta. \tag{298}$$

The north pole is specified as $(x, y, z) = (0, 0, 1)$ and the south pole is specified as $(x, y, z) = (0, 0, -1)$. The inner product $\mathbf{n} \cdot \mathbf{n}'$ in the case where $\mathbf{n}'$ is the north pole or the south pole is given by

$$\mathbf{n} \cdot \mathbf{n}_{\rm north} = \cos\theta, \quad \mathbf{n} \cdot \mathbf{n}_{\rm south} = -\cos\theta, \tag{299}$$

respectively, and we have

$$\sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}_{\rm north}) = \frac{2l+1}{4\pi} C_l^{1/2}(\cos\theta) = \frac{2l+1}{4\pi} P_l(\cos\theta), \tag{300}$$

$$\begin{aligned} \sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}_{\rm south}) &= \frac{2l+1}{4\pi} C_l^{1/2}(-\cos\theta) = \frac{2l+1}{4\pi} P_l(-\cos\theta) \\ &= \frac{2l+1}{4\pi} (-1)^l P_l(\cos\theta). \end{aligned} \tag{301}$$

Then, from Equation (296), we obtain

$$\delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm north}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}_{\rm north}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta), \tag{302}$$

$$\delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm south}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}_{\rm south}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)(-1)^l P_l(\cos\theta). \tag{303}$$

Through these expressions of the δ-functions, the first-order perturbation of the energy-momentum tensor (295) yields

$$\begin{aligned} {}^{(1)}\mathcal{T}_{ac} &= -\frac{1}{4r^2} y_{ab} \Big[-(2\alpha_1 M + C_1)\delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm north}) \\ &\qquad + (2\alpha_1 M - C_1)\delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm south}) \Big] \\ &= -\frac{1}{r^2} y_{ab} \Big[\mu_n \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm north}) + \mu_s \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\rm south}) \Big], \end{aligned} \tag{304}$$

where

$$\mu_n = -\frac{1}{4}(2\alpha_1 M + C_1), \quad \mu_s = \frac{1}{4}(2\alpha_1 M - C_1). \tag{305}$$

The first-order perturbation of the energy-momentum tensor (304) coincides with the energy-momentum tensor for a half-infinite string with constant line densities μ_n on the north half of the symmetry axis and μ_s on the south half of the symmetry axis. If we impose the regularity at the north pole, i.e., $\mu_n = 0$, we have $\mu_s = \alpha_1 M > 0$, which corresponds to the positive energy density of string. On the other hand, if we impose the regularity at the south pole, i.e., $\mu_s = 0$, we have $\mu_n = -\alpha_1 M < 0$, which corresponds to the negative energy density of string. These results are consistent with the stringy interpretation of the singularity of the C-metric [43]. Finally, we have to emphasize that the treatment of $l = 1$ mode perturbations in our derivation of the energy density of the C-metric is essentially different from those in ref. [43].

5. Summary and Discussion

In summary, after reviewing our general framework of the gauge-invariant perturbation theory and its application to the perturbation theory on the Schwarzschild background spacetime developed in refs. [12,14,15], we checked that our linearized solutions derived in refs. [12,14,15] realize the linearized LTB solution and the linearized C-metric around the

Schwarzschild background spacetime. These facts yield that our derived $l = 0, 1$ solutions to the linearized Einstein equation following Proposal 1 are physically reasonable. Then, we may say that Proposal 1 itself is also physically reasonable. Our general framework of the gauge-invariant perturbation theory developed in refs. [32–35] was applied to the cosmological perturbation theory in refs. [36,41]. On the other hand, in this series of our papers [12–15], we apply our general framework of the gauge-invariant perturbation theory to the perturbations on the Schwarzschild background spacetime. Thus, we may say that the applicability of our general framework of the gauge-invariant perturbation theory developed in refs. [32–35] is very wide.

Our general framework is based on the single non-trivial Conjecture 1. This conjecture is almost proved in ref. [34], except for the "zero-mode problem". In the proof in ref. [34], we introduced the Green functions for some elliptic derivative operators. This means that the kernel modes of these elliptical derivative operators were beyond our consideration. We call these kernel modes "zero modes", and the problem of finding gauge-invariant treatments for these kernel modes is the "zero-mode problem". To carry out the application to the perturbations on the Schwarzschild background spacetime, we have to propose a gauge-invariant treatment of $l = 0, 1$ mode perturbation on the Schwarzschild background spacetime because these modes correspond to the above "zero modes", and the gauge-invariant treatments were unclear until our proposal in refs. [12,14,15]. We should also note that such a "zero-mode problem" exists even in the cosmological perturbation theory developed in refs. [36,41].

In conventional perturbation theory on spherically symmetric background spacetimes, we use the spherical harmonics $S = Y_{lm}$ as the scalar harmonics and construct vector and tensor harmonics from the derivative of this scalar harmonics. However, in this construction of tensor harmonics, the set (17) of the scalar, vector, and tensor harmonics loses its linear independence as the basis of the tangent space on S^2 in $l = 0, 1$ mode. To recover this linear independence of the set (17), we introduced the singular harmonics for $l = 0, 1$ modes at once and proposed the strategy to construct gauge-invariant variables and derive the Einstein equation as Proposal 1. The conventional expansion using the spherical harmonic functions Y_{lm} is the restriction of the function space to the L^2 space on S^2. This restriction corresponds to the imposition of the regular boundary condition for the functions on S^2 at the starting point. On the other hand, our introduction of the singular harmonic functions at once and Proposal 1 state that the boundary condition on S^2 should be imposed when we solve the linearized Einstein equations. Owing to Proposal 1, we could prove Conjecture 1 for perturbations on the spherically symmetric background spacetime. Then, we reached the statement Theorem 1.

In actuality, following Proposal 1, we could construct gauge-invariant variables not only for $l \geq 2$ modes but also for $l = 0, 1$ modes. Furthermore, in ref. [14], we derived the solution of $l = 1$ odd-mode perturbations, and in ref. [15], we derived the solutions of $l = 0, 1$ even-mode perturbations.

This paper also reviewed the strategy to solve even-mode perturbation on the Schwarzschild background spacetime, including $l = 0, 1$ modes, which was discussed in the Part II paper [15]. Then, we showed that it is possible to confirm the realizations of the linearized LTB solutions and the linearized non-rotating C-metric through the even-mode solutions derived in the Part II paper [15]. Because the LTB solution is spherically symmetric, its linearized version should be realized by $l = 0$ even-mode perturbations. On the other hand, the non-rotating C-metric includes all $l \geq 0$ even-mode perturbations. This implies that the realization of the C-metric perturbation supports the fact that our derived solutions in the Part II paper [15] are reasonable, and then we may say that Proposal 1 is also physically reasonable.

The LTB solution is a spherically symmetric exact solution that describes the expanding universe with dust matter or the dust matter collapsing into a black hole. It is well known that the LTB solutions include the Schwarzschild spacetime as a special case. For this reason, we can regard this LTB solution as a black hole solution with perturbative collapsing dust matter. After reviewing the LTB exact solution, we considered the vacuum black hole solution, i.e., the Schwarzschild spacetime with the perturbative dust matter. We examined our $l = 0$ even-mode solution derived in the Part II paper [15], which describes this perturbative solution at the linear level. From the perturbative treatment, we confirmed the linearized continuity equations of the dust matter in terms of the static Schwarzschild coordinate. We also considered the linear metric perturbation on the Schwarzschild background spacetime of the LTB exact solution. In this realization, the perturbative arbitrary functions $f(R)$ and $\tau_0(R)$ in the LTB solutions, which have their physical meanings, were included in the term of the Lie derivative of the background metric g_{ab}. Therefore, we should regard that such terms of the Lie derivative of the background metric g_{ab} are physical. Thus, we confirmed that the $l = 0$ even-mode solution derived in the Part II paper [15] does describe this perturbative LTB solution.

Next, we considered the linearized C-metric with the Schwarzschild background spacetime, which may have $l = 1$ even-mode perturbations. As this linearized C-metric includes $l = 1$ even-mode perturbations, this solution is appropriate for checking whether our derived $l = 1$ even-mode perturbation is physically reasonable. After reviewing the non-rotating vacuum C-metric [38], in which conical singularities may occur both in the axis $\theta = 0$ and $\theta = \pi$, we considered the perturbative form of this solution on the Schwarzschild background spacetime. To consider the perturbative expression of the C-metric on the Schwarzschild background spacetime, we discussed the situation where the acceleration parameter α_1 is sufficiently small. Furthermore, we must consider the deficit/excess angle perturbation C_1. We must keep in mind that we may always change the point identification between the physical C-metric spacetime and the background Schwarzschild spacetime, i.e., we may change the second-kind gauge at any time, as in the LTB case.

Although we follow Proposal 1, we compare the result after imposing the regularity $\delta = 0$. We only consider the even $m = 0$-mode perturbations because the non-rotating C-metric does not have the ϕ-dependence or the odd-mode components of perturbations. We consider the mode decomposition of the linearized C-metric with the Schwarzschild background spacetime using the Legendre function $P_l(\cos\theta)$ as the scalar harmonics. Then, we could identify the linearized C-metric with the $l = 0, 1$- and $l \geq 2$-mode perturbations on the Schwarzschild background spacetime.

From the $l \geq 2$ metric perturbation of the linearized C-metric, we identify the components of the linear perturbations of the energy-momentum tensor for $l \geq 2$ modes. Then, we have obtained the linear perturbation of the energy-momentum tensor (230) with the constant λ_l, which is given by (231). Furthermore, we also checked the consistency of all components of the Einstein equation for $l \geq 2$ modes.

For $l = 0$ modes of the linearized C-metric, we have the additional mass parameter perturbation of the Schwarzschild spacetime and the deficit/excess angle perturbation. The additional mass parameter perturbation can always be added as the vacuum solution with the term of the Lie derivative of the background metric g_{ab}. In this case, the term of the Lie derivative of the background metric g_{ab} plays an important role. On the other hand, the deficit/excess perturbation is proportional to the metric on S^2. This perturbation is also expressed as the traceless part of the (t, r)-component and the term of the Lie derivative of the background metric g_{ab}. From this expression, we obtained the equation of the state of the linear perturbations of the energy-momentum tensor and the connection of the deficit/excess angle perturbation and the energy density. Then, we showed the formula of

the deficit/excess angle perturbation, and the energy density for $l \geq 2$ mode perturbations is also valid for $l = 0$ mode perturbations.

For $l = 1$ modes of the linearized C-metric, the equation of state for the linear perturbations of the energy-momentum tensor and the formula that gives the relation between the energy density and the deficit/excess perturbation in the cases $l \geq 2$ modes and $l = 0$ modes are also consistent even in the $l = 1$-mode case. Furthermore, we showed that the linearized C-metric perturbation is given from the $l = 1$-mode solution obtained in the Part II paper [15], apart from the Lie derivative of the background metric g_{ab}. Note that the term of the Lie derivative of the background metric g_{ab} plays quite an essential role even in $l = 1$ mode solution.

As a result of the above $l \geq 2$-mode, $l = 0$-mode, and $l = 1$-mode analyses, we obtained the expression (293) of the linear perturbation of the energy-momentum tensor with the relation (294) between the energy density and the acceleration parameter perturbation and the deficit/excess angle perturbation. Substituting Equation (294) into Equation (293), we obtained the δ-function expression at the north and south pole of S^2, which is given by Equation (304) with the string tension formulae (305). Thus, we confirmed that the linearized C-metric is realized by the linear perturbative solutions obtained in the Part II paper [15].

As a summary of the three papers [14,15], including this paper, we have formulated a gauge-invariant formulation of the linear perturbations on the Schwarzschild background spacetime. Our formulation includes gauge-invariant treatments for $l \geq 2$ modes and $l = 0, 1$ mode perturbations. To construct gauge-invariant formulation for $l = 0, 1$-mode perturbations, we introduce the singular harmonic function at once and propose Proposal 1 as the strategy to solve the linearized Einstein equations on the Schwarzschild background spacetime, which state that we eliminate the singular behavior of introduced singular harmonics when we solve the linearized Einstein equations. Following Proposal 1, we showed the strategy to solve the odd-mode perturbation in the Part I paper [14] and the even-mode perturbation in the Part II paper [15].

We also derived the $l = 0, 1$ mode solution of the linearized Einstein equations for the odd and even-mode perturbations in the Part I paper [14] and in the Part II paper [15], respectively. Furthermore, in this paper, we showed that the solutions for $l = 0, 1$-mode perturbations derived in the Part II paper [15] realize two linearized exact solutions. One is the LTB solution, and the other is the non-rotating C-metric. Thus, the results in this paper support our solutions derived in the Part II paper [15] and our proposal in the Part I paper [14]. In this sense, we may say that our strategy to solve the linearized Einstein equations on the Schwarzschild background spacetime proposed as Proposal 1 is physically reasonable. We also note that the gauge-invariant solutions derived in the Part I paper [14] and the Part II [15] paper include the terms of the Lie derivative of the background metric g_{ab}. It is well known that we have a "residual gauge degree of freedom" even if we employ some gauge fixing conditions as discussed in [16]. The terms of the Lie derivative of the background metric g_{ab} might correspond to this "residual gauge". On the other hand, in this series of papers, we distinguish the notion of the gauge of the first kind and the notion of the gauge of the second kind. Furthermore, we declare that the purpose of our gauge-invariant perturbation theory is to exclude not the gauge of the first kind but the gauge of the second kind. Moreover, our formulation completely excludes the second kind of gauge. Therefore, the term of the Lie derivative of the background metric g_{ab} in our derived solution should be regarded as the gauge degree of freedom of the first kind. Even in this Part III paper, these terms of Lie derivative played a crucial role. Therefore, we may say that taking into account these terms of Lie derivative is also important when we compare the two metrics' perturbations.

Owing to Proposal 1, we could treat $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime in a gauge-invariant manner. This implies that the "zero-mode problem" on our general framework of the gauge-invariant perturbation theory was resolved, at least in the perturbations on the spherically symmetric background spacetime. This also implies that we can apply our general framework of higher-order gauge-invariant perturbation theory to any-order perturbations on the spherically symmetric background spacetime. This extension to any-order perturbations was briefly discussed in our companion paper [13]. We leave further detailed discussions for future works.

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Abbreviations

The following abbreviations are used in this manuscript:

LTB solution	Lemaître–Tolman–Bondi solution

References

1. LIGO Scientific Collaboration. 2025. Available online: https://ligo.org (accessed on 23 January 2025).
2. Virgo. 2025. Available online: https://www.virgo-gw.eu (accessed on 23 January 2025).
3. KAGRA. 2025. Available online: https://gwcenter.icrr.u-tokyo.ac.jp (accessed on 23 January 2025).
4. LIGO INDIA. 2025. Available online: https://www.ligo-india.in (accessed on 23 January 2025).
5. Einstein Telescope. 2024. Available online: https://www.et-gw.eu (accessed on 13 November 2024).
6. Cosmic Explorer. 2025. Available online: https://cosmicexplorer.org/ (accessed on 23 January 2025).
7. LISA. 2025. Available online: https://lisa.nasa.gov/ (accessed on 23 January 2025).
8. Kawamura, S.; Ando, M.; Seto, N.; Sato, S.; Musha, M.; Kawano, I.; Hong, F.L. Current status of space gravitational wave antenna DECIGO and B-DECIGO. *Prog. Theor. Exp. Phys.* **2021**, *2021*, 05A105. [CrossRef]
9. Mei, J.; Bai, Y.; Bao, J.; Barausse, E.; Cai, L.; Canuto, E.; Cao, B.; Chen, W.; Chen, Y.; Ding, Y.-W.; et al. The TianQin project: Current progress on science and technology. *Prog. Theor. Exp. Phys.* **2020**, *2020*, 05A107.
10. Luo, Z.; Wang, Y.; Wu, Y.; Hu, W.; Jin, G. The Taiji program: A concise overview. *Prog. Theor. Exp. Phys.* **2020**, *2020*, 05A108.
11. Barack, L.; Pound, A. Self-force and radiation reaction in general relativity. *Rep. Prog. Phys.* **2019**, *82*, 016904. [CrossRef] [PubMed]
12. Nakamura, K. Proposal of a gauge-invariant treatment of $l = 0, 1$-mode perturbations on Schwarzschild background spacetime. *Class. Quantum Gravity* **2021**, *38*, 145010. [CrossRef]
13. Nakamura, K. Formal Solutions of Any-Order Mass, Angular-Momentum, and Dipole Perturbations on the Schwarzschild Background Spacetime. *Lett. High Energy Phys.* **2021**, *2021*, 215. [CrossRef]
14. Nakamura, K. A gauge-invariant perturbation theory on the Schwarzschild background spacetime Part I: —Formulation and odd-mode perturbations. *arXiv* **2021**, arXiv:2110.13508. [CrossRef]
15. Nakamura, K. Gauge-invariant perturbation theory on the Schwarzschild background spacetime Part II: —Even-mode perturbations. *arXiv* **2021**, arXiv:2110.13512. [CrossRef]
16. Regge, T.; Wheeler, J.A. Stability of a Schwarzschild Singularity. *Phys. Rev.* **1957**, *108*, 1063. [CrossRef]
17. Zerilli, F. Effective potential for even-parity Regge-Wheeler gravitational perturbation equations. *Phys. Rev. Lett.* **1970**, *24*, 737. [CrossRef]
18. Zerilli, F. Gravitational Field of a Particle Falling in a Schwarzschild Geometry Analyzed in Tensor Harmonics. *Phys. Rev. D* **1970**, *2*, 2141. [CrossRef]
19. Moncrief, V. Gravitational Perturbations of Spherically Symmetric Systems. I. The Exterior Problem. *Ann. Phys.* **1974**, *88*, 323. [CrossRef]
20. Moncrief, V. Gravitational Perturbations of Spherically Symmetric Systems. II. Perfect Fluid Interiors. *Ann. Phys.* **1974**, *88*, 343. [CrossRef]

21. Cunningham, C.T.; Price, R.H.; Moncrief, V. Radiation from collapsing relativistic stars. I. linearized odd-parity radiation. *Astrophys. J.* **1978**, *224*, 643. [CrossRef]
22. Chandrasekhar, S. *The Mathematical Theory of Black Holes*; Clarendon Press: Oxford, UK, 1983.
23. Gerlach, U.H.; Sengupta, U.K. Gauge-invariant perturbations on most general spherically symmetric space-times. *Phys. Rev. D* **1979**, *19*, 2268. [CrossRef]
24. Gerlach, U.H.; Sengupta, U.K. Junction conditions for odd-parity perturbations on most general spherically symmetric space-times. *Phys. Rev. D* **1979**, *20*, 3009. [CrossRef]
25. Gerlach, U.H.; Sengupta, U.K. Even parity junction conditions for perturbations on most general spherically symmetric space-times. *J. Math. Phys.* **1979**, *20*, 2540. [CrossRef]
26. Gerlach, U.H.; Sengupta, U.K. Gauge-invariant coupled gravitational, acoustical, and electromagnetic modes on most general spherical space-times. *Phys. Rev. D* **1980**, *22*, 1300. [CrossRef]
27. Nakamura, T.; Oohara, K.; Kojima, Y. General Relativistic Collapse to Black Holes and Gravitational Waves from Black Holes. *Prog. Theor. Phys. Suppl.* **1987**, *90*, 1–218. [CrossRef]
28. Gundlach, C.; Martín-García, J.M. Gauge-invariant and coordinate-independent perturbations of stellar collapse: The interior. *Phys. Rev. D* **2000**, *61*, 084024. [CrossRef]
29. Martín-García, J.M.; Gundlach, C. Gauge-invariant and coordinate-independent perturbations of stellar collapse. II. Matching to the exterior. *Phys. Rev. D* **2001**, *64*, 024012. [CrossRef]
30. Nagar, A.; Rezzolla, L. Gauge-invariant non-spherical metric perturbations of Schwarzschild black-hole spacetimes. *Class. Quantum Grav.* **2005**, *22*, R167; Erratum in *IBID* **2006**, *23*, 4297. [CrossRef]
31. Martel, K.; Poisson, E. Gravitational perturbations of the Schwarzschild spacetime: A practical covariant and gauge-invariant formalism. *Phys. Rev. D* **2005**, *71*, 104003. [CrossRef]
32. Nakamura, K. Gauge Invariant Variables in Two-Parameter Nonlinear Perturbations. *Prog. Theor. Phys.* **2003**, *110*, 723. [CrossRef]
33. Nakamura, K. Second-Order Gauge Invariant Perturbation Theory—Perturbative Curvatures in the Two-Parameter Case. *Prog. Theor. Phys.* **2005**, *113*, 481. [CrossRef]
34. Nakamura, K. Construction of gauge-invariant variables of linear metric perturbations on an arbitrary background spacetime. *Prog. Theor. Exp. Phys.* **2013**, *2013*, 043E02. [CrossRef]
35. Nakamura, K. Recursive structure in the definitions of gauge-invariant variables for any order perturbations. *Class. Quantum Grav.* **2014**, *31*, 135013. [CrossRef]
36. Nakamura, K. Second-Order Gauge Invariant Cosmological Perturbation Theory—Einstein Equations in Terms of Gauge Invariant Variables. *Prog. Theor. Phys.* **2007**, *117*, 17. [CrossRef]
37. Kinnersley, W.; Walker, M. Uniformly Accelerating Charged Mass in General Relativity. *Phys. Rev. D* **1970**, *2*, 1359. [CrossRef]
38. Griffiths, J.B.; Krtous, P.; Podolsky, J. Interpreting the C-metric. *Class. Quantum Grav.* **2006**, *23*, 6745. [CrossRef]
39. Sachs, R.K. Gravitational Radiation. In *Relativity, Groups and Topology*; DeWitt, C., DeWitt, B., Eds.; Gordon and Breach: New York, NY, USA, 1964.
40. Bruni, M.; Matarrese, S.; Mollerach, S.; Sonego, S. Perturbations of spacetime: Gauge transformations and gauge invariance at second order and beyond. *Class. Quantum Grav.* **1997**, *14*, 2585. [CrossRef]
41. Nakamura, K. Perturbations of matter fields in the second-order gauge-invariant cosmological perturbation theory. *Phys. Rev. D* **2009**, *80*, 124021. [CrossRef]
42. Landau, L.; Lifshitz, E. *The Classical Theory of Fields*; Addison-Wesley: Reading, MA, USA, 1962.
43. Kodama, H. Accelerating a Black Hole in Higher Dimensions. *Prog. Theor. Phys.* **2008**, *120*, 371. [CrossRef]
44. Szmytkowski, R. Closed forms of the Green's function and the generalized Green's function for the Helmholtz operator on the N-dimensional unit sphere. *J. Phys. A Math. Theor.* **2007**, *40*, 995–1009. [CrossRef]
45. Szmytkowski, R. Closed form of the generalized Green's function for the Helmholtz operator on the two-dimensional unit sphere. *J. Math. Phys.* **2006**, *47*, 063506. [CrossRef]

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