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Gauge-Invariant Perturbation Theory on the Schwarzschild Background Spacetime Part I: Formulation and Odd-Mode Perturbations

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Abstract: This article is Part I of our series of full papers on a gauge-invariant "linear" perturbation theory on the Schwarzschild background spacetime which was briefly reported in our short papers by the present author in 2021. We first review our general framework of the gauge-invariant perturbation theory, which can be easily extended to the "higher-order" perturbation theory. When we apply this general framework to perturbations on the Schwarzschild background spacetime, gauge-invariant treatments of $l = 0, 1$ mode perturbations are required. On the other hand, in the current consensus on the perturbations of the Schwarzschild spacetime, gauge-invariant treatments for $l = 0, 1$ modes are difficult if we keep the reconstruction of the original metric perturbations in our mind. Due to this situation, we propose a strategy of a gauge-invariant treatment of $l = 0, 1$ mode perturbations through the decomposition of the metric perturbations by singular harmonic functions at once and the regularization of these singularities through the imposition of the boundary conditions to the Einstein equations. Following this proposal, we derive the linearized Einstein equations for any modes of $l \geq 0$ in a gauge-invariant manner. We discuss the solutions to the odd-mode perturbation equations in the linearized Einstein equations and show that these perturbations include the Kerr parameter perturbation in these odd-mode perturbations, which is physically reasonable. In the Part II and Part III papers of this series of papers, we will show that the even-mode solutions to the linearized Einstein equations obtained through our proposal are also physically reasonable. Then, we conclude that our proposal of a gauge-invariant treatment for $l = 0, 1$-mode perturbations is also physically reasonable.

Keywords: black hole; Schwarzschild spacetime; perturbation theory; gauge-invariance

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1. Introduction

The study of gravitational-wave astronomy began at the first event GW150914 of the direct observation of gravitational waves in 2015 [1]. This event also marked the beginning of multi-messenger astronomy including gravitational waves [2]. We are now at the stage where we can directly measure gravitational waves and we can carry out scientific research through these gravitational-wave events. We can also expect that one future direction of gravitational-wave astronomy is the development as a "precise science" by the detailed studies of source science, the tests of general relativity, and the developments of the global network of gravitational-wave detectors [2–5]. In addition to the current network of ground-based detectors, as future ground-based gravitational-wave detectors, the projects of Einstein Telescope [6] and Cosmic Explorer [7] are also progressing to achieve more sensitive detection.

In addition to these ground-based detectors, some projects of space gravitational-wave antenna are also progressing [8–11]. Among them, the Extreme Mass Ratio Inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar mass object around a supermassive black hole, is a promising target of the Laser Interferometer Space Antenna [8]. To describe the gravitational wave from EMRIs, black hole perturbations are used [12]. Furthermore, the sophistication of higher-order black hole perturbation theories is required to support these gravitational-wave physics as a precise science. Very recently, the backaction effect of mass and angular momentum accretion on the Schwarzschild black hole due to the Blandford–Znajek process [13] was also discussed [14], which are higher-order effects of two-parameter perturbations [15,16]. The motivation of this paper is in the theoretical sophistication of black hole perturbation theories toward higher-order perturbations for very wide physical situations including the topic in ref. [14].

In the current situation of black hole perturbation theories, we may say that further sophistication is possible even in perturbation theories on the Schwarzschild background spacetime, although realistic black holes have their angular momentum and we have to consider the perturbation theory of a Kerr black hole for direct applications to EMRI. From the pioneering works by Regge and Wheeler [17] and Zerilli [18,19], there have been many studies on the perturbations in the Schwarzschild background spacetime [20–32]. They usually decompose the perturbations on the Schwarzschild spacetime using the spherical harmonics Y_{lm} and classify them into odd- and even-modes based on their parity, because the Schwarzschild spacetime has the spherical symmetry. However, in the current situations, $l = 0$ and $l = 1$ modes should be separately treated through a gauge-fixing procedure [29–32]. From the arguments in refs. [29–32], it is the current consensus that the constructions of "gauge-invariant" variables for $l = 0, 1$ mode perturbations are difficult if we keep the reconstruction of the original metric perturbations in our mind.

On the other hand, toward unambiguous sophisticated "nonlinear" general-relativistic perturbation theories, we have been developing the general formulation of a higher-order gauge-invariant perturbation theory on a generic background spacetime [15,16,33–36] and have been applying it to cosmological perturbations [37–39]. We review our framework of the linear gauge-invariant perturbation theory on generic background spacetime [15,16] in Section 2 of this paper. This framework can be easily extended to "higher-order" perturbations, since the reconstruction of the original metric is trivial. This framework starts from the distinction of the notions of the first- and the second-kind gauges. These two notions of gauges in perturbations are different from each other and this distinction of the first- and second-kind gauges is quite important to understand the development of perturbation theory in this series of our papers. We point out the fact that we often use the first-kind gauge transformation when we predict or interpret the measurement results of observations or experiments. Since the actual measurement results include the information of the detector directivity and the relative motion of the detector and observational targets, we exclude these information using the first-kind gauge transformation when we predict or interpret the experimental results. On the other hand, the second-kind gauge has nothing to do with the nature of physical spacetime and the second-kind gauge should be regarded as an unphysical mode. More details are described in Section 2.

The general framework of gauge-invariant perturbation theories developed in refs. [15,16,33–36] is based on a conjecture (Conjecture 1 below), which roughly states that "we already know the procedure to find gauge-invariant variables for linear-order metric perturbations". Throughout this series of papers and in refs. [15,16,33–36], we use the terminology "gauge-invariant variables" as the variables in which the gauge-degree of freedom of the "second kind" is completely excluded, if there is no possibility of any confusion. Owing to Conjecture 1, the reconstruction of the original metric from the gauge-invariant

variables is trivial. A proof of Conjecture 1 was already discussed in refs. [33–35]. In this proof, we had to introduce some Green functions for some elliptic derivative operators and ignored the kernel modes of these elliptic derivative operators due to a technical reason. We called these kernel modes "zero modes", and the treatment of these zero modes remained unclear. We also called the problem to find a gauge-invariant treatment of these zero modes as the "zero-mode problem". This zero-mode problem is a serious problem to be resolved when we develop higher-order gauge-invariant perturbation theory, since mode-coupling effects including the above "zero modes" occur in higher-order perturbations.

In the case of the perturbations on the Schwarzschild background spacetime, as we will see in Section 3, these "zero modes" correspond to the above $l = 0, 1$ modes. The above conventional special treatments of $l = 0, 1$ modes in many studies correspond to a partial gauge-fixing procedure. If arguments are completed within the linear perturbations on a single patch of the spacetime, this partial gauge-fixing procedure might be harmless, because there is no mode-coupling in the linear perturbation level. However, from the viewpoint of the application of our "higher-order" perturbation theory, the above special treatments of these modes become an obstacle when we develop nonlinear perturbation theory because the mode-couplings owing to the nonlinear effects make the couplings between linear-order $l = 0, 1$ modes and other modes, as mentioned above. Actually, higher-order $l = 0, 1$ modes are also created due to the mode-coupling owing to the nonlinear effects of Einstein equations [40]. Due to this mode-coupling, the special treatments by gauge-fixing for the linear $l = 0, 1$ modes in many studies make the "gauge covariance" of the higher-order perturbations unclear. Moreover, in the EMRI case, we separate the whole spacetime of the system into some regions and derive the perturbative solutions including $l = 0, 1$ mode in each region at once, then we construct global solutions through some matching method such as the matched asymptotic expansion. To exclude "gauge-ambiguity" in these matching, we have to carry out these matching procedure under the "same gauge". To guarantee that the matching procedure is under "same gauge", it is convenient to discuss the perturbation theory in which "gauge covariance" is manifest. Since this "gauge covariance" is already manifest for $l \geq 2$ modes of the perturbations on the Schwarzschild spacetime in the gauge-invariant perturbation theory, it is natural to hope that there is a gauge-invariant treatment for $l = 0, 1$-modes perturbations in spite of the current consensus mentioned above. Thus, the finding of a gauge-invariant treatment of $l = 0, 1$ modes in the perturbations on Schwarzschild background spacetime is not only a resolution of the above technical zero-mode problem in a specific background spacetime but also is quite physically crucial in the arguments of EMRI.

This paper is Part I of our series of full papers on the application of our gauge-invariant perturbation theory on generic background spacetime to that on the Schwarzschild background spacetime, which is already reported in our short papers [41,42]. This series of papers is the full paper version of our short paper [41]. In this Part I paper, we propose a gauge-invariant treatment of the $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime and show that Conjecture 1 is true even for these modes if we accept our proposal. If we consider the mode decompositions for $l = 0, 1$ modes by the spherical harmonic functions Y_{lm}, the vector and tensor harmonics vanish for $l = 0$ mode and the tensor harmonics vanish for $l = 1$ mode. This is the essential reason why we have to treat $l = 0, 1$ modes separately in the conventional approaches as explained in Section 3.1. The mode decomposition based on the conventional spherical harmonic function Y_{lm} corresponds to the imposition of the boundary condition due to the restriction of the functions to L^2-space at the starting point. Due to this regular boundary condition at the starting point, vector and tensor harmonics for $l = 0$ modes and tensor harmonics for $l = 1$ mode vanishes. This requires the special treatments of $l = 0, 1$ modes in the conventional approaches. In

Section 4, we also explained the explicit reason for the difficulties of the construction of gauge-invariant variables for $l = 0, 1$ modes through the gauge-transformation rules of the metric perturbations.

In contrast with these conventional approaches, in our proposal, we introduce singular harmonic functions at once to prepare the non-vanishing vector and tensor harmonics for $l = 0, 1$ mode. Owing to this introduction of the singular harmonic functions, we can treat $l = 0, 1$ modes of perturbations in the similar manner to the treatment of $l \geq 2$ modes in which the gauge-degree-of-freedom of the second kind is completely excluded. We can also construct the gauge-invariant variables for $l = 0, 1$-mode perturbations in the similar manner to those of $l \geq 2$-modes perturbations in which the reconstruction of the original metric from the gauge-invariant variables is trivial. This unified construction of gauge-invariant variables including $l = 0, 1$ modes enable us to define gauge-invariant variables for perturbations of any tensor fields of any order in our higher-order gauge-invariant perturbation theory [15,16,33–39], in which mode-couplings between $l = 0, 1$ modes and the other modes are naturally included. After the derivation of the linear-order Einstein equations in terms of these gauge-invariant variables, we eliminate the introduced singular behavior of harmonics by imposing the regularity of perturbations as the boundary conditions. This is the main scenario of our proposal in this paper.

In this paper, we show that we can resolve the above "zero-mode problem" if we accept the above proposal. This resolution will be an important step of the development of the higher-order gauge-invariant perturbation theory on the Schwarzschild background spacetime which includes the analyses of EMRI. In addition to the perturbation theory on a specific background spacetime, this resolution will become a clue to the perturbation theory on a generic background spacetime. We note that we do not intend to insist that this proposal is the unique resolution of the above "zero-mode problem". However, in the series of our papers, we derive the solutions to the linearized Einstein equation through our proposal and point out that these solutions are physically reasonable. In this Part I paper, we derive the odd-mode perturbative solutions which are physically reasonable. In the Part II paper [43], we will discuss the strategy to solve the even-mode perturbations following our Proposal 1 and derive their $l = 0, 1$-mode solutions. Then, we show these solutions are physically reasonable. Furthermore, in the Part III paper [44], we will discuss the realization of two exact solutions in terms of the linear perturbations on the Schwarzschild background spacetime. Owing to these supports, we may say that our proposal in this paper is also physically reasonable. A brief discussion on the extension to the higher-order perturbations are already given in ref. [42].

The organization of this Part I paper is as follows. In Section 2, we briefly review the framework of the general-relativistic gauge-invariant perturbation theory within the linear perturbation theory, as mentioned above. This framework can be easily extended to higher-order perturbations [15,16,33–36], since the reconstruction of the original metric is trivial through the Conjecture 1. In this Section 2, we emphasize that the distinction of the first-kind gauge and the second-kind gauge is an important premise of our gauge-invariant perturbation theory. In Section 3, we explain the situation in many studies why the special treatments of $l = 0, 1$ modes are required. Then, we propose a strategy for gauge-invariant treatments of $l = 0, 1$ modes. In Section 4, we construct gauge-invariant variables including $l = 0, 1$ modes through the proposal described in Section 3. This is a proof of Conjecture 1 for all modes of perturbations, $l \geq 0$, on the background spacetimes with spherical symmetry. In Section 5, we derive the Einstein equations for any mode perturbations following the proposal in Section 3. In Section 6, we show the strategy to solve the odd-mode perturbations and derive the explicit solutions for $l = 0, 1$ mode perturbations through the component treatment of gauge-invariant variables in the Einstein equations derived

in Section 5. The final Section 7 is devoted to the summary and discussions within this Part I paper.

Throughout this paper, we use the unit $G = c = 1$, where G is Newton's constant of gravitation, and c is the velocity of light.

2. Review of Our General-Relativistic Gauge-Invariant Perturbation Theory

In this section, we briefly review our general framework of the gauge-invariant perturbation theory [15,16]. Although the main purpose of the framework of the gauge-invariant perturbation theory developed in refs. [15,16] is the extension to the higher-order perturbation theory, in this review, we concentrate only on the linear perturbations. This is because we treat only the linear perturbations within this paper. Since we want to explain the gauge-invariant perturbation theory in general relativity, first of all, we have to explain the notions of "gauges" in general relativity [39].

General relativity is a theory with general covariance. This general covariance intuitively states that there is no preferred coordinate system in nature. This general covariance also introduces the notion of "gauge" in the theory. In the theory with general covariance, these "gauges" give rise to the unphysical degree of freedom and we have to fix the "gauges" or to extract some invariant quantities to obtain physical results. Therefore, treatments of "gauges" are crucial in general relativity and this situation becomes more delicate in general relativistic perturbation theories.

In 1964, Sachs [45] pointed out that there are two kinds of "gauges" in general relativity. Sachs called these two "gauges" as the first- and the second-kind gauges, respectively. Here, we review these concepts of "gauge", which are different from each other. Furthermore, the distinction of these "gauges" is important to understand the results of this paper and papers [43,44].

In Section 2.1, we first explain the notion of the first kind gauge. Second, we explain the notion of the second-kind gauge in Section 2.2. We expect that the reader can distinguish these two different notions of gauges in general relativity through these explanations. Then, we review our general framework of the general-relativistic gauge-invariant perturbation theory on generic background spacetimes in Section 2.3. We have to emphasize that the aim of our general formulation of general-relativistic gauge-invariant perturbation theory is to completely exclude the degree of freedom of the second-kind gauge.

2.1. First Kind Gauge

"The first kind gauge" is a coordinate system on a single manifold $\mathscr{M}$. This first kind gauge is not the "gauge" of our "gauge-invariant perturbation theory". However, we have to explain this first kind gauge to distinguish the notions of the first-kind gauge and the second-kind gauge, as emphasized above.

In standard textbooks of manifolds (for example, see [46]), the following property of a manifold is written, "On a manifold, we can always introduce a coordinate system as a diffeomorphism ψ_α from an open set $O_\alpha \subset \mathscr{M}$ to an open set $\psi_\alpha(O_\alpha) \subset \mathbb{R}^n$ ($n = \dim \mathscr{M}$)". This diffeomorphism ψ_α, i.e., coordinate system of the open set O_α, is called "gauge choice" (of the first kind). If we consider another open set in $O_\beta \subset \mathscr{M}$, we have another gauge choice $\psi_\beta : O_\beta \mapsto \psi_\beta(O_\beta) \subset \mathbb{R}^n$ for O_β. If these two open sets O_α and O_β have the intersection $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the diffeomorphism $\psi_\beta \circ \psi_\alpha^{-1}$. This diffeomorphism $\psi_\beta \circ \psi_\alpha^{-1}$ is just a coordinate transformation $\psi_\alpha(O_\alpha \cap O_\beta) \subset \mathbb{R}^n \mapsto \psi_\beta(O_\alpha \cap O_\beta) \subset \mathbb{R}^n$, which is called "gauge transformation" (of the first kind) in general relativity. These are depicted in Figure 1 which is a famous figure in many textbooks of the theory of manifolds.

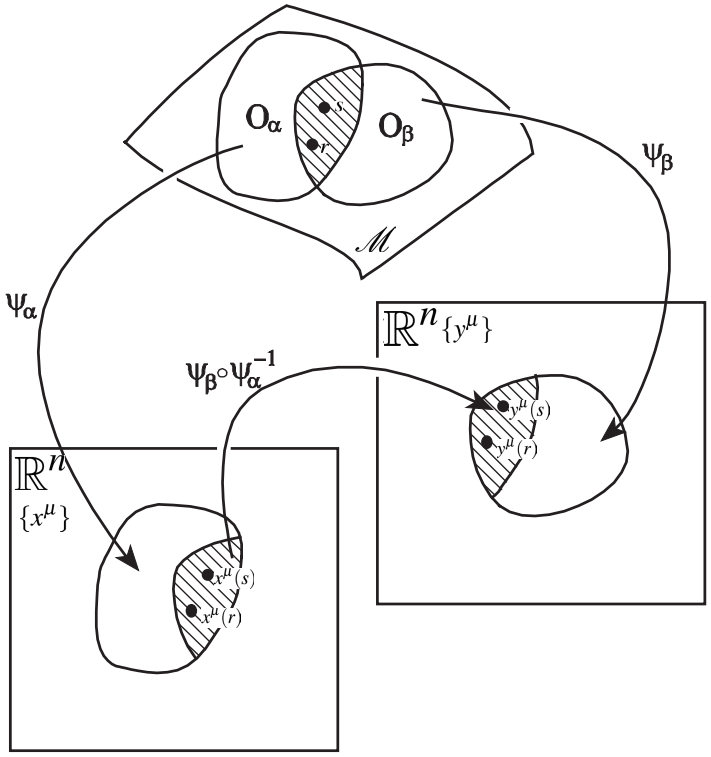

Figure 1. The first kind gauge is a coordinate system of a single manifold. The shaded regions are the intersection $O_\alpha \cap O_\beta$ and its images though the diffeomorphism ψ_α and ψ_β, respectively. The points r and s and its coordinates $\{x^\mu(s), x^\mu(r)\}$ and $\{y^\mu(s), y^\mu(r)\}$ are used in the explanations at the paragraph of Equation (1).

According to the theory of manifolds, coordinate systems are not on a manifold themselves, but we can always introduce a coordinate system as a map from an open set on the manifold $\mathscr{M}$ to an open set of $\mathbb{R}^n$. Furthermore, we may choose a different coordinate system through the different map from an open set in the manifold $\mathscr{M}$ to an open set of $\mathbb{R}^n$. We can always change the coordinate system as we want. This is a realization of the statement of the general covariance that "there is no preferred coordinate system in nature". For this reason, general covariance in general relativity is automatically included in the premise that our spacetime is regarded as a single manifold. The first kind gauge does arise due to this general covariance. The gauge issue of the first kind is usually represented by the question, "Which coordinate system is convenient?" The answer to this question depends on the problem which we are addressing, i.e., what we want to clarify. In some cases, this gauge issue of the first kind is important. On the other hand, in many cases, this gauge issue becomes harmless if we apply a covariant theory on the manifold.

We also note the fact that we often use this first-kind gauge transformation when we predict or interpret the measurement results in observations and experiments as mentioned in Section 1. In general, directly measured results in observations or experiments include the information of the detector directivity and the relative motion of the detector and observational targets. When we predict or interpret the results of these directly-measured results, we have to take into account the information of our detectors.

One typical example is the dipole mode in the fluctuations of the cosmic microwave background (CMB). It is well-known that the dipole mode of CMB is actually detected by the detectors. Usually, this detected dipole mode in CMB is interpreted as the relative motion of the detector against the last scattering surface of the universe. Then, this detected dipole mode is regarded as unimportant detected data when we want to discuss the primordial fluctuations in CMB which are generated in the early history of universe. Regarding

the reason of the detection of these dipole fluctuations in CMB is the proper motion of the detector against the last scattering surface; we use the coordinate transformation to eliminate our relative motion of the detector against the last scattering surface so that the dipole fluctuations disappear [47]. This coordinate transformation is a typical example of the first-kind gauge transformation. We can also give the inclination of rotating star or a binary system and the antenna pattern function of interferometric gravitational-wave detectors as examples of the first-kind gauges.

The final example of the first-kind gauge transformation is the most important one for general relativistic perturbation theories. This is the identification of the actual replacement of points within the single manifold $\mathscr{M}$ with an infinitesimal coordinate transformation [48]. To explain this, we consider the replacement of a points $r \in \mathscr{M}$ to the other point $s \in \mathscr{M}$ in a neighborhood r. This replacement $r \mapsto s$ is represented by a diffeomorphism $\Psi_\lambda : \mathscr{M} \to \mathscr{M}$ as $s = \Psi_\lambda(r)$, where λ is an infinitesimal parameter satisfying $\Psi_{\lambda=0}(r) = r$. The pullback Ψ_λ^* of any tensor field Q on $\mathscr{M}$ is given by

$$Q(s) = (\Psi_\lambda^* Q)(r) = Q(r) + \lambda \pounds_\xi Q\big|_{\lambda=0} + O(\lambda^2), \tag{1}$$

where ξ^a is the generator of the pull-back Ψ_λ^* and a vector field on the tangent space of $\mathscr{M}$. We consider this expression (1) by a coordinate transformation. To see this, we introduce the coordinate system $\{O_\alpha, \psi_\alpha\}$ on $\mathscr{M}$ as above and assume that $r, s \in O_\alpha \cap O_\beta \neq \emptyset$ as in Figure 1. Here, we denote the coordinates $\psi_\alpha : O_\alpha \subset \mathscr{M} \mapsto \mathbb{R}^n(\{x^\mu\})$ and $\psi_\beta : O_\beta \subset \mathscr{M} \mapsto \mathbb{R}^n(\{y^\mu\})$. Through these coordinate systems, we can assign the coordinate labels $(x^\mu(r), x^\mu(s)) \in \mathbb{R}^n(\{x^\mu\})$ and $(y^\mu(r), y^\mu(s)) \in \mathbb{R}^n(\{y^\mu\})$ for the points r and s as in Figure 1. When the variable Q is the coordinate function x^μ associated with the chart ψ_α, we obtain $x^\mu(s) = x^\mu(r) + \lambda\xi^\mu(r) + O(\lambda^2)$. Now, we consider the coordinate transformation $\psi_\beta \circ \psi_\alpha^{-1}$ so that $y^\mu(s) := x^\mu(s)$ and we have the relation between the different coordinates as

$$y^\mu(s) := x^\mu(r) + \lambda\xi^\mu(r) + O(\lambda^2). \tag{2}$$

As an example of tensor fields, we consider the metric g_{ab} on $\mathscr{M}$. Under the infinitesimal coordinate transformation (2), the metric at the point s is given by

$$\begin{aligned} g_{ab}(s) &= g_{\mu\nu}(x(s))(dx^\mu)_a(dx^\nu)_b\big|_s = g_{\mu\nu}(y(s))(dy^\mu)_a(dy^\nu)_b\big|_s \\ &= g_{\mu\nu}\Big(x(r) + \lambda\xi(r) + O(\lambda^2)\Big)\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}(dx^\rho)_a(dx^\sigma)_b\Big|_r \\ &= g_{ab}(r) + \lambda\left(\xi^\tau\partial_\tau g_{\rho\sigma} + g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\mu\right)(dx^\rho)_a(dx^\sigma)_b\big|_r + O(\lambda^2) \\ &= g_{ab}(r) + \lambda\pounds_\xi g_{ab}\big|_r + O(\lambda^2). \end{aligned} \tag{3}$$

Because of $g_{ab}(s) = \Psi_\lambda^* g_{ab}(r)$, Equation (3) is usually written as

$$(\Psi_\lambda^* g_{ab})(r) = g_{ab}(r) + \lambda\pounds_\xi g_{ab}\big|_r + O(\lambda^2). \tag{4}$$

This is just the definition of the Lie derivative and the realization of Equation (1) itself [1]. From the action of the coordinate transformation (2), the coordinate transformation should be regarded as the action of the diffeomorphism

$$\psi_\beta \circ \Psi_\lambda \circ \psi_\alpha^{-1} \tag{5}$$

rather than the simple coordinate transformation $\psi_\beta \circ \psi_\alpha^{-1}$. However, in our perturbation theory, we also regard the infinitesimal coordinate transformation (2) is the first-kind gauge transformation, since the above arguments are restricted within a single manifold $\mathscr{M}$.

Namely, the Taylor expansion through the infinitesimal parameter λ is to the tangential direction within the manifold $\mathscr{M}$.

We may write the metric g_{ab} as $g_{ab} = {}^{(0)}g_{ab} + \lambda\, h_{ab} + O(\lambda^2)$ within $\mathscr{M}$. We emphasize that the direction of this Taylor expansion through the infinitesimal parameter λ is still "tangential" to $\mathscr{M}$. In this case, Equation (3) yields

$$ {}^{(0)}g_{ab}(s) + \lambda h_{ab}(s) = {}^{(0)}g_{ab}(r) + \lambda \left(h_{ab}(r) + \pounds_{\bar{\xi}}\, {}^{(0)}g_{ab}\Big|_r \right) + O(\lambda^2). \tag{6} $$

In many studies, arguments start from the infinitesimal coordinate transformation (2) and reach the conclusion (6). For this reason, the term of Lie derivative of the background metric in the right-hand side in Equation (3) is understood as the "degree of freedom of coordinate transformations" and it is "unphysical degree of freedom", in many studies. However, the appearance of the Lie derivative of the background metric in Equation (6) is just due the change of the reference point within the single manifold $\mathscr{M}$ and this situation is same as the above example of CMB dipole measurement. For this reason, we regard this example as the appearance of the first-kind gauge. This example appears when we interpret our results in Section 6 of this paper.

We will be able to find many other examples of the first-kind gauges. All of these are interpreted as the changes of reference point within the single manifold. In some case, these change of reference point within the single manifold included in the measurement results in observations and experiments. For this reason, we do not regard this above first-kind gauge is "unphysical degree of freedom". On the other hand, the second-kind gauge which is explained in Section 2.2 has nothing to do with our physical spacetime but is included in the perturbative variables as explained below. We have to emphasize that this second-kind gauge is the "unphysical degree of freedom" which should be excluded in general relativistic perturbation theory.

2.2. Second Kind Gauge

"The second kind gauge" appears in perturbation theories as a theory with general covariance. To explain this, we have to remind the reader of our purpose regarding perturbation theories.

First, in any perturbation theories, we always treat two spacetime manifolds. One is the "physical spacetime" $\mathscr{M}_{\rm ph}$. We want to describe the properties of this physical spacetime $\mathscr{M}_{\rm ph}$ through perturbative analyses. This physical spacetime $\mathscr{M}_{\rm ph}$ is usually identified with our nature itself. The other is the "background spacetime" $\mathscr{M}$. This background spacetime has nothing to do with our nature and is a fictitious manifold which is introduced as a reference to carry out perturbative analyses by us. We emphasize that these two spacetime manifolds $\mathscr{M}_{\rm ph}$ and $\mathscr{M}$ are distinct. Let us denote the physical spacetime by $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$ and the background spacetime by $(\mathscr{M}, g_{ab})$, where $\bar{g}_{ab}$ is the metric on the physical spacetime manifold, $\mathscr{M}_{\rm ph}$, and g_{ab} is the metric on the background spacetime manifold, $\mathscr{M}$. Further, we formally denote the spacetime metric and the other physical tensor fields on $\mathscr{M}_{\rm ph}$ by Q and its background value on $\mathscr{M}$ by Q_0.

Second, in any perturbation theory, we always write equations for the perturbation of the variable Q as follows

$$ Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{7} $$

Equation (7) gives a relation between variables on different manifolds. Actually, $Q(\text{“}p\text{”})$ in the left-hand side of Equation (7) is a variable on $\mathscr{M}_{\rm ph}$, whereas $Q_0(p)$ and $\delta Q(p)$ in the right-hand side of Equation (7) are variables on $\mathscr{M}$. Because we regard Equation (7) as a field equation, Equation (7) includes an implicit assumption of the existence of a point identification map $\mathscr{M} \to \mathscr{M}_{\rm ph}$: $p \in \mathscr{M} \mapsto \text{“}p\text{”} \in \mathscr{M}_{\rm ph}$. This identification map is a "gauge

choice" in general-relativistic perturbation theories (see Figure 2). This is the notion of the "second-kind gauge" pointed out by Sachs [45]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on the single manifold which is explained in Section 2.1.

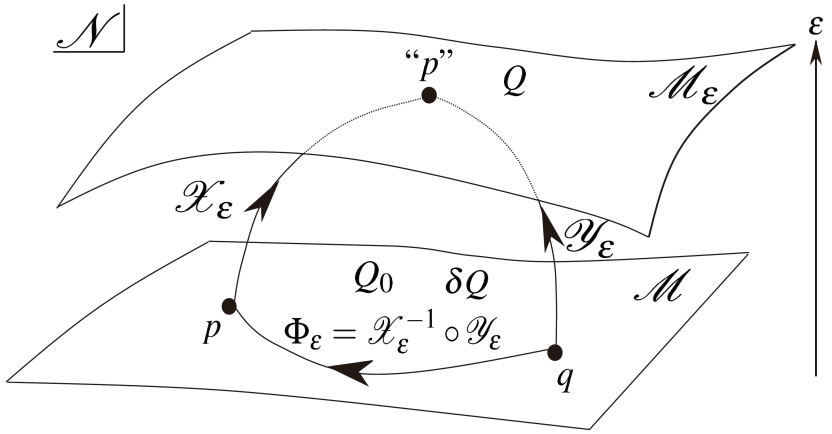

Figure 2. The second kind gauge is a point-identification between the physical spacetime $\mathscr{M}_{\rm ph} = \mathscr{M}_\epsilon$ and the background spacetime $\mathscr{M}$ on the extended manifold $\mathscr{N}$. Through Equation (7), we implicitly assume the existence of a point-identification map between $\mathscr{M}_\epsilon$ and $\mathscr{M}$. However, this point-identification is not unique by virtue of the general covariance in the theory. We may choose the gauge of the second kind so that $p \in \mathscr{M}$ and "p"$\in \mathscr{M}_\epsilon$ is same ($\mathscr{X}_\epsilon$). We may also choose the gauge so that $q \in \mathscr{M}_0$ and "p"$\in \mathscr{M}_\epsilon$ is same ($\mathscr{Y}_\epsilon$). These are different gauge choices. The gauge transformation $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$ is given by the diffeomorphism $\Phi_\epsilon = \mathscr{X}_\epsilon^{-1} \circ \mathscr{Y}_\epsilon$.

To develop this understanding of the "gauge of the second kind", we introduce an infinitesimal parameter ϵ for perturbations and $4+1$-dimensional manifold $\mathscr{N} = \mathscr{M}_{\rm ph} \times \mathbb{R}$ ($4 = \dim \mathscr{M}$) such that $\mathscr{M} = \mathscr{N}|_{\epsilon=0}$ and $\mathscr{M}_{\rm ph} = \mathscr{M}_\epsilon = \mathscr{N}|_{\mathbb{R}=\epsilon}$. On $\mathscr{N}$, the point-identification choice is regarded as a diffeomorphism $\mathscr{X}_\epsilon : \mathscr{N} \to \mathscr{N}$ such that $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon$. This point-identification is a gauge choice of the second kind [39,45,49–51]. Furthermore, we introduce a gauge choice $\mathscr{X}_\epsilon$ as an exponential map with a generator ${}^{\mathscr{X}}\eta^a$, which is chosen such that its integral curve in $\mathscr{N}$ is transverse to each $\mathscr{M}_\epsilon$ everywhere on $\mathscr{N}$. Points lying on the same integral curve are regarded as the "same point" by the gauge choice $\mathscr{X}_\epsilon$. Note that the action of $\mathscr{X}_\epsilon$ is transverse to each $\mathscr{M}_\epsilon$.

The first-order perturbation of the variable Q on $\mathscr{M}_\epsilon$ is defined as the pulled-back $\mathscr{X}_\epsilon^* Q$ on $\mathscr{M}$, which is induced by $\mathscr{X}_\epsilon$, and is expanded as

$$\mathscr{X}_\epsilon^* Q = Q_0 + \epsilon \pounds_{\mathscr{X}_\eta} Q\Big|_{\mathscr{M}} + O(\epsilon^2), \tag{8}$$

where $Q_0 = Q|_{\mathscr{M}}$ is the background value of Q and all terms in Equation (8) are evaluated on the background spacetime $\mathscr{M}$. Because Equation (8) is the perturbative expansion of $\mathscr{X}_\epsilon^* Q_\epsilon$, the first-order perturbation of Q is given by ${}^{(1)}_{\mathscr{X}}Q := \pounds_{\mathscr{X}_\eta} Q\Big|_{\mathscr{M}}$.

When we have two gauge choices $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$ with the generators ${}^{\mathscr{X}}\eta^a$ and ${}^{\mathscr{Y}}\eta^a$, respectively, and when these generators have different tangential components to each $\mathscr{M}_\epsilon$, $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$ are regarded as "different gauge choices". A "gauge-transformation" is regarded as the change of the point-identification $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$, which is given by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon : \mathscr{M} \to \mathscr{M}$. The diffeomorphism Φ_ϵ does change the point-identification. Here, Φ_ϵ induces a pull-back from the representation $\mathscr{X}_\epsilon^* Q_\epsilon$ to the representation $\mathscr{Y}_\epsilon^* Q_\epsilon$ as [2]

$$\mathscr{Y}_\epsilon^* Q_\epsilon(q) = \Phi_\epsilon^* \mathscr{X}_\epsilon^* Q_\epsilon(q) \tag{9}$$

for any point $q \in \mathscr{M}$. From general arguments of the Taylor expansion [52], the pull-back Φ_ϵ^* is expanded as

$$\mathscr{Y}_\epsilon^* Q_\epsilon = \mathscr{X}_\epsilon^* Q_\epsilon + \epsilon \pounds_{\xi_{(1)}} \mathscr{X}_\epsilon^* Q_\epsilon + O(\epsilon^2), \tag{10}$$

where $\xi^a_{(1)}$ is the generator of Φ_ϵ. From Equations (8) and (10), the linear-order gauge-transformation is given as

$$\overset{(1)}{\mathscr{Y}}Q(q) - \overset{(1)}{\mathscr{X}}Q(q) \quad = \quad \pounds_{\xi_{(1)}} Q_0(q) \tag{11}$$

for any point $q \in \mathscr{M}$. We also employ the "order-by-order gauge invariance" (of the second kind) as a concept of gauge invariance [38]. We call the kth-order perturbation $\overset{(k)}{\mathscr{X}}Q$ as gauge-invariant (of the second-kind) if and only if

$$\overset{(k)}{\mathscr{X}}Q(q) = \overset{(k)}{\mathscr{Y}}Q(q) \tag{12}$$

for any point $q \in \mathscr{M}$ and gauge choice $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$.

Here, we have to emphasize the importance of the gauge invariance of the second kind. As explained above, the second kind gauge has nothing to do with the properties of the physical spacetime. The physical spacetime is usually identified with our nature itself. We are living not on the background spacetime but on the physical spacetime. Any experiment and observation is carried out within the physical spacetime through the physical process within the physical spacetime. Therefore, measurement results of experiments and observations should have nothing to do with the background spacetime nor the gauge-degree of freedom of the second kind. For this reason, measurement results of experiments and observations should be gauge-invariant in the sense of the second kind. Keeping this premise in mind, the gauge-transformation rule (11) indicates that the first-order perturbation $^{(1)}Q$ for an arbitrary tensor field Q is transformed through the gauge-transformation, i.e., the change in the point identification of the points of the physical spacetime and the background spacetime in general. This implies that the first-order perturbation $^{(1)}Q$ includes the unphysical degree of freedom, i.e., the gauge degree of freedom in the second kind, in general. Thus, order-by-order gauge-invariant variables defined by Equation (12) does not include the gauge degree of freedom in the second kind and is quite important for perturbation theories in general relativity.

Finally, we comment on the difference between the notion of this second-kind gauge and the first-kind gauge especially the example in the paragraph which contains Equation (1) and in the next paragraph. First, we point out that the Taylor expansion through the infinitesimal parameter λ in Equations (1)–(6) is the expansion within the single manifold $\mathscr{M}$. Therefore, even if we include higher-order perturbations of the infinitesimal parameter λ, this Taylor expansion is still within the single manifold. On the other hand, the direction of the Taylor expansion (8) for the perturbative variable $\mathscr{X}_\epsilon^* Q$ is the transverse direction from the background spacetime $\mathscr{M}$ to the physical spacetime $\mathscr{M}_{ph}$ in the extended manifold $\mathscr{N}$. Although the action of the diffeomorphism Φ_ϵ^* is within the background spacetime, the Taylor expansion of $\mathscr{Y}^* Q_\epsilon$ and $\mathscr{X}_\epsilon Q_\epsilon$ through the infinitesimal parameter ϵ is the transverse direction to each manifold $\mathscr{M}_\epsilon$ in the extended manifold $\mathscr{N}$. Therefore, the metric perturbation in Equation (6) cannot direct to the physical spacetime $\mathscr{M}_{ph}$ from the background spacetime $\mathscr{M}$, but the perturbation in Equation (8) actually directs to the physical spacetime $\mathscr{M}_{ph}$ from the background spacetime $\mathscr{M}$. Therefore, the perturbation of h_{ab} in Equation (6) does not have any information of $\mathscr{M}_{ph}$ if the manifold $\mathscr{M}$ for Equation (1) is the background spacetime of perturbation, but $\overset{(1)}{\mathscr{X}}Q$ in Equation (8) should have the information of $\mathscr{M}_{ph}$.

However, Equation (11) indicates that the variables ${}^{(1)}_{\mathscr{X}}Q$ include the information of the second-kind gauge; thus, we have to exclude this second-kind gauge completely. This is accomplished by the construction of gauge-invariant variables (in the sense of the second-kind). The general-relativistic gauge-invariant perturbation theory explained below (in Section 2.3) automatically treats only gauge-invariant variables in the sense of the second-kind defined by Equation (12). Thus, the development of our gauge-invariant perturbation theory is crucially important in physics. Here, we emphasize the important fact that the gauge degree of freedom in perturbations to be excluded by the gauge-invariant perturbation theory is not the above first-kind gauge but the second-kind gauge as explained below.

2.3. The General-Relativistic Gauge-Invariant Linear Perturbation Theory

Based on the above setup, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations [15,16]. In this paper, we concentrate only on the explanations of the linear perturbations. First, we expand the metric on the physical spacetime $\mathscr{M}_\epsilon$, which was pulled back to the background spacetime $\mathscr{M}$ through a gauge choice $\mathscr{X}_\epsilon$ as

$$\mathscr{X}_\epsilon^* \bar{g}_{ab} \quad = \quad g_{ab} + \epsilon\, {}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \tag{13}$$

Although the expression (13) depends entirely on the gauge choice $\mathscr{X}_\epsilon$, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\epsilon$ in the expression if there is no possibility of confusion. The important premise of our proposal was the conjecture [15,16] for the linear metric perturbation h_{ab} as follows:

Conjecture 1. *If the gauge-transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab}, there then exists a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

We call $\mathscr{F}_{ab}$ and Y^a as the "gauge-invariant" and "gauge-dependent" parts of h_{ab}, respectively. In our higher-order gauge-invariant perturbation theory [15,16,33–39], Conjecture 1 plays an essential role in the derivation of the formula for the decomposition of any variables of higher-order perturbations into their gauge-invariant and gauge-dependent variables.

The proof of Conjecture 1 is highly non-trivial [33,35], and it was found that gauge-invariant variables are essentially non-local. Despite this non-triviality, once we accept Conjecture 1, we can construct gauge-invariant variables for the linear perturbation of an arbitrary tensor field ${}^{(1)}_{\mathscr{X}}Q$, whose gauge-transformation is given by Equation (11), through the gauge-dependent part of the metric perturbation Y_a in Conjecture 1 as

$${}^{(1)}\mathscr{Q} := {}^{(1)}_{\mathscr{X}}Q - \pounds_{\mathscr{X}Y} Q_0. \tag{14}$$

This definition implies that the linear perturbation ${}^{(1)}_{\mathscr{X}}Q$ of an arbitrary tensor field $\mathscr{X}_\epsilon^* Q$ is always decomposed into its gauge-invariant part ${}^{(1)}\mathscr{Q}$ and gauge-dependent part $\pounds_{\mathscr{X}Y} Q_0$ as

$${}^{(1)}_{\mathscr{X}}Q = {}^{(1)}\mathscr{Q} + \pounds_{\mathscr{X}Y} Q_0. \tag{15}$$

As examples, the linearized Einstein tensor ${}^{(1)}_{\mathscr{X}}G_a{}^b$ and the linear perturbation of the energy-momentum tensor ${}^{(1)}_{\mathscr{X}}T_a{}^b$ are also decomposed as

$$ {}^{(1)}_{\mathscr{X}}G_a{}^b = {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] + \pounds_{\mathscr{X}Y}G_a{}^b, \quad {}^{(1)}_{\mathscr{X}}T_a{}^b = {}^{(1)}\mathscr{T}_a{}^b[\mathscr{F},\phi] + \pounds_{\mathscr{X}Y}T_a{}^b, \tag{16} $$

where G_{ab} and T_{ab} are the background values of the Einstein tensor and the energy-momentum tensor, respectively, and "ϕ" in the gauge-invariant variable ${}^{(1)}\mathscr{T}_a{}^b[\mathscr{F},\phi]$ symbolically represents the matter degree of freedom. The gauge-invariant part ${}^{(1)}\mathscr{G}_a{}^b$ of the linear-order perturbation of the Einstein tensor is given by

$$ {}^{(1)}\mathscr{G}_a{}^b[A] := {}^{(1)}\Sigma_a{}^b[A] - \frac{1}{2}\delta_a{}^b{}^{(1)}\Sigma_c{}^c[A], \tag{17} $$

$$ {}^{(1)}\Sigma_a{}^b[A] := -2\nabla_{[a}H_{d]}{}^{bd}[A] - A^{cb}R_{ac}, \quad H_{ba}{}^c[A] := \nabla_{(a}A_{b)}{}^c - \frac{1}{2}\nabla^c A_{ab}. \tag{18} $$

Then, using the background Einstein equation $G_a{}^b = 8\pi T_a{}^b$, the linearized Einstein equation ${}^{(1)}_{\mathscr{X}}G_a{}^b = 8\pi {}^{(1)}_{\mathscr{X}}T_a{}^b$ is automatically given in the gauge-invariant form

$$ {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] = 8\pi {}^{(1)}\mathscr{T}_a{}^b[\mathscr{F},\phi], \tag{19} $$

even if the background Einstein equation is non-trivial. We also note that, in the case of a vacuum background case, i.e., $G_a{}^b = 8\pi T_a{}^b = 0$, Equation (16) shows that the linear perturbations of the Einstein tensor and the energy-momentum tensor are automatically gauge-invariant.

We can also derive the perturbation of the divergence of $\bar{\nabla}_a\bar{T}_b{}^a$ of the second-rank tensor $\bar{T}_b{}^a$ on $(\mathscr{M}_{\rm ph},\bar{g}_{ab})$. Through the gauge choice $\mathscr{X}_\epsilon$, the tensor $\bar{T}_b{}^a$ is pulled back to $\mathscr{X}_\epsilon^*\bar{T}_b{}^a$ on the background spacetime $(\mathscr{M},g_{ab})$, and the covariant derivative operator $\bar{\nabla}_a$ on $(\mathscr{M}_{\rm ph},\bar{g}_{ab})$ is pulled back to a derivative operator

$$ \bar{\nabla}_a := \mathscr{X}_\epsilon^*\bar{\nabla}_a(\mathscr{X}_\epsilon^{-1})^* \tag{20} $$

on $(\mathscr{M},g_{ab})$. Note that the derivative $\bar{\nabla}_a$ is the covariant derivative associated with the metric $\mathscr{X}_\epsilon\bar{g}_{ab}$, whereas the derivative ∇_a on the background spacetime $(\mathscr{M},g_{ab})$ is the covariant derivative associated with the background metric g_{ab}. Bearing in mind the difference in these derivatives, the first-order perturbation of $\bar{\nabla}_a\bar{T}_b{}^a$ is given by

$$ {}^{(1)}\big(\bar{\nabla}_a\bar{T}_b{}^a\big) = \nabla_a{}^{(1)}\mathscr{T}_b{}^a[\mathscr{F},\phi] + H_{ca}{}^a[\mathscr{F}]T_b{}^c - H_{ba}{}^c[\mathscr{F}]T_c{}^a + \pounds_Y\nabla_a T_b{}^a. \tag{21} $$

The derivation of the Formula (21) is given in ref. [16]. If the tensor field $\bar{T}_b{}^a$ is the Einstein tensor $\bar{G}_a{}^b$, Equation (21) yields the linear-order perturbation of the Bianchi identity

$$ \nabla_a{}^{(1)}\mathscr{G}_b{}^a[\mathscr{F}] + H_{ca}{}^a[\mathscr{F}]G_b{}^c - H_{ba}{}^c[\mathscr{F}]G_c{}^a = 0. \tag{22} $$

Furthermore, if the background Einstein tensor vanishes $G_a{}^b = 0$, we obtain the identity

$$ \nabla_a{}^{(1)}\mathscr{G}_b{}^a[\mathscr{F}] = 0. \tag{23} $$

By contrast, if the tensor field $\bar{T}_b{}^a$ is the energy-momentum tensor, Equation (21) yields the continuity equation of the energy-momentum tensor

$$ \nabla_a{}^{(1)}\mathscr{T}_b{}^a[\mathscr{F},\phi] + H_{ca}{}^a[\mathscr{F}]T_b{}^c - H_{ba}{}^c[\mathscr{F}]T_c{}^a = 0, \tag{24} $$

where we used the background continuity equation $\nabla_a T_b{}^a = 0$. If the background spacetime is vacuum $T_{ab} = 0$, Equation (24) yields a linear perturbation of the energy-momentum tensor given by

$$ \nabla_a{}^{(1)}\mathscr{T}_b{}^a[\phi] = 0. \tag{25} $$

Thus, starting from the Conjecture 1, we can develop the gauge-invariant perturbation theory through the above framework. Furthermore, this formulation can be extended to any order perturbations [15,16,33,36] from Conjecture 1. In this sense, the proof of the Conjecture 1 is crucial to this framework.

We should also note that the decomposition of the metric perturbation h_{ab} into its gauge-invariant part $\mathscr{F}_{ab}$ and into its gauge-dependent part Y^a is not unique [38,39]. For example, the gauge-invariant part $\mathscr{F}_{ab}$ has six components and we can create the gauge-invariant vector field Z^a through the component $\mathscr{F}_{ab}$ such that the gauge-transformation of the vector field Z^a is given by ${}_{\mathscr{Y}}Z^a - {}_{\mathscr{X}}Z^a = 0$. Using this gauge-invariant vector field Z^a, the original metric perturbation can be expressed as follows

$$h_{ab} = \mathscr{F}_{ab} - \pounds_Z g_{ab} + \pounds_{Z+Y} g_{ab} =: \mathscr{H}_{ab} + \pounds_X g_{ab}. \tag{26}$$

The tensor field $\mathscr{H}_{ab} := \mathscr{F}_{ab} - \pounds_Z g_{ab}$ is also regarded as the gauge-invariant part of the perturbation h_{ab} because ${}_{\mathscr{Y}}\mathscr{H}_{ab} - {}_{\mathscr{X}}\mathscr{H}_{ab} = 0$. Similarly, the vector field $X^a := Z^a + Y^a$ is also regarded as the gauge-dependent part of the perturbation h_{ab} because ${}_{\mathscr{Y}}X^a - {}_{\mathscr{X}}X^a = \xi^a_{(1)}$.

Equation (26) does show that the definition of the gauge-invariant variable $\mathscr{F}_{ab}$ is not unique. At the same time, this non-uniqueness of the definition of the gauge-invariant variable $\mathscr{F}_{ab}$ implies the symmetry of the linearized Einstein Equation (19). Through the same derivation of the Formula (16), we can also derive the linearized Einstein tensor ${}^{(1)}_{\mathscr{X}}G_a{}^b$ and the linear perturbation of the energy-momentum tensor ${}^{(1)}_{\mathscr{X}}T_a{}^b$ as

$${}^{(1)}_{\mathscr{X}}G_a{}^b = {}^{(1)}\mathscr{G}_a{}^b[\mathscr{H}] + \pounds_{\mathscr{X}X}G_a{}^b, \quad {}^{(1)}_{\mathscr{X}}T_a{}^b = {}^{(1)}\mathscr{T}_a{}^b[\mathscr{H},\phi] + \pounds_{\mathscr{X}X}T_a{}^b. \tag{27}$$

Then, through the same logic for the derivation of Equation (19), we reach the conclusion

$${}^{(1)}\mathscr{G}_a{}^b[\mathscr{H}] = 8\pi {}^{(1)}\mathscr{T}_a{}^b[\mathscr{H},\phi]. \tag{28}$$

Equations (19) and (28) indicate the symmetry of the linearized Einstein equation. Namely, if the gauge-invariant metric perturbation $\mathscr{F}_{ab}$ is a solution to the linearized Einstein Equation (19), the gauge-invariant metric perturbation $\mathscr{H}_{ab} := \mathscr{F}_{ab} - \pounds_Z g_{ab}$ is also a solution to the linearized Einstein equation. This symmetry of the linearized Einstein equation implies that solutions to the linearized Einstein equation may includes the term $\pounds_Z g_{ab}$ as a gauge-invariant arbitrary degree of freedom. Actually, we will see the fact that the gauge-invariant term $\pounds_Z g_{ab}$ appears in the solutions derived in Section 6.

Finally, we comment on the relation between the gauge-transformation Φ_ϵ and the coordinate transformation [39,53,54]. As mentioned above, the notion of the second-kind gauges above is different from the notion of the degree of freedom of the coordinate transformation on a single manifold which is called first-kind gauge. However, the gauge-transformation Φ_ϵ of the second kind induces the coordinate transformation. To see this, we introduce the coordinate system $\{O_\alpha, \psi_\alpha\}$ on the background spacetime $\mathscr{M}$, where O_α are open sets on the background spacetime and ψ_α are diffeomorphisms from O_α to $\mathbb{R}^4$ ($4 = \dim \mathscr{M}$) as depicted in Figure 3. The coordinate system $\{O_\alpha, \psi_\alpha\}$ is the set of collections of the pair of open sets O_α and diffeomorphism $O_\alpha \mapsto \mathbb{R}^4$. If we employ a gauge choice $\mathscr{X}_\epsilon$ of the second kind, we have the correspondence of the physical spacetime $\mathscr{M}_\epsilon = \mathscr{M}_{\rm ph}$ and the background spacetime $\mathscr{M}$. Together with the coordinate system ψ_α on $\mathscr{M}$, this correspondence between $\mathscr{M}_\epsilon$ and $\mathscr{M}$ induces the coordinate system on $\mathscr{M}_\epsilon$. Actually, $\mathscr{X}_\epsilon(O_\alpha)$ for each α is an open set of $\mathscr{M}_\epsilon$. Then, $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ becomes a diffeomorphism from an open set $\mathscr{X}_\epsilon(O_\alpha) \subset \mathscr{M}_\epsilon$ to $\mathbb{R}^4(\{x^\mu\})$. This diffeomorphism $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ induces a coordinate system of an open set on $\mathscr{M}_\epsilon$. When we have two different gauge choices $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$ of the second kind, $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1} \mapsto \mathbb{R}^4(\{x^\mu\})$ and $\psi_\alpha \circ \mathscr{Y}_\epsilon^{-1} \mapsto \mathbb{R}^4(\{y^\mu\})$ become

different coordinate systems on $\mathscr{M}_\epsilon$. We can also consider the coordinate transformation from the coordinate system $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ to another coordinate system $\psi_\alpha \circ \mathscr{Y}_\epsilon^{-1}$. Because the gauge transformation $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$ (of the second kind) is induced by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon$, this diffeomorphism Φ_ϵ induces the coordinate transformation as

$$y^\mu(q) := x^\mu(p) = \left((\Phi_\epsilon^{-1})^* x^\mu\right)(q) \tag{29}$$

in the passive point of view [15,52], where $p \in \mathscr{M}$, $\mathscr{X}_\epsilon(p) = "p" \in \mathscr{M}_{ph}$ and $q \in \mathscr{M}$, $\mathscr{Y}_\epsilon(q) = "p" \in \mathscr{M}_{ph}$. If we represent this coordinate transformation in terms of the Taylor expansion (10), we have the coordinate transformation

$$y^\mu(q) = x^\mu(q) - \epsilon \xi^\mu_{(1)}(q) + O(\epsilon^2). \tag{30}$$

We should emphasize that the coordinate transformation (30) is not the starting point of the gauge-transformation but a result of the above framework. Because our above framework of the gauge-invariant perturbation theory is constructed without a coordinate transformation (30), we "do not" use the coordinate transformation (30) in our formulation.

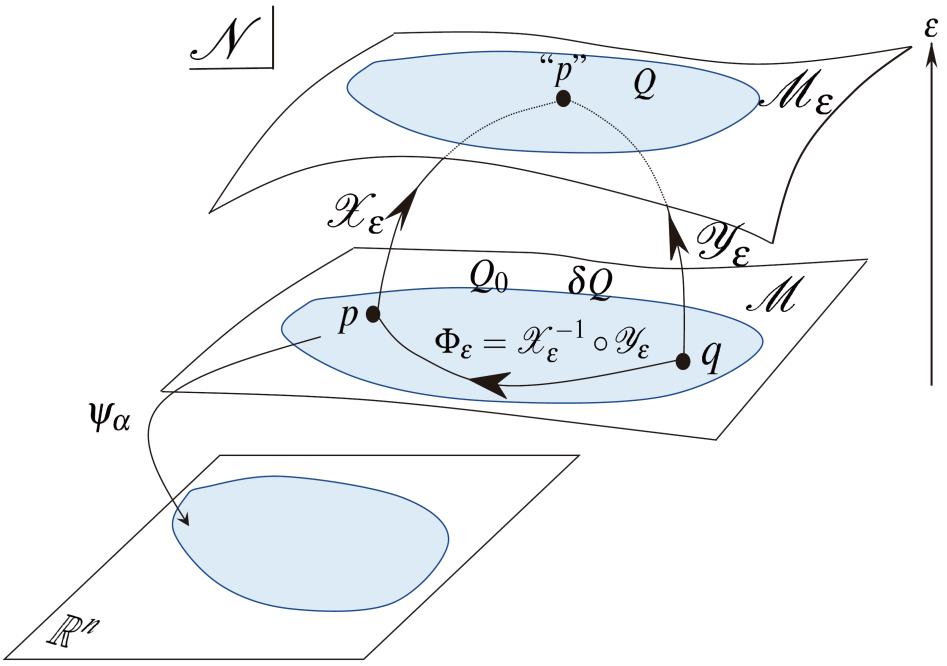

Figure 3. A second kind gauge transformation induces a coordinate transformation. The diffeomorphism $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ maps the open set $\mathscr{X}_\epsilon(O_\alpha) \subset \mathscr{M}_{\rm ph}$ to a open set on $\mathbb{R}^4$. The blue part on $\mathscr{M}$ is an open set includes p and q. This open set is mapped to $\mathbb{R}^n$ and $\mathscr{M}_\epsilon$ through ψ_α and $\mathscr{X}_\epsilon$ (and $\mathscr{Y}_\epsilon$), respectively. If we change the gauge choice from $\mathscr{X}_\epsilon$ to $\mathscr{Y}_\epsilon$, this change induces the coordinate transformation from $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ to $\psi_\alpha \circ \mathscr{Y}_\epsilon^{-1}$.

3. Linear Perturbations on Spherically Symmetric Background

Here, we consider the 2 + 2 formulation of perturbations of a spherically symmetric background spacetime, which originally proposed by Gerlach and Sengupta [24–27]. In this formulation, we pay attention to the symmetry of the background spacetime. Spherically symmetric spacetimes are characterized by the direct product $\mathscr{M} = \mathscr{M}_1 \times S^2$ and the metric on this spacetime is given by

$$g_{ab} = y_{ab} + r^2 \gamma_{ab}, \tag{31}$$

$$y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \tag{32}$$

where $x^A = (t, r)$, $x^p = (\theta, \phi)$, and γ_{pq} is the metric on the unit sphere. In the case of the Schwarzschild spacetime, the metric (31) is given by

$$y_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \quad f := 1 - \frac{2M}{r}, \tag{33}$$

$$\gamma_{ab} = (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b = \theta_a\theta_b + \phi_a\phi_b, \tag{34}$$

$$\theta_a = (d\theta)_a, \quad \phi_a = \sin\theta(d\phi)_a. \tag{35}$$

In Section 3.1, we review the conventional decomposition of the metric perturbation and its inverse relation and show that the conventional decomposition is essentially non-local and the two Green functions for the derivative operators are necessary to derive its inverse relation. The kernel modes of these derivative operators are $l = 0, 1$ modes. This is the reason why $l = 0, 1$ modes in the perturbations on the spherically symmetric background spacetime should be treated separately. In Section 3.2, we discuss a treatment in which the special treatments of these kernel modes are not necessary. To develop such treatment, we use the different scalar harmonic functions from the conventional spherical harmonic functions. We also summarize the conditions for the harmonic functions should be satisfied. In Section 3.3, we derive the explicit form of the mode functions. In Section 3.4, we propose a treatment of $l = 0, 1$ modes in perturbations on spherically symmetric background spacetimes.

3.1. Conventional Perturbation Decomposition and Its Inverse Relation

On the above background spacetime $(\mathscr{M}, g_{ab})$, the components of the metric perturbation are given by

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} + h_{pq}(dx^p)_a(dx^q)_b. \tag{36}$$

Here, we note that the components h_{AB}, h_{Ap}, and h_{pq} are regarded as components of scalar, vector, and tensor on S^2, respectively. In many studies, these components are decomposed through the decomposition [55–57] using the spherical harmonics $S = Y_{lm}$ as follows

$$h_{AB} = \sum_{l,m} \tilde{h}_{AB} S, \tag{37}$$

$$h_{Ap} = r \sum_{l,m} \left[\tilde{h}_{(e1)A} \hat{D}_p S + \tilde{h}_{(o1)A} \epsilon_{pq} \hat{D}^q S \right], \tag{38}$$

$$h_{pq} = r^2 \sum_{l,m} \left[\frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)} S + \tilde{h}_{(e2)} \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r \right) S + 2\tilde{h}_{(o2)} \epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S \right], \tag{39}$$

where $\hat{D}_p$ is the covariant derivative associated with the metric γ_{pq} on S^2, $\hat{D}^p = \gamma^{pq}\hat{D}_q$, $\epsilon_{pq} = \epsilon_{[pq]} = 2\theta_{[p}\phi_{q]}$ is the totally antisymmetric tensor on S^2. Here, we note that the covariant derivatives of the basis θ_p and ϕ_p on S^2 are given by

$$\hat{D}_p\theta_q = \cot\theta\phi_p\phi_q, \quad \hat{D}_p\phi_q = -\cot\theta\phi_p\theta_q. \tag{40}$$

Through these formulae, we can check $\hat{D}_r\epsilon_{pq} = 0$. We also note that the curvature tensors ${}^{(2)}\hat{R}_{pqrs}$ and ${}^{(2)}\hat{R}_{pr}$ associated with the metric γ_{pq} are given by Equation (A76) in Appendix B.

Although the matrix representations of the independent harmonic functions are used in the pioneer papers [17–19], these are equivalent to the covariant form (37)–(39) with the choice $S = Y_{lm}$. The choice $S = Y_{lm}$ is the starting point of the original 2 + 2 formulation proposed by Gerlach and Sengupta [24–27]. They showed the constructions of gauge-invariant variables for $l \geq 2$ modes and derived Einstein equations. If we apply the

decomposition (37)–(39) with $S = Y_{lm}$ to the metric perturbation h_{ab}, special treatments for $l = 0, 1$ modes are required [17–27,29–32]. This is due to the fact that the set of harmonic functions

$$\left\{ S, \hat{D}_p S, \epsilon_{pq}\hat{D}^q S, \frac{1}{2}\gamma_{pq}S, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S, 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S \right\} \tag{41}$$

loses its linear independence in $l = 0, 1$ cases. To clarify this situation, we consider the inverse relation of the decomposition Formulas (37)–(39), later. Furthermore, we see that the inverse-relation of the decomposition Formulas (37)–(39) requires the Green functions of the derivative operators $\hat{\Delta} := \hat{D}^r\hat{D}_r$ and $\hat{\Delta} + 2 := \hat{D}^r\hat{D}_r + 2$, respectively. The eigen modes of these operators are $l = 0$ and $l = 1$, respectively. Actually, for $l = 0$ modes, the basis in (41) vanish except for $\{S, \frac{1}{2}\gamma_{pq}S\}$. For $l = 1$ modes, we have $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0$. These are explicitly shown in Appendix A.

Note that the decomposition Formulas (37)–(39) with the spherical harmonic function Y_{lm} carry out two decompositions. The first one is the decomposition of the function space through the spherical harmonic function Y_{lm} as the bases of L^2-space on S^2. This corresponds to the imposition of the regular boundary conditions for the perturbations at the starting point. The second one is the decomposition of the tangent space on S^2 through the derivative of the scalar harmonic function $S = Y_{lm}$. The imposition of the boundary conditions at the starting point leads to the vanishing of vector and tensor harmonics in (41) for $l = 0$ modes and tensor harmonics in (41) for $l = 1$ modes. These vanishing vector and tensor harmonics leads to the failure of the decomposition of the tangent space for $l = 0, 1$ modes. This is the reason why the special treatments for these modes are required in many studies. At the same time, these vanishing mode functions are an essential reason for the fact that the proof of Conjecture 1 for perturbations on the Schwarzschild background spacetime including $l = 0, 1$ modes is difficult.

Now, we consider the derivation of the inverse relation of the decomposition (37)–(39). In this derivation, we use the orthogonality

$$\int_{S^2} d\Omega Y^*_{lm} Y_{l'm'} = \delta_{ll'}\delta_{mm'} \tag{42}$$

of the spherical harmonic function $S = Y_{lm}$, where $d\Omega = \sin\theta d\theta d\phi$. Therefore, we do not show the final expressions as the results of the application of Equation (42).

First, we consider the inverse relation of the decomposition (38). Taking the divergence of Equation (38), we obtain

$$\hat{D}^p h_{Ap} = r\sum_{l,m}\tilde{h}_{(e1)A}\hat{D}^p\hat{D}_p S = r\sum_{l,m,(l\neq 0)}\tilde{h}_{(e1)A}\hat{\Delta}S. \tag{43}$$

Thus, we should regard that the mode coefficient $\tilde{h}_{(e1)A}$ in Equation (38) does not include $l = 0$ mode. Using the Green function $\hat{\Delta}^{-1}$, we obtain

$$\sum_{l,m,(l\neq 0)}\tilde{h}_{(e1)A}S = \frac{1}{r}\hat{\Delta}^{-1}\hat{D}^p h_{Ap}. \tag{44}$$

Furthermore, using the orthogonal property (42) of the $S = Y_{lm}$ with $l \neq 0$, we obtain the mode coefficient $\tilde{h}_{(e1)A}$ for each mode, except for $l = 0$ mode. Similarly, taking the rotation of Equation (38), we obtain

$$\sum_{l,m,(l\neq 0)}\tilde{h}_{(o1)A}S = \frac{1}{r}\hat{\Delta}^{-1}\hat{D}_r\left(\epsilon^{rq}h_{Aq}\right) \tag{45}$$

and the mode coefficient $\tilde{h}_{(o1)A}$ for each mode, except for $l = 0$ mode, through the orthogonal property (42) of the $S = Y_{lm}$ with $l \neq 0$.

The explicit form of the Green function is given by refs. [58,59]. The expressions (44) and (45) indicates that the decomposition (38) is meaningless for the modes which belongs to the kernel $\hat{\Delta} := \hat{D}^r \hat{D}_r$, i.e., $l = 0$ mode.

Next, we consider the inverse relation of Equation (39). First, we note that the trace of Equation (39) yields

$$\sum_{l,m} \tilde{h}_{(e0)} S = \frac{1}{r^2} \gamma^{pq} h_{pq}, \tag{46}$$

and the traceless part of Equation (39) yields

$$\mathbb{H}_{pq}[h_{tu}] := h_{pq} - \frac{1}{2} \gamma_{pq} \gamma^{rs} h_{rs}, \tag{47}$$

$$= r^2 \sum_{l,m} \left[\tilde{h}_{(e2)} \left(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{\Delta} \right) S + 2 \tilde{h}_{(o2)} \epsilon_{r(p} \hat{D}_{q)} \hat{D}^r S \right]. \tag{48}$$

The mode coefficient $\tilde{h}_{(e0)}$ for each mode is obtained through the orthogonal property (42) of the spherical harmonics $S = Y_{lm}$ from the trace part (46) of h_{pq}. Therefore, we may concentrate on the traceless part (48) of h_{pq}. Taking the divergence of Equation (48), we obtain

$$\hat{D}^p \mathbb{H}_{pq}[h_{tu}] = r^2 \sum_{l,m} \left[\tilde{h}_{(e2)} \frac{1}{2} \hat{D}_q (\hat{\Delta} + 2) S + \tilde{h}_{(o2)} \epsilon_{rq} \hat{D}^r (\hat{\Delta} + 2) S \right] \tag{49}$$

$$= r^2 \sum_{l,m,(l \neq 1)} \left[\tilde{h}_{(e2)} \frac{1}{2} \hat{D}_q (\hat{\Delta} + 2) S + \tilde{h}_{(o2)} \epsilon_{rq} \hat{D}^r (\hat{\Delta} + 2) S \right]. \tag{50}$$

Equation (50) indicates that the mode coefficients $\tilde{h}_{(e2)}$ and $\tilde{h}_{(o2)}$ do not include $l = 1$ mode if $S = Y_{lm}$ because the $l = 1$ spherical harmonic function Y_{1m} is in the kernel of the derivative operator $\hat{\Delta} + 2$. Furthermore, we take the divergence of Equation (50), and obtain

$$\hat{D}^q \hat{D}^p \mathbb{H}_{pq}[h_{tu}] = r^2 \sum_{l,m,(l \neq 1)} \left[\frac{1}{2} \tilde{h}_{(e2)} \hat{\Delta} (\hat{\Delta} + 2) S \right] = r^2 \sum_{l,m,l \geq 2} \left[\frac{1}{2} \tilde{h}_{(e2)} \hat{\Delta} (\hat{\Delta} + 2) S \right]. \tag{51}$$

Equation (51) indicates that, in addition to the $l = 1$ mode, the mode coefficient $\tilde{h}_{(e2)}$ does not include the $l = 0$ mode which is the kernel mode of the derivative operator $\hat{\Delta}$. Then, through the Green functions of the derivative operators $\hat{\Delta}$ and $(\hat{\Delta} + 2)$, we obtain the solution to Equation (51) as

$$\sum_{l,m,l \geq 2} \tilde{h}_{(e2)} S = \frac{2}{r^2} [\hat{\Delta} + 2]^{-1} \hat{\Delta}^{-1} \hat{D}^q \hat{D}^p \mathbb{H}_{pq}[h_{tu}]. \tag{52}$$

From the orthogonal property (42) of the spherical harmonic function $S = Y_{lm}$ with $l \geq 2$, we obtain the mode coefficient $\tilde{h}_{(e2)}$.

On the other hand, multiplying ϵ^{qs} to Equation (50), we obtain

$$\epsilon^{qs} \hat{D}^p \mathbb{H}_{pq}[h_{tu}] = r^2 \sum_{l,m,(l \neq 1)} \left[\frac{1}{2} \tilde{h}_{(e2)} \epsilon^{qs} \hat{D}_q (\hat{\Delta} + 2) S + \tilde{h}_{(o2)} \hat{D}^s (\hat{\Delta} + 2) S \right], \tag{53}$$

and then, taking the divergence of Equation (53), we obtain

$$\epsilon^{qs}\hat{D}_s\hat{D}^p\mathbb{H}_{pq}[h_{tu}] = r^2\hat{\Delta}(\hat{\Delta}+2)\sum_{l,m,(l\neq 1)}\tilde{h}_{(o2)}S = r^2\hat{\Delta}(\hat{\Delta}+2)\sum_{l,m,(l\geq 2)}\tilde{h}_{(o2)}S. \tag{54}$$

Equation (54) indicates that, in addition to the $l=1$ mode, the mode coefficient $\tilde{h}_{(o2)}$ does not include the $l=0$ mode, which is the kernel mode of the derivative operator $\hat{\Delta}$. Through the Green functions of the derivative operators $\hat{\Delta}$ and $\hat{\Delta}+2$, we can solve Equation (54) as

$$\sum_{l,m,l\geq 2}\tilde{h}_{(o2)}S = \frac{1}{r^2}\left[\hat{\Delta}+2\right]^{-1}\hat{\Delta}^{-1}\epsilon^{qs}\hat{D}_s\hat{D}^p\mathbb{H}_{pq}[h_{tu}]. \tag{55}$$

From the orthogonality property (42) of the spherical harmonic function $S=Y_{lm}$ with $l\neq 0,1$, we obtain the mode coefficient $\tilde{h}_{(o2)}$.

Since the eigenvalue of the Laplacian operator $\hat{\Delta}$ on S^2 is $-l(l+1)$ with the non-negative integer l, the fact that we have to use the Green function of the operators Δ and $(\Delta+2)$ implies that the one-to-one correspondence between the set of variables $\{h_{pq}\}$ and the set of the variables $\{\tilde{h}_{(e0)},\tilde{h}_{(e2)},\tilde{h}_{(o2)}\}$ is not guaranteed for the kernel modes $l=0$ and $l=1$.

Finally, we also note that the operators $\hat{\Delta}^{-1}\hat{\Delta}$ and $\left[\hat{\Delta}+2\right]^{-1}\left[\hat{\Delta}+2\right]$ are not identity operators but should be regarded as the projection operators. We regard that the domains of the operators $\hat{\Delta}^{-1}\hat{\Delta}$ and $\left[\hat{\Delta}+2\right]^{-1}\left[\hat{\Delta}+2\right]$ is the L^2-space which is spanned by the spherical harmonics $\{Y_{lm}\}$. Since the operator $\hat{\Delta}$ eliminates the kernel

$$\mathscr{K}_{\hat{\Delta}} := \{f\in\mathscr{F}|\hat{\Delta}f=0\}, \tag{56}$$

where $\mathscr{F}$ is the function algebra, the range of the operator $\hat{\Delta}^{-1}\hat{\Delta}$ is the L^2-space which is spanned by the spherical harmonics $\{Y_{lm}|l\neq 0\}$, i.e.,

$$\{Y_{lm}|l\neq 0\} = L^2\backslash\mathscr{K}_{(\hat{\Delta})}. \tag{57}$$

Similarly, the domain of the operator $\left[\hat{\Delta}+2\right]^{-1}\left[\hat{\Delta}+2\right]$ is the L^2-space which is spanned by the spherical harmonics $\{Y_{lm}|l\geq 0\}$, while the kernel

$$\mathscr{K}_{\hat{\Delta}+2} := \{f\in\mathscr{F}|(\hat{\Delta}+2)f=0\} \tag{58}$$

is excluded in the range of the operator $\left[\hat{\Delta}+2\right]^{-1}\left[\hat{\Delta}+2\right]$, i.e., the range of this operator is

$$\{Y_{lm}|l\neq 1\} = L^2\backslash\mathscr{K}_{(\hat{\Delta}+2)}. \tag{59}$$

Namely, the operators $\hat{\Delta}^{-1}\hat{\Delta}$ and $[\hat{\Delta}+2]^{-1}[\Delta+2]$ are regarded as the projection operators as

$$\hat{\Delta}^{-1}\hat{\Delta} \quad : \quad L^2 \quad \mapsto \quad L^2\backslash\mathscr{K}_{(\hat{\Delta})} \tag{60}$$

$$[\hat{\Delta}+2]^{-1}[\Delta+2] \quad : \quad L^2 \quad \mapsto \quad L^2\backslash\mathscr{K}_{(\hat{\Delta}+2)}. \tag{61}$$

From Equations (60) and (61), we obtain the projection operator

$$\hat{\Delta}^{-1}[\hat{\Delta}+2]^{-1}[\Delta+2]\hat{\Delta} \quad : \quad L^2 \quad \mapsto \quad L^2\backslash\left(\mathscr{K}_{(\hat{\Delta})}\oplus\mathscr{K}_{(\hat{\Delta}+2)}\right). \tag{62}$$

This is one reason why we should discuss the treatments of the modes $l=0$ and $l=1$ separately if we choose $S=Y_{lm}$.

3.2. Treatments of the Kernel Modes

As seen in Section 3.1, the decomposition Formulas (37)–(39) with $S = Y_{lm}$ does not include the $l = 0, 1$ modes of the perturbations. In the general-relativistic gauge-invariant perturbation theory proposed in refs. [15,16], we assumed the separation of the linear-order metric perturbation into its gauge-invariant and gauge-dependent parts, i.e., Conjecture 1. In refs. [33,35], we discuss a scenario of the proof of Conjecture 1 on the generic background spacetime. In this scenario of the proof, we had to use the Green functions of some elliptic differential operators. In other words, we ignored the kernel modes of these elliptic differential operators in the scenario of the proof of Conjecture 1 in refs. [33,35]. The treatment of these kernel modes was unclear at that time. We called these kernel modes as "zero modes". Furthermore, we called the problem to find the treatment of these zero modes as the "zero-mode problem". In the case of the perturbations on the spherically symmetric background spacetimes, the $l = 0, 1$ modes correspond to the above zero mode in refs. [33,35]. This is also a well-known problem like the "$l = 0, 1$ mode problem" in the treatments of perturbations on spherically symmetric background spacetimes.

Here, we consider the resolution of this $l = 0, 1$ mode problem. To carry out this, we re-examine the derivation of the inverse relations of the decomposition Formulas (37)–(39), again. In this re-examination, we use the harmonic function $S = Y_{lm}$ for $l \geq 2$ model, because the set of the harmonic functions (41) has the linear independence at least for $l \geq 2$ mode. For $l = 0, 1$ mode, we change the harmonic function S from the spherical harmonic function Y_{00} and Y_{1m} to $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$, respectively, i.e., we use the harmonic functions S which are given by

$$S = S_\delta := \begin{cases} Y_{lm} & (l \geq 2); \\ k_{(\hat{\Delta}+2)} & (l = 1); \\ k_{(\hat{\Delta})} & (l = 0). \end{cases} \tag{63}$$

In this paper, we look for the explicit form of functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ within the constraints

$$k_{(\hat{\Delta})} \in \mathcal{K}_{(\hat{\Delta})}, \quad k_{(\hat{\Delta}+2)} \in \mathcal{K}_{(\hat{\Delta}+2)}, \tag{64}$$

respectively. Within these domains (64) of the kernel modes, we specify the conditions for the functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ to realize the independence of the set of the harmonic functions (41). These introductions of $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ correspond to the fact that we do not impose the regular boundary conditions as the function on S^2 before the construction of gauge-invariant variables, which was imposed in the conventional approach at the starting point.

3.2.1. h_{pq}

Here, we first consider the decomposition of the component h_{pq}. Previously, we considered the decomposition of the component h_{pq} as Equation (39)

$$h_{pq} = r^2 \sum_{l,m} \left[\frac{1}{2} \gamma_{pq} \tilde{h}_{(e0)} S + \tilde{h}_{(e2)} \left(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}_r \right) S + 2 \tilde{h}_{(o2)} \epsilon_{r(p} \hat{D}_{q)} \hat{D}^r S \right]. \tag{65}$$

As shown in Equation (46), we can separate the component h_{pq} into the trace part and the traceless part. The trace part of h_{pq} is given by Equation (46), which is also given by

$$\sum_{l,m,l\geq 2} \tilde{h}_{(e0,l\geq 2)} Y_{lm} + \sum_{m=-1,0,1} \tilde{h}_{(e0,l=1)} k_{(\hat{\Delta}+2)} + \tilde{h}_{(e0,l=0)} k_{(\hat{\Delta})} = \frac{1}{r^2} \gamma^{pq} h_{pq}. \tag{66}$$

Here, we note the effects (60) and (61) of the operators $\hat{\Delta}^{-1}\hat{\Delta}$ and $[\hat{\Delta}+2]^{-1}[\hat{\Delta}+2]$ as projection operators. If we apply the derivative operator $[\hat{\Delta}+2]$ to Equation (66), we obtain

$$\sum_{l,m,l\geq 2}\tilde{h}_{(e0,l\geq 2)}[\hat{\Delta}+2]Y_{lm}+2\tilde{h}_{(e0,l=0)}k_{(\hat{\Delta})}=\frac{1}{r^2}[\hat{\Delta}+2]\gamma^{pq}h_{pq}, \tag{67}$$

since we chose the functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ are eigenfunctions through Equation (64). Furthermore, applying the derivative operator $\hat{\Delta}$ to Equation (67) as

$$\sum_{l,m,l\geq 2}\tilde{h}_{(e0,l\geq 2)}\hat{\Delta}[\hat{\Delta}+2]Y_{lm}=\frac{1}{r^2}\hat{\Delta}[\hat{\Delta}+2]\gamma^{pq}h_{pq}. \tag{68}$$

The left- and right-hand sides of Equation (68) are in the domains of the Green functions $[\hat{\Delta}]^{-1}$ and $[\hat{\Delta}+2]^{-1}$. Therefore, we may apply the Green functions $[\hat{\Delta}]^{-1}$ and $[\hat{\Delta}+2]^{-1}$ to Equation (68) and obtain

$$\sum_{l,m,l\geq 2}\tilde{h}_{(e0,l\geq 2)}Y_{lm}=\frac{1}{r^2}[\hat{\Delta}+2]^{-1}\hat{\Delta}^{-1}\hat{\Delta}[\hat{\Delta}+2]\gamma^{pq}h_{pq}. \tag{69}$$

Through the orthogonal property (42) of the spherical harmonic function, we obtain

$$\tilde{h}_{(e0,l\geq 2)}=\frac{1}{r^2}\int_{S^2}d\Omega Y^{*}_{lm}[\hat{\Delta}+2]^{-1}\hat{\Delta}^{-1}\hat{\Delta}[\hat{\Delta}+2]\gamma^{pq}h_{pq}=:\tilde{h}_{(e0,l\geq 2)}[\![h_{pq}]\!]. \tag{70}$$

Thus, for $l\geq 2$, the mode coefficients $\tilde{h}_{(e0,l\geq 2)}$ is given by the functional of the original metric component h_{pq}.

Substituting Equation (70) into Equation (67), we obtain

$$\begin{aligned}2\tilde{h}_{(e0,l=0)}k_{(\hat{\Delta})} &= \frac{1}{r^2}[\hat{\Delta}+2]\gamma^{pq}h_{pq}-\sum_{l,m,l\geq 2}\tilde{h}_{(e0,l\geq 2)}[\![h_{pq}]\!][\hat{\Delta}+2]Y_{lm}\\ &=: 2\tilde{h}_{(e0,l=0)}[\![h_{pq}]\!]k_{(\hat{\Delta})}.\end{aligned} \tag{71}$$

Then, the mode coefficient $\tilde{h}_{(e0,l=0)}$ is obtained as a functional of the original metric perturbation h_{pq} if $k_{(\hat{\Delta})}\neq 0$. Furthermore, from Equations (66), (70) and (71), we obtain

$$\sum_{m=-1,0,1}\tilde{h}_{(e0.l=1)}k_{(\hat{\Delta}+2)}=\frac{1}{r^2}\gamma^{pq}h_{pq}-\sum_{l,m,l\geq 2}\tilde{h}_{(e0,l\geq 2)}[\![h_{pq}]\!]Y_{lm}-\tilde{h}_{(e0,l=0)}[\![h_{pq}]\!]k_{(\hat{\Delta})}. \tag{72}$$

To resolve the degeneracy of the modes with $m=0,\pm 1$ in Equation (72), we choose $k_{(\hat{\Delta}+2)}$ as

$$k_{(\hat{\Delta}+2)}=k_{(\hat{\Delta}+2)m}=\Theta_{1m}(\theta)e^{im\phi}. \tag{73}$$

Through the orthogonality condition

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi e^{i(m-m')\phi}=\delta_{mm'}, \tag{74}$$

we obtain

$$e^{+im\phi}\frac{1}{2\pi}\int_0^{2\pi}d\phi e^{-im'\phi}k_{(\hat{\Delta}+2)m}=k_{(\hat{\Delta}+2)m}\delta_{mm'}. \tag{75}$$

Applying the property (75) to Equation (72), we obtain

$$\begin{aligned}\tilde{h}_{(e0.l=1)}k_{(\hat{\Delta}+2)m} &= e^{+im\phi}\frac{1}{2\pi}\int_0^{2\pi}d\phi e^{-im\phi}\\ &\quad\times\left[\frac{1}{r^2}\gamma^{pq}h_{pq}-\sum_{l,m',(l\neq 0,1)}\tilde{h}_{(e0,l\geq 2)}[h_{pq}]Y_{lm'}-\tilde{h}_{(e0,l=0)}[h_{pq}]k_{(\hat{\Delta})}\right]\\ &=: \tilde{h}_{(e0.l=1)}[\![h_{pq}]\!]k_{(\hat{\Delta}+2)m}. \end{aligned} \tag{76}$$

Then, if $\Theta_{1m}(\theta)\neq 0$, i.e., $k_{(\hat{\Delta}+2)}\neq 0$, the mode coefficient $\tilde{h}_{(e0,l=1)}$ is given in the functional form of the original metric perturbation h_{pq}.

Thus, the mode decomposition of the trace part (66) of the metric perturbation h_{pq} is invertible. In this argument, we essentially used Equation (64) for the eigenfunctions and the ϕ-dependence (73) of the function $k_{(\hat{\Delta}+2)}$.

Next, we consider the traceless part (47) of h_{pq} as Equation (48). Taking the divergence of Equation (48), we obtain

$$\begin{aligned}\frac{1}{r^2}\hat{D}^p\mathbb{H}_{pq}[h_{tu}] &= \sum_{l,m}\left[\frac{1}{2}\tilde{h}_{(e2)}\hat{D}_q\left(\hat{\Delta}+2\right)Y_{lm}-\tilde{h}_{(o2)}\epsilon_{qr}\hat{D}^r\left(\hat{\Delta}+2\right)Y_{lm}\right]\\ &= \sum_{l,m,l\geq 2}\left[\frac{1}{2}\tilde{h}_{(e2,l\geq 2)}\hat{D}_q\left(\hat{\Delta}+2\right)Y_{lm}-\tilde{h}_{(o2,l\geq 2)}\epsilon_{qr}\hat{D}^r\left(\hat{\Delta}+2\right)Y_{lm}\right]\\ &\quad+\tilde{h}_{(e2,l=0)}\hat{D}_q k_{(\hat{\Delta})}-\tilde{h}_{(o2.l=0)}2\epsilon_{qr}\hat{D}^r k_{(\hat{\Delta})}, \end{aligned} \tag{77}$$

where we used Equation (A76) in Appendix B and (64). We have to emphasize that the $l=1$ mode does not appear in the expression (77). Taking the divergence of Equation (77), again, we have

$$\frac{1}{r^2}\hat{D}^q\hat{D}^p\mathbb{H}_{pq}[h_{tu}] = \frac{1}{2}\sum_{l,m,l\geq 2}\tilde{h}_{(e2,l\geq 2)}\left(\hat{\Delta}+2\right)\hat{\Delta}Y_{lm}, \tag{78}$$

where we used the property of the eigen equation for $k_{(\hat{\Delta})}$ in Equation (64). Through the Green functions $\hat{\Delta}^{-1}$ and $[\hat{\Delta}+2]^{-1}$ and the orthogonal property (42) of the spherical harmonics Y_{lm}, we obtain the same result as Equation (55) and the mode coefficient $\tilde{h}_{(e2,l\geq 2)}$ of each mode is given in a functional form of the original metric perturbation h_{tu} as

$$\tilde{h}_{(e2,l\geq 2)} = \frac{2}{r^2}\int_{S^2}d\Omega Y^*_{lm}[\hat{\Delta}]^{-1}[\hat{\Delta}+2]^{-1}\hat{D}^q\hat{D}^p\mathbb{H}_{pq}[h_{tu}] =: \tilde{h}_{(e2,l\geq 2)}[\![h_{tu}]\!]. \tag{79}$$

On the other hand, taking the rotation of Equation (77) and use the eigen equation for $k_{(\hat{\Delta})}$ in Equation (64), Green functions $[\hat{\Delta}]^{-1}$ and $[\hat{\Delta}+2]$, and the orthogonal properties (42) of the spherical harmonics Y_{lm}, we obtain the mode coefficient $\tilde{h}_{(o2,l\geq 2)}$ in the functional form of the original metric perturbation h_{tu} as

$$\tilde{h}_{(o2,l\geq 2)} = \frac{1}{r^2}\int_{S^2}d\Omega Y^*_{lm}[\hat{\Delta}]^{-1}[\hat{\Delta}+2]^{-1}\epsilon^{ps}\hat{D}_s\hat{D}^q\mathbb{H}_{pq}[h_{tu}] =: \tilde{h}_{(o2,l\geq 2)}[\![h_{tu}]\!]. \tag{80}$$

Substituting Equations (79) and (80) into Equation (77), we obtain

$$\begin{aligned}&\tilde{h}_{(e2,l=0)}\hat{D}_q k_{(\hat{\Delta})}-\tilde{h}_{(o2.l=0)}2\epsilon_{qr}\hat{D}^r k_{(\hat{\Delta})}\\ &= -\sum_{l,m,l\geq 2}\left[\frac{1}{2}\tilde{h}_{(e2,l\geq 2)}[\![h_{tu}]\!]\hat{D}_q\left(\hat{\Delta}+2\right)Y_{lm}-\tilde{h}_{(o2,l\geq 2)}[\![h_{tu}]\!]\epsilon_{qr}\hat{D}^r\left(\hat{\Delta}+2\right)Y_{lm}\right]\\ &\quad+\frac{1}{r^2}\hat{D}^p\mathbb{H}_{pq}[h_{tu}]. \end{aligned} \tag{81}$$

If $\hat{D}_q k_{(\hat{\Delta})} \neq 0$, the vectors $\hat{D}_q k_{(\hat{\Delta})}$ and $\epsilon_{qr}\hat{D}^r k_{(\hat{\Delta})}$ are orthogonal to each other. Then, we have

$$\begin{aligned}\tilde{h}_{(e2,l=0)} &= \tilde{h}_{(e2,l=0)}[\![h_{ut}]\!] \\ &:= \frac{\left(\hat{D}^q k_{(\hat{\Delta})}\right)}{\left(\hat{D}_s k_{(\hat{\Delta})}\right)\left(\hat{D}^s k_{(\hat{\Delta})}\right)}\Bigg[\frac{1}{r^2}\hat{D}^p \mathbb{H}_{pq}[h_{tu}] \\ &\qquad - \sum_{l,m,l\geq 2}\Bigg\{\frac{1}{2}\tilde{h}_{(e2,l\geq 2)}[\![h_{tu}]\!]\hat{D}_q\left(\hat{\Delta}+2\right)Y_{lm} \\ &\qquad\quad -\tilde{h}_{(o2,l\geq 2)}[\![h_{tu}]\!]\epsilon_{qr}\hat{D}^r\left(\hat{\Delta}+2\right)Y_{lm}\Bigg\}\Bigg] \end{aligned} \tag{82}$$

and

$$\begin{aligned}\tilde{h}_{(o2,l=0)} &= \tilde{h}_{(o2,l=0)}[\![h_{ut}]\!] \\ &:= \frac{\left(\epsilon^{qr}\hat{D}_r k_{(\hat{\Delta})}\right)}{\left(\hat{D}_s k_{(\hat{\Delta})}\right)\left(\hat{D}^s k_{(\hat{\Delta})}\right)}\Bigg[\frac{1}{r^2}\hat{D}^p \mathbb{H}_{pq}[h_{tu}] \\ &\qquad - \sum_{l,m,l\geq 2}\Bigg\{\frac{1}{2}\tilde{h}_{(e2,l\geq 2)}[\![h_{tu}]\!]\hat{D}_q\left(\hat{\Delta}+2\right)Y_{lm} \\ &\qquad\quad -\tilde{h}_{(o2,l\geq 2)}[\![h_{tu}]\!]\epsilon_{qr}\hat{D}^r\left(\hat{\Delta}+2\right)Y_{lm}\Bigg\}\Bigg]. \end{aligned} \tag{83}$$

Now, we return to the original definition (48) of the traceless part $\mathbb{H}_{pq}$. From Equations (47), (79), (80), (82) and (83), we obtain

$$\begin{aligned}&\sum_{m=-1,0,1}\left[\tilde{h}_{(e2,l=1,m)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)k_{(\hat{\Delta}+2)} + 2\tilde{h}_{(o2,l=1,m)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta}+2)}\right] \\ &= \frac{1}{r^2}\mathbb{H}_{pq}[h_{tu}] - \Bigg[\sum_{l,m,l\geq 2}\Bigg\{\tilde{h}_{(e2)}[\![h_{tu}]\!]\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)Y_{lm} + 2\tilde{h}_{(o2)}[\![h_{tu}]\!]\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r Y_{lm}\Bigg\} \\ &\qquad +\tilde{h}_{(e2,l=0)}[\![h_{tu}]\!]\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)k_{(\hat{\Delta})} + 2\tilde{h}_{(o2,l=0)}[\![h_{tu}]\!]\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta})}\Bigg] \\ &=: H_{(\hat{\Delta}+2)pq}[\![h_{tu}]\!]. \end{aligned} \tag{84}$$

To simplify the notation, we define

$$K_{(m)pq} := \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)k_{(\hat{\Delta}+2)m}, \quad J_{(m)pq} := 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta}+2)m}, \tag{85}$$

and we evaluate $K_{(m)pq}K^{pq}_{(m')}$, $J_{(m)pq}K^{pq}_{(m')}$, and $J_{(m)pq}J^{pq}_{(m')}$, which are given by

$$\begin{aligned} K_{(m)pq}K^{pq}_{(m')} &= \left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m'}\right) \\ &\quad -2\left(k_{(\hat{\Delta}+2)m}\right)\left(k_{(\hat{\Delta}+2)m'}\right), \end{aligned} \tag{86}$$

$$J_{(m)pq}K^{pq}_{(m')} = 2\epsilon^{rp}\hat{D}_r\hat{D}_q k_{(\hat{\Delta}+2)m}\hat{D}_p\hat{D}^q k_{(\hat{\Delta}+2)m'}, \tag{87}$$

$$\begin{aligned} J_{(m)pq}J^{pq}_{(m')} &= 4\Big[\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m}\right)\left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)m'}\right) \\ &\quad -2\left(k_{(\hat{\Delta}+2)m}\right)\left(k_{(\hat{\Delta}+2)m'}\right)\Big]. \end{aligned} \tag{88}$$

To carry out the resolution of the degeneracy in Equation (84), we use the property (73) of the function $k_{(\hat{\Delta}+2)}$. From the property (73), we have

$$\hat{D}_p k_{(\hat{\Delta}+2)m} = \left(\frac{d}{d\theta}\Theta_m(\theta)\right)e^{im\phi}\theta_p + \frac{im}{\sin\theta}\Theta_m(\theta)e^{im\phi}\phi_p \tag{89}$$

and

$$\begin{aligned}\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)m} &= \left(\frac{d^2}{d\theta^2}\Theta_m(\theta)\right)e^{im\phi}\theta_p\theta_q \\ &+\left[\left(\frac{d}{d\theta}\Theta_m(\theta)\right)\cot\theta - m^2\frac{1}{\sin^2\theta}\Theta_m(\theta)\right]e^{im\phi}\phi_p\phi_q \\ &+im\frac{1}{\sin\theta}\left[\frac{d}{d\theta}\Theta_m(\theta) - \cot\theta\Theta_m(\theta)\right]e^{im\phi}2\theta_{(p}\phi_{q)}.\end{aligned} \tag{90}$$

From Equation (90), we obtain

$$\begin{aligned}K_{(m)pq} &:= \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)k_{(\hat{\Delta}+2)m} \\ &= -\left(\theta_p\theta_q - \phi_p\phi_q\right)\left[\cot\theta\frac{d}{d\theta}\Theta_m(\theta) + \left(1 - \frac{m^2}{\sin^2\theta}\right)\Theta_m(\theta)\right]e^{im\phi} \\ &+2\theta_{(p}\phi_{q)}\frac{im}{\sin\theta}\left(\frac{d}{d\theta}\Theta_m(\theta) - \cot\theta\Theta_m(\theta)\right)e^{im\phi},\end{aligned} \tag{91}$$

where we used $(\hat{\Delta}+2)k_{(\hat{\Delta}+2)m} = 0$, i.e.,

$$\frac{d^2}{d\theta^2}\Theta_m(\theta) + \cot\theta\frac{d}{d\theta}\Theta_m(\theta) + \left(2 - \frac{m^2}{\sin^2\theta}\right)\Theta_m(\theta) = 0. \tag{92}$$

From the expression of the components $K_{(m')pq}$, $J_{(m')pq}$, θ_p, and ϕ_p, we can confirm

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-im\phi}K_{(m')pq} = K_{(m)pq}e^{-im\phi}\delta_{mm'}, \tag{93}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-im\phi}J_{(m')pq} = J_{(m)pq}e^{-im\phi}\delta_{mm'}. \tag{94}$$

Furthermore, straightforward calculations yield

$$K_{(m)pq}K^{pq}_{(m)} = \left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)m}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m}\right) - 2\left(k_{(\hat{\Delta}+2)m}\right)^2, \tag{95}$$

$$J_{(m)pq}K^{pq}_{(m)} = 0, \tag{96}$$

$$J_{(m)pq}J^{pq}_{(m)} = 4K_{(m)pq}K^{pq}_{(m)}. \tag{97}$$

Through Equations (93) and (94), we can consider the resolution of the m-degeneracy of $l = 1$ mode in Equation (84) as follows

$$\begin{aligned}&\frac{1}{2\pi}e^{+im\phi}\int_0^{2\pi} d\phi e^{-im\phi}H_{(\hat{\Delta}+2)pq}[\![h_{tu}]\!] \\ &= \sum_{m'=-1,0,1}\left[\tilde{h}_{(e2,l=1,m)}e^{+im\phi}\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-im\phi}K_{(m')pq}\right. \\ &\qquad\left.+\tilde{h}_{(o2,l=1,m)}e^{+im\phi}\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-im\phi}J_{(m')pq}\right] \\ &= \sum_{m'=-1,0,1}\left[\tilde{h}_{(e2,l=1,m)}K_{(m)pq}\delta_{mm'} + \tilde{h}_{(o2,l=1,m)}J_{(m)pq}\delta_{mm'}\right] \\ &= \tilde{h}_{(e2,l=1,m)}K_{(m)pq} + \tilde{h}_{(o2,l=1,m)}J_{(m)pq}.\end{aligned} \tag{98}$$

Furthermore, from Equations (95)–(97), we obtain

$$\begin{aligned}\tilde{h}_{(e2,l=1,m)} &= [K_{(m)pq}K^{pq}_{(m)}]^{-1}K^{pq}_{(m)}\frac{1}{2\pi}e^{+im\phi}\int_0^{2\pi}d\phi e^{-im\phi}H_{(\hat{\Delta}+2)pq}[\![h_{tu}]\!] \\ &=: \tilde{h}_{(e2,l=1,m)}[\![h_{tu}]\!], \end{aligned} \tag{99}$$

and

$$\begin{aligned}\tilde{h}_{(o2,l=1,m)} &= \frac{1}{4}[K_{(m)pq}K^{pq}_{(m)}]^{-1}J^{pq}_{(m)}\frac{1}{2\pi}e^{+im\phi}\int_0^{2\pi}d\phi e^{-im\phi}H_{(\hat{\Delta}+2)pq}[\![h_{tu}]\!] \\ &=: \tilde{h}_{(o2,l=1,m)}[\![h_{tu}]\!]. \end{aligned} \tag{100}$$

Thus, we have obtained the mode coefficients $\tilde{h}_{(e2,l=1,m)}$ and $\tilde{h}_{(o2,l=1,m)}$ in the functional forms of the original metric h_{tu}.

Here, we summarize the conditions for the eigenfunctions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ to obtain the inverse relation of the metric decomposition (65). To obtain the inverse relations of the mode decomposition of the trace and the traceless-part of Equation (65), we use the conditions

$$k_{(\hat{\Delta})} \in \mathscr{K}_{(\hat{\Delta})}, \quad k_{(\hat{\Delta}+2)} \in \mathscr{K}_{(\hat{\Delta}+2)}, \quad k_{(\hat{\Delta}+2)} = k_{(\hat{\Delta}+2)m} = \Theta_{1m}(\theta)e^{im\phi}, \tag{101}$$

$$\left(\hat{D}_p k_{(\hat{\Delta})}\right)\left(\hat{D}^p k_{(\hat{\Delta})}\right) \neq 0, \tag{102}$$

$$K_{(m)pq}K^{pq}_{(m)} = \left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)m}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m}\right) - 2\left(k_{(\hat{\Delta}+2)m}\right)^2 \neq 0. \tag{103}$$

The condition (103) implies the non-vanishing $K_{(m)pq}$ and $J_{(m)pq}$.

3.2.2. h_{Ap}

Next, we consider the inversion relation of the decomposition (38) taking account of the kernel modes $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$.

$$h_{Ap} = r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_p S + \tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^q S\right] \tag{104}$$

$$\begin{aligned} &= r\sum_{l,m,l\geq 2}\left[\tilde{h}_{(e1,l\geq 2)A}\hat{D}_p Y_{lm} + \tilde{h}_{(o1,l\geq 2)A}\epsilon_{pq}\hat{D}^q Y_{lm}\right] \\ &\quad + r\sum_{m}\left[\tilde{h}_{(e1,l=1)A}\hat{D}_p k_{(\hat{\Delta}+2)} + \tilde{h}_{(o1,l=1)A}\epsilon_{pq}\hat{D}^q k_{(\hat{\Delta}+2)}\right] \\ &\quad + r\left[\tilde{h}_{(e1,l=0)A}\hat{D}_p k_{(\hat{\Delta})} + \tilde{h}_{(o1,l=0)A}\epsilon_{pq}\hat{D}^q k_{(\hat{\Delta})}\right]. \end{aligned} \tag{105}$$

Taking the divergence of Equation (105) we obtain

$$\hat{D}^p h_{Ap} = r\sum_{l,m,l\geq 2}\tilde{h}_{(e1,l\geq 2)A}\hat{\Delta}Y_{lm} - 2r\sum_m \tilde{h}_{(e1,l=1,m)A}k_{(\hat{\Delta}+2)}. \tag{106}$$

Applying the derivative operator $\hat{\Delta}+2$ to Equation (106), we obtain

$$[\hat{\Delta}+2]\hat{D}^p h_{Ap} = r\sum_{l,m,l\geq 2}\tilde{h}_{(e1,l\geq 2)A}[\hat{\Delta}+2]\hat{\Delta}Y_{lm}. \tag{107}$$

Using the Green functions $[\hat{\Delta}+2]^{-1}$, $\hat{\Delta}^{-1}$, and the orthogonal property (42) of the spherical harmonics Y_{lm}, we obtain

$$\tilde{h}_{(e1,l\geq 2)A} = \frac{1}{r}\int_{S^2}d\Omega Y^*_{lm}\hat{\Delta}^{-1}[\hat{\Delta}+2]^{-1}[\hat{\Delta}+2]\hat{D}^p h_{Ap} =: \tilde{h}_{(e1,l\geq 2)A}[\![h_{Bs}]\!]. \tag{108}$$

Thus, the mode coefficient $\tilde{h}_{(e1)A}$ is given in the form of the functional of the original metric component h_{Ap}. Through Equation (108), Equation (106) is expressed as

$$\sum_m \tilde{h}_{(e1,l=1,m)A} k_{(\hat{\Delta}+2)} = \frac{1}{2} \sum_{l,m,l\geq 2} \tilde{h}_{(e1,l\geq 2)A} [\![h_{Br}]\!] \hat{\Delta} Y_{lm} - \frac{1}{2r} \hat{D}^p h_{Ap}. \tag{109}$$

To resolve the m-degeneracy of Equation (109), we use Equations (101) and (75). Then, we have

$$\begin{aligned} \tilde{h}_{(e1,l=1,m)A} &= \frac{e^{im\phi}}{k_{(\hat{\Delta}+2)m}} \frac{1}{4\pi} \int_0^{2\pi} d\phi e^{-im'\phi} \left[\sum_{l,m',l\geq 2} \tilde{h}_{(e1,l\geq 2)A} [\![h_{Br}]\!] \hat{\Delta} Y_{lm'} - \frac{1}{r} \hat{D}^p h_{Ap} \right] \\ &=: \tilde{h}_{(e1,l=1,m)A} [\![h_{Bs}]\!] \end{aligned} \tag{110}$$

On the other hand, taking the rotation of Equation (105), we have

$$\epsilon^{pq} \hat{D}_q h_{Ap} = r \sum_{l,m,l\geq 2} \left[\tilde{h}_{(o1,l\geq 2)A} \hat{\Delta} Y_{lm} \right] - 2r \sum_m \left[\tilde{h}_{(o1,l=1)A} k_{(\hat{\Delta}+2)} \right]. \tag{111}$$

As in the case of Equation (108), we have

$$\tilde{h}_{(o1,l\geq 2)A} = \frac{1}{r} \int_{S^2} d\Omega Y^*_{lm} \hat{\Delta}^{-1} [\hat{\Delta}+2]^{-1} [\hat{\Delta}+2] \epsilon^{pq} \hat{D}_q h_{Ap} =: \tilde{h}_{(o1,l\geq 2)A} [\![h_{Br}]\!], \tag{112}$$

$$\begin{aligned} \tilde{h}_{(o1,l=1)A} &= \frac{e^{im\phi}}{k_{(\hat{\Delta}+2)m}} \frac{1}{4\pi} \int_0^{2\pi} d\phi e^{-im'\phi} \left[\sum_{l,m',l\geq 2} \left\{ \tilde{h}_{(o1,l\geq 2)A} [\![h_{Br}]\!] \hat{\Delta} Y_{lm'} \right\} - \frac{1}{r} \epsilon^{pq} \hat{D}_q h_{Ap} \right] \\ &=: \tilde{h}_{(o1,l=1)A} [\![h_{Bs}]\!]. \end{aligned} \tag{113}$$

Through Equations (108), (110), (112) and (113), we obtain

$$\begin{aligned} &\tilde{h}_{(e1,l=0)A} \hat{D}_p k_{(\hat{\Delta})} + \tilde{h}_{(o1,l=0)A} \epsilon_{pq} \hat{D}^q k_{(\hat{\Delta})} \\ &\quad = \frac{1}{r} h_{Ap} - \sum_{l,m,l\geq 2} \left[\tilde{h}_{(e1,l\geq 2)A} [\![h_{Bs}]\!] \hat{D}_p Y_{lm} + \tilde{h}_{(o1,l\geq 2)A} [\![h_{Bs}]\!] \epsilon_{pq} \hat{D}^q Y_{lm} \right] \\ &\qquad - \sum_m \left[\tilde{h}_{(e1,l=1)A} [\![h_{Bs}]\!] \hat{D}_p k_{(\hat{\Delta}+2)} + \tilde{h}_{(o1,l=1)A} [\![h_{Bs}]\!] \epsilon_{pq} \hat{D}^q k_{(\hat{\Delta}+2)} \right] \qquad (114) \\ &\quad =: \mathbb{H}_{Ap} [\![h_{Bs}]\!]. \qquad (115) \end{aligned}$$

Here, we use the condition (102). Then, we have

$$\tilde{h}_{(e1,l=0)A} = \left[\left(\hat{D}^q k_{(\hat{\Delta})} \right) \left(\hat{D}_q k_{(\hat{\Delta})} \right) \right]^{-1} \hat{D}^p k_{(\hat{\Delta})} \mathbb{H}_{Ap} [\![h_{Bs}]\!] =: \tilde{h}_{(e1,l=0)A} [\![h_{Bs}]\!], \tag{116}$$

$$\tilde{h}_{(o1,l=0)A} = \left[\left(\hat{D}^r k_{(\hat{\Delta})} \right) \left(\hat{D}_r k_{(\hat{\Delta})} \right) \right]^{-1} \epsilon^{pq} \hat{D}_q k_{(\hat{\Delta})} \mathbb{H}_{Ap} [\![h_{Bs}]\!] =: \tilde{h}_{(o1,l=0)A} [\![h_{Bs}]\!]. \tag{117}$$

Thus, we have shown that the mode coefficients $\tilde{h}_{(e1)A}$ and $\tilde{h}_{(o1)A}$ for all $l \geq 0$ modes are given in the functional forms (108), (110), (112), (113), (116) and (117) of the original metric h_{Ap} under the conditions (101)–(103).

3.2.3. h_{AB}

Through the harmonic functions Y_{lm} ($l \geq 2$), $k_{(\hat{\Delta}+2)m}$, and $k_{(\hat{\Delta})}$, the component h_{AB} of the metric perturbation h_{ab} is decomposed as

$$h_{AB} = \sum_{l,m(l\geq 2)} \tilde{h}_{(l\geq 2)AB} S + \sum_{m=-1,0,1} \tilde{h}_{(l=1,m)AB} k_{(\hat{\Delta}+2)m} + \tilde{h}_{(l=0)AB} k_{(\hat{\Delta})}. \tag{118}$$

This decomposition has the same form as Equation (66) for the trace part of the component h_{pq}. Then, we obtain the inverse relations

$$\tilde{h}_{(l\geq 2)AB} = \int_{S^2} d\Omega Y^*_{lm}[\hat{\Delta}+2]^{-1}\hat{\Delta}^{-1}\hat{\Delta}[\hat{\Delta}+2]h_{AB} =: \tilde{h}_{(l\geq 2)AB}[\![h_{AB}]\!], \quad l \geq 2, \tag{119}$$

$$\begin{aligned}\tilde{h}_{(l=0)AB} &= \frac{1}{2k_{(\hat{\Delta})}}\left[[\hat{\Delta}+2]h_{AB} - \sum_{l,m,(l\neq 0,1)} \tilde{h}_{(l\geq 2)AB}[\![h_{AB}]\!][\hat{\Delta}+2]Y_{lm}\right] \\ &=: \tilde{h}_{(l=0)AB}[\![h_{AB}]\!],\end{aligned} \tag{120}$$

$$\begin{aligned}\tilde{h}_{(l=1,m)AB} &= \frac{1}{k_{(\hat{\Delta}+2)m}} e^{+im\phi}\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-im\phi} \\ &\quad \times\left[\frac{1}{r^2}h_{AB} - \sum_{l,m',(l\neq 0,1)} \tilde{h}_{(l\geq 2)AB}[\![h_{AB}]\!]Y_{lm'} - \tilde{h}_{(l=0)AB}[\![h_{AB}]\!]k_{(\hat{\Delta})}\right] \\ &=: \tilde{h}_{(l=1)AB}[\![h_{AB}]\!].\end{aligned} \tag{121}$$

which correspond to Equations (66), (71) and (76), respectively.

3.2.4. Summary of the Mode Decomposition Including $l = 0, 1$ Modes

Here, we summarize the mode decomposition by harmonic functions Y_{lm} ($l \geq 2$), $k_{(\hat{\Delta}+2)m}$, and $k_{(\hat{\Delta})}$. We decompose the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ of the metric perturbation h_{ab} as Equations (37)–(39) with

$$S = \begin{cases} Y_{lm} & \text{for} \quad l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} \quad l = 1; \\ k_{(\hat{\Delta})} & \text{for} \quad l = 0. \end{cases} \tag{122}$$

This decomposition is invertible for any l, m modes including $l = 0, 1$ if the conditions (101)–(103), i.e.,

$$k_{(\hat{\Delta})} \in \mathscr{K}_{(\hat{\Delta})}, \quad k_{(\hat{\Delta}+2)} \in \mathscr{K}_{(\hat{\Delta}+2)}, \quad k_{(\hat{\Delta}+2)} = k_{(\hat{\Delta}+2)m} = \Theta_{1m}(\theta)e^{im\phi}, \tag{123}$$

$$\left(\hat{D}_p k_{(\hat{\Delta})}\right)\left(\hat{D}^p k_{(\hat{\Delta})}\right) \neq 0, \tag{124}$$

$$K_{(m)pq}K^{pq}_{(m)} = \left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)m}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)m}\right) - 2\left(k_{(\hat{\Delta}+2)m}\right)^2 \neq 0 \tag{125}$$

are satisfied. As the inverse relation of Equations (37)–(39), the mode coefficients of these decompositions are given in the functional form of the metric components h_{AB}, h_{Ap}, and h_{pq} as Equations (70), (71), (76), (79), (80), (82), (83), (99), (100), (108), (110), (112), (113), (116), (117) and (119)–(121). From Equations (37)–(39), the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ vanish if all mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ vanish. On the contrary, from the obtained functional forms, all mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ vanish if the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ vanish. This indicates the linear independence of the set of the harmonic functions (41). Therefore, the conditions (123)–(125) guarantee the linear independence of the set of these harmonic functions (41).

We also note that the Green functions $\hat{\Delta}^{-1}$ and $[\hat{\Delta}+2]^{-1}$ which used above do not directly operate to the functions $k_{(\hat{\Delta})}$, nor $k_{(\hat{\Delta}+2)m}$. Therefore, the domains of these Green functions $\hat{\Delta}^{-1}$ and $[\hat{\Delta}+2]^{-1}$ may be regarded as the L^2-space spanned by $\{Y_{lm}|l \neq 0\}$ and $\{Y_{lm}|l \neq 1\}$, respectively. The explicit form of these Green functions are given in Refs. [58,59].

3.3. Explicit Form of the Mode Functions

Here, we consider the explicit expression of the mode functions $k_{\hat{\Delta}}$ and $k_{(\hat{\Delta}+2)}$ which satisfy the conditions (123)–(125). In Appendix A, we explicitly see that the choice $S = Y_{lm}$ for $l \geq 0$ does not satisfy these conditions. As the result of this, in the choice $S = Y_{lm}$, any vector and tensor harmonics vanish for $l = 0$ modes. On the other hand, for $l = 1$ modes, the vector harmonics have their vector value and the trace parts of the second-rank tensor of each modes have their tensor values, while all traceless even- and odd-mode harmonics identically vanish. Therefore, in the choice $S = Y_{lm}$, the set of harmonics (41) does not play the role of basis of tangent space on S^2 for $l = 0, 1$ mode. This situation already appeared in terms of the Green function $\hat{\Delta}^{-1}$ and $(\hat{\Delta}+2)^{-1}$ in the inverse relations in Section 3.1. For this reason, we seek an alternative choice of S which satisfy the conditions (123)–(125).

3.3.1. Explicit Form of $k_{(\hat{\Delta})}$

Here, we treat the modes which belong to the kernel of the derivative operator $\hat{\Delta}$, i.e.,

$$\hat{\Delta} k_{(\hat{\Delta})} = \frac{1}{\sqrt{\gamma}} \partial_p \left(\sqrt{\gamma} \gamma^{pq} \partial_q k_{(\hat{\Delta})} \right) = 0. \tag{126}$$

We look for the function which satisfies the conditions (123) and (124). We emphasize that we do not impose the regularity on the function $k_{(\hat{\Delta})}$ on S^2 itself in this selection of $k_{(\hat{\Delta})}$. Since the regularity is a kind of boundary condition for perturbations, this regularity may be imposed on the solutions when we solve the Einstein equations.

Our guiding principle is to look for the solution to Equation (126) with a simple modification from the conventional spherical harmonic functions. Although the conditions (123) and (124) do not restrict the ϕ-dependence for $k_{(\hat{\Delta})}$, we look for the solution to Equation (126) which is independent of ϕ as the original Y_{00} in the conventional spherical harmonics is so. Then, in terms of the coordinate system where γ_{ab} is given by Equation (34), Equation (126) yields

$$\frac{d^2}{dy^2} k_{(\hat{\Delta})} = 0, \tag{127}$$

where we introduced an independent variable y by

$$y = \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right)^{1/2}. \tag{128}$$

As the solution to Equation (127), we choose

$$k_{(\hat{\Delta})} = 1 + \delta y = 1 + \delta \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right)^{1/2}, \quad \delta \in \mathbb{R}. \tag{129}$$

If $\delta \neq 0$, we see that

$$\hat{D}_p k_{(\hat{\Delta})} (dx^p)_a = \delta (dy)_a = \frac{\delta}{\sin\theta} (d\theta)_a \neq 0, \tag{130}$$

and

$$\left(\hat{D}_p k_{(\Delta)} \right) \left(\hat{D}^p k_{(\Delta)} \right) = \frac{\delta^2}{\sin^2\theta} \neq 0. \tag{131}$$

Thus, $\hat{D}_p k_{(\hat{\Delta})}$ given by Equation (130) and $\epsilon_{pq}\hat{D}^q k_{(\hat{\Delta})}$ spans the vector space though their norm is singular at $\theta = 0, \pi$. The solution (129) to Equation (127) also yields

$$\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)k_{(\hat{\Delta})} = \hat{D}_p\hat{D}_q k_{(\hat{\Delta})} = \delta\frac{\cos\theta}{\sin^2\theta}\left(-\theta_p\theta_q + \phi_p\phi_q\right) \neq 0, \tag{132}$$

$$\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta})} = -2\delta\frac{\cos\theta}{\sin^2\theta}\theta_{(p}\phi_{q)} \neq 0. \tag{133}$$

Together with the trace part

$$\frac{1}{2}\gamma_{pq}k_{(\hat{\Delta})} = \frac{1}{2}\left(1 + \delta\ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}\right)(\theta_p\theta_q + \phi_p\phi_q), \tag{134}$$

the tensors (132) and (133) span the basis of the space of the second-rank tensor field though these are singular at $\theta = 0, \pi$.

3.3.2. Explicit Form of $k_{(\hat{\Delta}+2)}$

Here, we consider the kernel mode $k_{(\hat{\Delta}+2)}$ for the operator $\hat{\Delta}+2$. The condition (123) for $k_{(\hat{\Delta}+2)}$ is given by

$$(\hat{\Delta}+2)k_{(\hat{\Delta}+2)} = \frac{1}{\sqrt{\gamma}}\partial_p\left(\sqrt{\gamma}\gamma^{pq}\partial_q k_{(\hat{\Delta}+2)}\right) + 2k_{(\hat{\Delta}+2)} = 0. \tag{135}$$

We look for the function which satisfies the conditions (123) and (125). We emphasize that we do not impose the regularity on the function $k_{(\hat{\Delta}+2)}$ on S^2 itself as in the case of $k_{(\hat{\Delta})}$. To obtain the solution to Equation (135) which satisfies the conditions (123) and (125), we first consider the ϕ-dependence from the condition (123). Then, Equation (135) is given by

$$\sin\theta\partial_\theta(\sin\theta\partial_\theta\Theta_{1m}(\theta)) - m^2\Theta_{1m}(\theta) + 2\sin^2\theta\Theta_{1m}(\theta) = 0. \tag{136}$$

To solve Equation (136), we introduce the independent variable

$$z = \cos\theta, \quad dz = -\sin\theta d\theta. \tag{137}$$

In terms of the independent variable z, we obtain

$$\sin\theta\frac{d}{d\theta} = -(1-z^2)\frac{d}{dz}. \tag{138}$$

Then, Equation (136) is given by

$$(1-z^2)\frac{d^2}{dz^2}\Theta_{1m}(\theta) - 2z\frac{d}{dz}\Theta_{1m}(\theta) + \left(1(1+1) - \frac{m^2}{1-z^2}\right)\Theta_{1m}(\theta) = 0. \tag{139}$$

Suppose that we have obtained the solution to Equation (139) as

$$k_{(\hat{\Delta}+2)m} = \Theta_{1m}(\theta)e^{im\phi}. \tag{140}$$

Here, we introduce the ladder operator $\hat{L}_\pm$ [60] as

$$\hat{L}_\pm := -ie^{\pm i\phi}\left(\pm i\partial_\theta - \cot\theta\partial_\phi\right) \tag{141}$$

and examine the function defined by

$$\begin{aligned}\hat{L}_+ k_{(\hat{\Delta}+2,m)} &= -ie^{+i\phi}\left(+i\partial_\theta - \cot\theta\partial_\phi\right)\Theta_{1m}(\theta)e^{im\phi} \\ &= (\partial_\theta - m\cot\theta)\Theta_{1m}(\theta)e^{i(m+1)\phi}.\end{aligned} \tag{142}$$

Evidently, the function given by Equation (142) is the eigenfunction of the operator $-i\partial_\phi$ with the eigenvalue $m+1$ as follows

$$-i\partial_\phi \hat{L}_+ k_{(\hat{\Delta}+2,m)} = (m+1)\hat{L}_+ k_{(\hat{\Delta}+2,m)}. \tag{143}$$

Now, we consider the variable Φ_+ defined by

$$\begin{aligned} \Phi_+ &:= (\partial_\theta - m\cot\theta)\Theta_{1m} \\ &= -(1-z^2)^{-1/2}\left[(1-z^2)\frac{d}{dz}\Theta_{1,m} + mz\Theta_{1m}\right], \end{aligned} \tag{144}$$

and straightforward calculations using Equation (139) yields

$$(1-z^2)\frac{d^2}{dz^2}\Phi_+ - 2z\frac{d}{dz}\Phi_+ + \left(1(1+1) - \frac{(m+1)^2}{1-z^2}\right)\Phi_+ = 0. \tag{145}$$

This indicates

$$\Phi_+ = \Theta_{1,m+1}(\theta). \tag{146}$$

Therefore, we conclude that

$$\hat{L}_+ k_{(\hat{\Delta}+2,m)} = k_{(\hat{\Delta}+2,m+1)}. \tag{147}$$

On the other hand, we consider the operator $\hat{L}_-$ defined by

$$\begin{aligned} \hat{L}_- k_{(\hat{\Delta}+2,m)} &= -ie^{-i\phi}\left(-i\partial_\theta - \cot\theta\partial_\phi\right)\Theta_{1,m}e^{im\phi} \\ &= (-\partial_\theta - m\cot\theta)\Theta_{1,m}e^{i(m-1)\phi}. \end{aligned} \tag{148}$$

Evidently, the function given by Equation (148) is an eigenfunction of the operator $-i\partial_\phi$ with the eigenvalue $m-1$:

$$-i\partial_\phi \hat{L}_- k_{(\hat{\Delta}+2,m)} = (m-1)\hat{L}_- k_{(\hat{\Delta}+2,m)}. \tag{149}$$

Now, we consider

$$\Phi_- := (-\partial_\theta - m\cot\theta)\Theta_{1,m} \tag{150}$$

and straightforward calculations using Equation (139) yields

$$(1-z^2)\frac{d^2}{dz^2}\Phi_- - 2z\frac{d}{dz}\Phi_- + \left[1(1+1) - \frac{(m-1)^2}{1-z^2}\right]\Phi_- = 0. \tag{151}$$

This indicates

$$\Phi_- = \Theta_{1,m-1}(\theta). \tag{152}$$

Therefore, we conclude that

$$\hat{L}_- k_{(\hat{\Delta}+2)m} = k_{(\hat{\Delta}+2)m-1}. \tag{153}$$

From the above operator $\hat{L}_\pm$ and

$$\hat{L}_\pm k_{(\hat{\Delta}+2)m} = k_{(\hat{\Delta}+2)m\pm1}, \tag{154}$$

we may concentrate only on solving the $m = 0$ case. Corresponding $m = \pm 1$ modes with $l = 1$ can be derived from Equation (154). Since $k_{(\hat{\Delta}+2)m=0} = \Theta_{10}(\theta)$, the equation for $\Theta_{10}(\theta)$ is given by

$$(1-z^2)\frac{d^2}{dz^2}\Theta_{10}(\theta) - 2z\frac{d}{dz}\Theta_{10}(\theta) + 1(1+1)\Theta_{10}(\theta) = 0. \tag{155}$$

Here, we note that $\Theta_{10} = z \propto Y_{10}$ should be a solution to Equation (155). To obtain the other independent solution, we consider the solution in the form $\Theta_{10} = \Psi(z)z$. Substituting this into Equation (155), we can solve Equation (155) as

$$\Theta_{10} = z + \delta\left(\frac{1}{2}z\ln\frac{1+z}{1-z} - 1\right), \tag{156}$$

where we choose one of constant of integration as 1 and δ is another integration constant. Then, we obtain

$$k_{(\hat{\Delta}+2)m=0} = z + \delta\left(\frac{1}{2}z\ln\frac{1+z}{1-z} - 1\right) = P_1(z) + \delta Q_1(z), \tag{157}$$

where $P_1(z)$ is the Legendre polynomial and $Q_1(z)$ is the first order and the second kind Legendre function.

Since we have the explicit form (157) of $k_{(\hat{\Delta}+2)m=0}$ as

$$k_{(\hat{\Delta}+2)m} = \Theta_{1m}(\theta)e^{im\phi}, \tag{158}$$

we can derive the $m = \pm 1$ modes by applying the ladder operators $\hat{L}_\pm$ defined by Equation (141) as

$$\begin{aligned} k_{(\hat{\Delta}+2)m=\pm 1} &= \hat{L}_\pm k_{(\hat{\Delta}+2)m=0} \\ &= \left[\sqrt{1-z^2} + \delta\left(\frac{1}{2}\sqrt{1-z^2}\ln\frac{1+z}{1-z} + \frac{z}{\sqrt{1-z^2}}\right)\right]e^{\pm i\phi}. \end{aligned} \tag{159}$$

Equations (157) and (159) are summarized as

$$k_{(\hat{\Delta}+2,m=0)} = \cos\theta + \delta\left(\frac{1}{2}\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - 1\right), \quad \delta \in \mathbb{R}, \tag{160}$$

$$k_{(\hat{\Delta}+2,m=\pm 1)} = \left[\sin\theta + \delta\left(+\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} + \cot\theta\right)\right]e^{\pm i\phi}. \tag{161}$$

Here, we check the non-vanishing properties of $\hat{D}_p k_{(\hat{\Delta}+2)}$ and $\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)}$. For $m = 0$ modes, the vector $\hat{D}_p k_{(\hat{\Delta}+2,m=0)}$ is given by

$$\hat{D}_p k_{(\hat{\Delta}+2,m=0)} = -\left[1 + \frac{1}{2}\delta\left(\ln\frac{1+\cos\theta}{1-\cos\theta} + \frac{2\cos\theta}{\sin^2\theta}\right)\right]\sin\theta\theta_p. \tag{162}$$

Then $\hat{D}_p k_{(\hat{\Delta}+2,m=0)}$ and $\epsilon_{pq}\hat{D}^q k_{(\hat{\Delta}+2),m=0}$ span the basis of the tangent space on S^2.

Next, we consider the tensor $\hat{D}_q\hat{D}_p k_{(\hat{\Delta}+2,m=0)}$ as

$$\begin{aligned} &\hat{D}_q\hat{D}_p k_{(\hat{\Delta}+2,m=0)} \\ =\ & -\left[\cos\theta + \frac{1}{2}\delta\left(+\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - 4 - 2\cot^2\theta\right)\right]\theta_p\theta_q \\ & -\left[\cos\theta + \frac{1}{2}\delta\cos\theta\left(+\ln\frac{1+\cos\theta}{1-\cos\theta} + \frac{2\cos\theta}{\sin^2\theta}\right)\right]\phi_p\phi_q. \end{aligned} \tag{163}$$

This is not proportional to γ_{ab}. Therefore, we should have non-vanishing $K_{(m)pq}$ and $J_{(m)pq}$ defined by Equations (85). To confirm this, we evaluate the condition (103) as

$$\left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)}\right) - 2\left(k_{(\hat{\Delta}+2)}\right)^2 = \frac{2\delta}{\sin^4\theta}. \tag{164}$$

This indicates that we have non-vanishing $K_{(m)pq}$ and $J_{(m)pq}$ if $\delta \neq 0$. However, we should note that these tensor singular at $\theta = 0, \pi$.

For $m = \pm 1$ modes, the vector $\hat{D}_p k_{(\hat{\Delta}+2,m=\pm1)}$ is given by

$$\begin{aligned}\hat{D}_p k_{(\hat{\Delta}+2,m=\pm1)} &= \left[\cos\theta + \delta\left(+\frac{1}{2}\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - 1 - \frac{1}{\sin^2\theta}\right)\right]e^{\pm i\phi}\theta_p \\ &\quad +(\pm i)\left[1+\delta\left(+\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} + \frac{\cos\theta}{\sin^2\theta}\right)\right]e^{\pm i\phi}\phi_p.\end{aligned} \tag{165}$$

Finally, we evaluate the condition (103) as

$$\begin{aligned}\hat{D}_q\hat{D}_p k_{(\hat{\Delta}+2,m=\pm1)} &= \left[-\sin\theta + \delta\left(-\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - \frac{\cos\theta}{\sin\theta} + \frac{2\cos\theta}{\sin^3\theta}\right)\right]e^{\pm i\phi}\theta_p\theta_q \\ &\quad +\left[-\sin\theta + \delta\left(-\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - \frac{\cos\theta}{\sin\theta} - \frac{2\cos\theta}{\sin^3\theta}\right)\right]e^{\pm i\phi}\phi_p\phi_q \\ &\quad \mp\frac{4i\delta}{\sin^3\theta}e^{\pm i\phi}\theta_{(p}\phi_{q)}.\end{aligned} \tag{166}$$

This is not proportional to γ_{pq}. Therefore, we should have non-vanishing $K_{(m)pq}$ and $J_{(m)pq}$ defined by Equations (85). This is confirmed by verifying the condition (103) as

$$\left(\hat{D}_p\hat{D}_q k_{(\hat{\Delta}+2)}\right)\left(\hat{D}^p\hat{D}^q k_{(\hat{\Delta}+2)}\right) - 2\left(k_{(\hat{\Delta}+2)}\right)^2 = -\frac{8\delta^2}{\sin^4\theta}e^{\pm 2i\phi}. \tag{167}$$

Then, we have seen that if $\delta \neq 0$, the condition (103) is satisfied, though this norm is singular at $\theta = 0, \pi$. We also note that $K_{(m)pq}$ is orthogonal to $J_{(m)pq}$ as shown in Equations (95)–(97). Therefore, γ_{pq}, $K_{(m)pq}$, and $J_{(m)pq}$ span the basis of the second-rank tensor field on S^2.

3.4. Proposal of the Treatment of $l = 0, 1$-Mode Perturbations

As mentioned above, it is shown that the harmonic decomposition (37)–(39) have the one-to-one correspondence between the original metric perturbations $\{h_{AB}, h_{Ap}, h_{pq}\}$ and the mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ for any modes $l \geq 0$ through the employment of the scalar harmonic functions

$$S_\delta = \begin{cases} Y_{lm} & \text{for} \quad l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} \quad l = 1; \\ k_{(\hat{\Delta})} & \text{for} \quad l = 0, \end{cases} \tag{168}$$

where $k_{(\hat{\Delta})}$ is given by Equation (129), i.e.,

$$k_{(\hat{\Delta})} = 1 + \delta\ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}, \quad \delta \in \mathbb{R} \tag{169}$$

and $k_{(\hat{\Delta}+2)m}$ are given by Equations (157) and (159), i.e.,

$$k_{(\hat{\Delta}+2)m=0} = \cos\theta + \delta\left(\frac{1}{2}\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - 1\right), \tag{170}$$

$$k_{(\hat{\Delta}+2)m=\pm1} = \left[\sin\theta + \delta\left(\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} + \cot\theta\right)\right]e^{\pm i\phi}. \tag{171}$$

These mode functions $k_{(\hat{\Delta}+2)m}$ and $k_{(\hat{\Delta})}$ are parametrized by the single parameter δ. This choice satisfies the conditions (123)–(125) but singular at $\theta = 0, \pi$ if $\delta \neq 0$. When $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}$ and $k_{(\hat{\Delta}+2)m} \propto Y_{1m}$. In this decomposition, for each mode of any $l \geq 0$, the set of harmonic functions (41) is a linear-independent set in the both senses of the second-rank tensor field and the function on S^2.

Using the above harmonics functions S_δ in Equation (168), we propose the following strategy [3]:

Proposal 1. *We decompose the metric perturbation h_{ab} on the background spacetime with the metric (31)–(34) through Equations (37)–(39) with the harmonic function S_δ given by Equation (168). Then, Equations (37)–(39) become invertible including $l = 0, 1$ modes. After deriving the mode-by-mode field equations such as linearized Einstein equations by using the harmonic functions S_δ, we choose $\delta = 0$ as a regular boundary condition for solutions when we solve these field equations.*

Since the set of the mode functions (41) with $S = S_\delta$ have the linear-independence including $l = 0, 1$ modes, we can construct gauge-invariant variables and evaluate the field equations through the mode-by-mode analyses including $l = 0, 1$ modes through the choice of these mode functions.

4. Construction of Gauge-Invariant Variables

In this section, we construct gauge-invariant variables for perturbations on a spherically symmetric background with the metric (31) through Proposal 1. To construct gauge-invariant variables, we first discuss the gauge-transformation rule for the metric perturbation h_{ab}. We use the derivation of the gauge-transformation rules for the mode coefficient in the decomposition (37)–(39) with the harmonic function $S = S_\delta$ given by Equation (168). In this section, we use the relations of the covariant derivatives associated with the matrices g_{ab}, y_{ab}, and γ_{ab}, which are summarized in Appendix B. In Section 4.1, we derive the gauge-transformation rules for the mode coefficients of the metric perturbation in the decomposition (37)–(39) with the harmonic function $S = S_\delta$. In Section 4.2, we explicitly construct gauge-invariant variables for the metric perturbations through the mode-by-mode analyses. In Section 4.2.3, we summarize gauge-invariant and gauge-dependent variables in the four-dimensional form.

4.1. Gauge-Transformation Rules

Here, we consider the gauge-transformation rules for the linear-order metric perturbation h_{ab} following to Proposal 1. The gauge-transformation rule for linear-order metric perturbation is given by

$$\mathscr{Y}h_{ab} - \mathscr{X}h_{ab} = \pounds_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{172}$$

We rewrite this gauge-transformation rule in terms of 2 + 2 formulation. To do this, the generator of gauge-transformation rules is decomposed as

$$\xi_a = \xi_A (dx^A)_a + \xi_p (dx^p)_a. \tag{173}$$

Through the component-representations (36) and (173), the gauge-transformation rules (172) are given by

$$
\begin{aligned}
{}_{\mathscr{Y}}h_{AB} - {}_{\mathscr{X}}h_{AB} &= \nabla_A\xi_B + \nabla_B\xi_A = \bar{D}_A\xi_B + \bar{D}_B\xi_A, && (174)\\
{}_{\mathscr{Y}}h_{Ap} - {}_{\mathscr{X}}h_{Ap} &= \nabla_A\xi_p + \nabla_p\xi_A = \bar{D}_A\xi_p + \hat{D}_p\xi_A - \frac{2}{r}\bar{D}_A r\xi_p, && (175)\\
{}_{\mathscr{Y}}h_{pq} - {}_{\mathscr{X}}h_{pq} &= \nabla_p\xi_q + \nabla_q\xi_p = \hat{D}_p\xi_q + \hat{D}_q\xi_p + 2r\bar{D}^A r\gamma_{pq}\xi_A. && (176)
\end{aligned}
$$

Furthermore, through the mode-decomposition (37)–(39) and

$$
\begin{aligned}
\xi_A &=: \sum_{l,m}\zeta_A S_\delta, && (177)\\
\xi_p &=: r\sum_{l,m}\left[\zeta_{(e)}\hat{D}_p S_\delta + \zeta_{(o)}\epsilon_{pq}\hat{D}^q S_\delta\right] && (178)
\end{aligned}
$$

with the harmonic function S_δ, we can carry out the mode-by-mode analyses, since the set of the harmonic functions (41) has the linear-independence due to the choice $S = S_\delta$. From Equation (174), we obtain

$$
{}_{\mathscr{Y}}\tilde{h}_{AB} - {}_{\mathscr{X}}\tilde{h}_{AB} = 2\bar{D}_{(A}\zeta_{B)}. \tag{179}
$$

From Equation (175), we obtain

$$
\begin{aligned}
{}_{\mathscr{Y}}\tilde{h}_{(e1)A} - {}_{\mathscr{X}}\tilde{h}_{(e1)A} &= \frac{1}{r}\zeta_A + \bar{D}_A\zeta_{(e)} - \frac{1}{r}\bar{D}_A r\zeta_{(e)}, && (180)\\
{}_{\mathscr{Y}}\tilde{h}_{(o1)A} - {}_{\mathscr{X}}\tilde{h}_{(o1)A} &= \bar{D}_A\zeta_{(o)} - \frac{1}{r}\bar{D}_A r\zeta_{(o)}. && (181)
\end{aligned}
$$

Finally, the gauge-transformation rules (176) yield

$$
\begin{aligned}
{}_{\mathscr{Y}}\tilde{h}_{(e0)} - {}_{\mathscr{X}}\tilde{h}_{(e0)} &= \frac{4}{r}\left(-\frac{1}{2}l(l+1)\zeta_{(e)} + \bar{D}^A r\zeta_A\right), && (182)\\
{}_{\mathscr{Y}}\tilde{h}_{(e2)} - {}_{\mathscr{X}}\tilde{h}_{(e2)} &= \frac{2}{r}\zeta_{(e)}, && (183)\\
{}_{\mathscr{Y}}\tilde{h}_{(o2)} - {}_{\mathscr{X}}\tilde{h}_{(o2)} &= -\frac{1}{r}\zeta_{(o)}. && (184)
\end{aligned}
$$

We note that these gauge-transformation rules (179)–(184) are not only that for $l \geq 2$ modes but also $l = 0, 1$ modes.

When we use the usual spherical harmonics Y_{lm} as the scalar harmonics, i.e., $\delta = 0$ from the starting point, we only have Equations (179) and (182) with $l = 0$ for $l = 0$ mode perturbations and the other gauge-transformation rules (180), (181), (183) and (184) do not appear. In this case, it is difficult to construct gauge-invariant variables for $l = 0$-mode perturbations through the similar procedure to the $l \geq 2$-mode case. For this reason, we usually use the gauge-fixing procedure for $l = 0$ mode perturbations from the old paper by Zerilli [19]. Of course, the construction of gauge-invariant variables might be possible if we use the integral representations of the original metric perturbations. However, such gauge-invariant variables do not match to the statement of Conjecture 1. For this reason, we do not consider such integral representations here.

Furthermore, for $l = 1$ modes with $\delta = 0$ from the starting point, we do not have Equations (183) nor (184) but we have Equations (179)–(182) with non-vanishing $\zeta_{(e)}$ and $\zeta_{(o)}$. For $l = 1$ odd-mode perturbations, it is well-known that the variable defined by

$$
\Phi_{KIF} := \epsilon^{AB}\bar{D}_A\left(\frac{1}{r}\tilde{h}_{(o1)B}\right) = \frac{1}{r}\partial_t\tilde{h}_{(o1)r} - \partial_r\left(\frac{1}{r}\tilde{h}_{(o1)t}\right) \tag{185}
$$

is gauge-invariant under the gauge transformation rule (181) [61], where $\epsilon^{AB} = 2(\partial_t)^{[A}(\partial_r)^{B]}$ in the coordinate system (33). However, when we reconstruct the original metric pertur-

bations from this gauge-invariant variables for $l = 1$ odd-mode perturbation, we have to integrate these gauge-invariant variables and we have to carry out delicate arguments for the problem that the integration constants are gauge-degree of freedom or not. On the other hand, such arguments are not necessary for the gauge-invariant variables given by the statement of Conjecture 1. In this sense, the above gauge-invariant variables Φ_{KIF} for $l = 1$ odd-mode perturbations does not match to the statement of Conjecture 1.

Moreover, for $l = 1$ even-mode perturbations, it is difficult to eliminate $\zeta_{(e)}$ and ζ_A from the gauge-transformation of even-mode perturbations through the similar procedure to the $l \geq 2$-mode case as in the case of $l = 0$ modes. In the conventional approach, we use the gauge-fixing procedure for $l = 1$ mode perturbations from the old paper by Zerilli [19] due to this reason. Of course, the construction of gauge-invariant variables for $l = 1$ even-modes might be possible if we use the integral representations of the original metric perturbations. However, such gauge-invariant variables do not match the statement of Conjecture 1. For this reason, we do not consider such integral representation as in the case of $l = 0$ mode perturbation.

These situations for $l = 0, 1$ mode perturbations are the essential reason for our proposal of the introduction of the singular harmonics $S = S_{\delta \neq 0}$. As shown below, we can construct the gauge-invariant variables through the similar procedure to the $l \geq 2$-mode case if we accept the introduction of the singular harmonics $S = S_{\delta \neq 0}$ at the starting point and Proposal 1.

4.2. Gauge-Invariant and Gauge-Variant Variables

Inspecting gauge-transformation rules (179)–(184), we can define gauge-invariant variables.

4.2.1. Odd Modes

From gauge-transformation rules (181) and (184), we easily find that the following combination is gauge-invariant:

$$\tilde{h}_{(o1)A} - \bar{D}_A\left(-r\tilde{h}_{(o2)}\right) + \frac{1}{r}\bar{D}_A r\left(-r\tilde{h}_{(o2)}\right) = \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)} =: \tilde{F}_A. \tag{186}$$

We also note that the gauge-transformation rule (184) implies that

$$-r^2\mathscr{Y}\tilde{h}_{(o2)} + r^2\mathscr{X}\tilde{h}_{(o2)} \quad = \quad r\zeta_{(o)}. \tag{187}$$

4.2.2. Even Modes

Now, we note that the gauge-transformation rule (183) implies that

$$\frac{r^2}{2}\mathscr{Y}\tilde{h}_{(e2)} - \frac{r^2}{2}\mathscr{X}\tilde{h}_{(e2)} \quad = \quad r\zeta_{(e)}. \tag{188}$$

Inspecting gauge-transformation rules (180) and (183), we define the variable $\tilde{Y}_A$ as

$$\begin{aligned}\tilde{Y}_A \quad &:= \quad r\tilde{h}_{(e1)A} - r\bar{D}_A\left(\frac{r}{2}\tilde{h}_{(e2)}\right) + \bar{D}_A r\left(\frac{r}{2}\tilde{h}_{(e2)}\right) \\ &= \quad r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)}.\end{aligned} \tag{189}$$

We easily check that the gauge-transformation rules for the variable $\tilde{Y}_A$ is given by

$$\mathscr{Y}\tilde{Y}_A - \mathscr{X}\tilde{Y}_A = \zeta_A. \tag{190}$$

From the gauge-transformation rules (188) and (190), we easily define the gauge-invariant variables as follows. First, from the gauge-transformation rules (179) and (190), we find the combination

$$\tilde{F}_{AB} := \tilde{h}_{AB} - 2\bar{D}_{(A}\tilde{Y}_{B)} \tag{191}$$

is gauge-invariant. Second, from the gauge-transformation rules (182), (188) and (190), we can define the gauge-invariant variables F as follows

$$\begin{aligned} \tilde{F} &:= \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r + \frac{2}{r}\frac{r}{2}\tilde{h}_{(e2)}l(l+1) \\ &= \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r + \tilde{h}_{(e2)}l(l+1). \end{aligned} \tag{192}$$

4.2.3. Summary of Gauge-Invariant and Gauge-Dependent Variables

In summary, we have defined gauge-invariant variables as follows

$$\tilde{F}_A := \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)}, \tag{193}$$

$$\tilde{F} := \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r + \tilde{h}_{(e2)}l(l+1), \tag{194}$$

$$\tilde{F}_{AB} := \tilde{h}_{AB} - 2\bar{D}_{(A}\tilde{Y}_{B)}, \tag{195}$$

where we defined the variable $\tilde{Y}_A$ by

$$\tilde{Y}_A := r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)}. \tag{196}$$

The gauge-transformation rules for the variable $\tilde{Y}_A$ is given by

$$\mathscr{Y}\tilde{Y}_A - \mathscr{X}\tilde{Y}_A = \zeta_A. \tag{197}$$

We also note that the gauge-transformation rules (187) and (188), i.e.,

$$-r^2\mathscr{Y}\tilde{h}_{(o2)} + r^2\mathscr{X}\tilde{h}_{(o2)} = r\zeta_{(o)}. \tag{198}$$

$$\frac{r^2}{2}\mathscr{Y}\tilde{h}_{(e2)} - \frac{r^2}{2}\mathscr{X}\tilde{h}_{(e2)} = r\zeta_{(e)}. \tag{199}$$

Therefore, it is reasonable to define the variables $\tilde{Y}_{(o)}$ and $\tilde{Y}_{(e)}$ as follows

$$\tilde{Y}_{(o1)} := -r^2\tilde{h}_{(o2)}, \tag{200}$$

$$\tilde{Y}_{(e1)} := \frac{r^2}{2}\tilde{h}_{(e2)} \tag{201}$$

so that their gauge-transformation rules are given by

$$\mathscr{Y}\tilde{Y}_{(o1)} - \mathscr{X}\tilde{Y}_{(o1)} = r\zeta_{(o)}, \tag{202}$$

$$\mathscr{Y}\tilde{Y}_{(e1)} - \mathscr{X}\tilde{Y}_{(e1)} = r\zeta_{(e)}. \tag{203}$$

Furthermore, we define the variable

$$Y_a := \sum_{l,m}\tilde{Y}_A S_\delta (dx^A)_a + \sum_{l,m}\left(\tilde{Y}_{(e1)}\hat{D}_p S_\delta + \tilde{Y}_{(o1)}\epsilon_{pq}\hat{D}^q S_\delta\right)(dx^p)_a. \tag{204}$$

The gauge transformation rule for the variable Y_a is given by

$$\begin{aligned}
{}^{\mathscr{Y}}Y_a - {}^{\mathscr{X}}Y_a &= \sum_{l,m}\left({}^{\mathscr{Y}}\tilde{Y}_A - {}^{\mathscr{X}}\tilde{Y}_A\right)S_\delta (dx^A)_a \\
&\quad + \sum_{l,m}\Big(\Big({}^{\mathscr{Y}}\tilde{Y}_{(e)} - {}^{\mathscr{X}}\tilde{Y}_{(e)}\Big)\hat{D}_p S_\delta + \Big({}^{\mathscr{Y}}\tilde{Y}_{(o)} - {}^{\mathscr{X}}\tilde{Y}_{(o)}\Big)\epsilon_{pq}\hat{D}^q S_\delta\Big)(dx^p)_a \\
&= \sum_{l,m}\zeta_A S_\delta (dx^A)_a + \sum_{l,m}\Big(r\zeta_{(e)}\hat{D}_p S_\delta + r\zeta_{(o)}\epsilon_{pq}\hat{D}^q S_\delta\Big)(dx^p)_a \\
&= \xi_A (dx^A)_a + \xi_p (dx^p)_a \\
&= \xi_a, \qquad (205)
\end{aligned}$$

where we used Equations (177) and (178).

In terms of the gauge-invariant variables $\{\tilde{F}_A, \tilde{F}, \tilde{F}_{AB}\}$ defined by Equations (193)–(195) and gauge-dependent variables Y_a defined by Equation (204), we can express the original components $\{h_{AB}, h_{Ap}, h_{pq}\}$. First, we consider the component h_{AB} as follows

$$\begin{aligned}
h_{AB} &= \sum_{l,m}(\tilde{h}_{AB})S_\delta = \sum_{l,m}\Big(\tilde{F}_{AB} + 2\bar{D}_{(A}\tilde{Y}_{B)}\Big)S_\delta, \\
&= F_{AB} + 2\bar{D}_{(A}Y_{B)}, \qquad (206)
\end{aligned}$$

where we defined the gauge-invariant variable F_{AB} by

$$F_{AB} := \sum_{l,m}\tilde{F}_{AB}S_\delta. \qquad (207)$$

Next, we consider the component h_{Ap} as follows

$$\begin{aligned}
h_{Ap} &= r\sum_{l,m}\Big[\Big(\tilde{h}_{(e1)A}\Big)\hat{D}_p S_\delta + \Big(\tilde{h}_{(o1)A}\Big)\epsilon_{pq}\hat{D}^q S_\delta\Big] \\
&= rF_{Ap} + \hat{D}_p Y_A + \bar{D}_A Y_p - \frac{2}{r}\bar{D}_A r Y_p, \qquad (208)
\end{aligned}$$

where we defined

$$F_{Ap} := \sum_{l,m}\left[\tilde{F}_A\epsilon_{pq}\hat{D}^q S_\delta\right], \quad \hat{D}^p F_{Ap} = 0. \qquad (209)$$

Finally, we consider the component h_{pq} as follows

$$\begin{aligned}
h_{pq} &= r^2\sum_{l,m}\Big[\tilde{h}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \Big(\tilde{h}_{(e2)}\Big)\Big(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\Big)S_\delta \\
&\qquad + 2\Big(\tilde{h}_{(o2)}\Big)\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\Big] \\
&= \frac{1}{2}\gamma_{pq}r^2 F + 2r\gamma_{pq}\bar{D}^A r Y_A + \hat{D}_p Y_q + \hat{D}_q Y_p, \qquad (210)
\end{aligned}$$

where we have defined

$$F := \sum_{l,m}\tilde{F}S_\delta. \qquad (211)$$

Then, we have obtained

$$h_{AB} = F_{AB} + 2\bar{D}_{(A}Y_{B)}, \tag{212}$$

$$h_{Ap} = rF_{Ap} + \hat{D}_p Y_A + \bar{D}_A Y_p - \frac{2}{r}\bar{D}_A r Y_p, \tag{213}$$

$$h_{pq} = \frac{1}{2}\gamma_{pq} r^2 F + 2r\gamma_{pq}\bar{D}^A r Y_A + \hat{D}_p Y_q + \hat{D}_q Y_p. \tag{214}$$

Comparing with the gauge-transformation rules (174)–(176), the expressions (212)–(214) are summarized as

$$h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}, \tag{215}$$

where $\mathscr{F}_{ab}$ is the gauge-invariant part in the 2 + 2 formulation. The components of $\mathscr{F}_{ab}$ are given by

$$\mathscr{F}_{AB} = F_{AB} = \sum_{l,m} \tilde{F}_{AB} S_\delta, \tag{216}$$

$$\mathscr{F}_{Ap} = rF_{Ap} = r\sum_{l,m} \tilde{F}_A \epsilon_{pq}\hat{D}^q S_\delta, \quad \hat{D}^p \mathscr{F}_{Ap} = 0, \tag{217}$$

$$\mathscr{F}_{pq} = \frac{1}{2}\gamma_{pq} r^2 F = \frac{1}{2}\gamma_{pq} r^2 \sum_{l,m} \tilde{F} S_\delta. \tag{218}$$

Here, we note that the above arguments include not only $l \geq 2$ modes but also $l = 0, 1$ modes of metric perturbations. Equations (215)–(218) represent the complete proof of the Conjecture 1 for the perturbations on the spherically symmetric background spacetime and are valid even in the case of $\delta = 0$. Therefore, our general arguments on the gauge-invariant perturbation theory reviewed in Section 2 are applicable to perturbations on the Schwarzschild background spacetime without special treatment of $l = 0, 1$ modes. Thus, we have resolved the zero-mode problem in the perturbations on the Schwarzschild background spacetime.

We also note that we only used the forms (31) and (34) of the background metric and did not used the specific forms of the Schwarzschild metric (33). Therefore, our construction of the gauge-invariant and gauge-dependent part of the metric perturbation is also valid for the metric perturbations on any spherically symmetric spacetime. Thus, if we accept Proposal 1, we reach the following statement:

Theorem 1. *If the gauge-transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab} with spherically symmetry, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed into ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

5. Einstein Equations

Here, we consider the linearized Einstein Equation (19) on the spherically symmetric background spacetime with the metric (31). The gauge-invariant part of the linearized Einstein tensor ${}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}]$ is given by Equations (17) and (18). The components of the tensor fields $H_{abc}[\mathscr{F}]$, $H_{ab}{}^c[\mathscr{F}]$, and $H_a{}^{bc}[\mathscr{F}]$ in terms of the variables F_{AB}, F_{Ap} and F in Equations (216)–(218) are summarized in Appendix C. Through these formulae and the mode decomposition in Equations (216)–(218) with the harmonic functions S_δ defined by Equation (168), the components of the tensor ${}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}]$ are given by

$$
\begin{aligned}
{}^{(1)}\mathscr{G}_A{}^B = \frac{1}{2}\sum_{l,m}\Bigg[&\left(-\bar{D}_D\bar{D}^D + \frac{l(l+1)}{r^2} - \frac{2}{r}(\bar{D}_D r)\bar{D}^D\right)\tilde{F}_A{}^B + \left(\bar{D}_D\bar{D}_A + \frac{2}{r}(\bar{D}_D r)\bar{D}_A\right)\tilde{F}^{BD} \\
&+\left(\bar{D}^D\bar{D}^B + \frac{2}{r}(\bar{D}^D r)\bar{D}^B\right)\tilde{F}_{AD} - \bar{D}_A\bar{D}^B\tilde{F}_D{}^D \\
&-\left(\bar{D}_A\bar{D}^B + \frac{1}{r}(\bar{D}_A r)\bar{D}^B + \frac{1}{r}(\bar{D}^B r)\bar{D}_A\right)\tilde{F}\Bigg]S_\delta \\
+\frac{1}{2}\delta_A{}^B\sum_{l,m}\Bigg[&\left(\bar{D}_E\bar{D}^E - \frac{l(l+1)+1}{r^2} + \frac{2}{r}(\bar{D}_E r)\bar{D}^E + \frac{1}{r^2}(\bar{D}^E r)(\bar{D}_E r)\right)\tilde{F}_D{}^D \\
&-\left(\bar{D}_D\bar{D}_E + \frac{4}{r}(\bar{D}_D r)\bar{D}_E + \frac{2}{r^2}(\bar{D}_E r)(\bar{D}_D r)\right)\tilde{F}^{ED} \\
&+\left(\bar{D}_D\bar{D}^D - \frac{l(l+1)-2}{2r^2} + \frac{3}{r}(\bar{D}^D r)\bar{D}_D\right)\tilde{F}\Bigg]S_\delta,
\end{aligned}
\tag{219}
$$

$$
\begin{aligned}
{}^{(1)}\mathscr{G}_A{}^q = \frac{1}{2r^2}\sum_{l,m}\Bigg[&\left(-\bar{D}_A + \frac{1}{r}(\bar{D}_A r)\right)\tilde{F}_D{}^D + \bar{D}^D\tilde{F}_{AD} - \frac{1}{2}\bar{D}_A\tilde{F}\Bigg]\hat{D}^q S_\delta \\
+\frac{1}{2r}\sum_{l,m}\Bigg[&\left(-\bar{D}^D\bar{D}_D + \frac{l(l+1)}{r^2} - \frac{2}{r}(\bar{D}^D r)\bar{D}_D + \frac{3}{2r^2}\Big\{(\bar{D}^D r)(\bar{D}_D r) - 1\Big\}\right)\tilde{F}_A \\
&+\left(\bar{D}^D\bar{D}_A + \frac{3}{r}(\bar{D}^D r)\bar{D}_A - \frac{1}{r}(\bar{D}_A r)\bar{D}^D - \frac{2}{r^2}(\bar{D}_A r)(\bar{D}^D r)\right)\tilde{F}_D\Bigg]\epsilon^{qt}\hat{D}_t S_\delta,
\end{aligned}
\tag{220}
$$

$$
\begin{aligned}
{}^{(1)}\mathscr{G}_p{}^B = \frac{1}{2}\sum_{l,m}\Bigg[&\left(-\bar{D}^B + \frac{1}{r}(\bar{D}^B r)\right)\tilde{F}_D{}^D + \bar{D}_D\tilde{F}^{BD} - \frac{1}{2}\bar{D}^B\tilde{F}\Bigg]\hat{D}_p S_\delta \\
+\frac{r}{2}\sum_{l,m}\Bigg[&\left(-\bar{D}_D\bar{D}^D + \frac{l(l+1)}{r^2} - \frac{2}{r}(\bar{D}_D r)\bar{D}^D + \frac{3}{2r^2}\Big\{(\bar{D}^D r)(\bar{D}_D r) - 1\Big\}\right)\tilde{F}^B \\
&+\left(\bar{D}_D\bar{D}^B + \frac{3}{r}(\bar{D}_D r)\bar{D}^B - \frac{1}{r}(\bar{D}^B r)\bar{D}_D - \frac{2}{r^2}(\bar{D}^B r)(\bar{D}_D r)\right)\tilde{F}^D\Bigg]\epsilon_{pq}\hat{D}^q S_\delta,
\end{aligned}
\tag{221}
$$

$$
\begin{aligned}
{}^{(1)}\mathscr{G}_p{}^q = \sum_{l,m}\Bigg[&\frac{1}{2r^2}\bar{D}_D\left(r^2\bar{D}^D\tilde{F}\right) - \frac{1}{r^2}\bar{D}_D\left(r^2\bar{D}_E\tilde{F}^{ED}\right) \\
&+\left(\bar{D}_E\bar{D}^E + \frac{1}{r}(\bar{D}_E r)\bar{D}^E - \frac{l(l+1)}{2r^2}\right)\tilde{F}_D{}^D\Bigg]\frac{1}{2}\gamma_p{}^q S_\delta \\
+\frac{1}{2r^2}\sum_{l,m}\Bigg[&-\tilde{F}_D{}^D\left(\hat{D}_p\hat{D}^q - \frac{1}{2}\gamma_p{}^q\hat{D}^s\hat{D}_s\right)S_\delta \\
&-\bar{D}_D\left(r\tilde{F}^D\right)\left(\epsilon^{sq}\hat{D}_p\hat{D}_s + \epsilon_{sp}\hat{D}^q\hat{D}^s\right)S_\delta\Bigg],
\end{aligned}
\tag{222}
$$

where we used the fact that the background Ricci curvature vanishes and the background Einstein Equations (A84) and (A85) in Appendix B.

We also decompose the components of the linearized energy-momentum tensor ${}^{(1)}\mathscr{T}_a{}^b$ as follows

$$
{}^{(1)}\mathscr{T}_A{}^B = \sum_{l,m}\tilde{T}_A{}^B S_\delta, \tag{223}
$$

$$
{}^{(1)}\mathscr{T}_A{}^q = \frac{1}{r}\sum_{l,m}\Big\{\tilde{T}_{(e1)A}\hat{D}^q S_\delta + \tilde{T}_{(o1)A}\epsilon^{qr}\hat{D}_r S_\delta\Big\}, \tag{224}
$$

$$
{}^{(1)}\mathscr{T}_p{}^B = r\sum_{l,m}\Big\{\tilde{T}^B_{(e1)}\hat{D}_p S_\delta + \tilde{T}^B_{(o1)}\epsilon_{pr}\hat{D}^r S_\delta\Big\}, \tag{225}
$$

$$
\begin{aligned}
{}^{(1)}\mathscr{T}_p{}^q \;=\; \sum_{l,m}\Big\{ & \tilde{T}_{(e0)}\frac{1}{2}\gamma_p{}^q S_\delta + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}^q S_\delta - \frac{1}{2}\gamma_p{}^q\hat{D}_r\hat{D}^r S_\delta\right) \\
& + \tilde{T}_{(o2)}\left(\epsilon_{sp}\hat{D}^q\hat{D}^s S_\delta + \epsilon^{sq}\hat{D}_p\hat{D}_s S_\delta\right)\Big\}. \qquad (226)
\end{aligned}
$$

The linearized continuity Equation (25) for the energy-momentum tensor $\mathscr{T}_a{}^b$ is summarized as

$$\bar{D}^C\tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^D r)\tilde{T}_D{}^B - \frac{1}{r}l(l+1)\tilde{T}^B_{(e1)} - \frac{1}{r}(\bar{D}^B r)\tilde{T}_{(e0)} = 0, \qquad (227)$$

$$\bar{D}^C\tilde{T}_{(e1)C} + \frac{3}{r}(\bar{D}^C r)\tilde{T}_{(e1)C} + \frac{1}{2r}\tilde{T}_{(e0)} - \frac{1}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0, \qquad (228)$$

$$\bar{D}^C\tilde{T}_{(o1)C} + \frac{3}{r}(\bar{D}^D r)\tilde{T}_{(o1)D} + \frac{1}{r}(l-1)(l+2)\tilde{T}_{(o2)} = 0. \qquad (229)$$

Through the components (219)–(222) for the linearized Einstein tensor and the components (223)–(226) for the linearized energy-momentum tensor, we evaluate the linearized Einstein Equation (19). Due to the linear-independence of the set of harmonics (41), we can carry out the mode-by-mode analyses including $l = 0, 1$ modes. Since the odd-mode perturbations and the even-mode perturbations are decoupled with each other, we consider these perturbations separately.

5.1. Odd Mode Perturbation Equations

From the linearized Einstein Equation (19) through Equations (219)–(222) and Equations (223)–(226), the odd-mode part in the linearized Einstein equations are simplified as the constraint equation

$$\bar{D}_D(r\tilde{F}^D) = -16\pi r^2\tilde{T}_{(o2)}, \qquad (230)$$

and the evolution equation

$$
\begin{aligned}
& -\left[\bar{D}^D\bar{D}_D - \frac{l(l+1)}{r^2}\right](r\tilde{F}_A) - \frac{2}{r^2}(\bar{D}^D r)(\bar{D}_A r)(r\tilde{F}_D) + \frac{2}{r}(\bar{D}^D r)\bar{D}_A(r\tilde{F}_D) \\
= \;& 16\pi r\left(\tilde{T}_{(o1)A} + r\bar{D}_A\tilde{T}_{(o2)}\right). \qquad (231)
\end{aligned}
$$

Furthermore, we have the continuity Equation (229) for the odd-mode matter perturbation which is derived from the divergence of the first-order perturbation of the energy-momentum tensor. The explicit strategy to solve these odd-mode perturbations and $l = 0, 1$ mode solutions will be discussed in Section 6 in this paper.

5.2. Even Mode Perturbation Equations

Here, we consider the even-mode perturbations from Equations (219)–(222) and (223)–(226). The traceless even part of the (p, q)-component of the linearized Einstein Equation (19) is given by

$$\tilde{F}_D{}^D = -16\pi r^2\tilde{T}_{(e2)}. \qquad (232)$$

Using this equation, the even part of the (A, q)-component, equivalent to the (p, B)-component, of the linearized Einstein Equation (19) yields

$$\bar{D}^D\tilde{\mathbb{F}}_{AD} - \frac{1}{2}\bar{D}_A\tilde{F} = 16\pi\left[r\tilde{T}_{(e1)A} - \frac{1}{2}r^2\bar{D}_A\tilde{T}_{(e2)}\right] =: 16\pi S_{(ec)A} \qquad (233)$$

through the definition of the traceless part $\tilde{\mathbb{F}}_{AB}$ of the variable $\tilde{F}_{AB}$ defined by

$$\tilde{\mathbb{F}}_{AB} := \tilde{F}_{AB} - \frac{1}{2}y_{AB}\tilde{F}_C{}^C. \qquad (234)$$

Using Equations (232), (233), and the component (A84) of background Einstein equation in Appendix B, the trace part of (p,q)-component of the linearized Einstein Equation (19) is given by

$$\bar{D}^D\tilde{T}_{(e1)D}+\frac{3}{r}(\bar{D}^D r)\tilde{T}_{(e1)D}+\frac{1}{2r}\tilde{T}_{(e0)}-\frac{(l-1)(l+2)}{2r}\tilde{T}_{(e2)}=0. \tag{235}$$

This coincides with the component (228) of the continuity equation for the linearized energy-momentum tensor. Next, we consider the (A,B)-components of the linearized Einstein Equation (19).

Through Equations (232) and (233), the trace part of the (A,B)-component of the linearized Einstein Equation (19) is given by

$$\left(\bar{D}_D\bar{D}^D+\frac{2}{r}(\bar{D}^D r)\bar{D}_D-\frac{(l-1)(l+2)}{r^2}\right)\tilde{F}-\frac{4}{r^2}(\bar{D}_C r)(\bar{D}_D r)\tilde{\mathbb{F}}^{CD}=16\pi S_{(F)}, \tag{236}$$

$$S_{(F)}:=\tilde{T}_C{}^C+4(\bar{D}_D r)\tilde{T}^D_{(e1)}-2r(\bar{D}_D r)\bar{D}^D\tilde{T}_{(e2)}-(l(l+1)+2)\tilde{T}_{(e2)}. \tag{237}$$

On the other hand, the traceless part of the (A,B)-component of the linearized Einstein Equation (19) is given by

$$\begin{aligned}&\left[-\bar{D}_D\bar{D}^D-\frac{2}{r}(\bar{D}_D r)\bar{D}^D+\frac{4}{r}(\bar{D}^D\bar{D}_D r)+\frac{l(l+1)}{r^2}\right]\tilde{\mathbb{F}}_{AB}\\&\quad+\frac{4}{r}(\bar{D}^D r)\bar{D}_{(A}\tilde{\mathbb{F}}_{B)D}-\frac{2}{r}(\bar{D}_{(A}r)\bar{D}_{B)}\tilde{F}\\&=16\pi S_{(\mathbb{F})AB},\end{aligned} \tag{238}$$

$$\begin{aligned}S_{(\mathbb{F})AB}:=\;&T_{AB}-\frac{1}{2}y_{AB}T_C{}^C-2\left(\bar{D}_{(A}(r\tilde{T}_{(e1)B)})-\frac{1}{2}y_{AB}\bar{D}^D(r\tilde{T}_{(e1)D})\right)\\&+2\left((\bar{D}_{(A}r)\bar{D}_{B)}-\frac{1}{2}y_{AB}(\bar{D}^D r)\bar{D}_D\right)(r\tilde{T}_{(e2)})\\&+r\left(\bar{D}_A\bar{D}_B-\frac{1}{2}y_{AB}\bar{D}^D\bar{D}_D\right)(r\tilde{T}_{(e2)})\\&+2\left((\bar{D}_A r)(\bar{D}_B r)-\frac{1}{2}y_{AB}(\bar{D}^C r)(\bar{D}_C r)\right)\tilde{T}_{(e2)}\\&+2y_{AB}(\bar{D}^C r)\tilde{T}_{(e1)C}-ry_{AB}(\bar{D}^C r)\bar{D}_C\tilde{T}_{(e2)},\end{aligned} \tag{239}$$

where we used the background Einstein Equation (A85) in Appendix B.

Equations (232), (233), (236) and (238) are all independent equations of the linearized Einstein equation for even-mode perturbations. These equations are coupled equations for the variables $\tilde{F}_C{}^C$, F, and $\tilde{\mathbb{F}}_{AB}$ and the energy-momentum tensor for the matter field. When we solve these equations, we have to take into account of the continuity Equations (227) and (228) for the matter fields. We note that these equations are valid not only for $l\geq 2$ modes but also $l=0,1$ modes in our formulation.

The explicit strategy to solve these Einstein equations for even modes, and the explicit solution for $l=0,1$ mode perturbations are discussed in the Part II paper [43].

6. Component Treatment for the Odd-Mode Perturbations of the Einstein Equations

6.1. Strategy to Solve Odd-Mode Perturbations

Here, we consider the component treatment for the odd-mode perturbations based on the old paper by Regge and Wheeler [17], and Zerilli [18,19]. We introduce the component of $r\tilde{F}^D$ as

$$r\tilde{F}_D =: X_{(o)}(dt)_D + Y_{(o)}(dr)_D, \quad r\tilde{F}^D = -f^{-1}X_{(o)}(\partial_t)^D + fY_{(o)}(\partial_r)^D, \tag{240}$$

where the background metric is given by Equations (31)–(34). In terms of the components (240), Equation (230) is given by

$$-\partial_t X_{(o)} + ff'Y_{(o)} + f^2\partial_r Y_{(o)} = -16\pi r^2 f\tilde{T}_{(o2)}, \tag{241}$$

where $f' = \partial_r f$. The components of Equation (231) are summarized as follows

$$\begin{aligned}
&\frac{1}{f}\partial_t^2 X_{(o)} - f\partial_r^2 X_{(o)} - \frac{2(1-f)}{r^2}X_{(o)} + \frac{l(l+1)}{r^2}X_{(o)} - \frac{1-3f}{r}\partial_t Y_{(o)} \\
&\quad = 16\pi r\left(\tilde{T}_{(o1)t} + r\partial_t\tilde{T}_{(o2)}\right), \qquad (242)\\
&\partial_t^2 Y_{(o)} - f\partial_r(f\partial_r Y_{(o)}) + \frac{2(2f-1)f}{r}\partial_r Y_{(o)} + \frac{(l-1)(l+2)}{r^2}fY_{(o)} + \frac{(1-f)(5f-1)}{r^2}Y_{(o)} \\
&\quad = 16\pi r\left(f\tilde{T}_{(o1)r} + rf\partial_r\tilde{T}_{(o2)} + (1-f)\tilde{T}_{(o2)}\right). \qquad (243)
\end{aligned}$$

In addition to these equations, the odd-mode perturbation (229) of the divergence of the energy-momentum tensor is also relevant.

Here, we consider Equation (243). We define the dependent variable $Z_{(o)}$ by

$$Y_{(o)} =: \frac{r}{f}Z_{(o)} \tag{244}$$

and we have obtained the famous equation which is called the Regge–Wheeler equation

$$\partial_t^2 Z_{(o)} - f\partial_r(f\partial_r Z_{(o)}) + \frac{1}{r^2}f[l(l+1) - 3(1-f)]Z_{(o)} = 16\pi f\left[f\tilde{T}_{(o1)r} + r\partial_r\left(f\tilde{T}_{(o2)}\right)\right]. \tag{245}$$

We can solve Equation (245) with appropriate boundary conditions and obtain the variable $Y_{(o)}$ through Equation (244). For the $l \geq 2$ case, the analytic solutions to Equation (245) are constructed by the formulation proposed by Mano, Suzuki, and Takasugi [62–65] (MST formulation). However, this is a partial solution to the odd-mode Einstein equations. We cannot regard such solutions as the solution to the total Einstein equation for odd-mode perturbations, because we have other two equations of the Einstein Equation (242) and the constraint Equation (241). To obtain the solution to the total Einstein equations for odd-mode perturbations, we have to discuss Equations (241), (242) and (229), i.e.,

$$-\frac{1}{f}\partial_t\tilde{T}_{(o1)t} + f\partial_r\tilde{T}_{(o1)r} + f'\tilde{T}_{(o1)r} + \frac{3}{r}f\tilde{T}_{(o1)r} + \frac{1}{r}(l-1)(l+2)\tilde{T}_{(o2)} = 0. \tag{246}$$

in addition to Equation (245).

To obtain the solution to the total Einstein equations for odd-mode perturbations, it is convenient to introduce the Cunningham–Price–Moncrief variable $\Phi_{(o)}$ [22] by

$$\begin{aligned}
\Phi_{(o)} &:= 2r\left[r^2\partial_r\left(\frac{X_{(o)}}{r^2}\right) - \partial_t Y_{(o)}\right] \qquad (247)\\
&= 2r\partial_r X_{(o)} - 4X_{(o)} - 2r\partial_t Y_{(o)}. \qquad (248)
\end{aligned}$$

Here, we consider the time derivative of $\Phi_{(o)}$ and use Equations (241), (243), and the background Einstein Equation (A82) as

$$\begin{aligned}\partial_t\Phi_{(o)} &= 2\frac{(l-1)(l+2)}{r}fY_{(o)} - 32\pi r^2 f\tilde{T}_{(o1)r} \\ &= 2(l-1)(l+2)Z_{(o)} - 32\pi r^2 f\tilde{T}_{(o1)r}. \end{aligned} \tag{249}$$

The relation (249) indicates that the variable $Z_{(o)}$ is related to $\Phi_{(o)}$ for $l \neq 1$ modes, while the time derivative of $\Phi_{(o)}$ is just the matter degree of freedom $\tilde{T}_{(o1)r}$ for the $l = 1$ mode. This relation also gives the relation with the metric perturbation $Y_{(o)}$ as

$$(l-1)(l+2)Y_{(o)} = \frac{r}{2f}\partial_t\Phi_{(o)} + 16\pi r^3\tilde{T}_{(o1)r}. \tag{250}$$

On the other hand, using Equations (241) and (242), the r-derivative of $\Phi_{(o)}$ through Equation (248) is given by

$$\partial_r\Phi_{(o)} = -\frac{1}{r}\Phi_{(o)} + \frac{2}{rf}(l-1)(l+2)X_{(o)} - 32\pi\frac{r^2}{f}\tilde{T}_{(o1)t}. \tag{251}$$

Then, we obtain the relation

$$(l-1)(l+2)X_{(o)} = \frac{f}{2}\left(r\partial_r\Phi_{(o)} + \Phi_{(o)}\right) + 16\pi r^3\tilde{T}_{(o1)t}. \tag{252}$$

From Equations (249) and (251) and the constraint (241), we obtain

$$\begin{aligned}\partial_r\partial_t\Phi_{(o)} - \partial_t\partial_r\Phi_{(o)} &= \partial_r\left[+2\frac{(l-1)(l+2)}{r}fY_{(o)} - 32\pi r^2 f\tilde{T}_{(o1)r}\right] \\ &\quad -\partial_t\left[-\frac{1}{r}\Phi_{(o)} + \frac{1}{r}2(l-1)(l+2)\frac{1}{f}X_{(o)} - 32\pi r^2\frac{1}{f}\tilde{T}_{(o1)t}\right] \\ &= -32\pi r^2\left[-\frac{1}{f}\partial_t\tilde{T}_{(o1)t} + f'\tilde{T}_{(o1)r} + f\partial_r\tilde{T}_{(o1)r}\right. \\ &\quad \left.+\frac{3}{r}f\tilde{T}_{(o1)r} + \frac{1}{r}(l-1)(l+2)\tilde{T}_{(o2)}\right] \\ &= 0. \end{aligned} \tag{253}$$

The final equality comes from the odd-mode perturbation (246) of the divergence of the energy-momentum tensor. Thus, Equations (249) and (251) are integrable under the constraint (241) and the continuity Equation (246).

We emphasize that the relations (250) and (252) give the relations of the metric components ($X_{(o)}$,$Y_{(o)}$) and the master variable $\Phi_{(o)}$ only for $l \neq 1$ mode. In the case of the $l = 1$ mode, these equations give the constraint of the master variable $\Phi_{(o)}$ and the matter degree of freedom. Furthermore, in the derivation of the relation (252), we used Equation (241) and (242), which means that the relation (252) carries the information of Equation (242).

From Equation (249), we evaluate the second time-derivative of the master variable $\Phi_{(o)}$. On the other hand, from Equation (251) we also evaluate the second derivative of $\Phi_{(o)}$ with respect to the tortoise coordinate $f\partial_r$. Furthermore, using Equations (248) and (252), we obtain

$$\begin{aligned}&\partial_t^2\Phi_{(o)} - f\partial_r\left[f\partial_r\Phi_{(o)}\right] + \frac{1}{r^2}f[l(l+1) - 3(1-f)]\Phi_{(o)} \\ &= 32\pi rf\left[\partial_r(r\tilde{T}_{(o1)t}) - r\partial_t\tilde{T}_{(o1)r}\right]. \end{aligned} \tag{254}$$

This has the same form as Equation (245), but we have different source terms from Equation (245). For the $l \geq 2$ case, the analytic solution to Equation (254) is also constructed by the MST formulation [62–65]. In the vacuum case, Equation (249) with $l \neq 1$ implies that the component $Y_{(o)}$ of the metric perturbation corresponds to the time-derivative of the variable $\Phi_{(o)}$. This indicates that Equation (254) corresponds to the time-integration of Equation (245) in the vacuum case. However, there is no degree of freedom of the integration constant in Equation (254). Therefore, we may say that the initial conditions for Equation (254) are restricted more than that of Equation (245).

Here, we note that Equation (250) is derived from Equations (241) and (245). This means that the relation (250) does not includes the information (242). On the other hand, the relation (252) is derived from Equations (241) and (242). This means that the relation (252) does not includes the information of Equation (245). In other words, we may consider the relation (250) to be a result of Equation (245), while Equation (251) is a result of Equation (242). Therefore, we obtain the two Equations (250) and (252) from the three Equations (241), (245) and (242). On the other hand, we have derived Equation (254) from Equations (241), (245) and (242), which is independent of Equations (250) and (252). Thus, we may regard that all information of the set of three Equations (241), (245) and (242) is included in the set of three Equations (250), (252) and (254). In addition to these equations, we have to take into account of the continuity Equation (246) for the odd-mode perturbations of the matter field.

However, as emphasized above, these arguments are not valid for $l = 1$ mode. Therefore, we have to reconsider the derivation of equations in the case of $l = 1$ mode, separately. Here, we examine the $l = 1$ modes. In this case, Equation (249) is still valid, though this equation does not give the component $Y_{(o)}$ of the metric perturbations. In this case, the time-derivative of the variable $\Phi_{(o)}$ is given by

$$\partial_t \Phi_{(o)} = -32\pi r^2 f \tilde{T}_{(o1)r}, \tag{255}$$

which indicates that $\partial_t \Phi_{(o)}$ is determined by the matter degree of freedom. Similarly, Equation (251) is also valid even in the case of $l = 1$ mode, though this equation does not give the component $X_{(o)}$ of the metric perturbations. In this case, we obtain

$$f\partial_r \Phi_{(o)} = -\frac{1}{r} f \Phi_{(o)} - 32\pi r^2 \tilde{T}_{(o1)t}. \tag{256}$$

This equation indicates that the $\partial_r \Phi_{(o)}$ is also determined by the matter degree of freedom. From Equations (255) and (256), we can confirm that the variable $\Phi_{(o)}$ satisfies Equation (254) with $l = 1$. However, we do not have to solve Equation (254) with $l = 1$ in this case, because we can directly integrate Equations (255) and (256). Actually, the integrability condition $\partial_t \partial_r \Phi_{(o)} = \partial_r \partial_t \Phi_{(o)}$ of Equations (255) and (256) can be checked through the continuity Equation (246) with $l = 1$.

Since we obtain the variable $\Phi_{(o)}$ by the direct integration of Equations (255) and (256), we can obtain the relation between the components $X_{(o)}$ and $Y_{(o)}$ of the metric perturbations through the definition (248). In addition to the solution $\Phi_{(o)}$, if we have a solution to $Z_{(o)} = \frac{f}{r} Y_{(o)}$, independently, we obtain the components $X_{(o)}$ and $Y_{(o)}$ of the metric perturbations through the above relation between $X_{(o)}$ and $Y_{(o)}$. Note that $Z_{(o)} = \frac{f}{r} Y_{(o)}$ can be determined through the integration of Equation (245) with $l = 1$ with appropriate boundary conditions. In this case, the continuity Equation (246) for odd-mode matter perturbations is used as a consistency check of the solutions.

6.2. Odd-Mode Solutions

Since the construction of solutions for $l \geq 2$ mode is accomplished by the MST formulation [62–65], we discuss the $l = 0, 1$-mode solutions for odd-mode perturbations along Proposal 1 and the strategy discussed in Section 6.1.

6.2.1. $l = 0$ Odd Mode

We choose Equation (129) as the harmonic function $k_{(\hat{\Delta})}$ and used the set $\{\hat{D}_p k_{(\hat{\Delta})}, \epsilon_{pr}\hat{D}^r k_{(\hat{\Delta})}, \hat{D}_p\hat{D}_q k_{(\hat{\Delta})}, 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta})}\}$ as the basis of the vector and tensors on S^2. The bases of the odd-mode perturbations are $\epsilon_{pr}\hat{D}^r k_{(\hat{\Delta})}$ and $2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta})}$. Following Proposal 1, we choose $\delta = 0$ as the regularity of solutions when we solve the linearized Einstein equations. As shown in Equations (131) and (133), $\epsilon_{pr}\hat{D}^r k_{(\hat{\Delta})} = 0 = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta})}$. Then, we conclude that there is no non-trivial solution for odd-mode perturbations with $l = 0$.

6.2.2. $l = 1$ Odd-Mode Vacuum Solution

Following the strategy to solve the $l = 1$ odd-mode perturbation given in Section 6.1, we consider the Equations (245), (246) with $l = 1$, (248), (255) and (256). To derive the non-vacuum solution to the linearized Einstein equations for $l = 1$ odd-mode perturbations, it is instructive to consider the vacuum case in which $\tilde{T}_{(o1)t} = \tilde{T}_{(o1)r} = \tilde{T}_{(o2)} = 0$. From Equations (255) and (256), we obtain the solution to these equations as [4]

$$\Phi_{(o)} = \frac{\alpha}{r}, \tag{257}$$

where α is constant of integration.

On the other hand, $Y_{(o)}$ is obtained as the solution to the $l = 1$ version of the Regge–Wheeler Equation (245) without source terms through Equation (244). Here, we consider the case $Y_{(o)} = 0$, at first. The derivations of solutions under the assumption $Y_{(o)} = 0$ is an instructive lesson for the derivation of the general solutions of the $l = 1$ odd-mode perturbations. Through the definition (247) of the variable $\Phi_{(o)}$ and Equation (257), we obtain

$$\frac{\alpha}{r} = 2r\left[r^2\partial_r\left(\frac{X_{(o)}}{r^2}\right)\right]. \tag{258}$$

The solution to Equation (258) together with the assumption $Y_{(o)} = 0$ is a special solution to the linearized Einstein equations for $l = 1$ odd-mode perturbations as follows:

$$X_{(o)} = -\frac{\alpha}{6r} + \beta_1 r^2, \quad Y_{(o)} = 0, \tag{259}$$

where β_1 is constant [5]. From Equations (217) and (240), we can derive the gauge-invariant metric perturbation $\mathscr{F}_{Ap}$ which corresponds to the solution (259). In the $l = 1$ modes, there are $m = 0, \pm 1$ modes. In this paper, we only consider the $m = 0$-mode perturbation, since the generalization to $m = \pm 1$ modes is straightforward. If we choose $\delta = 0$ in the mode function (162), we obtain

$$\mathscr{F}_{Ap} = rF_{Ap} = r\tilde{F}_A\epsilon_{pq}\hat{D}^q k_{(\hat{\Delta}+2,m=0)}, \quad \epsilon_{pq}\hat{D}^q k_{(\hat{\Delta}+2,m=0)} = \sin^2\theta(d\phi)_p. \tag{260}$$

Then, we have

$$\begin{aligned} 2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} &= 2rF_{(Ap)}(dx^A)_{(a}(dx^p)_{b)} \\ &= 2X_{(o)}\sin^2\theta(dt)_{(a}(d\phi)_{b)} + 2Y_{(o)}\sin^2\theta(dr)_{(a}(d\phi)_{b)} \quad (261) \\ &= \left(-\frac{\alpha}{3r} + 2\beta_1 r^2\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)}. \quad (262) \end{aligned}$$

Here, the term $\beta_1 r^2$ is diverged as $r \to \infty$. At this moment, we choose the arbitrary function $\beta_1 = 0$ to derive a special solution. Then, we have obtained

$$2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = -\frac{\alpha}{3r}\sin^2\theta(dt)_{(a}(d\phi)_{b)}. \tag{263}$$

Equation (263) is the linearized Kerr solution. Actually, the Kerr solution with the Kerr parameter a is expressed as [66,67]

$$\begin{aligned} g_{ab} &= -\left[1-\frac{2Mr}{\Sigma}\right](dt)_a(dt)_b - \frac{2aMr\sin^2\theta}{\Sigma}(dt)_{(a}(d\phi)_{b)} + \frac{\Sigma}{\Delta}(dr)_a(dr)_b \\ &\quad +\Sigma(d\theta)_a(d\theta)_b + \left[r^2+a^2+\frac{2Mr}{\Sigma}a^2\sin^2\theta\right]\sin^2\theta(d\phi)_a(d\phi)_b, \end{aligned} \tag{264}$$

where

$$\Sigma := r^2 + a^2\cos^2\theta, \quad \Delta := r^2 + a^2 - 2Mr. \tag{265}$$

In the metric (264), we replace $a \to \epsilon a$, where ϵ is the parameter for the perturbative expansion. Then, when the Kerr metric (264) is expressed as

$$g_{ab} = y_{ab} + r^2\gamma_{ab} + \epsilon\left(-\frac{2aM}{r}\sin^2\theta(dt)_{(a}(d\phi)_{b)}\right) + O(\epsilon^2). \tag{266}$$

Comparing Equations (263) and (266), the constant of integration α in Equation (263) is identified as the angular momentum perturbation in Kerr solution by choosing

$$\frac{\alpha}{3} = 2aM =: 2a_{10}M. \tag{267}$$

Thus, we have seen that the solution (262) is given using the Kerr parameter a_{10} as

$$2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = 2\left(-\frac{a_{10}M}{r} + \beta_1 r^2\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)}. \tag{268}$$

Next, we consider the physical meaning of the constant β_1 in the solution (262). If we consider the frame with the rigid rotation

$$t = t', \quad \phi = \varphi + \epsilon\omega t'. \tag{269}$$

In terms of (t', φ), the background metric (31) with Equations (33) and (34) is given by

$$\begin{aligned} g_{ab} &= -f(dt')_a(dt')_b + f^{-1}(dr)_a(dr)_b + r^2\left[(d\theta)_a(d\theta)_b + \sin^2\theta(d\varphi)_a(d\varphi)_b\right] \\ &\quad +2\epsilon\omega r^2\sin^2\theta(dt')_{(a}(d\varphi)_{b)} + O(\epsilon^2). \end{aligned} \tag{270}$$

Comparing Equations (270) and (268), we can see that the arbitrary function β_1 corresponds to

$$\beta_1 = \omega. \tag{271}$$

Thus, we may interpret the integration constant β_1 as non-inertia term due to the rigidly rotating frame with the angular velocity ω.

Finally, we consider the general solution for $l = 1$ odd-mode perturbations which includes the case $Y_{(o)} \neq 0$ through Equations (244) and (245). Here, we consider the situation $Y_{(o)} \neq 0$ and introduce the variable $W_{(o)}$ by the equation

$$Y_o =: r^2 \partial_r W_{(o)}, \quad Z_{(o)} = \frac{f}{r} Y_{(o)} = r f \partial_r W_{(o)}. \tag{272}$$

Through the solution (257) with Equation (267) and the definition (247) of the variable $\Phi_{(o)}$, we obtain the equation

$$\frac{6a_{10}M}{r} = 2r\left[r^2\partial_r\left(\frac{X_{(o)}}{r^2}\right) - r^2\partial_t\partial_r W_{(o)}\right]. \tag{273}$$

Integrating this equation, we obtain

$$X_{(o)} = -\frac{a_{10}M}{r} + \beta_1 r^2 + r^2\partial_t W_{(o)}. \tag{274}$$

Through Equations (217) and (240), we obtain

$$\begin{aligned} 2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} &= 2\left(-\frac{a_{10}M}{r} + r^2\beta_1 + r^2\partial_t W_{(o)}\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)} \\ &\quad +2r^2\partial_r W_{(o)}\sin^2\theta(dr)_{(a}(d\phi)_{b)}. \end{aligned} \tag{275}$$

Note again that the variable $Z_{(o)} = rf\partial_r W_{(o)}$ satisfy the Regge–Wheeler Equation (245) with $l = 1$.

The above interpretation of the arbitrary function β_1 as the inertia force on the rigidly rotation frame is instructive to consider the interpretation of the odd-mode vacuum solution (275). To see this, we consider the component expression of $\pounds_V g_{ab}$, where V^a is constructed from gauge-invariant variables, which is discussed in Section 2. To obtain the components of $\pounds_V g_{ab}$, the explicit components of the Christoffel symbol $\Gamma_{ab}{}^c$ for the background metric (31) with Equations (33) and (34) are convenient, which are summarized in Equation (A88) in Appendix B. Here, we assume that $V_a = V_\phi(d\phi)_a$, then the non-vanishing components of $\pounds_V g_{ab}$ are given by

$$\pounds_V g_{t\phi} = \partial_t V_\phi, \quad \pounds_V g_{r\phi} = \partial_r V_\phi - \frac{2}{r}V_\phi, \quad \pounds_V g_{\theta\phi} = \partial_\theta V_\phi - 2\cot\theta V_\phi. \tag{276}$$

Comparing Equations (275) and (276), we obtain

$$V_a = \Big(\beta_1 t + \beta_0 + W_{(o)}(t,r)\Big)r^2\sin^2\theta(d\phi)_a, \tag{277}$$

$$\begin{aligned} \pounds_V g_{ab} &= \partial_t\Big(\beta_1 t + \beta_0 + W_{(o)}(t,r)\Big)r^2\sin^2\theta 2(dt)_{(a}(d\phi)_{b)} \\ &\quad +\Big(\partial_r W_{(o)}(t,r)\Big)r^2\sin^2\theta 2(dr)_{(a}(d\phi)_{b)}, \end{aligned} \tag{278}$$

where β_0 is constant. This coincides with the perturbation (275) with the condition of the vanishing Kerr parameter $a_{10} = 0$. Then, we have

$$2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = -\frac{2a_{10}M}{r}\sin^2\theta(dt)_{(a}(d\phi)_{b)} + \pounds_V g_{ab}, \tag{279}$$

$$V_a = \Big(\beta_1 t + \beta_0 + W_{(o)}(t,r)\Big)r^2\sin^2\theta(d\phi)_a. \tag{280}$$

Here, we note that the vector field V_a and $\pounds_V g_{ab}$ are gauge-invariant. The interpretation of this term $\pounds_V g_{ab}$, which is gauge invariant in the sense of the second kind, is extensively discussed in Section 7.

6.3. Odd Mode Non-Vacuum $l = 1$ Solution

Inspecting the derivation of the vacuum solution for $l = 1$ modes in Section 6.2.2, we consider the non-vacuum solution for $l = 1$ modes. For $l = 1$ modes, the linearized Einstein equations for the master variable $\Phi_{(o)}$ defined by Equation (247) are given by Equations (255) and (256). As mentioned in Section 6.1, the integrability condition for these equations is guaranteed by the continuity Equation (246) with $l = 1$. Inspecting Equations (257) and (267), we consider the solution in the form

$$\Phi_{(o)} = \frac{6Ma_1(t,r)}{r}. \tag{281}$$

Substituting Equation (281) into Equations (255) and (256), we obtain

$$\partial_t a_1(t,r) = -\frac{16\pi}{3M} r^3 f \tilde{T}_{(o1)r}, \quad \partial_r a_1(t,r) = -\frac{16\pi}{3M} r^3 \frac{1}{f} \tilde{T}_{(o1)t}. \tag{282}$$

The integrability of Equation (282) is equivalent to the integrability of Equations (255) and (256) which is guaranteed by the continuity Equation (246) with $l = 1$. Then, we may integrate Equation (282) as

$$\begin{aligned} a_1(t,r) &= -\frac{16\pi}{3M} r^3 f \int dt \tilde{T}_{(o1)r} + a_{10} \\ &= -\frac{16\pi}{3M} \int dr r^3 \frac{1}{f} \tilde{T}_{(o1)t} + a_{10}, \end{aligned} \tag{283}$$

where a_{10} is the constant which corresponds to the Kerr parameter a in Equation (264) as shown in the vacuum case.

Similar arguments to those in Section 6.2.2, which lead the results (279) and (280), also leads

$$2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = 6Mr^2 \left[\int dr \frac{a_1(t,r)}{r^4} \right] \sin^2\theta (dt)_{(a}(d\phi)_{b)} + \pounds_V g_{ab}, \tag{284}$$

$$V_a = \Big(\beta_1 t + \beta_0 + W_{(o)}(t,r) \Big) r^2 \sin^2\theta (d\phi)_a. \tag{285}$$

Here, we note that the vector field V_a and $\pounds_V g_{ab}$ are gauge-invariant in the sense of the second kind. The term $\pounds_V g_{ab}$ may always appear due to the symmetry of the linearized Einstein equation as pointed out through Equation (28). However, it is also true that we can eliminate the term $\pounds_V g_{ab}$ by an infinitesimal coordinate transformation (2) at any time. The interpretation of the term $\pounds_V g_{ab}$ will be discussed in Section 7.

7. Summary and Discussions

In summary, after reviewing our general framework of the gauge-invariant perturbation theory, we discussed a resolution of the "zero-mode problem" in perturbations on the Schwarzschild background spacetime. The "zero-mode problem" in the context of our general framework of the gauge-invariant perturbation theory corresponds to the $l = 0, 1$ mode problem in perturbations of the Schwarzschild background spacetime. In the review of our general framework of the gauge-invariant perturbation theory, we emphasize the importance of the distinction of the first- and the second-kind gauge in general relativity. It should be also emphasized that our general framework for the gauge-invariant perturbation theory is a formulation to exclude the second-kind gauge degree of freedom, but we do not exclude first-kind gauge degrees of freedom.

As emphasized in Section 2, Conjecture 1 is the non-trivial and an important premise of our general framework of gauge-invariant perturbation theories. If Conjecture 1 is

actually true, we can develop gauge-invariant perturbation theory on general background spacetime and we can also extend this gauge-invariant perturbation theory to higher-order perturbation theory. For this reason, the gauge-invariant treatment of the $l = 0, 1$ modes in perturbations of the Schwarzschild background spacetime is important not only for the development of the linear perturbations but also for the development of the higher-order perturbation theory on the Schwarzschild background spacetime.

To find the gauge-invariant treatments of the $l = 0, 1$ mode perturbations on the Schwarzschild background spacetime, we first reviewed 2 + 2 formulation in which the decomposition Formulas (37)–(39) with the spherical harmonic functions Y_{lm} as the scalar harmonic function S and explained why $l = 0, 1$ modes should be separately treated in conventional perturbation theory on the Schwarzschild background spacetime. The special treatment in the conventional formulation caused by the loss of the linear independence of the set (41) of the tensor harmonic functions on S^2, i.e., vector and/or tensor harmonic functions vanishes in $l = 0, 1$ modes and does not play a role of the bases of tangent space on S^2.

To recover this situation, instead of the spherical harmonics Y_{00} and Y_{1m} for $l = 0, 1$ modes, we introduce the mode functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$, which belong to the kernel of the derivative operator $\hat{\Delta}$ and $\hat{\Delta} + 2$, respectively. We also derive the sufficient condition for which the decomposition Formulas (37)–(39) with the harmonic function $S = S_\delta$ defined by Equation (63) is invertible not only for $l \geq 2$ modes but also $l = 0, 1$ modes. As a result, we showed that the mode functions (169)–(171) with the parameter δ for $l = 0, 1$ modes satisfy this sufficient condition. These mode functions realize the conventional spherical harmonic functions Y_{00} and Y_{1m} when $\delta = 0$. However, in this case, the set of harmonic functions (41) loses the linear independence as the bases of the tangent space on S^2 as the conventional case, nevertheless the set $\{Y_{lm}\}$ of the spherical harmonics is a complete bases set of the L^2-space of scalar functions on S^2. On the other hand, when $\delta \neq 0$, the set of the mode functions (41) has the linear-independence as the bases of the tangent space on S^2. However, the mode functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$ with $\delta \neq 0$ are singular functions.

Due to this situation, we proposed Proposal 1 as a strategy to define the gauge-invariant variables for $l = 0, 1$ modes and to derive and solve the linearized Einstein equation. Following Proposal 1, we can construct gauge-invariant and gauge-dependent variables for linear metric perturbation through the similar manner to the case of the $l \geq 2$ modes. This construction is a proof of Conjecture 1 for the perturbations on the spherically symmetric background spacetime. Then, we reach to the statement Theorem 1. Owing to Theorem 1, we can develop gauge-invariant perturbation theory on spherically symmetric background spacetimes including $l = 0, 1$ modes. Furthermore, Theorem 1 yields that we can develop higher-order gauge-invariant perturbation theory on any spherically symmetric background spacetimes, although this development is beyond the current scope of this paper. A brief discussion of this development to higher-order perturbations was already given in ref. [42].

In addition to the discussion on the extension to the higher-order perturbation theory, it is also true that we are proposing a different procedure from the conventional one as Proposal 1. The difference is in the timing of the imposition of the boundary conditions on the functions on S^2 to solve the Einstein equations. In conventional treatments, we restrict the function on S^2 to the L^2-space through the mode decomposition using the spherical harmonics Y_{lm} from the starting point. In Proposal 1 in this paper, we do not impose the regular boundary condition on the functions S^2 at the starting point, but we impose the regular boundary condition $\delta = 0$ after the construction of the gauge-invariant variables and the derivation of the mode-by-mode Einstein equations. Physically, this different

timing of the imposition of the boundary condition should not affect the physical properties of the solution to the Einstein equations. Therefore, we have to confirm that the solutions to the Einstein equation derived by Proposal 1 are physically reasonable. To check this, we derived the linearized Einstein equations on the Schwarzschild background spacetime following Proposal 1. We consider the mode decomposition of the general expression of the linearized energy-momentum tensor as the source term of the linearized Einstein equations. To solve the derived linearized Einstein equations, the linearized perturbations of the continuity equation of the energy-momentum tensor should be taken into account. The metric perturbations on the Schwarzschild spacetime are classified into the odd-mode and the even-mode perturbations. In this Part I paper, we concentrate only on the odd-mode perturbations and derive the $l = 0, 1$-mode solutions following Proposal 1.

For odd-mode perturbations, we examined the strategy to solve the linearized Einstein equations for any l modes following the Proposal 1, though we take care of the structure of equations for $l = 1$ mode perturbations. As is well-known, to solve the odd-mode perturbations, Einstein equations for the $l \geq 2$ odd-mode perturbations are reduced to the Regge–Wheeler equation. Furthermore, the solutions to the Regge–Wheeler equation for $l \geq 2$ modes are constructed through the MST formulation [62–65]. Therefore, we concentrated on the $l = 0, 1$ mode perturbations.

Following Proposal 1, for $l = 0$ odd-mode perturbations, we reached the conclusion that there is no non-trivial solution to the linearized Einstein equation as expected. Then, we carefully examined the solutions to the Einstein equations for $l = 1$ odd-mode perturbations. We first consider the vacuum solution to the linearized Einstein equation in which the linear perturbation of the energy-momentum tensor vanishes. Then, we obtain the linearized Kerr parameter perturbation with the term given in the form of the Lie derivative of the background metric g_{ab}. Through the variation of constant, we derived the general solutions to non-vacuum linearized Einstein equations for the $l = 1$ odd-mode perturbations. Since we use the constant Kerr parameter in the variation of constant, we can expect that the obtained general solution describes the spin-up or the spin-down of the black hole due to the effect of the linearized energy-momentum tensor.

In addition to the Kerr parameter perturbations, we obtain the term which has the form of the Lie derivative of the background metric g_{ab} in our derived solution. The appearance of such term is natural consequence due to the symmetry of the linearized Einstein equations as discussed in Section 2.3. Actually, gauge-invariant variables defined through Conjecture 1 are not unique as pointed out by Equation (26) in Section 2. It is easy to show that new gauge-invariant variable $\mathscr{H}_{ab}$ defined by Equation (26) is also a solution to the linearized Einstein Equation (19) through Equations (16) and the background Einstein Equation $G_a{}^b = 8\pi T_a{}^b$ if the original gauge-invariant variable $\mathscr{F}_{ab}$ in Equation (26) is a solution to the linearized Einstein Equation (19). This is a diffeomorphism symmetry of the linearized Einstein equations.

The appearance of the term which has the form of the Lie derivative of the background metric g_{ab} in the derived solution is a natural consequence in the sense of the above diffeomorphism symmetry of the linearized Einstein equation. In the case where the conventional expansion through the spherical harmonics Y_{lm} at the starting point and the gauge-fixing method are used, the appearance of this type of solutions is well-known as the residual gauge degree of freedom. It might be able to regard that the term of the Lie derivative of the background metric g_{ab} in Equations (279) and (284) corresponds to these "residual gauge" solutions. On the other hand, we are using the gauge-invariant perturbation theory in which the gauge degree of freedom of the second kind is completely excluded. Therefore, the term which has the form of the Lie derivative is "not" the gauge degree of freedom of the second kind. On the other hand, in our gauge-invariant perturbation theory,

we do not exclude the gauge degree of freedom of the first kind as carefully explained in Sections 2.1 and 2.2. The term of the Lie derivative of the background metric g_{ab} in Equations (279) and (284) appears even if we completely excluded the gauge degree of freedom of the second kind. Therefore, we should regard that the term of the Lie derivative of the background metric g_{ab} in Equations (279) and (284) as the gauge degree of freedom of the first kind which is represented in Equation (6). Actually, we can interpret the term of the Lie derivative of the background metric g_{ab} can be eliminate by the infinitesimal coordinate transformation (2) on the background spacetime at any time. As an example, in Section 6.2.2, we explained that the constant β_1 in the solution (268) can be regarded as the degree of freedom of the infinitesimal coordinate transformation by Equation (269).

Now, we confirm the geometrical meaning of the gauge degree of freedom of the first kind in the context of the perturbation theory through Figure 4. Here, we consider the n-dimensional physical manifolds $\mathscr{M}_\epsilon$ and the background manifold $\mathscr{M}$. As depicted in Figure 4, we show that we may introduce the coordinate transformation on the physical spacetime $\mathscr{M}_\epsilon$, even if we completely fix the second-kind gauge as $\mathscr{X}_\epsilon$. Actually, we may introduce the diffeomorphism ψ_α, i.e., a coordinate system on $O_\alpha \subset \mathscr{M}_\epsilon$, from the open set O_α to an open set on $\mathbb{R}^n$ and the diffeomorphism ψ_β, i.e., a coordinate system on $O_\beta \subset \mathscr{M}_\epsilon$, from the open set O_β to an open set on the other $\mathbb{R}^n$. If $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the coordinate transformation $\psi_\beta \circ \psi_\alpha^{-1}$ which transforms the coordinate system (O_α, ψ_α) to (O_β, ψ_β). This is the first-kind gauge on $\mathscr{M}_\epsilon$ as shown in Figure 1. If we choose the gauge-choice of the second-kind by $\mathscr{X}_\epsilon$ as depicted in Figure 4, this gauge-choice induce the diffeomorphisms $\mathscr{X}_\epsilon^{-1} : O_\alpha \to \mathscr{X}_\epsilon^{-1} O_\alpha \subset \mathscr{M}$ and $\mathscr{X}_\epsilon^{-1} : O_\beta \to \mathscr{X}_\epsilon^{-1} O_\beta \subset \mathscr{M}$. Then, the coordinate systems (O_α, ψ_α) and (O_β, ψ_β) on $\mathscr{M}_\epsilon$ induce the coordinate systems $\{\mathscr{X}_\epsilon^{-1} O_\alpha, \psi_\alpha \circ \mathscr{X}_\epsilon\}$ and $\{\mathscr{X}_\epsilon^{-1} O_\beta, \psi_\beta \circ \mathscr{X}_\epsilon\}$ on $\mathscr{M}$. Actually, $\psi_\alpha \circ \mathscr{X}_\epsilon$ is a diffeomorphism which maps from $\mathscr{X}_\epsilon^{-1} O_\alpha \subset \mathscr{M}$ to $\mathbb{R}^n$ and $\psi_\beta \circ \mathscr{X}_\epsilon$ is a diffeomorphism which maps from $\mathscr{X}_\epsilon^{-1} O_\beta \subset \mathscr{M}$ to $\mathbb{R}^n$. Furthermore, the coordinate transformation is given by $(\psi_\beta \circ \mathscr{X}_\epsilon) \circ (\psi_\alpha \circ \mathscr{X}_\epsilon)^{-1} = \psi_\beta \circ \mathscr{X}_\epsilon \circ \mathscr{X}_\epsilon^{-1} \circ \psi_\alpha^{-1} = \psi_\beta \circ \psi_\alpha^{-1}$. This indicates that the first-kind gauge transformation on the physical spacetime $\mathscr{M}_\epsilon$ coincides with that on the background spacetime $\mathscr{M}$. Thus, even if we fix the gauge choice $\mathscr{X}_\epsilon$ of the second kind, the gauge degree of freedom of the first kind on the background spacetime $\mathscr{M}$ is induced by the gauge degree of freedom of the first kind on the physical spacetime $\mathscr{M}_\epsilon$. This induced gauge degree of freedom of the first-kind depends entirely on the gauge choice $\mathscr{X}_\epsilon$. Actually, the gauge choice $\psi_\alpha \circ \mathscr{X}_\epsilon$ of the first kind does depend on the gauge choice $\mathscr{X}_\epsilon$ of the second kind. However, the first-kind gauge transformation rule $(\psi_\beta \circ \mathscr{X}_\epsilon) \circ (\psi_\alpha \circ \mathscr{X}_\epsilon)^{-1} = \psi_\beta \circ \psi_\alpha^{-1}$ is independent of the gauge choice $\mathscr{X}_\epsilon$ of the second kind.

The above geometrical arguments indicate that even if we completely exclude the gauge-degree of freedom of the second kind, the gauge-degree of freedom of the first kind still remains. This situation supports the existence of the term of the Lie derivative of the background metric g_{ab} in the solution (284) of the linear metric perturbation. Actually, we may consider the point replacement $s = \Psi_\lambda(r)$ as Equation (5) on the physical spacetime $\mathscr{M}_{ph} = \mathscr{M}_\epsilon$. If we express the point replacement Ψ_λ through the point identification $\mathscr{X}_\epsilon$ to the background spacetime $\mathscr{M}$, the diffeomorphism Ψ_λ should be regarded as $\mathscr{X}_\epsilon^{-1}(s) = \mathscr{X}_\epsilon^{-1} \circ \Psi_\lambda \circ \mathscr{X}_\epsilon(\mathscr{X}_\epsilon^{-1}(r))$. This point replacement $\mathscr{X}_\epsilon^{-1} \circ \Psi_\lambda \circ \mathscr{X}_\epsilon : \mathscr{X}_\epsilon^{-1}(r) \mapsto \mathscr{X}_\epsilon^{-1}(s)$ on the background spacetime $\mathscr{M}$ completely depends on the second-kind gauge choice $\mathscr{X}_\epsilon$. However, if we use the coordinate systems $\{\mathscr{X}_\epsilon^{-1} O_\alpha, \psi_\alpha \circ \mathscr{X}_\epsilon\}$ and $\{\mathscr{X}_\epsilon^{-1} O_\beta, \psi_\beta \circ \mathscr{X}_\epsilon\}$ on the background spacetime $\mathscr{M}$, which are induced from the coordinate system on physical spacetime $\mathscr{M}_\epsilon$, the action (5) of the diffeomorphism is given by

$$\begin{aligned} & (\psi_\beta \circ \mathscr{X}_\epsilon) \circ \mathscr{X}_\epsilon^{-1} \circ \Psi_\lambda \circ \mathscr{X}_\epsilon \circ (\psi_\alpha \circ \mathscr{X}_\epsilon)^{-1} \\ = \; & \psi_\beta \circ \mathscr{X}_\epsilon \circ \mathscr{X}_\epsilon^{-1} \circ \Psi_\lambda \circ \mathscr{X}_\epsilon \circ \mathscr{X}_\epsilon^{-1} \circ \psi_\alpha^{-1} \\ = \; & \psi_\beta \circ \Psi_\lambda \circ \psi_\alpha^{-1}. \end{aligned} \tag{286}$$

This is just the "coordinate transformation" (5) and does not depend on the gauge choice $\mathscr{X}_\epsilon$ of the second-kind, i.e., is the gauge-invariant in the sense of the second-kind. Therefore, the coordinate transformation (286) may be regarded as the representation of the coordinate transformation (5), i.e., the replacement of points $r \mapsto s$ on the physical spacetime $\mathscr{M}_\epsilon$.

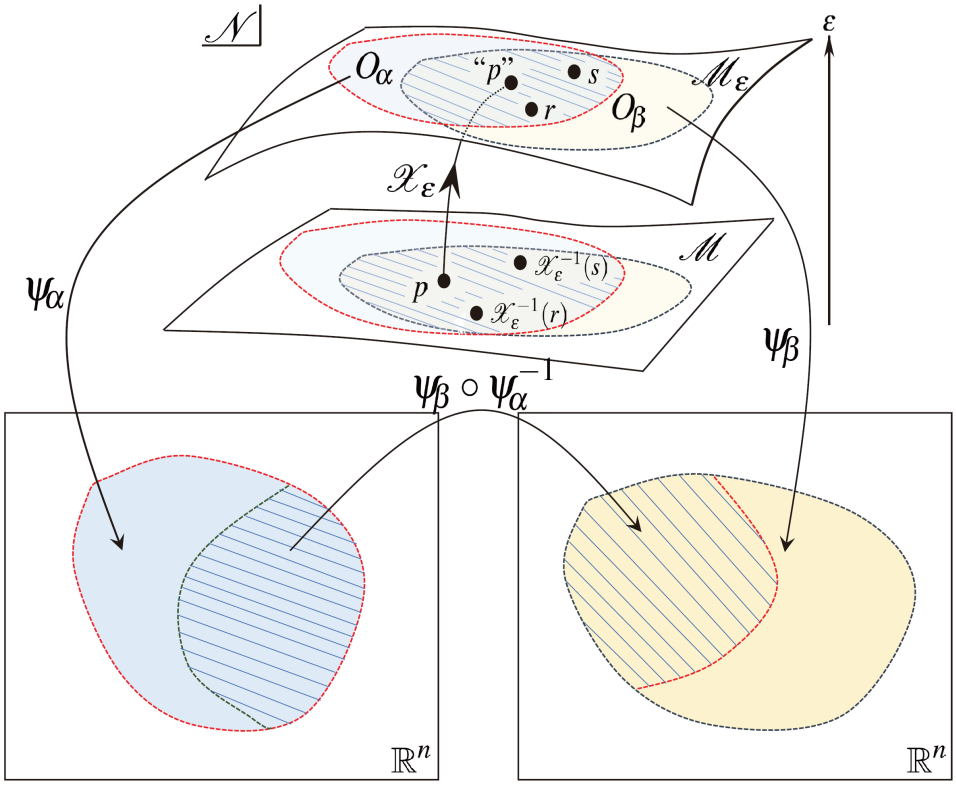

Figure 4. Consider the n-dimensional physical manifolds $\mathscr{M}_\epsilon$ and the background $\mathscr{M}$. We may introduce the coordinate transformation on the physical spacetime $\mathscr{M}_\epsilon$, even if we completely fix the second-kind gauge as $\mathscr{X}_\epsilon$. Actually, we may introduce the diffeomorphism ψ_α from the open set O_α (the blue region on $\mathscr{M}_\epsilon$) to an open set on $\mathbb{R}^n$ and the diffeomorphism ψ_β from the open set O_β (the yellow region on $\mathscr{M}_\epsilon$) to an open set on the other $\mathbb{R}^n$. If $O_\alpha \cap O_\beta \neq \emptyset$ (the shaded region on $\mathscr{M}_\epsilon$), we can consider the coordinate transformation $\psi_\beta \circ \psi_\alpha^{-1}$ which transforms the coordinate system (O_α, ψ_α) to (O_β, ψ_β). If we choose the gauge-choice of the second-kind by $\mathscr{X}_\epsilon$, this gauge-choice induce the coordinate systems $\{\mathscr{X}_\epsilon^{-1}O_\alpha, \psi_\alpha \circ \mathscr{X}_\epsilon\}$ and $\{\mathscr{X}_\epsilon^{-1}O_\beta, \psi_\beta \circ \mathscr{X}_\epsilon\}$ on $\mathscr{M}$. Furthermore, the coordinate transformation is given by $(\psi_\beta \circ \mathscr{X}_\epsilon) \circ (\psi_\alpha \circ \mathscr{X}_\epsilon)^{-1} = \psi_\beta \circ \psi_\alpha^{-1}$.

The solution (284) is gauge-invariant in the sense of the second kind, i.e., the degree of freedom of the point-identifications between the physical spacetime $\mathscr{M}_\epsilon$ and the background spacetime $\mathscr{M}$ is completely excluded. However, in this gauge-invariant solutions in the sense of the second kind, there still exists the term $\pounds_V g_{ab}$. As noted in Section 2.3, such terms may be included in the solution to the linearized Einstein equation due to the symmetry of the linearized Einstein equation as the gauge-invariant terms in the sense of the second-kind. Therefore, the term $\pounds_V g_{ab}$ in Equation (284) is no longer regarded as the gauge degree of the second kind, but we should regard this term as the gauge degree of freedom of the first kind as discussed above. Actually, the coordinate transformation (269)

should be regarded as the "coordinate transformation" (286), because β_1 is gauge-invariant in the sense of the second-kind. Furthermore, we note that the infinitesimal "coordinate transformation" which eliminates the term $\pounds_V g_{ab}$ in the solution (284) should be regarded as the "coordinate transformation" (286) due to the same reason. As explained in Section 2.1, the coordinate transformation (5) is regarded as the first-kind gauge degree of freedom. Then, the term $\pounds_V g_{ab}$ in the solution (284) should be regarded as the degree of freedom of the first-kind gauge. As pointed out in Section 2.1, the first kind gauge is often used to predict or to interpret the measurement results in observations and experiments. In this sense, this term of the Lie derivative of the background metric g_{ab} in the solution (284) should have their physical meaning. This is the reason why we emphasized the importance of the distinction of the notions of the first-kind gauge and the second-kind gauge.

We have to emphasize that this conclusion is the consequence of our complete exclusion of the second-kind gauge degree of freedom which includes not only $l \geq 2$ modes but also $l = 0, 1$ modes of perturbations and our Proposal 1. From the view point of the gauge-invariant perturbation theory developed in this paper, the conventional gauge-fixing procedure corresponds to the partial gauge-fixing. Therefore, it will be difficult to reach the above conclusion through the conventional gauge-fixing procedure. Furthermore, in the conventional approach, there is no distinction between the first- and the second-kind gauge and all terms which have the form $\pounds_V g_{ab}$ may be regarded as the "gauge-degree of freedom" and these are "unphysical degree of freedom" because we can always eliminate these terms through the infinitesimal coordinate transformation. If the concept of "the complete gauge fixing" corresponds to the standing point that all terms which have the form $\pounds_V g_{ab}$ are "unphysical degree of freedom", this concept of "the complete gauge-fixing" is stronger restriction of the metric perturbation than the concept of "gauge-invariant of the second kind" in this paper. Thus, we may say that these conceptual discussion is an important result comes from the realization of the gauge-invariant formulation including $l = 0, 1$ modes in this paper. Similar results are also obtained in even-mode perturbations which will be shown in the Part II paper [43].

Apart from these terms of the Lie derivative of the background metric g_{ab}, in the vacuum case, the only non-trivial solutions in $l = 1$ odd-mode perturbation are the Kerr parameter perturbations. These will be related to the uniqueness of the Kerr solution in the vacuum Einstein equations in the local sense [14], though the assertion of the uniqueness theorem of the Kerr solution includes topological statements [68]. In addition to the relation of the uniqueness theorem of Kerr black hole, at least, we may say that the derived vacuum solution for $l = 0, 1$ odd-mode perturbations is physically reasonable. In the paper [43], we derive the $l = 0, 1$ even-mode solution to the linearized Einstein equation which also includes the terms of the Lie derivative of the background metric. In the Part III paper [44], we show that the derived solutions in ref. [43] realize the linearized Lemaître–Tolman–Bondi solution and the linearized non-rotating C-metric. Due to these facts, we may say that our solutions derived through Proposal 1 are physically reasonable. In this sense, we may say that Proposal 1 is also physically reasonable.

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Appendix A. Explicit Form of Conventional Spherical Harmonics on S^2

First, we summarize the properties of the conventional spherical harmonic functions Y_{lm}. The spherical harmonic functions $Y_{lm}(\theta, \phi)$ satisfy the equations

$$\left[\hat{\Delta} + l(l+1)\right] Y_{lm} = 0, \tag{A1}$$

$$\partial_\phi Y_{lm} = im Y_{lm}. \tag{A2}$$

To be explicit, they are expressed in terms of the Legendre functions as

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi}. \tag{A3}$$

For $l = 0, 1$ modes, the spherical harmonic functions $Y_{lm} = Y_{l,m}$ are explicitly given by

$$Y_{00} = \sqrt{\frac{1}{4\pi}}, \tag{A4}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_{11} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}, \quad Y_{1-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}. \tag{A5}$$

Employing these spherical harmonic functions (A3) as the scalar harmonics, we construct the set of the tensor harmonics on S^2. Since the dimension of S^2 is two, we have enough number of tensor harmonic functions as bases of tangent space on S^2.

On the unit sphere, any vector field v^p is written in terms of two scalar functions v and w as

$$v^p = \hat{D}^p v + \epsilon^{pq} \hat{D}_q w. \tag{A6}$$

Here, $\hat{D}^p v$ is even part and $\epsilon^{pq} \hat{D}_q w$ is the odd part, which corresponds to $\hat{D}_p S$ and $\epsilon_{pq} \hat{D}^q S$ in Equation (38), respectively. If we choose $S = Y_{lm}$, these vectors are given by

$$\hat{D}_p Y_{lm}, \quad \epsilon_{pq} \hat{D}^q Y_{lm}. \tag{A7}$$

For $l = 0$ modes, the spherical harmonic function Y_{00} is constant as in Equation (A4) and corresponding vector harmonics vanish:

$$\hat{D}_p Y_{00} = 0, \quad \epsilon_{pq} \hat{D}^q Y_{00} = 0. \tag{A8}$$

On the other hand, for $l = 1$ modes, vector harmonics has the vector values as

$$\hat{D}_p Y_{10} = -\sqrt{\frac{3}{4\pi}} \sin\theta \theta_p, \tag{A9}$$

$$\hat{D}_p Y_{11} = \sqrt{\frac{3}{8\pi}} e^{i\phi} (\cos\theta \theta_p + i\phi_p), \quad \hat{D}_p Y_{1-1} = \sqrt{\frac{3}{8\pi}} e^{-i\phi} (-\cos\theta \theta_p + i\phi_p) \tag{A10}$$

and

$$\epsilon_{pq} \hat{D}^q Y_{10} = \sqrt{\frac{3}{4\pi}} \sin\theta \phi_p, \tag{A11}$$

$$\epsilon_{pq} \hat{D}^q Y_{11} = \sqrt{\frac{3}{8\pi}} e^{i\phi} (-\cos\theta \phi_p + i\theta_p), \quad \epsilon_{pq} \hat{D}^q Y_{1-1} = \sqrt{\frac{3}{8\pi}} e^{-i\phi} (\cos\theta \phi_p + i\theta_p). \tag{A12}$$

Thus, vector harmonics has its vector value for $l = 1$ modes, while does not for $l = 0$ mode.

Any smooth symmetric second-rank tensor field t^{pq} on the unit sphere can be expressed in terms of its trace $t = t^p_p$ and two scalar fields v and w as

$$t^{pq} = \frac{1}{2} t \gamma^{pq} + \left(\hat{D}^p \hat{D}^q - \frac{1}{2} \gamma^{pq} \hat{\Delta} \right) v + 2 \epsilon^{r(q} \hat{D}^{p)} D_r w. \tag{A13}$$

These three terms correspond to the terms proportional to $\frac{1}{2}\gamma_{pq} S$, $\left(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma^{pq} \hat{\Delta} \right) S$, and $2\epsilon_{r(p} \hat{D}_{q)} \hat{D}^r S$ in Equation (39). As in the case of vector harmonics above, for $l = 0$ modes, the spherical harmonic function Y_{00} is constant as in Equation (A4) and the only non-vanishing harmonics is its trace part

$$\frac{1}{2} \gamma_{pq} Y_{00} = \frac{1}{2} \gamma_{pq} \sqrt{\frac{1}{4\pi}} \tag{A14}$$

and the other traceless even and odd parts vanish. For $l = 1$ modes, from Equation (A5), the trace parts are trivially given by

$$\frac{1}{2} \gamma_{pq} Y_{10} = \frac{1}{2} \sqrt{\frac{3}{4\pi}} \cos\theta \gamma_{pq}, \tag{A15}$$

$$\frac{1}{2} \gamma_{pq} Y_{11} = \frac{1}{2} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \gamma_{pq}, \tag{A16}$$

$$\frac{1}{2} \gamma_{pq} Y_{1-1} = -\frac{1}{2} \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \gamma_{pq}. \tag{A17}$$

On the other hand, the traceless even and odd parts for $\left(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma^{pq} \hat{\Delta} \right) Y_{1m}$, and $2\epsilon_{r(p} \hat{D}_{q)} \hat{D}^r Y_{1m}$ identically vanish for all $m = -1, 0, 1$.

As a summary of $S = Y_{lm}$ cases, for $l = 0$ mode, any vector and tensor harmonics vanish, and these do not play roles of bases of the tangent space on S^2. On the other hand, for $l = 1$ modes, the vector harmonics have their vector value and play roles of bases of the tangent space on S^2. The trace parts of the second-rank tensor of each modes have their tensor values, while all traceless even and odd mode harmonics identically vanish and does not play roles of bases of the tangent space on S^2.

Appendix B. Covariant Derivatives in 2 + 2 Formulation and Background Curvatures

In this Appendix, we summarize the relation between the covariant derivatives ∇_a associated with the metric g_{ab}, $\bar{D}_A$ associated with the metric y_{ab}, and $\hat{D}_p$ associated with the metric γ_{ab}. These formulae are convenient to derive the gauge-transformation rules, linearized Einstein equations, and so on. Here, the metrices g_{ab}, y_{ab}, and γ_{ab} are given by Equation (31). We assume that y_{ab} depends on $\{x^A\}$ and $r = r(x^A)$. We also assume that γ_{ab} depends only on $\{x^p\}$. Under these assumptions, the Christoffel symbol ${\Gamma_{ab}}^c$ are given by

$${\Gamma_{ab}}^c = \frac{1}{2} g^{cd} (\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}), \tag{A18}$$

$${\Gamma_{AB}}^C = \frac{1}{2} y^{CD} (\partial_A y_{DB} + \partial_B y_{DA} - \partial_D y_{AB}) =: {\bar{\Gamma}_{AB}}^C, \tag{A19}$$

$${\Gamma_{pB}}^C = 0, \tag{A20}$$

$${\Gamma_{pq}}^C = -r(\bar{D}^C r) \gamma_{pq}, \tag{A21}$$

$${\Gamma_{AB}}^p = 0, \tag{A22}$$

$$\Gamma_{qA}{}^{p} = \frac{1}{r}(\bar{D}_A r)\gamma_q{}^{p}, \tag{A23}$$

$$\Gamma_{qr}{}^{p} = \frac{1}{2}\gamma^{pd}\left(\partial_q\gamma_{dr} + \partial_r\gamma_{dq} - \partial_d\gamma_{qr}\right) =: \hat{\Gamma}_{qr}{}^{p}. \tag{A24}$$

Here, we note that

$$\hat{D}_p\bar{D}_A t_B = \bar{D}_A\hat{D}_p t_B, \tag{A25}$$

and

$$\hat{D}_p\bar{D}_A t_q = \partial_p\bar{D}_A t_q - \hat{\Gamma}_{qp}{}^{r}\bar{D}_A t_q = \bar{D}_A\hat{D}_p t_q, \tag{A26}$$

since

$$\partial_p\bar{\Gamma}_{AB}{}^{C} = 0, \quad \partial_A\hat{\Gamma}_{pq}{}^{r} = 0. \tag{A27}$$

Then, we obtain the formulae for the covariant derivatives $\nabla_a v_b$ and $\nabla_a t^b$ as

$$\nabla_A v_B = \bar{D}_A v_B, \tag{A28}$$

$$\nabla_A v_p = \bar{D}_A v_p - \frac{1}{r}\bar{D}_A r v_p, \tag{A29}$$

$$\nabla_p v_A = \hat{D}_p v_A - \frac{1}{r}\bar{D}_A r v_p, \tag{A30}$$

$$\nabla_p v_q = \hat{D}_p v_q + r\bar{D}^A r\gamma_{pq}v_A, \tag{A31}$$

$$\nabla_A t^B = \bar{D}_A t^B, \tag{A32}$$

$$\nabla_A t^p = \partial_A t^p + \frac{1}{r}\bar{D}_A r t^p, \tag{A33}$$

$$\nabla_p t^A = \hat{D}_p t^A - r\bar{D}^A r\gamma_{pq}t^q, \tag{A34}$$

$$\nabla_p t^q = \hat{D}_p t^q + \frac{1}{r}\bar{D}_A r\gamma_p{}^{q}t^A. \tag{A35}$$

Here, we also summarize the expression of $\nabla_a T_{bc}$ for an arbitrary tensor T_{bc} in terms of the covariant derivatives $\bar{D}_A$ and $\hat{D}_p$ which are associated with the metric y_{AB} and γ_{pq}, respectively, from

$$\nabla_a T_{bc} = \partial_a T_{bc} - \Gamma_{ba}{}^{d}T_{dc} - \Gamma_{ca}{}^{d}T_{bd}. \tag{A36}$$

These are given by

$$\nabla_A T_{BC} = \bar{D}_A T_{BC}, \tag{A37}$$

$$\nabla_A T_{Bp} = \bar{D}_A T_{Bp} - \frac{1}{r}\bar{D}_A r T_{Bp}, \tag{A38}$$

$$\nabla_A T_{pC} = \bar{D}_A T_{pC} - \frac{1}{r}\bar{D}_A r T_{pC}, \tag{A39}$$

$$\nabla_p T_{BC} = \hat{D}_p T_{BC} - \frac{1}{r}\bar{D}_B r T_{pC} - \frac{1}{r}\bar{D}_C r T_{Bp}, \tag{A40}$$

$$\nabla_p T_{qC} = \hat{D}_p T_{qC} + r\bar{D}^D r\gamma_{qp}T_{DC} - \frac{1}{r}\bar{D}_C r T_{qp}, \tag{A41}$$

$$\nabla_p T_{Bq} = \hat{D}_p T_{Bq} - \frac{1}{r}\bar{D}_B r T_{pq} + r\bar{D}^D r\gamma_{qp}T_{BD}, \tag{A42}$$

$$\nabla_A T_{pq} = \bar{D}_A T_{pq} - \frac{2}{r}\bar{D}_A r T_{pq}, \tag{A43}$$

$$\nabla_p T_{qr} = \hat{D}_p T_{qr} + r\bar{D}^D r\gamma_{qp}T_{Dr} + r\bar{D}^D r\gamma_{rp}T_{qD}. \tag{A44}$$

Furthermore, the derive the linearized Einstein equation, we have to derive the components of

$$\nabla_a H_c{}^{bd} = \partial_a H_c{}^{bd} - \Gamma_{ca}{}^e H_e{}^{bd} + \Gamma_{ea}{}^b H_c{}^{ed} + \Gamma_{ea}{}^d H_c{}^{be}. \tag{A45}$$

Then, these are summarized as

$$\nabla_A H_C{}^{BD} = \bar{D}_A H_C{}^{BD}, \tag{A46}$$
$$\nabla_A H_C{}^{Bs} = \bar{D}_A H_C{}^{Bs} + \frac{1}{r}\bar{D}_A r H_C{}^{Bs}, \tag{A47}$$
$$\nabla_A H_C{}^{qD} = \bar{D}_A H_C{}^{qD} + \frac{1}{r}\bar{D}_A r H_C{}^{qD}, \tag{A48}$$
$$\nabla_A H_C{}^{qs} = \bar{D}_A H_C{}^{qs} + \frac{2}{r}\bar{D}_A r H_C{}^{qs}, \tag{A49}$$
$$\nabla_A H_r{}^{BD} = \bar{D}_A H_r{}^{BD} - \frac{1}{r}\bar{D}_A r H_r{}^{BD}, \tag{A50}$$
$$\nabla_A H_r{}^{Bs} = \bar{D}_A H_r{}^{Bs}, \tag{A51}$$
$$\nabla_A H_r{}^{qD} = \bar{D}_A H_r{}^{qD}, \tag{A52}$$
$$\nabla_A H_r{}^{qs} = \bar{D}_A H_r{}^{qs} + \frac{1}{r}\bar{D}_A r H_r{}^{qs}, \tag{A53}$$

$$\nabla_p H_C{}^{BD} = \hat{D}_p H_C{}^{BD} - \frac{1}{r}\bar{D}_C r H_p{}^{BD} - r\bar{D}^B r\gamma_{tp} H_C{}^{tD} - r\bar{D}^D r\gamma_{tp} H_C{}^{Bt}, \tag{A54}$$
$$\nabla_p H_C{}^{Bs} = \hat{D}_p H_C{}^{Bs} - \frac{1}{r}\bar{D}_C r H_p{}^{Bs} + \frac{1}{r}\bar{D}_E r\gamma_p{}^s H_C{}^{BE} - r\bar{D}^B r\gamma_{tp} H_C{}^{ts}, \tag{A55}$$
$$\nabla_p H_C{}^{qD} = \hat{D}_p H_C{}^{qD} - \frac{1}{r}\bar{D}_C r H_p{}^{qD} + \frac{1}{r}\bar{D}_E r\gamma_p{}^q H_C{}^{ED} - r\bar{D}^D r\gamma_{tp} H_C{}^{qt}, \tag{A56}$$
$$\nabla_p H_C{}^{qs} = \hat{D}_p H_C{}^{qs} - \frac{1}{r}\bar{D}_C r H_p{}^{qs} + \frac{1}{r}\bar{D}_E r\gamma_p{}^q H_C{}^{Es} + \frac{1}{r}\bar{D}_E r\gamma_p{}^s H_C{}^{qE}, \tag{A57}$$
$$\nabla_p H_r{}^{BD} = \hat{D}_p H_r{}^{BD} + r\bar{D}^E r\gamma_{rp} H_E{}^{BD} - r\bar{D}^B\gamma_{tp} H_r{}^{tD} - r\bar{D}^D\gamma_{tp} H_r{}^{Bt}, \tag{A58}$$
$$\nabla_p H_r{}^{Bs} = \hat{D}_p H_r{}^{Bs} + r\bar{D}^E r\gamma_{rp} H_E{}^{Bs} - r\bar{D}^B r\gamma_{tp} H_r{}^{ts} + \frac{1}{r}\bar{D}_E r\gamma_p{}^s H_r{}^{BE}, \tag{A59}$$
$$\nabla_p H_r{}^{qD} = \hat{D}_p H_r{}^{qD} + r\bar{D}^E r\gamma_{rp} H_E{}^{qD} - r\bar{D}^D r\gamma_{tp} H_r{}^{qt} + \frac{1}{r}\bar{D}_E r\gamma_p{}^q H_r{}^{ED}, \tag{A60}$$
$$\nabla_p H_r{}^{qs} = \hat{D}_p H_r{}^{qs} + r\bar{D}^E r\gamma_{rp} H_E{}^{qs} + \frac{1}{r}\bar{D}_E r\gamma_p{}^q H_r{}^{Es} + \frac{1}{r}\bar{D}_E r\gamma_p{}^s H_r{}^{qE}. \tag{A61}$$

Next, we summarize the components of the background curvatures induced by the metric Equation (31). We derive these components through the components of the connection (A19)–(A24) and the formula of the Riemann curvature

$$R_{abc}{}^d = \partial_b \Gamma_{ac}{}^d - \partial_a \Gamma_{bc}{}^d + \Gamma_{ac}{}^e \Gamma_{eb}{}^d - \Gamma_{bc}{}^e \Gamma_{ea}{}^d. \tag{A62}$$

To derive the components of this curvature, we use

$$\bar{D}_A \gamma_{pq} = 0 = \hat{D}_p y_{AB}, \quad \hat{D}_p r = 0. \tag{A63}$$

The components of the non-vanishing Riemann curvature are summarized as

$$R_{ABC}{}^D = {}^{(2)}\bar{R}_{ABC}{}^D, \tag{A64}$$
$$R_{pBr}{}^D = -r(\bar{D}_B \bar{D}^D r)\gamma_{pr}, \tag{A65}$$
$$R_{pBC}{}^s = \frac{1}{r}(\bar{D}_B \bar{D}_C r)\gamma_p{}^s, \tag{A66}$$
$$R_{pqr}{}^s = {}^{(2)}\hat{R}_{pqr}{}^s - 2(\bar{D}^E r)(\bar{D}_E r)\gamma_{r[p}\gamma_{q]}{}^s. \tag{A67}$$

The components of the Ricci curvature are summarized as

$$R_{AC} = {}^{(2)}\bar{R}_{AC} - \frac{2}{r}(\bar{D}_A\bar{D}_C r), \tag{A68}$$
$$R_{Ar} = 0, \tag{A69}$$
$$R_{pr} = {}^{(2)}\hat{R}_{pr} - \Big[r(\bar{D}_E\bar{D}^E r) + (\bar{D}^E r)(\bar{D}_E r)\Big]\gamma_{pr}. \tag{A70}$$

The Ricci scalar curvature is given by

$$R = g^{ac}R_{ac} = {}^{(2)}\bar{R} + \frac{1}{r^2}{}^{(2)}\hat{R} - \frac{4}{r}(\bar{D}^C\bar{D}_C r) - \frac{2}{r^2}(\bar{D}^E r)(\bar{D}_E r). \tag{A71}$$

Next, we derive the components of the Einstein tensor

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R \tag{A72}$$

and its components are summarized as

$$G_{AB} = -\frac{2}{r}(\bar{D}_A\bar{D}_B r) + \frac{1}{r^2}y_{AB}\Big[-1 + 2r(\bar{D}^C\bar{D}_C r) + (\bar{D}^E r)(\bar{D}_E r)\Big], \tag{A73}$$
$$G_{Aq} = 0, \tag{A74}$$
$$G_{pq} = \gamma_{pq}\left[r(\bar{D}^C\bar{D}_C r) - \frac{1}{2}r^{2\,(2)}\bar{R}\right], \tag{A75}$$

where we used the two-dimensional Einstein tensors are identically vanish and the fact that the metric γ_{pq} is the maximally symmetric space with positive curvature, i.e.,

$${}^{(2)}\hat{R}_{pqrs} = 2\gamma_{p[r}\gamma_{s]q}, \quad {}^{(2)}\hat{R}_{pr} = \gamma_{pr}, \quad {}^{(2)}\hat{R} = 2. \tag{A76}$$

Here, we consider the static solution whose metric is given by

$$y_{AB} = -f(dt)_A(dt)_B + f^{-1}(dr)_A(dr)_B, \tag{A77}$$

where $f = f(r)$. Due to the Birkhoff theorem [69], the vacuum solution with the spherically symmetric spacetime must be the Schwarzschild spacetime. We check this fact from Equations (A73)–(A75) with the substitution (A77). Actually, we obtain

$$\bar{D}_B r = (dr)_B, \quad \bar{D}^B r = f\left(\frac{\partial}{\partial r}\right)^B, \quad \bar{D}_A\bar{D}_B r = \frac{f'}{2}y_{AB}. \tag{A78}$$

Then, we have

$$(\bar{D}^B r)(\bar{D}_B r) = f, \quad \bar{D}^A\bar{D}_B r = \frac{f'}{2}\delta_B{}^A, \quad \bar{D}^C\bar{D}_C r = f'. \tag{A79}$$

From Equation (A73) as

$$y^{AB}G_{AB} = \frac{2}{r}\left(f' - \frac{1-f}{r}\right) = 0, \quad G_{AB} - \frac{1}{2}y_{AB}G_{AB} = 0. \tag{A80}$$

The solution to Equation (A80) is given by

$$f = 1 - \frac{2M}{r}, \tag{A81}$$
$$f' = \frac{1-f}{r}, \tag{A82}$$

where M is the constant of integration. This is the Schwarzschild metric. We also evaluate the component $G_{pq} = 0$ through Equation (A75) using Equation (A81) as

$$ {}^{(2)}\bar{R} = \frac{2}{r}(\bar{D}^C \bar{D}_C r) \tag{A83} $$

As the summary of the background vacuum Einstein equations, we have

$$ r(\bar{D}^C \bar{D}_C r) + (\bar{D}^E r)(\bar{D}_E r) = 1, \tag{A84} $$

$$ (\bar{D}_A \bar{D}_B r) = \frac{1}{2} y_{AB} (\bar{D}^C \bar{D}_C r), \tag{A85} $$

$$ {}^{(2)}\bar{R} = \frac{2}{r}(\bar{D}^C \bar{D}_C r). \tag{A86} $$

Equation (A84) is equivalent to Equation (A82). Since the two-dimensional curvature ${}^{(2)}\bar{R}_{DAEC}$ has only one independent component, ${}^{(2)}\bar{R}_{DAEC}$ is written as

$$ {}^{(2)}\bar{R}_{DAEC} = \frac{2}{r}(\bar{D}^F \bar{D}_F r) y_{D[E} y_{C]A}, \quad {}^{(2)}\bar{R}_{DE} = \frac{1}{r}(\bar{D}^F \bar{D}_F r) y_{DE}, \quad {}^{(2)}\bar{R} = \frac{2}{r}(\bar{D}^F \bar{D}_F r). \tag{A87} $$

The above formulae are expressed the covariant form of the 2 + 2 formulation. However, the explicit components of $\Gamma_{ab}{}^{c}$ are also convenient to leads the results in Section 6.2. From Equations (A19)–(A24) and the background metric (31) with Equations (33) and (34), non-vanishing components of $\Gamma_{ab}{}^{c}$ are summarized as

$$ \begin{aligned} &\Gamma_{tr}{}^{t} = \frac{f'}{2f}, \quad \Gamma_{tt}{}^{r} = \frac{1}{2} f f', \quad \Gamma_{rr}{}^{r} = -\frac{f'}{2f}, \quad \Gamma_{\theta\theta}{}^{r} = -rf, \\ &\Gamma_{\phi\phi}{}^{r} = -rf \sin^2\theta, \quad \Gamma_{r\theta}{}^{\theta} = \frac{1}{r}, \quad \Gamma_{\phi\phi}{}^{\theta} = -\sin\theta\cos\theta, \\ &\Gamma_{r\phi}{}^{\phi} = \frac{1}{r}, \quad \Gamma_{\phi\theta}{}^{\phi} = \cot\theta. \end{aligned} \tag{A88} $$

Appendix C. Summary of the 2 + 2 Representations of the Tensor $H_{abc}[\mathscr{F}]$, $H_{ab}{}^{c}[\mathscr{F}]$, $H_{a}{}^{bc}[\mathscr{F}]$

Here, we summarize the components of $H_{abc}[\mathscr{F}]$ through the expressions (216)–(218):

$$ H_{ABC} = \bar{D}_{(A} F_{B)C} - \frac{1}{2} \bar{D}_C F_{AB}, \tag{A89} $$

$$ H_{pBC} = \frac{1}{2}\left(\hat{D}_p F_{BC} + r\bar{D}_B F_{Cp} - r\bar{D}_C F_{Bp} - (\bar{D}_B r) F_{Cp} - (\bar{D}_C r) F_{Bp}\right), \tag{A90} $$

$$ H_{pqC} = \frac{1}{2}\left(2r\hat{D}_{(p} F_{q)C} - \frac{1}{2}\gamma_{pq} r^2 \bar{D}_C F - r(\bar{D}_C r)\gamma_{pq} F + 2r(\bar{D}^D r)\gamma_{pq} F_{DC}\right), \tag{A91} $$

$$ H_{ABr} = r\bar{D}_{(A} F_{B)r} + (\bar{D}_{(A} r) F_{B)r} - \frac{1}{2}\hat{D}_r F_{AB}, \tag{A92} $$

$$ H_{pBr} = \frac{1}{2}\left(r\hat{D}_p F_{rB} - r\hat{D}_r F_{pB} + \frac{1}{2} r^2 \gamma_{pr} \bar{D}_B F\right), \tag{A93} $$

$$ H_{pqr} = \frac{1}{2} r^2 \gamma_{r(q} \hat{D}_{p)} F - \frac{1}{4} r^2 \gamma_{pq} \hat{D}_r F + r^2 \bar{D}^D r \gamma_{pq} F_{Dr}. \tag{A94} $$

Next, we summarize the components of $H_{ab}{}^{c}[\mathscr{F}]$ through the expressions (216)–(218) as follows

$$
\begin{aligned}
H_{AB}{}^{C} &= \bar{D}_{(A}F_{B)}{}^{C} - \frac{1}{2}\bar{D}^{C}F_{AB}, && \text{(A95)}\\
H_{pB}{}^{C} &= \frac{1}{2}\Big(\hat{D}_{p}F_{B}{}^{C} + r\bar{D}_{B}F_{p}{}^{C} - r\bar{D}^{C}F_{Bp} - (\bar{D}_{B}r)F_{p}{}^{C} - (\bar{D}^{C}r)F_{Bp}\Big), && \text{(A96)}\\
H_{pq}{}^{C} &= \frac{1}{2}\left(2r\hat{D}_{(p}F_{q)}{}^{C} - \frac{1}{2}\gamma_{pq}r^{2}\bar{D}^{C}F - r(\bar{D}^{C}r)\gamma_{pq}F + 2r(\bar{D}^{D}r)\gamma_{pq}F_{D}{}^{C}\right), && \text{(A97)}\\
H_{AB}{}^{r} &= \frac{1}{r}\bar{D}_{(A}F_{B)}{}^{r} + \frac{1}{r^{2}}(\bar{D}_{(A}r)F_{B)}{}^{r} - \frac{1}{2r^{2}}\hat{D}^{r}F_{AB}, && \text{(A98)}\\
H_{pB}{}^{r} &= \frac{1}{2r}\hat{D}_{p}F_{B}{}^{r} - \frac{1}{2r}\hat{D}^{r}F_{Bp} + \frac{1}{4}\gamma_{p}{}^{r}\bar{D}_{B}F, && \text{(A99)}\\
H_{pq}{}^{r} &= \frac{1}{2}\gamma_{(q}{}^{r}\hat{D}_{p)}F - \frac{1}{4}\gamma_{pq}\hat{D}^{r}F + (\bar{D}^{D}r)\gamma_{pq}F_{D}{}^{r}. && \text{(A100)}
\end{aligned}
$$

Finally, we summarize the component $H_{a}{}^{bc}[\mathscr{F}]$ through the expression (216)–(218) as follows:

$$
\begin{aligned}
H_{A}{}^{BC} &= \frac{1}{2}\Big(\bar{D}_{A}F^{BC} + \bar{D}^{B}F_{A}{}^{C} - \bar{D}^{C}F_{A}{}^{B}\Big), && \text{(A101)}\\
H_{A}{}^{Br} &= \frac{1}{2r}\bar{D}_{A}F^{Br} + \frac{1}{2r}\bar{D}^{B}F_{A}{}^{r} + \frac{1}{2r^{2}}(\bar{D}_{A}r)F^{Br} + \frac{1}{2r^{2}}(\bar{D}^{B}r)F_{A}{}^{r} - \frac{1}{2r^{2}}\hat{D}^{r}F_{A}{}^{B}, && \text{(A102)}\\
H_{A}{}^{qC} &= \frac{1}{2r^{2}}\Big(\hat{D}^{q}F_{A}{}^{C} + r\bar{D}_{A}F^{qC} - r\bar{D}^{C}F_{A}{}^{q} - (\bar{D}_{A}r)F^{qC} - (\bar{D}^{C}r)F_{A}{}^{q}\Big), && \text{(A103)}\\
H_{A}{}^{qr} &= \frac{1}{2r^{3}}\left[\hat{D}^{q}F_{A}{}^{r} - \hat{D}^{r}F_{A}{}^{q} + \frac{1}{2}r\gamma^{qr}\bar{D}_{A}F\right], && \text{(A104)}\\
H_{p}{}^{BC} &= \frac{1}{2}\Big(\hat{D}_{p}F^{BC} + r\bar{D}^{B}F_{p}{}^{C} - r\bar{D}^{C}F_{p}{}^{B} - (\bar{D}^{B}r)F_{p}{}^{C} - (\bar{D}^{C}r)F_{p}{}^{B}\Big), && \text{(A105)}\\
H_{p}{}^{Br} &= \frac{1}{2r}\hat{D}_{p}F^{Br} - \frac{1}{2r}\hat{D}^{r}F_{p}{}^{B} + \frac{1}{4}\gamma_{p}{}^{r}\bar{D}^{B}F, && \text{(A106)}\\
H_{p}{}^{qC} &= \frac{1}{2r^{2}}\left(r\hat{D}_{p}F^{qC} + r\hat{D}^{q}F_{p}{}^{C} - \frac{1}{2}\gamma_{p}{}^{q}r^{2}\bar{D}^{C}F - r(\bar{D}^{C}r)\gamma_{p}{}^{q}F + 2r(\bar{D}^{D}r)\gamma_{p}{}^{q}F_{D}{}^{C}\right), && \text{(A107)}\\
H_{p}{}^{qr} &= \frac{1}{r^{2}}\left(\frac{1}{4}\gamma^{qr}\hat{D}_{p}F + \frac{1}{4}\gamma_{p}{}^{r}\hat{D}^{q}F - \frac{1}{4}\gamma_{p}{}^{q}\hat{D}^{r}F + (\bar{D}^{D}r)\gamma_{p}{}^{q}F_{D}{}^{r}\right). && \text{(A108)}
\end{aligned}
$$

Notes

1 In the derivation of the Lie derivative in § 94 of ref. [48], the coordinate transformation $x'^{i} = x^{i} + \xi^{i}$ is performed, at first, and the comparison inverse matrices $g'^{ik}(x'^{l})$ and $g^{ik}(x^{l})$ at the "same coordinate value" x^{l} is carried out. The comparison at the "same coordinate value" x^{l} under the coordinate transformation $x'^{i} = x^{i} + \xi^{i}$ means the comparison the inverse metrics at the "different points" on the same manifold as shown in Equation (3).

2 As depicted in Figure 2, the action of the diffeomorphism $\Phi_{\epsilon} := \mathscr{X}_{\epsilon}^{-1} \circ \mathscr{Y}_{\epsilon}$ is the replacement of $\Phi_{\epsilon}(q) = p$. However, the evaluations of the both-side of Equation (9) are carried out at the same point on the background spacetime $\mathscr{M}$ and Equation (11) is also evaluated at the same point on the background spacetime $\mathscr{M}$ as the result, while Equation (6) represents the difference between the tensor field at different points on the same manifold. To explain this, we consider the points "p" $\in \mathscr{M}_{ph}$, "q" $\in \mathscr{M}_{ph}$ ("p" $\neq$ "q"), and $q \in \mathscr{M}$ and the action of the diffeomorphisms $\mathscr{Y}_{\epsilon}$, and $\mathscr{X}_{\epsilon}$ so that "p" $= \mathscr{Y}_{\epsilon}(q)$ and "q" $= \mathscr{X}_{\epsilon}(q)$. Through this setup, Equation (9) derived as

$$
\begin{aligned}
Q(\text{“}p\text{”}) &= Q(\mathscr{Y}_{\epsilon}(q))\\
&= \mathscr{Y}_{\epsilon}^{*}Q(q) = \mathscr{Y}_{\epsilon}^{*}Q(\mathscr{X}_{\epsilon}^{-1}(\text{“}q\text{”})) = \mathscr{Y}_{\epsilon}^{*} \circ (\mathscr{X}_{\epsilon}^{-1})^{*}Q(\text{“}q\text{”}) = \mathscr{Y}_{\epsilon}^{*} \circ (\mathscr{X}_{\epsilon}^{-1})^{*}Q(\mathscr{X}_{\epsilon}(q))\\
&= \mathscr{Y}_{\epsilon}^{*} \circ (\mathscr{X}_{\epsilon}^{-1})^{*} \circ \mathscr{X}_{\epsilon}^{*}Q(q) = \left(\mathscr{X}_{\epsilon}^{-1} \circ \mathscr{Y}_{\epsilon}\right)^{*} \circ \mathscr{X}_{\epsilon}^{*}Q(q)\\
&= \Phi_{\epsilon}^{*}\mathscr{X}_{\epsilon}^{*}Q(q). && \text{(A109)}
\end{aligned}
$$

Then, through Equations (8) and (10), we reach to the gauge-transformation rule (11) at the same point, which should be regarded as ${}^{(1)}_{\mathscr{Y}}Q(q) - {}^{(1)}_{\mathscr{X}}Q(q) = \pounds_{\xi_{(1)}} Q_0(q)$.

[3] This statement of the Proposal 1 actually indicates that at once we ignore the neighborhood of the region where the harmonic function S_δ diverges but we apply the analytic extension of the linearized solution to these regions when we choose $\delta = 0$.

[4] From Equation (257) and the descriptions in ref. [61], readers might regard that the extension to $l = 1$ mode case of the Cunningham-Price-Moncrief variable $\Phi_{(o)}$ is the same variable as the gauge-invariant variable Φ_{KIF} defined by Equation (185). Actually, if we can identify $\tilde{h}_{(o1)A}$ with $\tilde{F}_A$, the extension to $l = 1$ mode case of the Cunningham-Price-Moncrief variable $\Phi_{(o)}$ coincides with the definition of Φ_{KIF} and there is the description in ref. [61] which is similar to Equation (257). However, this identification is not appropriate, since $\tilde{F}_A$ is gauge-invariant in the sense of the second-kind but $\tilde{h}_{(o1)A}$ is not gauge-invariant. We actually take $\delta = 0$ in the singular harmonic when we solve the mode-by-mode Einstein equations. However, this does not mean $\tilde{h}_{(o2)} = 0$, nevertheless the term $\tilde{h}_{(o2)}$ in the metric perturbation disappear since the singular harmonic function vanishes due to the choice $\delta = 0$. This difference also appears when we obtain the gauge-invariant relation between the components of $\tilde{F}_A$ and the extension to $l = 1$ mode case of Cunningham-Price-Moncrief variable $\Phi_{(o)}$ by integrating the linearized Einstein equations. In this integration, the integration constants appear in the relation between the components $\tilde{F}_A$ and $\Phi_{(o)}$. This integration "constants" are automatically gauge-invariant in the sense of second-kind. On the other hand, when we integrate Ψ_{KIF} to obtain the explicit relation with $\tilde{h}_{(o1)A}$, there is no guarantee that the integration "constants" are gauge-invariant, because $\tilde{h}_{(o1)A}$ is not gauge-invariant.

[5] Although the simple integration of Equation (258) yields the time-dependence of β_1, this time-dependence is inconsistent with Equation (241). This inconsistency is due to the fact that we just use the constraint (241) in the form $\partial_r\left(r^2(241)\right)$ when we derive Equation (249).

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