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Tensor Amplitudes for Partial Wave Analysis of $\psi \to \Delta\bar{\Delta}$ within Helicity Frame

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Abstract: We have derived the tensor amplitudes for partial wave analysis of $\psi \to \Delta\bar{\Delta}$, $\Delta \to p\pi$ within the helicity frame, as well as the amplitudes for the other decay sequences with same final states. These formulae are practical for the experiments measuring ψ decaying into $p\bar{p}\pi^+\pi^-$ final states, such as BESIII with its recently collected huge J/ψ and $\psi(2S)$ data samples.

Keywords: ψ decay; baryon; partial wave analysis; tensor formalism

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1. Introduction

Recently, many measurements of J/ψ or $\psi(2S)$ decays into baryon pairs have been reported by BESIII collaboration, such as $\psi(2S) \to \Omega^-\bar{\Omega}^+$ [1], $\psi(2S) \to p\bar{p}(n\bar{n})$ [2], $J/\psi(\psi(2S)) \to \Sigma^+\bar{\Sigma}^-$ [3], $J/\psi(\psi(2S)) \to \Xi^0\bar{\Xi}^0$ [4], $J/\psi(\psi(2S)) \to \Xi^-\bar{\Xi}^+$ [5], $J/\psi \to \Xi(1530)^-\bar{\Xi}^+$ [6], $\psi(2S) \to \Xi(1530)^-\Xi(1530)^+(\Xi(1530)^-\Xi^+)$ [7] etc. These measurements are based on the huge J/ψ or $\psi(2S)$ samples collected at the BESIII detector, which is operating at the electron position collider BEPCII. Taking advantage of the improved statistics, more precise measurements have been achieved, and some insights on the physics have been provided, such as the polarization parameters of baryons, relative phase between electric–magnetic and strong amplitudes, decay mechanism of ψ decaying into baryon pairs, and search for excited baryon states, etc. However, there is no measurement of $\psi \to \Delta\bar{\Delta}$ among these new exciting experimental results. A single ψ will represent both J/ψ and $\psi(2S)$ states later in this paper if not specified. The latest measurement of $\psi \to \Delta\bar{\Delta}$ was conducted by BESII about 20 years ago [8]. The lack of a $\Delta\bar{\Delta}$ study does not mean this channel is not essential. In contrast, a careful analysis of $\psi \to \Delta\bar{\Delta}$ would provide a lot of important information on the decay mechanism of the vector charmonia to the pair of Δ, the first discovered resonance in particle physics, as well as the line shape of the invariant mass of the final states of this resonance. The main difficulty of this measurement is that Δ has a much broader width than other baryon states. From PDG [9], the Breit–Wigner width of the $\Delta(1232)^{++}$ is about 117 MeV, and the widths of other excited Δ states are at the same level. It makes the interference effect significant in the measurement, but that has been ignored in previous measurements [8,10]. When we try to measure $\psi \to \Delta\bar{\Delta}$ via final states protons and pions, which are the products of the strong decays of Δ, the interference between $\Delta^{++}\Delta^{--}$ and $\Delta^0\bar{\Delta}^0$, as well as the excited Δ states and even (excited-)nucleons, is expected to be large. This large interference makes a simple measurement of a two-body decay impossible since the signal yield cannot be extracted by a naive fit with only signal and background components, and needs to be considered carefully.

A partial wave analysis (PWA) is necessary to consider this interference correctly, while suitable formulae to describe these complex processes are still in short supply. Till now, most studies of Δ have been based on πN or γN scattering [11–15], where the PWA is applied but not suitable for $\psi \to \Delta\bar{\Delta}$ studies since there are sequential decaysin the latter

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case. Compared with the abundant PWA formulae on the ψ or heavy vectors decaying into final meson states [16,17], including radiative decays [18], similar results on the baryons are relatively rare. To the best of our knowledge, there are only two papers that have discussed the ψ decays containing final baryon states. One has studied $\psi \to \omega p\bar{p}$ [19], and the other one has studied $\psi \to NN^*M$ [20], but none of them can satisfy our request completely. This situation encourages us to write down the PWA formulae ourselves.

While there are many frames and formalisms in PWA, we choose to derive our formulae within the helicity frame with the covariant tensor formalism [21–24]. This choice is based on the following considerations. Firstly, the helicity frame is suitable for processes of sequential decays. In this frame, all the helicity states are defined in the rest frame of the mother particles, and the whole amplitude is a product of the amplitudes of sequentialprocesses. It reduces the complexity of the derivation of the formulae compared with the PWA based on canonical forms, where an L-S coupling method is used, such as that in [19,20,25]. Secondly, the advantage of the tensor formalism, compared to the helicity formalism, is that a spin tensor can couple to any four-momentum or another spin tensor to form the Lorentz invariant amplitudes, and the momentum dependence is separated explicitly from couplings. The momentum independence is important since it distinguishes the kinematics of an interaction from the dynamics that is really crucial. On the other hand, the helicity formalism takes simpler formulae by hiding this dependence in the so-called couplings, actually form factors. This paper is organized as follows. After this introduction, we discuss all the possible processes, i.e, the intermediate resonances involved in our partial wave analysis in Section 2. Then, we give the Lorentz invariant amplitudes in Section 3, composed of the constructions of each state's wave function, concerns of the parity and charge conjugation translations, and the explicit covariant tensor formulae. Finally, we discuss some validations of these formulae and their possible applications on the present and future electron–positron colliding experiments.

2. Intermediate Resonances and Their J^P

We only consider two kinds of first level decays of ψ, which are $\psi \to \Delta\bar{\Delta}$ and $\psi \to NN^*$. Since our original physics motivation is to measure $\psi \to \Delta^{++}(1232)\Delta^{--}(1232)$ with $\Delta^{++}(1232)$ decaying with almost 100% branching fraction to p and π^+, the suitable final states in experiments are $p\bar{p}\pi^+\pi^-$. The final states determine that there are three possible decay chains, that are $\psi \to \Delta\bar{\Delta}$, $\Delta \to p\pi$, $\psi \to \bar{p}N^*$, $N^* \to \Delta\pi$, $\Delta \to p\pi$, and $\psi \to \bar{p}N^*$, $N^* \to p\rho_0(\sigma)$, $\rho(\sigma) \to \pi^+\pi^-$ as shown in Figures 1–3. The reason why we do not consider the process of $\psi \to \phi X(1870)$, $\phi \to \pi^+\pi^-$, $X(1870) \to p\bar{p}$ is that the mass of $X(1870)$ is near the threshold of the $p\bar{p}$ invariant mass spectrum. Then, the relevant events can be removed easily during the data analysis. In the present and later descriptions, we shall omit the discussion on the charge-conjugated modes for compactness when no ambiguity is caused. Notice there are many possible combinations even in the first-level decays. For example, ψ can decay into the following $\Delta\bar{\Delta}$ pairs if the phase space allows: $\Delta(1232)\bar{\Delta}(1620)$, $\Delta(1620)\bar{\Delta}(1620)$, $\Delta(1232)\bar{\Delta}(1910)$, $\Delta(1620)\bar{\Delta}(1910)$ with Δ representing both Δ^{++} and Δ^0 here. Δ^+ will not be considered since it cannot decay into both charged p and π. For the $\psi \to \bar{p}N^*$ mode, the decay product at the first level can be, for example, $\bar{p}N(1440)$, $\bar{p}N(1520)$, $\bar{p}N(1675)$, etc. Notice that we only consider the relevant nucleon and anti-nucleon pairs that are necessary for the $\Delta\bar{\Delta}$ analysis, not including all possible nucleon and anti-nucleon pairs.

From the physical consideration, complex spin and parity combination sets are obtained. At the first level, decaying from a vector mother particle, i.e., 1^{--}, the J^P of the decay products could be $\frac{1}{2}^+\frac{1}{2}^+$, $\frac{1}{2}^+\frac{1}{2}^-$, $\frac{1}{2}^+\frac{3}{2}^+$, $\frac{1}{2}^+\frac{3}{2}^-$, $\frac{3}{2}^+\frac{3}{2}^+$, $\frac{3}{2}^+\frac{3}{2}^-$, $\frac{5}{2}^+\frac{1}{2}^+$, $\frac{5}{2}^+\frac{1}{2}^-$, $\frac{5}{2}^+\frac{3}{2}^+$, $\frac{5}{2}^+\frac{3}{2}^-$, $\frac{7}{2}^+\frac{3}{2}^+$, and $\frac{7}{2}^+\frac{3}{2}^-$; at the second level, if the decays are from Δ, the final states are always $\frac{1}{2}^+0^-$, while the J^P of the Δ can be $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^+$, $\frac{3}{2}^-$, $\frac{5}{2}^+$, $\frac{5}{2}^-$, $\frac{7}{2}^+$, and $\frac{7}{2}^-$; if a decay chain contains N^* then there is another decay mode with final states $\Delta\pi$, i.e., $J^P = \frac{3}{2}^+0^-$, while the J^P of the initial N^* can be $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^+$, $\frac{3}{2}^-$, $\frac{5}{2}^+$, $\frac{5}{2}^-$ $\frac{7}{2}^+$, and $\frac{7}{2}^-$. In the processes

involving the combination sets mentioned above, the orbital momenta should take all possible values to ensure conservation of the total angular momentum and parity.

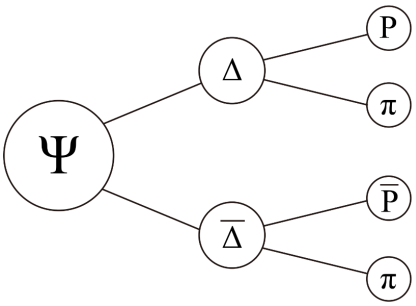


Figure 1. Illustration of decay $\psi \to \Delta\bar{\Delta}$, $\Delta \to p\pi$.

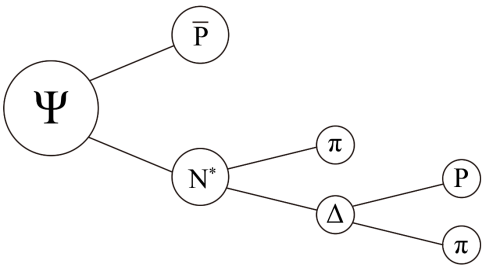


Figure 2. Illustration of decay $\psi \to \bar{p}N^*$, $N^* \to \Delta\pi$, $\Delta \to p\pi$.

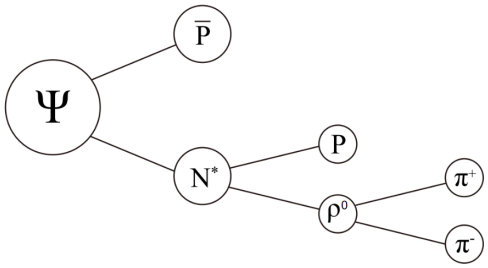


Figure 3. Illustration of decay $\psi \to \bar{p}N^*$, $N^* \to p\rho_0(\sigma)$, $\rho(\sigma) \to \pi^+\pi^-$.

3. Preparations

3.1. Wave Functions

To construct the covariant tensor amplitudes, we need the tensor wave functions describing relativistic particles with arbitrary spin (helicity) and satisfying the Rarita–Schwinger formalism [26]. We will follow the method proposed by Auvil and Brehm [27] to write down these wave functions explicitly. The wave function of a scalar particle is constant. We start with a particle of spin-1 at rest, whose wave functions may be expressed explicitly as column vectors:

$$e(\pm 1) = \mp\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm\mathrm{i}\\ 0\end{pmatrix}, \quad e(0) = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{1}$$

Since a particle at rest cannot have the energy component if the spin (helicity) wave function is presented in the momentum space, the four-vector describing a spin-1 particle at rest is defined as

$$e^{\mu}(0,\lambda) = \{0, \vec{e}(\lambda)\}, \quad e_{\mu}(0,\lambda) = \{0, -\vec{e}(\lambda)\}. \tag{2}$$

Then, the general helicity state vector with four-momentum p can be obtained by performing a Lorentz boost in the z direction and a proper rotation:

$$e^{\mu}(p,0)=\frac{E}{m}\begin{pmatrix}-p/E\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix},\quad e^{\mu}(p,\pm1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \mp\cos\theta\cos\phi+\mathrm{i}\sin\phi\\ \mp\cos\theta\sin\phi-\mathrm{i}\cos\phi\\ \pm\sin\theta\end{pmatrix}, \tag{3}$$

where m, E, p, and (θ,ϕ) are the invariant mass, energy, momentum, and helicity (polar and azimuthal) angles of the particle, respectively, and satisfy $E^2=p^2+m^2$ with the natural unit. The wave function describing spin-2 particles can be constructed out of the wave functions of spin-1 via C-G coefficients as follows:

$$e^{\mu\nu}(p,2\lambda)=\sum_{\lambda_1\lambda_2}(1\lambda_1 1\lambda_2|2\lambda)e^{\mu}(p,\lambda_1)e^{\nu}(p,\lambda_2). \tag{4}$$

Then, the wave function of spin-n ($n\in$ I) particles can be derived in a cumulative way as tensor $e^{\mu_1\cdots\mu_n}(p,n\lambda)$ [28]:

$$e^{\mu_1\cdots\mu_n}(p,n\lambda)=\sum_{\lambda_1,\lambda_2}(n\text{–}1\lambda_1 1\lambda_2|n\lambda)e^{\mu_1\cdots\mu_{n-1}}(p,n\text{–}1\lambda_1)e^{\mu_n}(p,\lambda_2)\,. \tag{5}$$

Therefore, the wave functions should satisfy the Rarita–Schwinger conditions:

$$p^{\mu_1}e_{\mu_1\cdots\mu_n}=0\,, \tag{6}$$

$$e_{\cdots\mu_i\cdots\mu_j\cdots}=e_{\cdots\mu_j\cdots\mu_i\cdots}\,, \tag{7}$$

$$g^{\mu_1\mu_2}e_{\mu_1\cdots\mu_n}=0\,. \tag{8}$$

We shall turn to the Fermion particles now. Consider a spin-1/2 particle at rest and its basis vectors may be given by the spinors or two-dimensional column vectors:

$$\chi(+\frac{1}{2})=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi(-\frac{1}{2})=\begin{pmatrix}0\\1\end{pmatrix}. \tag{9}$$

We adopt the four-component Dirac spinor to describe the relativistic spin 1/2 particles. Then, the two-component spinors $\chi(\lambda)$ are generalized to the four-component spinors $u(0,\lambda)$:

$$u(0,\lambda)=\begin{pmatrix}\chi(\lambda)\\0\end{pmatrix}, \tag{10}$$

where $\lambda=\pm\frac{1}{2}$. After a boost, the helicity wave functions' explicit expressions can be cast into the form

$$u(p,+1/2)=\frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{E+m}\cos(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E+m}\sin(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\\ \sqrt{E-m}\cos(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E-m}\sin(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\end{pmatrix},u(p,-1/2)=\frac{1}{\sqrt{2m}}\begin{pmatrix}-\sqrt{E+m}\sin(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E+m}\cos(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\\ \sqrt{E-m}\sin(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ -\sqrt{E-m}\cos(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\end{pmatrix}. \tag{11}$$

The "adjoint" spinor is defined as

$$\bar{u}(p,\lambda)=u^{\dagger}(p,\lambda)\gamma^0\,. \tag{12}$$

The helicity wave functions for the anti-Fermion can be written down similarly as

$$v(p,+1/2)=\frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{E-m}\cos(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E-m}\sin(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\\ \sqrt{E+m}\cos(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E+m}\sin(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\end{pmatrix},v(p,-1/2)=\frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{E-m}\sin(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ -\sqrt{E-m}\cos(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\\ -\sqrt{E+m}\sin(\theta/2)[\cos(\phi/2)-\mathrm{i}\sin(\phi/2)]\\ \sqrt{E+m}\cos(\theta/2)[\cos(\phi/2)+\mathrm{i}\sin(\phi/2)]\end{pmatrix}. \tag{13}$$

Following Auvil and Brehm, wave functions corresponding to particles of spin $j = n + 1/2$ can be constructed by the spin-1/2 wave functions and spin-n wave functions with C-G coefficients as the following:

$$u^{\mu_1\cdots\mu_n}(p, j\lambda) = \sum_{\lambda_1\lambda_2} (n\lambda_1 \frac{1}{2}\lambda_2|j\lambda) e^{\mu_1\cdots\mu_n}(p, n\lambda_1) u(p, \lambda_2) \,. \tag{14}$$

The spin-j wave function Equation (14) is a four-component spinor with the four-vector indices $\mu_1 \cdots \mu_n$. Since it describes a state of spin j, it can have only $(2j + 1)$ independent components. The desired supplementary conditions are just the Rarita–Schwinger equations:

$$(\gamma^\mu p_\mu - m) u_{\mu_1\cdots\mu_n} = 0 \,, \tag{15}$$

$$u_{\cdots\mu_i\cdots\mu_j\cdots} = u_{\cdots\mu_j\cdots\mu_i\cdots} \,, \tag{16}$$

$$p^{\mu_1} u_{\mu_1\cdots\mu_n} = 0 \,, \tag{17}$$

$$\gamma^{\mu_1} u_{\mu_1\cdots\mu_n} = 0 \,, \tag{18}$$

$$g^{\mu_1\mu_2} u_{\mu_1\cdots\mu_n} = 0 \,. \tag{19}$$

where m is the mass of the spin-j particle and p is its four momentum.

3.2. Effective Vertices

We also need effective vertices to construct various partial wave amplitudes. The principle idea is that the decay mechanism has been considered an effective interaction. Then, all the loops in the Feynman diagrams are absorbed into the effective vertices [29]. A general form of any amplitude in a single decay chain can be expressed as

$$A = \bar{u}_1 \Gamma u_2 e B, \tag{20}$$

in which $\bar{u}_1$, u_2, and e are wave functions of two baryons and a meson, respectively; B is considered the kernel of the propagator and usually parameterized as a Breit–Wigner function; and Γs are tensors representing effective vertices, which are composed by $\mathbb{I}$, p^μ, γ^5, γ^μ, $\sigma^{\mu\nu}$, $g^{\mu\nu}$, $\epsilon^{\mu\nu\alpha\beta}$. The main target of this paper is to find all the independent effective vertices. Before that, it is worth taking some time to consider the constraints on them due to symmetry. Since, in these decays, the strong interaction is dominant, the conservation is expected under the transformation of parity P and charge conjugate C, respectively [30]. However, $\bar{u}_1$ and u_2 do not correspond to precisely a particle and its anti-particle, and the charged particles do not have determined C-parity, so we only consider the symmetry of parity, and its conservation requires

$$\eta_1^* \eta_2 \eta_A \Gamma_P = 1, \tag{21}$$

where Γ_P is the transformation property of different tensors, and η_1, η_2, and η_A are the parities of the two baryons and one meson, respectively. The properties of different tensors under the P transformation are listed in Table 1. And we also notice the transformation property of wave function $e^{\mu_1\cdots\mu_n}$ will be $\prod_{i=1}^{n}(-1)^{\mu_i}$ as p^μ and $\epsilon^{\mu\nu\alpha\beta}$ since each of them only takes Lorentz indexes without the Dirac ones.

Table 1. The parity transformation properties of some tensors, the shorthand $(-1)^\mu \equiv 1$ for $\mu = 0$ while -1 for $\mu = 1, 2, 3$ is used as in [31].

	$\mathbb{I}$	γ^5	γ^μ	$\gamma^\mu\gamma^5$	$\sigma^{\mu\nu}$	p^μ	$\sigma^{\mu\nu}\gamma^5$
Γ_P	1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$

4. Decay Amplitudes in Tensor Formalism

Now we are ready to derive the covariant invariant amplitudes in the tensor formalism for resonance decays. Let us consider a resonance of spin parity J^P and mass m, decaying into two particles 1 and 2:

$$J \to 1 + 2\,.$$

In the rest frame of the mother resonance, let p be the momentum of particle 1 with helicity angles (θ, ϕ). Usually, the amplitude A describing the decay process as two particles with helicity λ_1 and λ_2 may be written as

$$A(\lambda_1, \lambda_2, \theta, \phi) = N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M,\lambda_1-\lambda_2}(\phi, \theta, 0) \tag{22}$$

in the helicity formalism, where N_J is the normalization factor, $F^J_{\lambda_1\lambda_2}$ is the helicity coupling, and $D^{J*}_{M,\lambda_1-\lambda_2}$ is the D function, with M being the z-component of the mother's spin. However, with the explicit wave functions and effective vertices, we can construct the independent covariant invariant amplitudes in the tensor formalism:

4.1. $\psi \to \Delta\bar{\Delta}$

$$\begin{aligned}\psi \to \Delta(\tfrac{1}{2}^+)\bar{\Delta}(\tfrac{1}{2}^-): \quad & g_1\bar{u}(p_1, \tfrac{1}{2}\lambda_1)\gamma^\mu v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_2\bar{u}(p_1, \tfrac{1}{2}\lambda_1)\sigma^{\mu\nu}P_\nu v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{23}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{1}{2}^+)\bar{\Delta}(\tfrac{1}{2}^+): \quad & g_1\bar{u}(p_1, \tfrac{1}{2}\lambda_1)\gamma^\mu\gamma^5 v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_2\bar{u}(p_1, \tfrac{1}{2}\lambda_1)\sigma^{\mu\nu}P_\nu\gamma^5 v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{3}{2}^+)\bar{\Delta}(\tfrac{1}{2}^-): \quad & g_1\bar{u}^\mu(p_1, \tfrac{3}{2}\lambda_1)v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_2\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)\gamma^\mu P^\nu v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_3\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)p_2^\mu P^\nu v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{25}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{3}{2}^+)\bar{\Delta}(\tfrac{1}{2}^+): \quad & g_1\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)\gamma^5 v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_2\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_3\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)p_2^\mu P^\nu\gamma^5 v(p_2, \tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{26}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{3}{2}^+)\bar{\Delta}(\tfrac{3}{2}^-): \quad & g_1\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)\gamma^\mu v^\nu(p_2, \tfrac{3}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_2\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)p_2^\mu v^\nu(p_2, \tfrac{3}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_3\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)p_2^\nu v^\mu(p_2, \tfrac{3}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_4\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)\gamma^\mu P^\nu P_\alpha v^\alpha(p_2, \tfrac{3}{2}\lambda_2)e_\mu(P,\lambda) + \\ & g_5\bar{u}_\nu(p_1, \tfrac{3}{2}\lambda_1)p_1^\mu P^\nu P_\alpha v^\alpha(p_2, \tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{27}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{3}{2}^+)\bar{\Delta}(\tfrac{3}{2}^+): \quad & g_1\bar{u}_\nu(p_1,\tfrac{3}{2}\lambda_1)\gamma^\mu\gamma^5 v^\nu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}_\nu(p_1,\tfrac{3}{2}\lambda_1)p_2^\mu\gamma^5 v^\nu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}_\nu(p_1,\tfrac{3}{2}\lambda_1)p_2^\nu\gamma^5 v^\mu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_4\bar{u}_\nu(p_1,\tfrac{3}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu P_\alpha v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_5\bar{u}_\nu(p_1,\tfrac{3}{2}\lambda_1)p_1^\mu P^\nu P_\alpha\gamma^5 v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{28}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{5}{2}^+)\bar{\Delta}(\tfrac{1}{2}^-): \quad & g_1\bar{u}_{\mu\nu}(p_1,\tfrac{5}{2}\lambda_1)P^\nu v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}_{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu P^\nu P^\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}_{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P^\nu P^\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{29}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{5}{2}^+)\bar{\Delta}(\tfrac{1}{2}^+): \quad & g_1\bar{u}_{\mu\nu}(p_1,\tfrac{5}{2}\lambda_1)P^\nu\gamma^5 v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}_{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu P^\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}_{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P^\nu P^\alpha\gamma^5 v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{30}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{5}{2}^+)\bar{\Delta}(\tfrac{3}{2}^-): \quad & g_1\bar{u}^{\mu\nu}(p_1,\tfrac{5}{2}\lambda_1)v_\nu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu P_\nu v_\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P_\nu v_\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_4\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)P^\nu P_\alpha v^\mu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_5\bar{u}^{\nu\beta}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu P_\nu P_\beta P_\alpha v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_6\bar{u}^{\nu\beta}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P_\nu P_\beta P_\alpha v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{31}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{5}{2}^+)\bar{\Delta}(\tfrac{3}{2}^+): \quad & g_1\bar{u}^{\mu\nu}(p_1,\tfrac{5}{2}\lambda_1)\gamma^5 v_\nu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu\gamma^5 P_\nu v_\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P_\nu\gamma^5 v_\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_4\bar{u}^{\nu\alpha}(p_1,\tfrac{5}{2}\lambda_1)P^\nu P_\alpha\gamma^5 v^\mu(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_5\bar{u}^{\nu\beta}(p_1,\tfrac{5}{2}\lambda_1)\gamma^\mu\gamma^5 P_\nu P_\beta P_\alpha v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_6\bar{u}^{\nu\beta}(p_1,\tfrac{5}{2}\lambda_1)p_1^\mu P_\nu P_\beta P_\alpha\gamma^5 v^\alpha(p_2,\tfrac{3}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{32}$$

$$\begin{aligned}\psi \to \Delta(\tfrac{7}{2}^+)\bar{\Delta}(\tfrac{1}{2}^-): \quad & g_1\bar{u}^{\mu\nu\alpha}(p_1,\tfrac{7}{2}\lambda_1)P_\nu P_\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_2\bar{u}_{\nu\beta\alpha}(p_1,\tfrac{7}{2}\lambda_1)\gamma^\mu P^\nu P^\beta P^\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\ & g_3\bar{u}_{\nu\beta\alpha}(p_1,\tfrac{7}{2}\lambda_1)p_1^\mu P^\nu P^\beta P^\alpha v(p_2,\tfrac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,\end{aligned} \tag{33}$$

$$\begin{aligned}
\psi \to \Delta(\frac{7}{2}^+)\bar{\Delta}(\frac{1}{2}^+): \quad & g_1\bar{u}^{\mu\nu\alpha}(p_1,\frac{7}{2}\lambda_1)P_\nu P_\alpha\gamma^5 v(p_2,\frac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_2\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu P^\beta P^\alpha v(p_2,\frac{1}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_3\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)p_1^\mu P^\nu P^\beta P^\alpha\gamma^5 v(p_2,\frac{1}{2}\lambda_2)e_\mu(P,\lambda)\,,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\psi \to \Delta(\frac{7}{2}^+)\bar{\Delta}(\frac{3}{2}^-): \quad & g_1\bar{u}^{\mu\nu\alpha}(p_1,\frac{7}{2}\lambda_1)P_\nu v_\alpha(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_2\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)\gamma^\mu P^\nu P^\alpha v^\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_3\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)p_1^\mu P^\nu P_\alpha v^\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_4\bar{u}_{\mu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)P^\nu P^\beta P^\alpha v_\nu(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_5\bar{u}_{\nu\delta\alpha}(p_1,\frac{7}{2}\lambda_1)\gamma^\mu P^\nu P^\delta P^\alpha P^\beta v_\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_6\bar{u}_{\nu\delta\alpha}(p_1,\frac{7}{2}\lambda_1)p_1^\mu P^\nu P^\delta P^\alpha P^\beta v_\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)\,,
\end{aligned} \tag{35}$$

$$\begin{aligned}
\psi \to \Delta(\frac{7}{2}^+)\bar{\Delta}(\frac{3}{2}^+): \quad & g_1\bar{u}^{\mu\nu\alpha}(p_1,\frac{7}{2}\lambda_1)P_\nu\gamma^5 v_\alpha(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_2\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu P^\alpha v^\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_3\bar{u}_{\nu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)p_1^\mu P^\nu P_\alpha\gamma^5 v^\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_4\bar{u}_{\mu\beta\alpha}(p_1,\frac{7}{2}\lambda_1)P^\nu P^\beta P^\alpha\gamma^5 v_\nu(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_5\bar{u}_{\nu\delta\alpha}(p_1,\frac{7}{2}\lambda_1)\gamma^\mu\gamma^5 P^\nu P^\delta P^\alpha P^\beta v_\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)+\\
& g_6\bar{u}_{\nu\delta\alpha}(p_1,\frac{7}{2}\lambda_1)p_1^\mu P^\nu P^\delta P^\alpha P^\beta\gamma^5 v_\beta(p_2,\frac{3}{2}\lambda_2)e_\mu(P,\lambda)\,.
\end{aligned} \tag{36}$$

4.2. $\Delta \to p\pi$

$$\Delta(\frac{1}{2}^+) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)\gamma^5 u(P,\frac{1}{2}\lambda)\,, \tag{37}$$

$$\Delta(\frac{1}{2}^-) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)u(P,\frac{1}{2}\lambda)\,, \tag{38}$$

$$\Delta(\frac{3}{2}^+) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu\gamma^5 u_\mu(P,\frac{3}{2}\lambda)\,, \tag{39}$$

$$\Delta(\frac{3}{2}^-) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu u_\mu(P,\frac{3}{2}\lambda)\,, \tag{40}$$

$$\Delta(\frac{5}{2}^+) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu\gamma^5 u_{\mu\nu}(P,\frac{5}{2}\lambda)\,, \tag{41}$$

$$\Delta(\frac{5}{2}^-) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu u_{\mu\nu}(P,\frac{5}{2}\lambda)\,, \tag{42}$$

$$\Delta(\frac{7}{2}^+) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha\gamma^5 u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)\,, \tag{43}$$

$$\Delta(\frac{7}{2}^-) \to p(\frac{1}{2}^+)\pi(0^-): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)\,. \tag{44}$$

4.3. $\psi \to N^* \bar{N}$

This type of partial wave formula for $\psi \to N^* \bar{N}$ where the spin of $\bar{N}$ is $\frac{1}{2}^-$ has been included in $\psi \to \Delta\bar{\Delta}$ and will not be repeated.

4.4. $N^* \to \Delta\pi$

$$N^*(\frac{1}{2}^+) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad g\bar{u}_\mu(p_1, \frac{3}{2}\lambda_1)P^\mu\gamma^5 u(P, \frac{1}{2}\lambda)\,, \tag{45}$$

$$N^*(\frac{1}{2}^-) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad g\bar{u}_\mu(p_1, \frac{3}{2}\lambda_1)P^\mu u(P, \frac{1}{2}\lambda)\,, \tag{46}$$

$$\begin{aligned} N^*(\frac{3}{2}^+) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}_\mu(p_1, \frac{3}{2}\lambda_1)\gamma^5 u^\mu(P, \frac{3}{2}\lambda) + \\ & g_2\bar{u}_\nu(p_1, \frac{3}{2}\lambda_1)p_1^\mu P^\nu\gamma^5 u_\mu(P, \frac{3}{2}\lambda)\,, \end{aligned} \tag{47}$$

$$\begin{aligned} N^*(\frac{3}{2}^-) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}_\mu(p_1, \frac{3}{2}\lambda_1)u^\mu(P, \frac{3}{2}\lambda) + \\ & g_2\bar{u}_\nu(p_1, \frac{3}{2}\lambda_1)p_1^\mu P^\nu u_\mu(P, \frac{3}{2}\lambda)\,, \end{aligned} \tag{48}$$

$$\begin{aligned} N^*(\frac{5}{2}^+) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}^\mu(p_1, \frac{3}{2}\lambda_1)p_1^\mu\gamma^5 u_{\mu\nu}(P, \frac{5}{2}\lambda) + \\ & g_2\bar{u}_\alpha(p_1, \frac{3}{2}\lambda_1)p_1^\mu p_1^\nu P^\alpha\gamma^5 u_{\mu\nu}(P, \frac{5}{2}\lambda)\,, \end{aligned} \tag{49}$$

$$\begin{aligned} N^*(\frac{5}{2}^-) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}^\nu(p_1, \frac{3}{2}\lambda_1)P^\mu u_{\mu\nu}(P, \frac{5}{2}\lambda) + \\ & g_2\bar{u}_\alpha(p_1, \frac{3}{2}\lambda_1)p_1^\mu p_1^\nu P^\alpha u_{\mu\nu}(P, \frac{5}{2}\lambda)\,, \end{aligned} \tag{50}$$

$$\begin{aligned} N^*(\frac{7}{2}^+) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}^\mu(p_1, \frac{3}{2}\lambda_1)p_1^\nu p_1^\alpha\gamma^5 u_{\mu\nu\alpha}(P, \frac{7}{2}\lambda) + \\ & g_2\bar{u}_\beta(p_1, \frac{3}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha P^\beta\gamma^5 u_{\mu\nu\alpha}(P, \frac{7}{2}\lambda)\,, \end{aligned} \tag{51}$$

$$\begin{aligned} N^*(\frac{7}{2}^-) \to \Delta(\frac{3}{2}^+)\pi(0^-): \quad & g_1\bar{u}^\mu(p_1, \frac{3}{2}\lambda_1)p_1^\nu p_1^\alpha u_{\mu\nu\alpha}(P, \frac{7}{2}\lambda) + \\ & g_2\bar{u}_\beta(p_1, \frac{3}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha P^\beta u_{\mu\nu\alpha}(P, \frac{7}{2}\lambda)\,. \end{aligned} \tag{52}$$

4.5. $N^* \to p\rho^0$

$$\begin{aligned} N^*(\frac{1}{2}^+) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad & g_1\bar{u}(p_1, \frac{1}{2}\lambda_1)\gamma^\mu\gamma^5 u(P, \frac{1}{2}\lambda)e^*_\mu(p_2, \lambda_2) + \\ & g_2\bar{u}(p_1, \frac{1}{2}\lambda_1)P^\mu\gamma^5 u(P, \frac{1}{2}\lambda)e^*_\mu(p_2, \lambda_2)\,, \end{aligned} \tag{53}$$

$$\begin{aligned} N^*(\frac{1}{2}^-) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad & g_1\bar{u}(p_1, \frac{1}{2}\lambda_1)\gamma^\mu u(P, \frac{1}{2}\lambda)e^*_\mu(p_2, \lambda_2) + \\ & g_2\bar{u}(p_1, \frac{1}{2}\lambda_1)P^\mu u(P, \frac{1}{2}\lambda)e^*_\mu(p_2, \lambda_2)\,, \end{aligned} \tag{54}$$

$$\begin{aligned} N^*(\frac{3}{2}^+) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad & g_1\bar{u}(p_1, \frac{1}{2}\lambda_1)\gamma^5 u^\mu(P, \frac{3}{2}\lambda)e^*_\mu(p_2, \lambda_2) + \\ & g_2\bar{u}(p_1, \frac{1}{2}\lambda_1)p_1^\mu\gamma^\nu\gamma^5 u_\mu(P, \frac{3}{2}\lambda)e^*_\nu(p_2, \lambda_2) + \\ & g_3\bar{u}(p_1, \frac{1}{2}\lambda_1)p_1^\mu P^\nu\gamma^5 u_\mu(P, \frac{3}{2}\lambda)e^*_\nu(p_2, \lambda_2)\,, \end{aligned} \tag{55}$$

$$N^*(\frac{3}{2}^-) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad g_1\bar{u}(p_1,\frac{1}{2}\lambda_1)u^\mu(P,\frac{3}{2}\lambda)e^*_\mu(p_2,\lambda_2)+ g_2\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu\gamma^\nu u_\mu(P,\frac{3}{2}\lambda)e^*_\nu(p_2,\lambda_2)+ g_3\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu P^\nu u_\mu(P,\frac{3}{2}\lambda)e^*_\nu(p_2,\lambda_2)\,, \tag{56}$$

$$N^*(\frac{5}{2}^+) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad g_1\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu\gamma^5 u_{\mu\nu}(P,\frac{5}{2}\lambda)e^{*\nu}(p_2,\lambda_2)+ g_2\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu\gamma^\alpha\gamma^5 u_{\mu\nu}(P,\frac{5}{2}\lambda)e^*_\alpha(p_2,\lambda_2)+ g_3\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu P^\alpha\gamma^5 u_{\mu\nu}(P,\frac{5}{2}\lambda)e^*_\alpha(p_2,\lambda_2)\,, \tag{57}$$

$$N^*(\frac{5}{2}^-) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad g_1\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu u_{\mu\nu}(P,\frac{5}{2}\lambda)e^{*\nu}(p_2,\lambda_2)+ g_2\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu\gamma^\alpha u_{\mu\nu}(P,\frac{5}{2}\lambda)e^*_\alpha(p_2,\lambda_2)+ g_3\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu P^\alpha u_{\mu\nu}(P,\frac{5}{2}\lambda)e^*_\alpha(p_2,\lambda_2)\,, \tag{58}$$

$$N^*(\frac{7}{2}^+) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad g_1\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)\gamma^5 e^{*\alpha}(p_2,\lambda_2)+ g_2\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu p_1^\beta\gamma^\alpha u_{\mu\nu\beta}(P,\frac{7}{2}\lambda)\gamma^5 e^*_\alpha(p_2,\lambda_2)+ g_3\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu p_2^\beta P^\alpha u_{\mu\nu\beta}(P,\frac{7}{2}\lambda)\gamma^5 e^*_\alpha(p_2,\lambda_2)\,, \tag{59}$$

$$N^*(\frac{7}{2}^-) \to p(\frac{1}{2}^+)\rho^0(1^-): \quad g_1\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)e^{*\alpha}(p_2,\lambda_2)+ g_2\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu p_1^\beta\gamma^\alpha u_{\mu\nu\beta}(P,\frac{7}{2}\lambda)e^*_\alpha(p_2,\lambda_2)+ g_3\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_2^\nu p_2^\beta P^\alpha u_{\mu\nu\beta}(P,\frac{7}{2}\lambda)e^*_\alpha(p_2,\lambda_2)\,. \tag{60}$$

4.6. The Formula of $N^* \to p\sigma$

$$N^*(\frac{1}{2}^+) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)u(P,\frac{1}{2}\lambda)\,, \tag{61}$$

$$N^*(\frac{1}{2}^-) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)\gamma^5 u(P,\frac{1}{2}\lambda)\,, \tag{62}$$

$$N^*(\frac{3}{2}^+) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_{1\mu}u^\mu(P,\frac{3}{2}\lambda)\,, \tag{63}$$

$$N^*(\frac{3}{2}^-) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)\gamma^5 u^\mu(P,\frac{3}{2}\lambda)p_{1\mu}\,, \tag{64}$$

$$N^*(\frac{5}{2}^+) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu u_{\mu\nu}(P,\frac{5}{2}\lambda)\,, \tag{65}$$

$$N^*(\frac{5}{2}^-) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu\gamma^5 u_{\mu\nu}(P,\frac{5}{2}\lambda)\,, \tag{66}$$

$$N^*(\frac{7}{2}^+) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)\,, \tag{67}$$

$$N^*(\frac{7}{2}^-) \to p(\frac{1}{2}^+)\sigma(0^+): \quad g\bar{u}(p_1,\frac{1}{2}\lambda_1)p_1^\mu p_1^\nu p_1^\alpha\gamma^5 u_{\mu\nu\alpha}(P,\frac{7}{2}\lambda)\,, \tag{68}$$

4.7. $\rho^0(\sigma) \to \pi^+\pi^-$

For completeness of the given formulae, we give the amplitudes of $\sigma(0^+) \to \pi^+\pi^-$ and $\rho^0 \to \pi^+\pi^-$. The amplitude formula of $\sigma(0^+) \to \pi^+\pi^-$ is 1. The amplitude formula of $\rho^0 \to \pi^+\pi^-$ is

$$\rho^0(1^-) \to \pi^+(0^-)\pi^-(0^-): \quad g p_1^\mu e_\mu(P,\lambda)\,. \tag{69}$$

5. Discussion

5.1. Comparison to Helicity Formalism

Compared with the amplitudes in helicity formalism Equation (22), it is obvious that the helicity coupling $F^J_{\lambda_1\lambda_2}$ ought to have momentum dependence but not expressed explicitly as that in the tensor formalism. When the measurements deal with narrow resonances such as Λ and Σ, this momentum dependence can be safely ignored. However, when the target resonance is the broad Δ, separating the momentum dependence, i.e., the kinematic effect, from the dynamic part is crucial for the study of the Δ line shape that reflects the internal interactions. As a cross check, we also calculate the number of $F^J_{\lambda_1\lambda_2}$ considering the relevant symmetry, and it is found to be consistent with the number of g_i in the tensor formalism.

5.2. Amplitude for Sequential Decays

In the previous section, we gave the sub-amplitudes for each interaction vertex. The whole amplitude of a decay chain then can be obtained in a straightforward way by the product of each sub-amplitude in the cascading decays. Thus, the cross-section can be written as

$$\sigma = \int \mathrm{d}\Omega |\mathcal{M}|^2 \tag{70}$$

with

$$\mathcal{M} = \sum_\Lambda \prod_i M_i(\Lambda) = \sum_\Lambda \prod_i BW_i \sum_j g_i^j A_i^j(\Lambda)\,, \tag{71}$$

where Ω and Λ indicate collectively all angular and helicity variables of decay chains and of the intermediate and final states. BW_i are the propagators describing the dynamic interactions of each decay process, usually expressed in a Breit–Wigner form. Notice that in the helicity frame, the direction of the polarization of a state would be different when this state is in different decay chains. So an alignment, usually a rotation, is required to align the directions. This kind of alignment has been handled by experimentalists already [32,33] and been discussed further by new means in [34–36].

6. Summary

In this paper, we have derived the helicity amplitudes in the tensor formalism needed for the measurement of $\psi \to \Delta\bar{\Delta}$, $\Delta \to p\pi$, as well as the formulae for the sequential decaysof the other processes with the same final states. The covariant tensor formalism is adopted instead of the helicity formalism to make the momentum dependence explicit then more suitable to the measurements with broad resonances such as Δ and its excited states. The number of independent terms in each decay chain has been checked and confirmed to be the same as the number of helicity amplitudes, and of course, the same as the number of partial waves in the decay too. These formulae are prepared for the measurements of ψ decaying into $p\bar{p}\pi^+\pi^-$ final states, and also can be extended to the final states, such as $p\bar{p}K^+K^-$ since their spins and parities are same. Experiments which collected large J/ψ and $\psi(2S)$ data samples, such as BESIII, would benefit from this study. Furthermore, as soon as the transitions of Δ into nucleon and pion are well known phenomenologically and experimentally, some numerical analyses of the cascade decay of the charmonia states will be useful for readers.

It should be noticed only P-parity but not C-parity conservation is considered during the formula derivation since the fermions do not have certain C-parity. But in some special

cases, where the daughter states are particles and anti-particles to each other, the C-parity is determined to be $(-1)^{L+S}$. It means if the L-S coupling scheme has been adopted, there is a chance to check the C violation by measuring the amplitude of the corresponding wave. However, the construction of pure spin wave functions based on high-fractional spin states such as two 3/2-spin states is beyond our ability. It also should be noticed that sometimes, the momentum notations P, p_1, and p_2 are replaceable in the formulae since the constructed terms are dependent on each other, so the choice between them is somewhat just arbitrary. Breit–Wigner functions and projection operators are important in the experimental measurements too. But the projection operators can be obtained by the construction of wave functions mentioned in this paper straightforwardly. So we ignore a discussion of them. Furthermore, a complete discussion of the Breit–Wigner functions is beyond this paper's scope, and their choices are left to experimentalists for when some specific forms are needed for specific resonances or decay channels.

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