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Article

Anti-Evaporation of Black Holes in Bigravity

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Abstract: We review properties of solutions in bigravity theory for a specific case where two metric tensors, $g_{\mu\nu}$ and $f_{\mu\nu}$, satisfy proportional relation $f_{\mu\nu} = C^2 g_{\mu\nu}$. For this condition, we find that the solutions describing the asymptotically de Sitter space-time can be obtained and investigate the perturbation around the Schwarzschild–de Sitter solutions and corresponding anti-evaporation. We discuss the stability under special perturbations related to the anti-evaporation and the importance of the non-diagonal components of the metric in bigravity.

Keywords: modified gravity; bigravity; quantum field theory

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1. Introduction

Recently, much attention has been paid to bigravity theory [1,2], which includes two independent metric tensor fields, $g_{\mu\nu}$ and $f_{\mu\nu}$. Bigravity contains a massive spin-2 propagating mode in addition to the ordinary massless spin-2 graviton, and it has been successfully constructed as the generalization of de Rham–Gabadadze–Tolley (dRGT) massive gravity [3–5], which describes a ghost-free massive spin-2 field theory. In this section, we review the history from free massive spin-2 field theory to bigravity [6,7].

The basics of the massive spin-2 field theory were established by Fierz and Pauli [8]. They constructed the consistent free massive spin-2 theory by adding a tuned mass term to free massless spin-2 field theory on flat space-time. However, it was shown that the Fierz–Pauli theory cannot recover general relativity in the massless limit, known as the van Dam–Veltman–Zakharov (vDVZ) discontinuity [9,10]. Here, arbitrary interactions can be added to the theory because the massive spin-2 theory does not have any gauge symmetry. Since the massless spin-2 theory is given by the perturbative expansion of the Einstein–Hilbert action, a straightforward way to extend the Fierz–Pauli theory to interacting theory is to use the Ricci scalar instead of the kinetic terms by Fierz and Pauli. As a result, the vDVZ discontinuity can be screened by a non-linear effect coming from the Ricci scalar, which is called the Vainshtein mechanism [11]. On the other hand, the non-linear terms leads to ghost called the Boulware–Deser (BD) ghost [12]. This problem of the ghost mode had been discussed for a long time (see, for instance, [13]), and the problem was solved finally as dRGT massive gravity by introducing a new form of mass terms.

Next, we consider interacting theory, which includes several spin-2 fields. It has been shown that there is no consistent interacting theory where all of the spin-2 fields are massless [14]; thus, the massive spin-2 field always appears in the interacting theory. The theory describing the interaction between two spin-2 fields, where one field is massless and another should be massive, is called bi-metric or bigravity theory. This theory was probably first proposed by Rosen [15–17] and had been studied as f-g gravity or strong gravity theory [18–20]. Finally, the ghost-free interaction between massless and massive spin-2 fields was established as a generalization of ghost-free massive spin-2 field theory, that is the dRGT massive gravity. The dRGT massive gravity is formulated with two metric tensors, where one is a dynamical metric and another is non-dynamical. When we extend the theory to make both of the two metrics dynamical, the bigravity theory can be obtained [1,2].

Extensions of the massive gravity and bigravity were under lively discussion while the fundamental idea was established. The ghost-free interaction in the massive gravity might be generalized to add new interaction terms without generating any ghost mode. New models with derivative and non-derivative interaction terms have been proposed [21–25]. There is also another extension that modifies the kinetic term of the gravitational action. The Einstein–Hilbert action is used to generate the kinetic term of the massive spin-2 propagating mode in the massive gravity and bigravity. On the other hand, it has been proposed that the Ricci scalar can be generalized to the function of it, from the point of view of $F(R)$ gravity [26,27]. Some cosmological models in $F(R)$ bigravity have been argued [28–31].

Some people expect that the new degrees of freedom introduced by another metric can solve remaining problems in cosmology, that is dark energy and dark matter problems. The cosmological constant could be effectively produced from the interactions between two metric tensors [32–36], and the massive spin-2 fields and matter fields coupled to the metric $f_{\mu\nu}$ can be candidates for dark matter [37–40]. If we regard the bigravity as an alternative theory of gravity, it is interesting and significant that we apply this theory to other phenomena in cosmology or astrophysics and find the differences from general relativity. For instance, many kinds of cosmological solutions have been investigated, and some people expect that the difference of gravitational waves in bigravity from that in general relativity could be constrained by forthcoming experiments [41]. In this paper, I review some properties of a special class of solutions in bigravity [42] and show that the specific family of Schwarzschild–de Sitter is stable for a special class of perturbation [43].

2. Bigravity Theory

2.1. The Action and Equation of Motion

In this section, I give a brief review of bigravity theory. The action of the bigravity is given by:

$$S_{\text{bigravity}} = M_g^2 \int d^4x \sqrt{-\det(g)} R(g) + M_f^2 \int d^4x \sqrt{-\det(f)} R(f) \\ - 2m_0^2\, M_{\text{eff}}^2 \int d^4x \sqrt{-\det(g)} \sum_{n=0}^{4} \beta_n e_n \left(\sqrt{g^{-1}f}\right) \tag{1}$$

Here, g and f are dynamical variables and rank-two tensor fields, which have properties as metrics, $R(g)$ and $R(f)$ are the Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, M_g and M_f are the two Plank mass scales for $g_{\mu\nu}$ and $f_{\mu\nu}$, as well, and the scale M_{eff} is the effective Plank mass scale defined by $1/M_{\text{eff}}^2 = 1/M_g^2 + 1/M_f^2$. The constants β_n's and m_0 are free parameters; the former defines the form of interactions, and the latter expresses the mass of the massive spin-2 field. The matrix $\sqrt{g^{-1}f}$ is defined by the square root of $g^{\mu\rho} f_{\rho\nu}$. For general matrix $\mathbf{X}$, $e_n(\mathbf{X})$'s are polynomials of the eigenvalues of X:

$$\begin{aligned} e_0(\mathbf{X}) =& 1\,,\ e_1(\mathbf{X}) = [\mathbf{X}]\,,\ e_2(\mathbf{X}) = \frac{1}{2}\left([\mathbf{X}]^2 - [\mathbf{X}^2]\right) \\ e_3(\mathbf{X}) =& \frac{1}{6}\left([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3]\right)\,,\ e_4(\mathbf{X}) = \det(\mathbf{X})\,,\ e_k(\mathbf{X}) = 0 \text{ for } k > 4 \end{aligned} \tag{2}$$

where the square brackets denote the traces of the matrices, that is $[X] = X^\mu_\mu$.

Here, we show that there appear one massless spin-2 mode and one massive spin-2 mode in this theory by following the paper by Hassan and Rosen [1]. For simplicity, we assume the minimal model, where the parameters in the interaction term β_n are chosen as:

$$\beta_0 = 3, \quad \beta_1 = -1, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 1 \tag{3}$$

and the interaction terms are given by:

$$2m^2\, M_{\text{eff}}^2 \int d^4x \sqrt{-g} \left(3 - \text{tr}\sqrt{g^{-1}f} + \det\sqrt{g^{-1}f}\right) \tag{4}$$

When we expand both $g_{\mu\nu}$ and $f_{\mu\nu}$ around the same fixed background $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}, \quad f_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu} \tag{5}$$

we obtain the linearized action up to second order,

$$S = \int d^4x \left(h_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} h_{\alpha\beta} + l_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} l_{\alpha\beta}\right) \\ - \frac{m_0^2 M_{\text{eff}}^2}{4} \int d^4x \left[\left(\frac{h^\mu_{\ \nu}}{M_g} - \frac{l^\mu_{\ \nu}}{M_f}\right)^2 - \left(\frac{h}{M_g} - \frac{l}{M_f}\right)^2\right] \tag{6}$$

Here, $\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$ is an operator for massless spin-2 propagating mode. In order to diagonalize the action, we redefine the fields $h_{\mu\nu}$ and $l_{\mu\nu}$ as follows:

$$\frac{1}{M_{\text{eff}}} u_{\mu\nu} = \frac{1}{M_f} h_{\mu\nu} + \frac{1}{M_g} l_{\mu\nu}, \quad \frac{1}{M_{\text{eff}}} v_{\mu\nu} = \frac{1}{M_g} h_{\mu\nu} - \frac{1}{M_f} l_{\mu\nu} \tag{7}$$

Then, linearized Action (6) can be rewritten as:

$$S = \int d^4x u_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} u_{\alpha\beta} + \int d^4x \left[v_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} v_{\alpha\beta} - \frac{1}{4} m_0^2 \left(v^{\mu\nu} v_{\mu\nu} - v^2 \right) \right] \tag{8}$$

This shows that $u_{\mu\nu}$ is the massless spin-2 mode, and $v_{\mu\nu}$ is the massive spin-2 mode. Additionally, if we regard the massless spin-2 mode as the usual graviton, bigravity describes the interaction between gravitational force and the massive spin-2 field. Note that we can find that there is no BD ghost, even at the non-linear level, if we study Hamiltonian constraints by using the ADM formalism.

Now, we consider the variation of Action (1) with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$. The obtained equations of motion are given by:

$$0 = R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} + \frac{1}{2} \left(\frac{m_0 M_{\mathrm{eff}}}{M_g} \right)^2 \sum_{n=0}^{3} (-1)^n \beta_n \left\{ g_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{g^{-1}f}) + g_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{g^{-1}f}) \right\} \tag{9}$$

$$0 = R_{\mu\nu}(f) - \frac{1}{2} R(f) f_{\mu\nu} + \frac{1}{2} \left(\frac{m_0 M_{\mathrm{eff}}}{M_f} \right)^2 \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left\{ f_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{f^{-1}g}) \right\} \tag{10}$$

Here, for a matrix $\mathbf{X}$, $Y_n(\mathbf{X})$'s are defined by:

$$Y_0(\mathbf{X}) = \mathbf{1}\,,\ Y_1(\mathbf{X}) = \mathbf{X} - \mathbf{1}[\mathbf{X}]\,,\ Y_2(\mathbf{X}) = \mathbf{X}^2 - \mathbf{X}[\mathbf{X}] + \frac{1}{2}\mathbf{1}\left([\mathbf{X}]^2 - [\mathbf{X}^2] \right)$$

$$Y_3(\mathbf{X}) = \mathbf{X}^3 - \mathbf{X}^2[\mathbf{X}] + \frac{1}{2}\mathbf{X}\left([\mathbf{X}]^2 - [\mathbf{X}^2] \right) - \frac{1}{6}\mathbf{1}\left([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3] \right) \tag{11}$$

In the above case, we do not consider the energy-momentum tensor for the ordinary matter fields. Matter coupling is usually considered to be a minimal one with respect to two metrics [1], since the non-minimal couplings could lead to the ghost propagating mode again. Appropriate degrees of freedom in massive gravity or bigravity are supported by constraints on the system, and the non-minimal couplings would break the constraint. There is much discussion about non-minimal coupling and the ghost problem [44,45].

2.2. Proportional Solutions

Now, we consider the specific solutions under the following ansatz,

$$f_{\mu\nu}(x) = C^2 g_{\mu\nu}(x) \tag{12}$$

where C is a constant. The ansatz of this form is implied by the equations of motion, Equations (9) and (10), since $\sqrt{f^{-1}g}$ in the interaction terms is reduced to just an identity matrix, and also, this assumption makes it simple to solve the equations of motion, because we have only to determine one tensor field and one constant, rather than two tensor fields. Furthermore, considering the interaction between two metric tensors, it might be reasonable to assume that the metrics could be dynamically proportional to each other.

By using Assumption (12), we obtain two Einstein equations with cosmological constant as follows:

$$0 = R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} + \Lambda_g(C) g_{\mu\nu} \tag{13}$$

$$0 = R_{\mu\nu}(f) - \frac{1}{2} R(f) f_{\mu\nu} + \Lambda_f(C) f_{\mu\nu} \tag{14}$$

and two cosmological constants are defined as follows:

$$\Lambda_g(C) = \left(\frac{m_0 M_{\text{eff}}}{M_g}\right)^2 \left[\beta_0 + 3|C|\beta_1 + 3C^2\beta_2 + C^2|C|\beta_3\right] \tag{15}$$

$$\Lambda_f(C) = \left(\frac{m_0 M_{\text{eff}}}{M_f}\right)^2 \frac{1}{C^2|C|} \left[\beta_1 + 3|C|\beta_2 + 3C^2\beta_3 + C^2|C|\beta_4\right] \tag{16}$$

Here, the dynamics of two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ are separated from each other, and the Bianchi identity is automatically satisfied. This structure of dynamics means that if $f_{\mu\nu} = C^2 g_{\mu\nu}$, the solutions of bigravity are those of general relativity, and we can use the solutions in general relativity. Note that the metric can be diagonalized simultaneously because of the assumption.

Now, we express five β_n's in terms of two free parameters α_3 and α_4 as follows:

$$\beta_0 = 6 - 4\alpha_3 + \alpha_4, \quad \beta_1 = -3 + 3\alpha_3 - \alpha_4, \quad \beta_2 = 1 - 2\alpha_3 + \alpha_4, \quad \beta_3 = \alpha_3 - \alpha_4, \quad \beta_4 = \alpha_4 \tag{17}$$

The interaction terms are equivalently written by another matrix defined as $K \equiv \sqrt{g^{-1}f} - 1$,

$$\sum_{n=0}^{4} \beta_n e_n(\sqrt{g^{-1}f}) = \sum_{n=0}^{4} \alpha_n e_n(K) \tag{18}$$

If we use Equation (17), the interaction terms are given by:

$$-2m_0^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det(g)}\left[e_2(K) + \alpha_3 e_3(K) + \alpha_4 e_4(K)\right] \tag{19}$$

Note that five parameters β_n can be reduced to two parameters α_3, α_4 by requiring two conditions. The first condition is that the theory has the solution that describes the massive spin-2 field on the Minkowski space-time in the massive gravity limit, corresponding to $\alpha_0 = \alpha_1 = 0$. The second condition is that the interaction terms produce the Fierz–Pauli mass term for small fluctuation around the background space-time, corresponding to $\alpha_2 = 1$. One of the reasons that we chose the above parametrization is merely to make the analysis simpler than the case of the five-parameter family. However, as we will see later, there are rich pictures, even if we restrict the five parameters to two parameters, which gives us better understanding of the theory and results in interesting properties of the solutions.

Furthermore, we can take $C > 0$ without loss of generality, because Equations (15) and (16) remain invariant under changing C to $-C$. For consistency, both Equations (13) and (14) should be identical to each other. By putting $f_{\mu\nu} = C^2 g_{\mu\nu}$, we find $R_{\mu\nu}(f) = R_{\mu\nu}(g)$, $R(f)f_{\mu\nu} = R(g)g_{\mu\nu}$. Then, we find $\Lambda_g = C^2\Lambda_f$, and this leads to the quartic equation as follows:

$$\begin{aligned} 0 =& (C-1)\left[M_{\text{ratio}}^2(\alpha_3 - \alpha_4)C^3 + \{-5M_{\text{ratio}}^2\alpha_3 + (2M_{\text{ratio}}^2 - 1)\alpha_4 + 3M_{\text{ratio}}^2\}C^2\right. \\ &\left. + \{(4M_{\text{ratio}}^2 - 3)\alpha_3 - (M_{\text{ratio}}^2 - 2)\alpha_4 - 6M_{\text{ratio}}^2\}C + (3\alpha_3 - \alpha_4 - 3)\right] \end{aligned} \tag{20}$$

where we define $M_{\text{ratio}} \equiv M_f/M_g$.

Apparently, we can find that the general model with arbitrary α_3 and α_4 has solution where $f_{\mu\nu} = g_{\mu\nu}$, that is $C = 1$, and therefore, two cosmological constants vanish, which tells us that the model in the two-parameter family of bigravity has the solution $g_{\mu\nu} = f_{\mu\nu}$, which is asymptotically flat solution in general relativity. Now, we concentrate on the cubic part in Equation (20) and classify two parameters α_3

and α_4 when $C \neq 1$. If there is no solution that satisfies $C > 0$ and $C \neq 1$, we do not have a non-trivial solution in bigravity.

In order to classify the parameter region corresponding to the non-trivial solution, we assume $M_{\text{ratio}} = 1$ for simplicity. Then, we find the non-trivial solutions and corresponding parameter region (Figure 1). For instance, the minimal model $(\alpha_3, \alpha_4) = (1, 1)$ has only asymptotically flat solutions although the next to the minimal models $(\alpha_3, \alpha_4) = (1, -1), (-1, 1), (-1, -1)$ have asymptotically de Sitter solutions. Note that the combination (α_3, α_4) for non-trivial solutions deviates when we consider the case $M_{\text{ratio}} \neq 1$ [42]. The magnitude of cosmological constants is proportional to the square of the mass, m_0^2, and the sign depends on α_3, α_4 and corresponding C. Furthermore, the two cosmological constants are related to each other by the equation $\Lambda_g = C^2 \Lambda_f$, and they have the same sign.

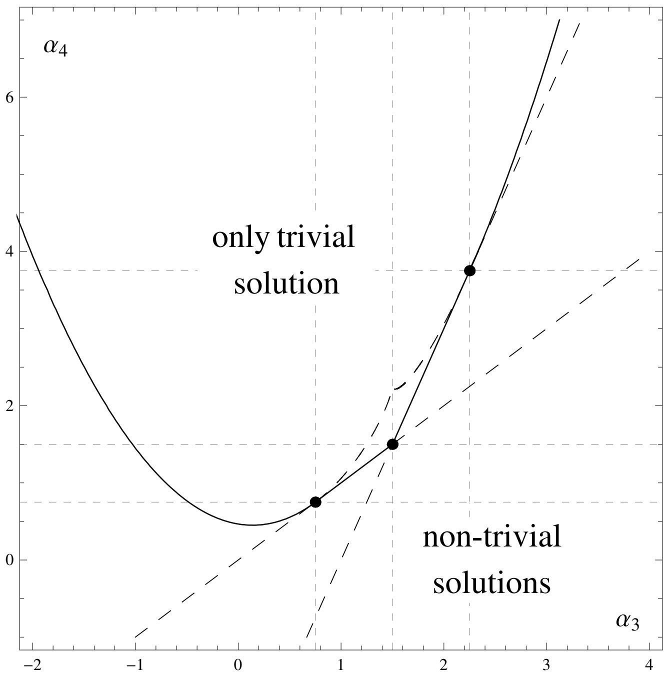

Figure 1. Classification of the parameters α_3 and α_4 is shown. We obtain only asymptotically-flat solutions in the region "only trivial solution", although asymptotically non-flat (de Sitter and/or anti-de Sitter) solutions are realized in "non-trivial solutions". The black line is the border of the two regions, and the dashed lines relate to criteria for the number of non-trivial solutions (see the Appendix in [42]).

3. Stability and Anti-Evaporation of the Schwarzschild–de Sitter Space-Time

3.1. Nariai Space-Time and Anti-Evaporation

It is well known that the horizon radius of a black hole in a vacuum usually decreases by the Hawking radiation, which is called the black hole evaporation. However, Bousso and Hawking have observed a phenomenon that a perturbation around the specific Nariai space-time leads to the increase of the black hole horizon in general relativity if one takes the quantum corrections from the radiation into account [46]. This phenomenon is called anti-evaporation of black holes, and it has been implied that the phenomenon relates to the abundance of primordial black holes in the current Universe, because the increase of the black hole horizon extends the lifetime of the black hole. Note that the origin of the anti-evaporation is the modification of the equation of motion caused by the quantum correction from the radiation.

On the other hand, it has been shown that the anti-evaporation may occur even on the classical level in $F(R)$ gravity theories [47–49], although the quantum corrections play an important role in general relativity. The anti-evaporation without the quantum corrections could be due to the change of field equations from the Einstein equation, because the behavior of perturbations depends on the equations of motion. $F(R)$ gravity indeed modifies the classical Einstein–Hilbert action, as well as the quantum corrections in general relativity; thus, it might be interesting if the anti-evaporation at the classical level were a general phenomenon in modified gravity.

Therefore, we consider the possibility of anti-evaporation in bigravity at the classical level, because the contribution from the interaction between two metric tensors is not so trivial. In the previous section, we found that the dynamics of two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ are described by the Einstein equations under the assumption that $f_{\mu\nu} = C^2 g_{\mu\nu}$. Note that, however, this picture is just for the background solution, and perturbations can be free from the proportional relation. In other words, the perturbations of $f_{\mu\nu}$ are independent of those of $g_{\mu\nu}$. Therefore, it could be important to analyze the stability of perturbation even at the classical level, In this section, I give a brief review of the anti-evaporation in general relativity, following the paper by Bousso and Hawking.

At first, we introduce the Nariai space-time as a family of the Schwarzschild–de Sitter space-time. The Schwarzschild–de Sitter solution is expressed in the following form:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2\,,\ V(r) = 1 - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2 \tag{21}$$

Here, μ is a mass parameter, and Λ is a positive cosmological constant. For $0 < \mu < \frac{1}{3}\Lambda^{-1/2}$, $V(r)$ has two positive roots r_c and r_b, corresponding to the cosmological and black hole horizon, respectively. In the limit $\mu \to \frac{1}{3}\Lambda^{-1/2}$, the radius of the black hole horizon coincides with that of the cosmological horizon. Here, the coordinate system in Equation (21) becomes inappropriate because $V(r) \to 0$ between the two horizons. Then, it is useful to introduce a new coordinate system as follows,

$$t = \frac{1}{\epsilon\sqrt{\Lambda}}\psi\,, \quad r = \frac{1}{\sqrt{\Lambda}}\left(1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right) \tag{22}$$

where ϵ is the parameter defined as $9\mu^2\Lambda = 1 - 3\epsilon^2$, and $\epsilon \to 0$ corresponds to the degeneracy of the two horizons. In the above coordinate, the black hole horizon corresponds to $\chi = 0$; the cosmological horizon corresponds to $\chi = \pi$; and the metric takes the following form:

$$ds^2 = -\frac{1}{\Lambda}\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi d\psi^2 + \frac{1}{\Lambda}\left(1 - \frac{2}{3}\epsilon\cos\chi\right)d\chi^2 + \frac{1}{\Lambda}(1 - 2\epsilon\cos\chi)d\Omega^2 \tag{23}$$

In the degenerate case, $\epsilon = 0$, the space-time is called the Nariai solution. Note that the topology of the space-like sections of the Schwarzschild–de Sitter space-time (and the Nariai space-time) is $S^1 \times S^2$, while that of the ordinary black hole solution is S^2 in four dimensions.

Next, we introduce the Hawking radiation from the black hole horizon. It is well known that there is radiation by the quantum effects of matter fields around the black hole horizon, which is called the

Hawking radiation. For the massless scalar field as the radiation around black hole horizon, we consider the following action,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}\sum_{i=1}^{n}(\nabla f_i)^2\right] \tag{24}$$

where f_i are N scalar fields that carry the quantum radiation. The quantum corrections by the scalar field lead to the trace anomaly of the energy-momentum tensor. When we reduce the four-dimensional space-time to the two-dimensional one in a spherically-symmetric way,

$$ds^2 = \sum_{\mu,\nu=t,r} g_{\mu\nu}dx^\mu dx^\nu + e^{-2\phi}d\Omega^2 \tag{25}$$

the trace anomaly can be expressed by the following effective action [50,51],

$$S_{\text{eff}} = -\frac{1}{48\pi G}\int d^2x\sqrt{-g}\left[\frac{1}{2}R\frac{1}{\Box}R - 6(\nabla\phi)^2\frac{1}{\Box}R - \omega\phi R\right] \tag{26}$$

Here, ω is the redundant parameter corresponding to the renormalization scheme. We can render the effective Action (26) local by introducing the scalar field Z [52] and integrate out the classical solution, $f_i = 0$. Then, the action with the trace anomaly can be expressed by the following effective action:

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\left[\left(e^{-2\phi} + \frac{\kappa}{2}(Z + \omega\phi)\right)R - \frac{\kappa}{4}(\nabla Z)^2 + 2 + 2e^{-2\phi}(\nabla\phi)^2 - 2e^{-2\phi}\Lambda\right] \tag{27}$$

Here, $\kappa \equiv 2N/3$.

We now consider the large N limit, $\kappa \gg 1$, where the quantum fluctuations of metric are dominated by the contribution from the N scalar fields. We also assume that the quantum correction itself should be small, that is $b \equiv \kappa\Lambda \ll 1$. Then, we consider the perturbation around the Nariai space-time in general relativity. According to the topology, $S^1 \times S^2$, we make a spherically-symmetric metric ansatz as follows,

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\rho(t,x)}\left(-dt^2 + dx^2\right) + e^{-2\phi(t,x)}d\Omega^2 \tag{28}$$

Here, the two-dimensional metric, corresponding to t and x components, is written in the conformal gauge, and x is the coordinate system on the one-sphere and has the period of 2π. One can obtain the equations of motion for ρ, ϕ and Z by substituting Ansatz (28) into Action (27) and then finding the following solution:

$$e^{2\rho} = \frac{1}{\Lambda_1}\frac{1}{\cos^2 t}\,,\; e^{2\phi} = \Lambda_2 \quad \text{where } \frac{1}{\Lambda_1} = \frac{1}{\Lambda}\left(1 - \frac{\omega b}{4}\right),\; \Lambda_2 = \Lambda\left(1 - \frac{b}{2}\right) \tag{29}$$

until the first order of b.

Finally, we perturb this solution, so that the two-sphere radius, $e^{-\phi}$, varies along the one-sphere coordinate, x. We assume the perturbation in the following form:

$$e^{2\phi} = \Lambda_2\left[1 + 2\epsilon\sigma(t)\cos x\right]\,,\; |\epsilon| \ll 1 \tag{30}$$

We now trace the time evolution of the black hole horizon, The condition for a horizon is $(\nabla\delta\varphi)^2 = 0$, which requires that the gradient of the two-sphere size is null. Here, perturbation Ansatz (28) yields:

$$\delta\dot{\phi} = \epsilon\dot{\sigma}\cos x\,, \quad \delta\phi' = -\epsilon\sigma\sin x \tag{31}$$

From the above conditions, the locations of the black hole horizon x_b and cosmological horizon x_c are found as follows:

$$x_b = \arctan\left|\frac{\dot{\sigma}}{\sigma}\right|\,, \quad x_c = \pi - x_b \tag{32}$$

Therefore, the radius of the black hole horizon, r_b, is given by:

$$r_b^{-2}(t) = e^{2\phi(t,x_b)} = \Lambda_2\left\{1 + 2\epsilon\delta(t)\right\} \tag{33}$$

where we define the perturbation for the horizon $\delta(t)$,

$$\delta(t) \equiv \sigma(t)\cos x_b = \sigma\left\{1 + \left(\frac{\dot{\sigma}}{\sigma}\right)^2\right\}^{-1/2} \tag{34}$$

For the semi-classical case, $\kappa > 0$, one cannot find the analytic solution, because the quantum corrections from the matter field lead to the modification of the equation for $\sigma(t)$. However, one can solve the equation of motion as a power series in t for the early Universe. The horizon perturbation is given by:

$$\delta(t) \approx \sigma_0\left(1 - \frac{1}{2}bt^2\right) \tag{35}$$

This result implies that the black hole perturbation shrinks from its initial value, and the size of black hole horizon increases at least initially. This phenomenon is called anti-evaporation.

3.2. Perturbations and Stability of Bi-Diagonal Narial Solution

As we discussed in Section 2.2, we obtain the asymptotically de Sitter solution for the specific combinations of parameters under Ansatz (12). Therefore, the Schwarzschild–de Sitter space-time can be the solutions if we impose spherical symmetry, and we can obtain the Nariai black hole solution by the degeneracy limit. In the following analysis, we assume that the background solutions for both metrics are the Nariai space-time.

At first, we consider the spherically-symmetric metric ansatz for two metrics,

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\rho_1(t,x)}\left(-dt^2 + dx^2\right) + e^{-2\varphi_1(t,x)}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{36}$$
$$f_{\mu\nu}dx^\mu dx^\nu = e^{2\rho_2(t,x)}\left(-dt^2 + dx^2\right) + e^{-2\varphi_2(t,x)}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{37}$$

Here, the black hole and cosmological horizons are located at the same place, respectively. In the coordinate system of Equations (36) and (37), $\rho(t,x)$'s and $\varphi(t,x)$'s corresponding to the Nariai solutions are given by:

$$\rho_1 = -\frac{1}{2}\log\Lambda - \log(\cos t)\,,\ \varphi_1 = \frac{1}{2}\log\Lambda$$
$$\rho_2 = \log C - \frac{1}{2}\log\Lambda - \log(\cos t)\,,\ \varphi_2 = -\log C + \frac{1}{2}\log\Lambda \tag{38}$$

Here, we choose $M_g = M_f$, and the effective Plank mass scale is given by $M_{\text{eff}}^2 = \frac{1}{2}M_g^2 = \frac{1}{2}M_f^2$. Additionally, we assume that the β_n's are chosen to realize the asymptotically de Sitter space-time.

Next, we define the perturbations as follows:

$$\rho_1 \equiv \bar{\rho}_1 + \delta\rho_1(t,x)\,,\ \varphi_1 \equiv \bar{\varphi}_1 + \delta\varphi_1(t,x)\,, \rho_2 \equiv \bar{\rho}_2 + \delta\rho_2(t,x)\,,\ \varphi_2 \equiv \bar{\varphi}_2 + \delta\varphi_2(t,x) \tag{39}$$

Here, $\bar{\rho}$'s and $\bar{\varphi}$'s correspond to the unperturbed Nariai space-time Equation (38) , and $\delta\rho$'s and $\delta\varphi$'s are the perturbations. Note that these perturbations are not general, but keep the space-time isometry to be $S^1 \times S^2$. By substituting the above expressions into Equations (36) and (37), we find the metric perturbations $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ in the first order,

$$\begin{aligned}\delta g_{\mu\nu} \equiv&\text{diag}\left(-2e^{2\bar{\rho}_1}\delta\rho_1, 2e^{2\bar{\rho}_1}\delta\rho_1, -2e^{-2\bar{\varphi}_1}\delta\varphi_1, -2e^{-2\bar{\varphi}_1}\delta\varphi_1\sin^2\theta\right)\\ \delta f_{\mu\nu} \equiv&\text{diag}\left(-2e^{2\bar{\rho}_2}\delta\rho_2, 2e^{2\bar{\rho}_2}\delta\rho_2\,, -2e^{-2\bar{\varphi}_2}\delta\varphi_2, -2e^{-2\bar{\varphi}_2}\delta\varphi_2\sin^2\theta\right)\end{aligned} \tag{40}$$

We now evaluate the equations of motion for the perturbation. For the convention, we express the equations of motion, (9) and (10), as follows:

$$G_{\mu\nu}(g) + I^{\lambda}_{\ \nu}(\mathbf{A})g_{\mu\lambda} = 0 \tag{41}$$

$$G_{\mu\nu}(f) + I^{\lambda}_{\ \nu}(\mathbf{B})f_{\mu\lambda} = 0 \tag{42}$$

where $G_{\mu\nu}$ is the Einstein tensor and $I^{\lambda}_{\ \nu}$'s are the sum of Y_n's. When we consider the perturbation up to first order, the above equations are divided by the background part and deviation part, and the equations for the deviation take the following forms:

$$\delta G_{\mu\nu}(g) + \delta I^{\lambda}_{\ \nu}(\mathbf{A})g_{\mu\lambda} + I^{\lambda}_{\ \nu}(\mathbf{B})\delta g_{\mu\lambda} = 0 \tag{43}$$

$$\delta G_{\mu\nu}(f) + \delta I^{\lambda}_{\ \nu}(\mathbf{B})f_{\mu\lambda} + I^{\lambda}_{\ \nu}(\mathbf{A})\delta f_{\mu\lambda} = 0 \tag{44}$$

Here, we define $\mathbf{A} = \sqrt{g^{-1}f}$ and $\mathbf{B} = \sqrt{f^{-1}g}$. These two matrices are expressed as follows:

$$\mathbf{A} = \text{diag}\left(e^{-\zeta}, e^{-\zeta}, e^{\xi}, e^{\xi}\right)\ ,\ \mathbf{B} = \text{diag}\left(e^{\zeta}, e^{\zeta}, e^{-\xi}, e^{-\xi}\right) \tag{45}$$

where $\zeta \equiv \rho_1 - \rho_2, \xi \equiv \varphi_1 - \varphi_2$. Additionally, the deviations of ζ and ξ are given by $\delta\zeta = \delta\rho_1 - \delta\rho_2$ and $\delta\xi = \delta\varphi_1 - \delta\varphi_2$, respectively. After the short calculation, we can obtain the deviation of the Einstein tensor $\delta G_{\mu\nu}$. Regarding the interaction terms δI^{λ}_{ν}, we find:

$$\delta I(\mathbf{A}) = -\frac{1}{2}m_0^2\left[\beta_1 C + 2\beta_2 C^2 + \beta_3 C^3\right]\mathbf{Z} \tag{46}$$

$$\delta I(\mathbf{B}) = \frac{1}{2}m_0^2\left[\beta_3 C^{-1} + 2\beta_2 C^{-2} + \beta_1 C^{-3}\right]\mathbf{Z} \tag{47}$$

where we define the tensor $\mathbf{Z}$ as follows,

$$\mathbf{Z} = \text{diag}\left(\delta\zeta - 2\delta\xi, \delta\zeta - 2\delta\xi, 2\delta\zeta - \delta\xi, 2\delta\zeta - \delta\xi\right) \tag{48}$$

Finally, we determine the evolution of black holes due to the perturbations. In the following, we consider the black hole horizon for $g_{\mu\nu}$ at first. Let us specify the form of perturbations according to the original procedure by Hawking and Bousso:

$$e^{2\varphi_1} = \Lambda\left\{1 + 2\epsilon\sigma_1(t)\cos x\right\}\ ,\delta\varphi_1 \equiv \epsilon\sigma_1(t)\cos x \tag{49}$$

Substituting the above form of perturbation into the (t, x) component of (43), we obtain:

$$\dot{\sigma_1} = \sigma_1 \tan t \tag{50}$$

With the boundary condition, $\dot{\sigma_1} = 0$ at $t = 0$, the solution is:

$$\sigma_1(t) = \frac{\sigma_g}{\cos t} \tag{51}$$

Then, we find the horizon perturbation Equation (34) as follows:

$$\delta(t) = \sigma_g = \text{const} \tag{52}$$

This result means that no anti-evaporation takes place, as well as in the classical case in general relativity. Furthermore, if we consider the perturbation in the same form for $\delta\varphi_2$ as that for $\delta\varphi_1$, we obtain the same results, because the equations have the same form as that of φ_1. Then, one can find that anti-evaporation does not occur for two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ at the classical level.

We now focus on the problem of how we can identify the difference between the case in general relativity and in bigravity. When we substitute the perturbations into the (t, t) and (x, x) components of Equation (43), we obtain $\delta\zeta - 2\delta\xi = 0$. Thus, the contributions from the interaction terms in the (t, t), (x, x) and (t, x) components vanish, and the equations for $\delta\varphi$'s are the same as that in general relativity. The deviations of the interaction term Equations (46) and (47) exactly vanish if $\delta\zeta = 0$ or $\delta\xi = 0$. When we define the perturbation for $f_{\mu\nu}$ as:

$$e^{2\varphi_2} = \frac{\Lambda}{C^2} \left\{1 + 2\epsilon\sigma_2(t) \cos x\right\} , \quad \sigma_2(t) = \frac{\sigma_f}{\cos t} \tag{53}$$

$\delta\xi$ vanishes in the case where the amplitude of the perturbations are identical, $\sigma_g = \sigma_f$. This means that the two sets of metric perturbations are proportional to each other, and the relation between the perturbations is not changed from the background, $\delta f_{\mu\nu} = C^2 \delta g_{\mu\nu}$. In this case, whole metrics, including the perturbations, are proportional, and this does not lead to a difference from general relativity. Therefore, we cannot distinguish bigravity theory from general relativity for this case.

4. Results and Discussion

We have studied the possibility of the anti-evaporation at the classical level in the bigravity. For the assumption $f_{\mu\nu} = C^2 g_{\mu\nu}$, particular parameters β_n's and the Plank mass scales $M_g = M_f$, we obtained the asymptotically de Sitter space-time. When we considered the perturbations around the Nariai space-time, the size of the black hole horizon does not change, which implies that the Nariai space-time is stable. Additionally, we have found that the anti-evaporation does not take place at the classical level, although the equations of motion are different from general relativity.

We may expect that there could occur anti-evaporation if we include the quantum correction of matter fields, as in the case of the general relativity. The explicit calculation could be pretty complicated, but an interesting problem could be to study if we need to introduce the quantum corrections only for one of the two metrics or both of them. In the bigravity, we may assume two kinds of matter fields $\Psi_g(x)$ and $\Psi_f(x)$ coupled to $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Thus, there are potentially quantum radiations and the corrections from the two kinds of matter. For instance, if we were to find that the anti-evaporation occurs by including the quantum corrections only coupled to $f_{\mu\nu}$, the black hole radius could increases even though the dynamics of $g_{\mu\nu}$ is exactly classical. The abundance of primordial black holes can also be an important problem if we can find the way to realize anti-evaporation in bigravity.

There could be another way to realize anti-evaporation by modification to $F(R)$ bigravity theory [28–31]. This theory modify the kinetic terms of bigravity, from the Ricci scalar to the function of it. In $F(R)$ bigravity, we find a similar problem to introducing the quantum corrections. That is, we need to study if the modification is required for only one metric or both metrics.

In contrast to our result, it has been shown that the bi-Schwarzschild solutions are classically unstable [53,54]. In these papers, the authors concluded that spherically-symmetric perturbation leads to the instability even in Schwarzschild de–Sitter spacetime. The perturbations that we considered are, however, not general spherically symmetric, but the specific one restricted to keeping the background space-time isometry $S^1 \times S^2$. Thus, the stability of the Nariai solution in our work could relate to the symmetry of the space-time.

When we assume the perturbation (49), Equation (51) is derived from the (t, x) component of Equation (43). However, the non-diagonal components of Equation (43) take the forms identical to those in general relativity, because the interaction terms do not modify the non-diagonal components. Note that this outcome depends on the special configuration of background solutions, that is simultaneously-diagonalized metrics. In this condition, the non-diagonal components of Y_n's vanish, and as a result, the size of the black hole horizon does not increase, as is the case in general relativity.

It is interesting that our approach may be generalized to the case of other background solutions. As I mentioned above, we took the background $\bar{g}_{\mu\nu}$ and $\bar{f}_{\mu\nu}$ as the Nariai space-time, and these metrics are diagonalized because of the proportional relation between two metrics $f_{\mu\nu} = C^2 g_{\mu\nu}$. In general, however, two metrics cannot be simultaneously diagonalized, because we have only one set of diffeomorphism for two independent metrics in the bigravity [55,56]. If we remove the assumption $f_{\mu\nu} = C^2 g_{\mu\nu}$ and we can find the non-diagonal solution for $g_{\mu\nu}$ and/or $f_{\mu\nu}$, the interaction terms modify the non-diagonal components for the equations of perturbations, and these modifications lead to nontrivial contributions.

From the point of view of specifying the difference from general relativity, non-diagonal components of the metric are of great interest. For instance, non-diagonal solutions, even for the spherically-symmetric space-time, are permitted, because of one set of diffeomorphisms for two metrics. Therefore, if we can detect the phenomena that stem from such solutions in the cosmological and astrophysical observation, this leads us to the possibility of distinguishing or restricting the bigravity theory.

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Conflicts of Interest

The authors declare no conflict of interest.

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