Universe **2015**, *1*, 82-91; doi:10.3390/universe1010082

OPEN ACCESS

universe

ISSN 2218-1997

www.mdpi.com/journal/universe

Article

Arctan-Gravity Model

Sergey I. Kruglov

Department of Chemical and Physical Sciences, University of Toronto, 3359 Mississauga Road North, Mississauga, ON L5L 1C6, Canada; E-Mail: serguei.krouglov@utoronto.ca

Academic Editor: Kazuharu Bamba and Sergei D. Odintsov

Received: 28 March 2015 / Accepted: 15 May 2015 / Published: 22 May 2015

Abstract: A new gravity model with the function $F(R) = (1/\beta)\arctan(\beta R - \beta^2 R^2)$ instead of the Ricci scalar in the Einstein–Hilbert action, describing inflation of the Universe, is suggested and analyzed. We obtain constant curvature solutions of the model in the Jordan frame. Performing the conformal transformation of the metric, the potential and the mass of a scalar degree of freedom in the Einstein frame are found. The slow-roll and cosmological parameters of the model are evaluated. It was demonstrated that the index of the scalar spectrum power law, n_s, is in agreement with the PLANCK data.

Keywords: Einstein–Hilbert action; modified gravity; Ricci scalar; Jordan and Einstein frames; cosmological parameters; Planck's data

PACS Classifications: 04.50.Kd; 98.80.Es

1. Introduction

In modern cosmology, the inflationary scenario may solve the initial conditions problem and explain the formation of galaxies and irregularities in the microwave background. The Λ Cold Dark Matter (ΛCDM) model [1] describes correctly the inflationary epoch in accordance with the experimental data, and is a good candidate for a description of dark energy (DE) and observational data of the accelerated universe. Nevertheless, the ΛCDM model has difficulty with the theoretical proof of the cosmological constant smallness. Accelerated universe at the present time can be described by dynamical DE introducing the scalar field [2]. At the same time $F(R)$ gravity theories modifying general relativity (GR), without introduction of the fundamental scalar field, may be an alternative to the ΛCDM model. In such models the Ricci scalar, R, in the Einstein–Hilbert action, is replaced by the function $F(R)$ [3,4]. It should be noted that $F(R)$ gravity models effectively contain a scalar degree of freedom (scalaron) and can be represented in a scalar-tensor form by using the conformal transformation and can describe the inflation and late time acceleration. Viable $F(R)$ gravity models were suggested in [5–11]. Such models can unify DE with inflation in a natural way. Some models of $F(R)$ gravity theories were given in [3,4,12–24] and in other publications. The condition of classical stability is $F''(R) > 0$ (primes denote the derivatives with respect to the argument) and it means that the scalar field is not a tachyon and a ghost. The requirement of quantum stability $F'(R) > 0$ assures that a graviton is not a ghost [8]. The $F(R)$ gravity models are phenomenological effective models which can describe inflation and current acceleration. The problem is to derive the $F(R)$ function from first principles (probably from string or M theories) to describe the present and primordial DE and the evolution of our Universe. Such derivation is absent and, therefore, different $F(R)$ gravity models describing early time inflation and late time Universe acceleration are of great interest. In this paper we suggest and analyze the particular model of modified gravity which is the modification of arctan-gravity model introduced in [15]. Such modified model contains only one dimensional parameter β.

The paper has the structure as follows. In Section 2 the Lagrangian of the model (possessing the classical stability) is formulated. We obtain the constant curvature solutions of field equations. In Section 3, after performing the conformal transformation of the metric, the scalar-tensor form of the model in the Einstein frame is found. The potential and the mass of a scalar degree of freedom (scalaron) are obtained and the plots of the functions $\phi(\beta R)$, $V(\beta R)$, and $m^2_\phi(\beta R)$ are presented. In Section 4 the slow-roll and cosmological parameters of the model are given and the plots of $\epsilon(\beta R)$, $\eta(\beta R)$, and $n_s(\beta R)$ are represented. In Section 5 we draw the conclusion.

The Minkowski metric $\eta_{\mu\nu}$ = diag$(-1, 1, 1, 1)$ is explored, and we use $c = \hbar = 1$.

2. The Model

We suggest the $F(R)$ gravity theory with the function

$$F(R) = (1/\beta)\arctan\left(\beta R - \beta^2 R^2\right), \tag{1}$$

where the constant β ($\beta > 0$) has the dimension of (length)^2. The theory based on Equation (1) is the modification of the model studied in [15]. The action in the Jordan frame is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}F(R) + \mathcal{L}_m\right], \tag{2}$$

where $\kappa = M^{-1}_{Pl}$ (M_{Pl} is the reduced Planck mass), and $\mathcal{L}_m$ is the matter Lagrangian density. Here we investigate a pure gravitation field and put $\mathcal{L}_m = 0$. If $\beta R \ll 1$, we move from Equation (1) to the Lagrangian density

$$\mathcal{L} \approx R - \beta R^2, \tag{3}$$

which was considered in [3]. From Equation (1) we obtain the derivatives of the function (1)

$$F'(R) = \frac{1 - 2\beta R}{1 + (\beta R - \beta^2 R^2)^2},$$

$$F''(R) = -2\beta\frac{1 + (\beta R - \beta^2 R^2)(1 - 3\beta R + 3\beta^2 R^2)}{\left[1 + (\beta R - \beta^2 R^2)^2\right]^2}. \tag{4}$$

The condition $F''(R) > 0$ ensures the classical stability of the solution at high curvature and the condition $F'(R) > 0$ leads to quantum stability. The condition $F'(R) > 0$ gives $0 < \beta R < 0.5$, and the condition $F''(R) > 0$ leads to

$$\beta R > \frac{1}{2}\left(\sqrt{1 + \frac{2}{3}(\sqrt{13} - 1)} + 1\right) \approx 1.3272. \tag{5}$$

In the high curvature regime, the model violates the condition $F'(R) > 0$, which leads to a negative effective Newton's constant, $G_{eff} = G/F'(R)$, and antigravity. Thus, it is impossible to satisfy the requirements of the classical and quantum stabilities simultaneously. Therefore, we consider the case of the classical stability, $F''(R) > 0$.

2.1. Constant Curvature Solutions

Now we find constant curvature solutions to the equations of motion [25]

$$2F(R) - RF'(R) = 0. \tag{6}$$

With the help of Equations (1), (4) and (6), one obtains

$$2\arctan(\beta R - \beta^2 R^2) = \frac{\beta R(1 - 2\beta R)}{1 + (\beta R - \beta^2 R^2)^2}. \tag{7}$$

Equation (7) has the solution $R_0 = 0$, corresponding to flat spacetime (the Minkowski spacetime). There is also non-trivial solution to Equation (7): $\beta R_1 \approx 1.6846$. For this value, $F''(R_1) > 0$ and the condition of classical stability is satisfied. We will see later that the value $\beta R_1 \approx 1.6846$ corresponds to the minimum of the effective potential of the scalar degree of freedom in Einstein's frame.

3. The Scalar-Tensor Form

Making a conformal transformation of the metric [26], we come to the Einstein frame corresponding to the scalar-tensor theory of gravity,

$$\widetilde{g}_{\mu\nu} = F'(R) g_{\mu\nu} = \frac{1 - 2\beta R}{1 + [\beta R - (\beta R)^2]^2} g_{\mu\nu}. \tag{8}$$

Then action (2), at $\mathcal{L}_m = 0$, becomes

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}\widetilde{R} - \frac{1}{2}\widetilde{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right], \tag{9}$$

where $\widetilde{R}$ is calculated using the new metric (8) and ∇_μ is the covariant derivative. The field ϕ, corresponding to the scalar degree of freedom (scalaron), and the potential $V(\phi)$ are given by

$$\phi = -\frac{\sqrt{3}\ln F'(R)}{\sqrt{2}\kappa} = \frac{\sqrt{3}}{\sqrt{2}\kappa}\ln\left(\frac{1 + [\beta R - (\beta R)^2]^2}{1 - 2\beta R}\right), \tag{10}$$

$$V(\phi) = \frac{RF'(R) - F(R)}{2\kappa^2 F'^2(R)}$$

$$= \frac{1+[\beta R-(\beta R)^2]^2}{2\kappa^2\beta(1-2\beta R)}\left[\beta R-\frac{(1+[\beta R-(\beta R)^2]^2)\arctan[\beta R-(\beta R)^2]}{1-2\beta R}\right]. \tag{11}$$

In Equation (11) the curvature R is the function of ϕ given in Equation (10). The function $\kappa\phi(R)$ is given in Figure 1.

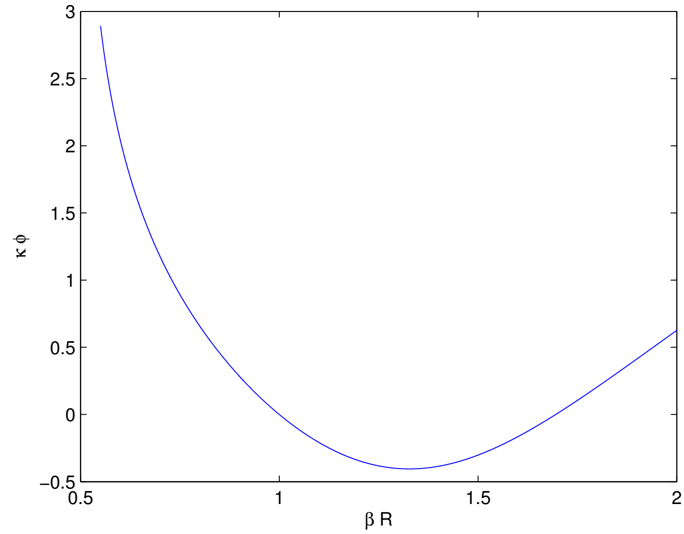

Figure 1. *$\kappa\phi$ versus βR.*

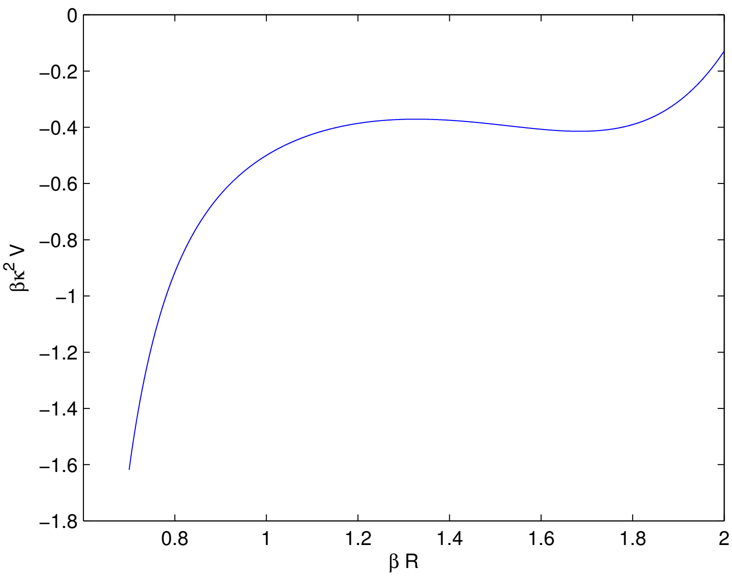

Figure 2. *$\beta\kappa^2V$ versus βR.*

The function $\phi(R)$ possesses a singularity at $\beta R = 0.5$ and thus we imply that $\beta R > 0.5$. It should be noted that the extremum of the potential (11), $V'(\phi) = 0$, is realized by the constant curvature solutions to Equation (7) and by the condition $F''(R) = 0$. The condition $F''(R) = 0$ gives the value $\beta R \approx 1.3272$ ($\kappa\phi \approx -0.405$) and the constant curvature solution $\beta R_1 \approx 1.6846$ ($\kappa\phi \approx -0.020$) corresponds to the minimum of the potential. The plot of the function $V(R)$ (11) is given in Figure 2.

The plot of the function $V(\kappa\phi)$ is represented in Figure 3.

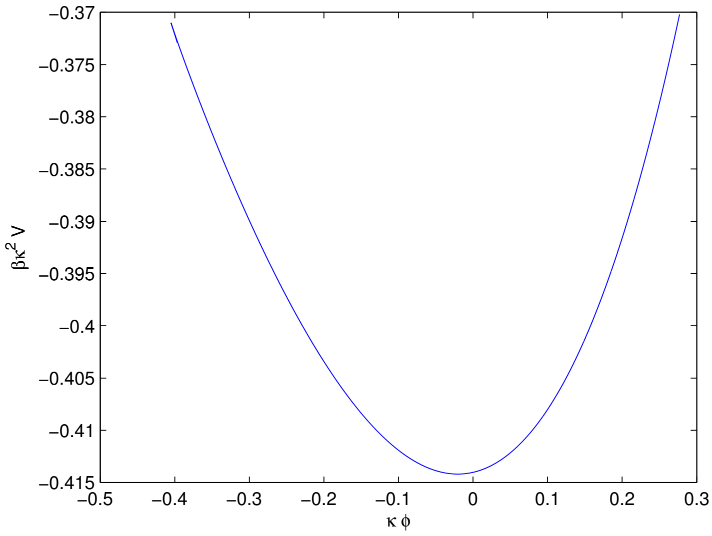

Figure 3. $\beta\kappa^2 V$ *versus* $\kappa\phi$.

It follows from Figure 3 that the constant curvature solution $\kappa\phi \approx -0.020$ realizes the minimum of the potential. The shape of the potential in Figure 3 is similar to one describing chaotic inflation [2]. The mass squared of a scalar degree of freedom is given by

$$m_\phi^2 = \frac{d^2V}{d\phi^2} = \frac{1}{3}\left(\frac{1}{F''(R)} + \frac{R}{F'(R)} - \frac{4F(R)}{F'^2(R)}\right)$$
$$= \frac{1+(x-x^2)^2}{3\beta}\left[-\frac{1+(x-x^2)^2}{2[1+(x-x^2)(1-3(x-x^2))]}\right.$$
$$\left.+\frac{x(1-2x)-4[1+(x-x^2)^2]\arctan(x-x^2)}{(1-2x)^2}\right], \tag{12}$$

where $x = \beta R$. From Equation (12), at the minimum of the potential, $\beta R_1 \approx 1.6846$ ($\kappa\phi \approx -0.020$), we obtain $\beta m_\phi^2 = 0.7706 > 0$, which indicates the stability of the de Sitter phase. Therefore, this phase describes the case of eternal inflation [2]. The plot of the function m_ϕ^2 is given in Figure 4.

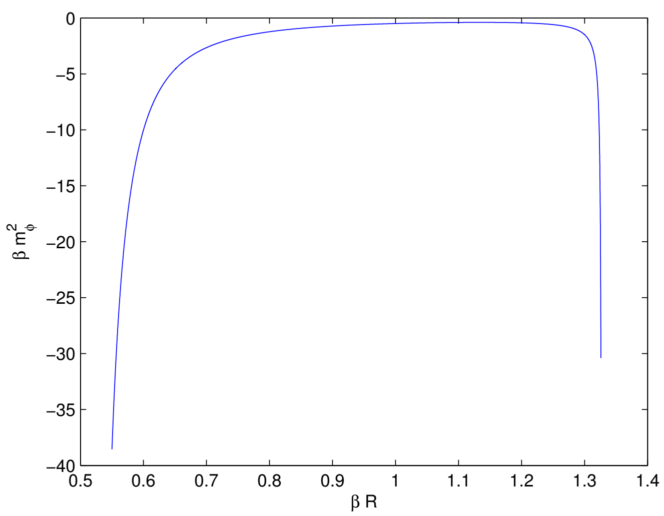

Figure 4. βm_ϕ^2 *versus* βR.

The value m_ϕ^2 is negative, and corresponds to unstable states for the range $0.5 < \beta R < 1.3272$. This behavior of the function $m_\phi^2(\beta R)$ is in accordance with the shape of the potential $V(\beta R)$ (see Figure 2). The stability of solutions in $F(R)$ gravity was discussed in [27]. At the values of $\beta R \approx 0.5$ and $\beta R \approx 1.3272$, the function m_ϕ^2 approaches $-\infty$, and corrections to Newton's law are negligible.

4. Cosmological Parameters

The slow-roll parameters are given by [28]

$$\epsilon(\phi) = \frac{1}{2} M_{Pl}^2 \left(\frac{V'(\phi)}{V(\phi)} \right)^2 = \frac{1}{3} \left[\frac{RF'(R) - 2F(R)}{RF'(R) - F(R)} \right]^2 ,$$

$$\eta(\phi) = M_{Pl}^2 \frac{V''(\phi)}{V(\phi)} = \frac{2}{3} \left[\frac{F'^2(R) + F''(R)\left[RF'(R) - 4F(R)\right]}{F''(R)\left[RF'(R) - F(R)\right]} \right] , \tag{13}$$

where $M_{Pl} = \kappa^{-1}$ is the reduced Planck mass. The slow-roll approximation takes place if the conditions $\epsilon(\phi) < 1$, $| \eta(\phi) | < 1$ are satisfied. One can obtain the slow-roll parameters expressed through curvature from Equations (1), (4) and (13). The plots of the functions $\epsilon(\beta R)$, $\eta(\beta R)$ are presented in Figures 5 and 6 respectively.

One can verify from Equation (13) that the slow-roll condition $\epsilon < 1$ is satisfied at $1.87453 > \beta R > 0.579892$. The second condition $|\eta| < 1$ holds at $1.15265 > \beta R > 1$. Thus, both inequalities $\epsilon < 1$, $|\eta| < 1$ take place at $1.15265 > \beta R > 1$. Inflation ends when $\epsilon = 1$ or $|\eta| = 1$. As a result, the slow-roll approximation is justified in the model suggested.

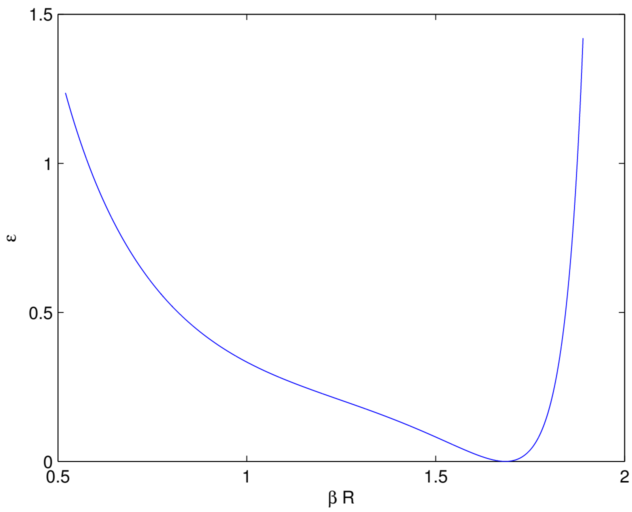

Figure 5. The function ϵ *versus* βR.

The index of the scalar spectrum power law due to density perturbations is as follows [28]:

$$n_s = 1 - 6\epsilon + 2\eta. \tag{14}$$

With the help of Equations (13) and (14), the function of n_s *versus* βR is given by Figure 7.

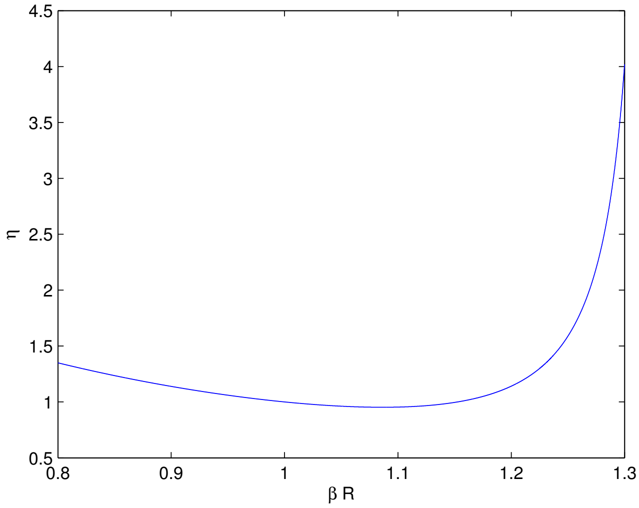

Figure 6. The function η *versus* βR.

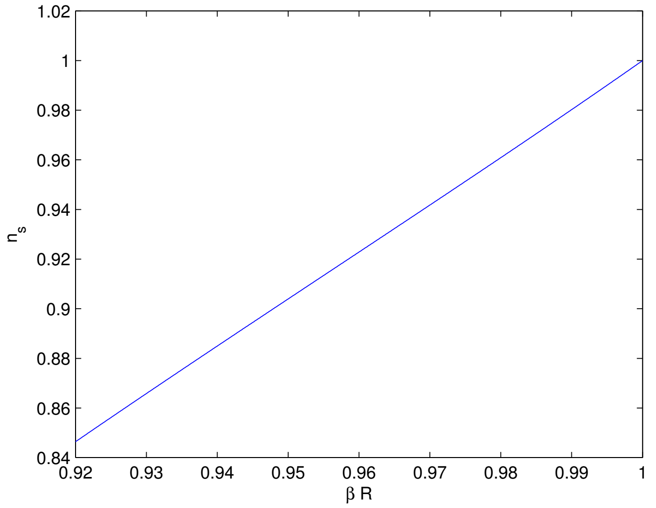

Figure 7. The function n_s *versus* βR.

The tensor-to-scalar ratio is [28] $r = 16\epsilon$. The PLANCK experiment gives the values [29]

$$n_s = 0.9603 \pm 0.0073, \quad r < 0.11. \tag{15}$$

From Equation(14) (see Figure 7) we obtain the experimental value of n_s (15) at $\beta R \approx 0.9797$. The condition $r < 0.11$ is satisfied at $1.64644 < \beta R < 1.71575$. Thus, conditions (15) can not be satisfied simultaneously. The model can explain the PLANCK experimental data for the index of the scalar spectrum power law due to density perturbations at $\beta R \approx 0.9797$, but the bound $r < 0.11$ is violated. We note that this constraint was challenged by BICEP2 experimental data [32] yielding an upper limit $r < 0.12$.

5. Conclusions

It should be noted that the model considered, as well as any $F(R)$ gravity models, are effective models that are not renormalizable. The quantum gravity corrections (one-loop divergences) include a Ricci scalar squared, R^2, and a Ricci tensor squared, $R_{\mu\nu}R^{\mu\nu}$ [30,31]. The justification of the particular $F(R)$ functions, describing modified gravity, is absent, and has to follow from the fundamental theory (quantum gravity). The arctan-gravity model suggested admits the constant curvature solution to Equation (7)—$\beta R_1 \approx 1.6846$—that is a stable state corresponding to the de Sitter spacetime. This solution matches the minimum of the effective potential and corresponds to the de Sitter phase describing the eternal inflation with the cosmic acceleration. The scenario of inflation, due to the form of the potential function, can be described in the similar way as chaotic inflation [33]. The parameter β introduced is connected with the fundamental length $l = \sqrt{\beta}$, probably arising from quantum gravity, so that the value βR is dimensionless. At $\beta R \to 0$, action (2) ($\mathcal{L}_m = 0$) approaches the EH action, and as GR passes local tests the parameter β should be small.

We show that the cosmological parameter evaluated, n_s (at $\beta R \approx 0.9797$), agrees with the observed PLANCK experiment data but the constraint $r < 0.11$ is violated. The model may be observationally acceptable and GR can be an approximation to the intermediate cosmic time. To describe all the cosmological periods and to verify the viability of the model, one needs further investigation. We leave such a study for the future.

Conflicts of Interest

The author declares no conflict of interest.

References

1. Frieman, J.A.; Turner, M.S.; Huterer, D. Dark Energy and the Accelerating Universe. *Annu. Rev. Astron. Astrophys.* **2008**, *46*, 385–432.
2. Linde, A.D. *Particle Physics and Inflationary Cosmology*; Harwood Academic Publishers: Chur, Switzerland, 1992.
3. Capozziello, S.; Faraoni, V. *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*; Springer: New York, NY, USA, 2011.
4. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144.
5. Starobinsky, A.A. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B* **1980**, *91*, 99–102 .
6. Hu, W.; Sawicki, I. Models of f(R) Cosmic Acceleration that Evade Solar-System Tests. *Phys. Rev. D* **2007**, *76*, 064004.
7. Appleby, S.A.; Battye, R.A. Do consistent F(R) models mimic General Relativity plus? *Phys. Lett. B* **2007**, *654*, 7–12.
8. Starobinsky, A.A. Disappearing cosmological constant in f(R) gravity. *JETP Lett.* **2007**, *86*, 157–163.
9. Nojiri, S.; Odintsov, S.D. Unifying inflation with ΛCDM epoch in modified $f(R)$ gravity consistent with Solar System tests. *Phys. Lett. B* **2007**, *657*, 238.
10. Nojiri, S.; Odintsov, S.D. Modified $f(R)$ gravity unifying R^m inflation with ΛCDM epoch. *Phys. Rev. D* **2008**, *77*, 026007.

11. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Sebastiani, L.; Zerbini, S. A Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion. *Phys. Rev. D* **2008**, *77*, 046009.
12. Deser, S.; Gibbons, G.W. Born-Infeld-Einstein actions? *Class. Quant. Grav.* **1998**, *15*, L35–L39.
13. Kruglov, S.I. Born-Infeld-like modified gravity. *Int. J. Theor. Phys.* **2013**, *52*, 2477–2484.
14. Kruglov, S.I. On exponential modified gravity. *Int. J. Mod. Phys. A* **2013**, *28*, 1350119.
15. Kruglov, S.I. Modified arctan-gravity model mimicking a cosmological constant. *Phys. Rev. D* **2014**, *89*, 064004.
16. Sebastiani, L.; Cognola, G.; Myrzakulov, R.; Odintsov, S. D.; Zerbini, S. Nearly Starobinsky inflation from modified gravity. *Phys. Rev. D* **2014**, *89*, 023518.
17. Bamba, K; Nojiri, S.; Odintsov, S.D.; Diego, S.-G. Inflationary universe from perfect fluid and $F(R)$ gravity and its comparison with observational data. *Phys. Rev. D* **2014**, *90*, 124061.
18. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Zerbini S. One-loop $f(R)$ gravity in de Sitter universe. *J. Cosmo. Astropart. Phys.* **2005**, *2005*, 010.
19. Nojiri, S.; Odintsov, S.D. Modified gravity with negative and positive powers of the curvature: Unification of the inflation and of the cosmic acceleration. *Phys. Rev. D* **2003**, *68*, 123512.
20. Carroll, S.M.; Duvvuri, V.; Trodden, M.; Turner, M.S. Is cosmic speed - up due to new gravitational physics? *Phys. Rev. D* **2004**, *70*, 043528.
21. Amendola, L.; Gannouji, R.; Polarski, D.; Tsujikawa, S. Conditions for the cosmological viability of f(R) dark energy models. *Phys. Rev. D* **2007**, *75*, 083504.
22. Linder, E.V. Exponential Gravity. *Phys. Rev. D* **2009**, *80*, 123528.
23. Bañados, M.; Ferreira, P.G. Eddington's theory of gravity and its progeny. *Phys. Rev. Lett.* **2010**, *105*, 011101; *Erratum-ibid.* **2014**, *113*, 119901.
24. Pani, P.; Cardoso, V.; Delsate, T. Compact stars in Eddington inspired gravity. *Phys. Rev. Lett.* **2011**, *107*, 031101.
25. Barrow, J.D.; Ottewill, A.C. The Stability of General Relativistic Cosmological Theory. *J. Phys. A* **1983**, *16*, 2757.
26. Magnano, G.; Sokolowski, L.M. On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field. *Phys. Rev. D* **1994**, *50*, 5039–5059.
27. Müller, V.; Schmidt, H.J.; Starobinsky, A.A. The Stability of the De Sitter Space-time in Fourth Order Gravity. *Phys. Lett. B* **1988**, *202*, 198.
28. Liddle, A.R.; Lyth, D.H. *Cosmological Inflation and Large-Scale Structure*; Cambridge University Press: Cambridge, UK, 2000.
29. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; *et al.* (Planck Collaboration, 2013). Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.* **2014**, *571*, A16.
30. Stelle, K.S. Renormalization of Higher Derivative Quantum Gravity. *Phys. Rev. D* **1977**, *16*, 953.
31. Buchbinder, I.L.; Odintsov, S.D.; Shapiro, I.L. *Effective Action in Quantum Gravity*; IOP Publishing Ltd: Bristol, UK, 1992.

32. Ade, P. A. R.; Aghanim, N.; Ahmed, Z.; Aikin, R.W.; Alexander, K.D.; Arnaud, M.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barkats , D.; *et al.* Joint Analysis of BICEP2/Keck Array and Planck Data. *Phys. Rev. Lett.* **2015**, *114*, 101301.
33. Linde, A.D. Inflationary Cosmology. *Lect. Notes Phys.* **2008**, *738*, 1–54.