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H_∞ Consensus Control for Heterogeneous Multi-Agent via Output under Markov Switching Topologies

Guoying Miao [1,2,*], Gang Li [1], Tao Li [1,2] and Yunping Liu [1,2]

[1] School of Automation, Nanjing University of Information Science and Technology, Nanjing 210044, China; 13851506174@163.com (G.L.); litaojia@163.com (T.L.); liuyunping@nuist.edu.cn (Y.L.)

[2] Jiangsu Collaborative Innovation Center on Atmospheric Environment and Equipment Technology, Nanjing 210044, China

* Correspondence: mgyss66@163.com; Tel.: +86-255-873-1276

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Abstract: The paper investigates H_∞ consensus problem of heterogeneous multi-agent systems including agents with first- and second-order integrators in the presence of disturbance and communication time delays under Markov switching topologies. Based on current messages, outdated information stored in memory and communication time delay information from neighbors, a more general kind of distributed consensus algorithm is proposed, which is faster consensus convergence. By applying stochastic stability analysis, model transformation techniques and graph theory, sufficient conditions of mean square consensus and H_∞ consensus are obtained, respectively. Finally, simulation examples are given to illustrate the effectiveness of obtained theoretical results.

Keywords: consensus; heterogeneous multi-agent; time delays

1. Introduction

In recent years, coordination control of multi-agent systems has become an interesting and attractive topic in control community. Moreover, consensus problem is a basic and critical issue of coordination control.

At present, there have been many results on consensus reported in literatures [1–27]. In [1], authors provided consensus preliminary work of multi-agent systems, where several basic consensus algorithms were established and convergence analysis was also carried out. Then, consensus problems were taken into account for continuous- and discrete-time multi-agent systems in [2] and some useful graph lemmas were introduced. Based on sampled data approach, consensus protocols with sampled interval for the second-order multi-agent systems were proposed in [3]. In order to accelerate consensus convergence, outdated and current positions information was utilized to design controllers for the first-order multi-agent systems in [4]. Moreover, in the context of complex environment, agents have to be separated into small groups to execute different tasks. Then, we say that it is group consensus. By use of models transformation, it is shown that group consensus of heterogeneous multi-agent systems was addressed in [5]. Furthermore, asynchronous group consensus was achieved if the union graph had a spanning tree in [6].

It is well known that time delay widely exists in practical networks. Form the viewpoint of systems stability, researchers mainly focus on time delays how much impact on networks [7–12]. For example, when time delay was changing in [7], average consensus could be reached if there were balanced and connected communication graphs. In [8], output consensus problem was discussed for discrete-time multi-agent systems including output time delays. By applying Laplace transformation approaches, authors in [9] were concerned with consensus analysis of high-order multi-agent systems with different time delays. Since there was input time delays in [10], a kind of group consensus algorithms were proposed for heterogeneous multi-agent systems and stability analysis of closed loop system was discussed in frequency domain. Moreover, due to external disturbance, H infinity control for multi-agent systems with time delays was investigated in [11]. Furthermore, authors in [12] dealt with H infinity consensus problem by applying periodic sampled data approach and graph theory under switching leader-follower topologies. Additionally, other results of networks can be found in [13–20], to name a few.

Since there are some factors like packet dropouts or limited communication distance in practical systems, communication links between agents may be dynamically different, which leads to stochastic switching topologies. Moreover, if they satisfy the property of a Markov chain, it is worth studying consensus problems of multi-agent systems [21–27]. For example, assume that some of transition probabilities were unknown for Markov jump systems, authors in [21] adopted two methods to realize H infinity performance by use of Finsler's Lemma and iterative theory, respectively. In view of an active leader, authors in [22] applied sampled data method to derive necessary and sufficient consensus conditions for discrete-time multi-agent systems under switching topologies driven by a Markov chain. Based on event-triggered idea, the filter was design for discrete-time systems in [23] and made the estimated signal satisfy H infinity performance with disturbance. Moreover, by using systems transformations, mean square consensus problems were turned to be stability analysis of closed-loop systems with stochastic switching signals in [24]. Furthermore, if communication link failures occurred, consensus problems with and without leader of heterogeneous multi-agent systems were studied in [25]. In [26], authors modeled dynamics of the second-order multi-agent systems with input controller. In [27], mean square consensus with saturation was considered. Based on consensus idea, controller laws for network were proposed in [28] for realistic cases. In addition, sleep scheduling method in [29] and separation of time scale in [30] were adopted to deal with network problems, respectively.

The main contribution of the paper is summarized as follows. (1) Compared with literatures [1–7], a more general kind of consensus algorithms is proposed based on three types of output messages. However, proposed algorithms in the paper are different from ones with output information in the literature [8,17–20]. (2) In contrast to similar algorithms in literatures [11,16], proposed consensus algorithms are faster consensus convergence, which is shown by simulation example in Section 4. (3) Based on model transformation techniques and graph theory, mean square consensus is changed to be the problem of stochastic stability of closed-loop systems. Moreover, continuous-time heterogeneous multi-agent systems were discussed under the fixed topology in [11], while we investigate consensus of discrete-time cases under directed Markovian switching topologies.

2. Graph Theory

Before we give the main results, fundamental graph theory used in the article is introduced ahead. Suppose that there are $n_1 + n_2$ agents, which consists n_1 first-order agents and n_2 second-order agents. We use $\mathcal{G} = (\mathcal{E}, \mathcal{A})$ to express the graph corresponding to communication topology, where $\mathcal{E}$ is a collection of communication edges and $\mathcal{A} = [a_{ij}]_{n_1+n_2,n_1+n_2}$ is the adjacency matrix. Note that if there exists communication information exchange from the ith agent to the jth agent, we define $a_{ij} > 0$; otherwise $a_{ij} = 0$. Moreover, if messages transmission is directional for any two agents, $\mathcal{G}$ is called a directed graph. Then, Laplacian matrix is defined $L = [l_{ij}]_{n_1+n_2,n_1+n_2}$ with $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = -\sum_{j=1,j\neq i}^{n_1+n_2} a_{ij}$ with $i = j$.

In the graph $\mathcal{G}$, if a node could not receive any information from others but could transfer messages to at least a node, it is called the root node. The directed graph $\mathcal{G}$ has a spanning tree if there exists at least a path from the root node to others.

3. Main results

3.1. Mean Square Consensus of Heterogeneous Multi-Agent Systems without Disturbance

The dynamical equations of the first-order agents are modeled as follows:

$$x_i(k+1) = mx_i(k) + u_i(k)T + g_{1i}w_{1i}(k) \tag{1}$$

$$y_i = c_i x_i(t), i = 1, \dots, n_1 \tag{2}$$

where $x_i(k) \in R^1$, $u_i(k) \in R^1$, $y_i(t) \in R^1$ represent position, input control and output of the ith agent, respectively. And c_i is a constant, R^1 is the set of vectors, $w_{1i}(k) \in \mathcal{L}_2[0, \infty)$ denotes external disturbance, $\mathcal{L}_2[0, \infty)$ is the set of square-integrable functions, $g_{1i} > 0$ is a constant, m is a constant coefficient, T is a sampling time.

The dynamics of the second-order agents are considered as follows

$$x_i(k+1) = mx_i(k) + v_i(k)T + g_{2i}w_{2i}(k) \tag{3}$$

$$v_i(k+1) = v_i(k) + u_i(k)T + g_{3i}w_{3i}(k) \tag{4}$$

$$y_i(t) = c_i x_i(t) \tag{5}$$

$$\widehat{y}_i(t) = d_i v_i(t),\ i = n_1+1, \dots, n_1+n_2 \tag{6}$$

where $x_i(k) \in R^1$, $v_i(k) \in R^1$, $u_i(k) \in R^1$ denote the ith agent's position, velocity, input control, respectively. And $y_i(t)$, $\widehat{y}_i(t)$ denote the ith agent's output, $g_{2i} > 0$ and $g_{3i} > 0$ are constants, $w_{2i}(k) \in \mathcal{L}_2[0, \infty)$ and $w_{3i}(k) \in \mathcal{L}_2[0, \infty)$ are external disturbances. m is the same to the one in (1).

In some practical situations, information of positions and velocities could not be directly obtained at some times. However, output messages are easily measured. Then, based on output $y_i(t)$ and $\widehat{y}_i(t)$, consensus algorithms with communication time delays are proposed as follows

$$u_i(k) = \begin{cases} k_1 \sum\limits_{j=1}^{n_1+n_2} a_{ij}[y_j(k-\tau) - y_i(k-\tau)] - \alpha \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij}\widehat{y}_j(k-\tau) - k_1\varepsilon_i y_i(k-\tau), \quad i = 1, \dots n_1, \\ k_2 \sum\limits_{j=1}^{n_1+n_2} a_{ij}\left[y_j(k-\tau) - y_i(k-\tau)\right] + \beta \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij}\left[\widehat{y}_j(k-\tau) - \widehat{y}_i(k-\tau)\right] - k_3\widehat{y}_i(k), i = n_1+1 \dots n_1+n_2, \end{cases} \tag{7}$$

where $\tau > 0$ is communication time delay among agents, $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $\alpha > 0$, $\beta > 0$, $\varepsilon_1 > 0$ are constants and $\varepsilon_i = 0$ for $i = 2, \dots, n_1$. Moreover, $y_j(k-\tau)$ and $\widehat{y}_j(k-\tau)$ denote received messages of the ith agent from the jth agent at time kT, $y_i(k-\tau)$ and $\widehat{y}_i(k-\tau)$ are the outdated information stored in memory of the ith agent at time $(k-\tau)T$, $\widehat{y}_i(k)$ is current output of the ith agent.

Replacing $y_i(t)$ and $\widehat{y}_i(t)$ in (7) by ones in (2), (5), (6), we have

$$u_i(k) = \begin{cases} k_1 \sum\limits_{j=1}^{n_1+n_2} a_{ij}[c_j x_j(k-\tau) - c_i x_i(k-\tau)] - \alpha \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij} d_j v_j(k-\tau) - k_1\varepsilon_i c_i x_i(k-\tau), \quad i = 1, \dots n_1, \\ k_2 \sum\limits_{j=1}^{n_1+n_2} a_{ij}\left[c_j x_j(k-\tau) - c_i x_i(k-\tau)\right] + \beta \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij}\left[d_j v_j(k-\tau) - d_i v_i(k-\tau)\right] - k_3 d_i v_i(k), i = n_1+1 \dots n_1+n_2. \end{cases} \tag{8}$$

Remark 1. *In [1–7], authors used positions and velocities to design consensus algorithms. In order to overcome the difficulty of directly obtaining the above information at certain times, consensus algorithms with output information are proposed in this paper, which are different from those with output signals in [8,17–20]. Moreover, authors in [4] applied history information to make agents converge much faster. Mainly inspired by the idea in [4], we utilize outdated data to construct consensus algorithms. And (7) is made up of three parts, which is time delay messages of the jth agent, outdated memory and current information of the ith agent.*

Remark 2. *Note that authors directly used position information to design controllers and behavior of every agent was identical in literature [4], while output information is utilized to construct algorithms and dynamic of every agent is different in this paper. In addition, directed stochastic switching topologies are consider in the paper.*

The estimated signal is denoted by $z(k)$ as follows

$$z(k) = \begin{bmatrix} \overline{D}_1 & D_2 \end{bmatrix} \begin{bmatrix} \xi(k) \\ \widehat{y}(t) \end{bmatrix} \tag{9}$$

where $\overline{D}_1$ and D_2 are constant matrices, $\widehat{y}(t) = \widehat{D}v(t)$, $\xi(k)$, $\widehat{D}$, $v(k)$ are identical to ones in (11).

Definition 1. *[24] Under the proposed algorithms (7), systems (1), (3), (4) reach mean square consensus without disturbance if $\lim\limits_{k\to\infty} E(x_i(k) - x_j(k))^2 = 0, \quad i = 1, \ldots, n_1,$*

$$\lim_{k\to\infty} E(v_i(k) - v_j(k))^2 = 0, \quad i = n_1 + 1, \ldots, n_1 + n_2$$

Definition 2. *[12,21,23,31] H_∞ consensus problem investigated in this paper is to design the consensus algorithm (7), such that the following conditions are satisfied*

(1) Under the consensus algorithm (7), systems (1), (3), (4) reach mean square consensus with $w(k) = 0$;
(2) Under zero initial condition, for any nonzero $w(k) \neq 0$, the following inequality holds for $\gamma > 0$

$$E\left\{\sum_{k=0}^{\infty} z^T(k)z(k)\right\} \leq \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k)$$

Lemma 1. *[2,3] The Laplacian matrix L has exactly one zero eigenvalues and all other eigenvalues have positive real parts, if and only if the directed graph associated with L has a spanning tree.*

Set $x_f(k) = \begin{bmatrix} x_1^T(k) & \ldots & x_{n_1}^T(k) \end{bmatrix}^T$, $x_s(k) = \begin{bmatrix} x_{n_1+1}^T(k) & \ldots & x_{n_1+n_2}^T(k) \end{bmatrix}^T$. In order to express clearly, we divide L into the form $\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$, where $L_{11} \in R^{n_1 \times n_1}$, $L_{12} \in R^{n_1 \times n_2}$, $L_{22} \in R^{n_2 \times n_2}$, $R^{n_1 \times n_1}$ is the set of $n_1 \times n_1$ dimensional matrices. Moreover, $\overline{L}$ is denoted by $\begin{bmatrix} \overline{L}_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$ with $\overline{L}_{11} = L_{11} + diag\{\varepsilon_1, \ldots, 0\}$.

Then, combining (1)–(6) with (7) yields

$$\begin{aligned} \begin{bmatrix} x_f(k+1) \\ x_s(k+1) \end{bmatrix} &= \begin{bmatrix} mI & 0 \\ 0 & mI \end{bmatrix} \begin{bmatrix} x_f(k) \\ x_s(k) \end{bmatrix} + \begin{bmatrix} -k_1\overline{L}_{11}T & -k_1L_{12}T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1x_f(k-\tau) \\ C_2x_s(k-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ v(k)T \end{bmatrix} \\ &+ \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} + \begin{bmatrix} \alpha L_{12}\widehat{D}T \\ 0 \end{bmatrix} v(k-\tau) \end{aligned} \tag{10}$$

where

$$w_1(k) = \begin{bmatrix} w_{11}(k) \\ \vdots \\ w_{1,n_1}(k) \end{bmatrix}, w_2(k) = \begin{bmatrix} w_{2,n_1+1}(k) \\ \vdots \\ w_{2,n_1+n_2}(k) \end{bmatrix}, v(k) = \begin{bmatrix} v_{n_1+1}(k) \\ \vdots \\ v_{n_1+n_2}(k) \end{bmatrix}, C_1 = diag\{c_1, \ldots, c_{n_1}\},$$

$g_1 = diag\{g_{11}, \ldots, g_{1,n_1}\}$, $g_2 = diag\{g_{2,n_1+1}, \ldots, g_{2,n_1+n_2}\}$, $C_2 = diag\{c_{n_1+1}, \ldots, c_{n_1+n_2}\}$, $\widehat{D} = diag\{d_{n_1+1}, \ldots, d_{n_1+n_2}\}$. And I is an identity matrix with appropriate dimensions. 0 is a suitable dimensional matrix (or vector) from the context with every element being zero.

We let $\xi(k)=\overline{L}\begin{bmatrix} C_1x_f(k)\\ C_2x_s(k)\end{bmatrix}$. It follows from (10) that

$$\begin{aligned}
&\xi(k+1)=\overline{L}\begin{bmatrix} C_1x_f(k+1)\\ C_2x_s(k+1)\end{bmatrix}\\
&=\overline{L}\begin{bmatrix} mI & 0\\ 0 & mI\end{bmatrix}\begin{bmatrix} C_1x_f(k)\\ C_2x_s(k)\end{bmatrix}+\overline{L}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\begin{bmatrix} -k_1\overline{L}_{11}T & -k_1L_{12}T\\ 0 & 0\end{bmatrix}\begin{bmatrix} C_1x_f(k-\tau)\\ C_2x_s(k-\tau)\end{bmatrix}+\overline{L}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\begin{bmatrix} 0\\ v(k)T\end{bmatrix}\\
&+\overline{L}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\begin{bmatrix} g_1 & 0\\ 0 & g_2\end{bmatrix}\begin{bmatrix} w_1(k)\\ w_2(k)\end{bmatrix}+\overline{L}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\begin{bmatrix} \alpha L_{12}\widehat{D}T\\ 0\end{bmatrix}v(k-\tau)\\
&=\begin{bmatrix} mI & 0\\ 0 & mI\end{bmatrix}\overline{L}\begin{bmatrix} C_1x_f(k)\\ C_2x_s(k)\end{bmatrix}-\overline{L}\begin{bmatrix} k_1C_1T & 0\\ 0 & 0\end{bmatrix}\overline{L}\begin{bmatrix} C_1x_f(k-\tau)\\ C_2x_s(k-\tau)\end{bmatrix}+\begin{bmatrix} L_{12}C_2T\\ L_{22}C_2T\end{bmatrix}v(k)+F\begin{bmatrix} w_1(k)\\ w_2(k)\end{bmatrix}+\widehat{L}v(k-\tau)\\
&=\begin{bmatrix} mI & 0\\ 0 & mI\end{bmatrix}\xi(k)-\widetilde{L}\xi(k-\tau)+\begin{bmatrix} L_{12}C_2T\\ L_{22}C_2T\end{bmatrix}v(k)+F\begin{bmatrix} w_1(k)\\ w_2(k)\end{bmatrix}+\widehat{L}v(k-\tau)
\end{aligned}\tag{11}$$

where $\widetilde{L}=\overline{L}\begin{bmatrix} k_1C_1T & 0\\ 0 & 0\end{bmatrix}$, $F=\overline{L}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\begin{bmatrix} g_1 & 0\\ 0 & g_2\end{bmatrix}$, $\widehat{L}=\begin{bmatrix} \alpha\overline{L}_{11}C_1L_{12}\widehat{D}T\\ \alpha L_{21}C_1L_{12}\widehat{D}T\end{bmatrix}$.

Together (4) with (8), it implies

$$\begin{aligned}
v(k+1)&=(I-k_3\widehat{D}T)v(k)-k_2T\begin{bmatrix} L_{21} & L_{22}\end{bmatrix}\begin{bmatrix} C_1x_f(k-\tau)\\ C_2x_s(k-\tau)\end{bmatrix}-\beta L_{22}\widehat{D}Tv(k-\tau)+g_3w_3(k)\\
&=(I-k_3\widehat{D}T)v(k)-\begin{bmatrix} 0 & k_2T\end{bmatrix}\begin{bmatrix} \overline{L}_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix} C_1x_f(k-\tau)\\ C_2x_s(k-\tau)\end{bmatrix}-\beta L_{22}\widehat{D}Tv(k-\tau)+g_3w_3(k)\\
&=(I-k_3\widehat{D}T)v(k)-\begin{bmatrix} 0 & k_2T\end{bmatrix}\xi(k-\tau)-\beta L_{22}\widehat{D}Tv(k-\tau)+g_3w_3(k),
\end{aligned}\tag{12}$$

where $g_3=diag\{g_{3,n_1+1},\ldots,g_{3,n_1+n_2}\}$, $w_3(k)=\begin{bmatrix} w^T_{3,n_1+1}(k) & \ldots & w^T_{3,n_1+n_2}(k)\end{bmatrix}^T$.

Changing (11) and (12) into a matrix form results in

$$\begin{bmatrix} \xi(k+1)\\ v(k+1)\end{bmatrix}=\begin{bmatrix} \begin{bmatrix} mI & 0\\ 0 & mI\end{bmatrix} & \begin{bmatrix} L_{12}C_2T\\ L_{22}C_2T\end{bmatrix}\\ 0 & I-k_3\widehat{D}T\end{bmatrix}\begin{bmatrix} \xi(k)\\ v(k)\end{bmatrix}+\begin{bmatrix} -\widetilde{L} & \widehat{L}\\ \begin{bmatrix} 0 & -k_2T\end{bmatrix} & -\beta L_{22}\widehat{D}T\end{bmatrix}\begin{bmatrix} \xi(k-\tau)\\ v(k-\tau)\end{bmatrix}+\begin{bmatrix} F & 0\\ 0 & g_3I\end{bmatrix}\begin{bmatrix} w_1(k)\\ w_2(k)\\ w_3(k)\end{bmatrix}$$

Set $\eta(k)=\begin{bmatrix} \xi(k)\\ v(k)\end{bmatrix}$. Rewriting the above equation yields

$$\eta(k+1)=R\eta(k)+B\eta(k-\tau)+\overline{C}w(k),\tag{13}$$

where

$$R=\begin{bmatrix} \begin{bmatrix} mI & 0\\ 0 & mI\end{bmatrix} & \begin{bmatrix} L_{12}C_2T\\ L_{22}C_2T\end{bmatrix}\\ 0 & I-k_3\widehat{D}T\end{bmatrix},\quad B=\begin{bmatrix} -\widetilde{L} & \widehat{L}\\ \begin{bmatrix} 0 & -k_2T\end{bmatrix} & -\beta L_{22}\widehat{D}T\end{bmatrix},\quad \overline{C}=\begin{bmatrix} F & 0\\ 0 & g_3I\end{bmatrix},\quad w(k)=\begin{bmatrix} w_1(k)\\ w_2(k)\\ w_3(k)\end{bmatrix}.$$

Considering the case that communication switching topologies satisfy property of the Markov chain, (13) can be expressed as follows

$$\eta(k+1)=R_{\sigma(k)}\eta(k)+B_{\sigma(k)}\eta(k-\tau)+\overline{C}_{\sigma(k)}w(k),\sigma(k)\in=\begin{Bmatrix} 1, & \ldots & s_1\end{Bmatrix}\tag{14}$$

where $\sigma(k)$ is a switching signal driven by a Markov chain, which takes vales in the set $\begin{Bmatrix} 1, & \ldots & s_1\end{Bmatrix}$ with s_1 being a positive number. Then, we use $\pi=\left[\pi_{ij}\right]_{s_1\times s_1}$ to represent the transition probability matrix with $\Pr\{\sigma(k+1) = \mathcal{G}_j | \sigma(k) = \mathcal{G}_j\} = \pi_{ij}$ where $\pi_{ij} \geq 0$ denotes the transition rate from the topology associated with graph $\mathcal{G}_i$ to the one corresponding to graph $\mathcal{G}_i$ and satisfies $\sum_{j=1}^{s_1} \pi_{ij} = 1$ for $i \in Z$.

After above models transformations, mean square consensus problems of systems (1), (3), (4) turns to be stability analysis of system (14).

Theorem 1. *Assume that each communication graph $\mathcal{G}_i$ contains a directed spanning tree for $i = 1, \ldots, s_1$. When $w(k) = 0$, if there exist positive definite matrices P_i, Q for constants $k_1 > 0$, $k_2 > 0$, $\tau > 0$, $T > 0$, $\alpha > 0$, $\beta > 0$ and $\varepsilon_1 > 0$, such that the following inequalities hold*

$$\begin{bmatrix} R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i + \tau Q - P_i & R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i \\ * & B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i - \tau Q \end{bmatrix} < 0, \quad i = 1, \ldots, s_1. \tag{15}$$

Then, under the proposed consensus algorithm (7), systems (1), (3), (4) can reach mean square consensus.

Proof. Choose the Lyapunov function modified based on ones in [21,23,31]

$$V(\eta(k), \sigma(k)) = \eta^T(k) P_i \eta(k) + \tau \sum_{l=k-\tau}^{k-1} \eta^T(l) Q \eta(l) \tag{16}$$

where P_i, Q are positive definite matrices. Taking difference of (16) in the stochastic process sense, we have

$$\begin{aligned}
&E[\Delta V(k)] = \eta^T(k+1) \sum_{j=1}^{s_1} \pi_{ij} P_j \eta(k+1) - \eta^T(k) P_i \eta(k) + \tau \sum_{l=k-\tau+1}^{(k-1)+1} \eta^T(l) Q \eta(l) - \tau \sum_{l=k-\tau}^{k-1} \eta^T(l) Q \eta(l) \\
&= \eta^T(k) R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i \eta(k) + 2\eta^T(k-\tau) B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i \eta(k) + \eta^T(k-\tau) B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i \eta(k-\tau) - \eta^T(k) P_i \eta(k) \\
&+ \tau[\eta^T(k) Q \eta(k) + \eta^T(k-1) Q \eta(k-1) + \ldots + \eta^T(k-\tau+1) Q \eta(k-\tau+1)] \\
&- \tau[\eta^T(k-1) Q \eta(k-1) + \ldots + \eta^T(k-\tau+1) Q \eta(k-\tau+1) + \eta^T(k-\tau) Q \eta(k-\tau)] \\
&= \begin{bmatrix} \eta(k) & \eta(k-\tau) \end{bmatrix} \begin{bmatrix} R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i \eta(k) + \tau Q - P_i & R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i \\ * & B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i - \tau Q \end{bmatrix} \begin{bmatrix} \eta(k) \\ \eta(k-\tau) \end{bmatrix}
\end{aligned}$$

From condition (15), we can see $E(\Delta V(k)) < 0$. Thus, we obtain $\lim_{k \to \infty} E(\eta^T(k)\eta(k)) = 0$. In view of $\eta(k) = \begin{bmatrix} \xi^T(k) & v^T(k) \end{bmatrix}^T$, $\lim_{k \to \infty} E(\xi^T(k)\xi(k)) = 0$ and $\lim_{k \to \infty} E(v^T(k)v(k)) = 0$ are derived. Since every communication graph has a spanning tree, according to Lemma 1, L has only one zero eigenvalue, which implies $rank(L) = n_1 + n_2 - 1$. Then, there exists an invertible matrix $\overset{\frown}{P}$ to make the following transformation,

$$\begin{aligned}
&\overset{\frown}{P}\{L + diag\{\varepsilon_1, 0, \ldots, 0\}\}\overset{\frown}{P}^{-1} \\
&= \overset{\frown}{P} L \overset{\frown}{P}^{-1} + \overset{\frown}{P} diag\{\varepsilon_1, 0, \ldots, 0\} \overset{\frown}{P}^{-1} \\
&= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_r \end{bmatrix} + \overset{\frown}{P} diag\{\varepsilon_1, 0, \ldots, 0\} \overset{\frown}{P}^{-1},
\end{aligned} \tag{17}$$

where J_i is a Jordan normal matrix for $i = 1, \ldots, r$ with $r \leq n_1 + n_2 - 1$.

Noting (17), we have

$$\begin{aligned} rank(\overline{L}) &= rank(L + diag\{\varepsilon_1, 0, \ldots, 0\}) \\ &= rank(\widehat{P}[L + diag\{\varepsilon_1, 0, \ldots, 0\}]\widehat{P}^{-1}). \end{aligned}$$

From above analysis, we can see $rank(\overline{L}) = n_1 + n_2$. It is obvious that $\overline{L}$ is a nonsingular matrix. Due to $\xi(k) = \overline{L}\begin{bmatrix} C_1^T x_f^T(k) & C_2^T x_s^T(k) \end{bmatrix}^T$ with nonsingular matrices C_1 and C_2, it implies $\lim\limits_{k\to\infty} E(x_i(k))^2 = 0$. Furthermore, we have $\lim\limits_{k\to\infty} E(x_i(k) - x_j(k))^2 = 0$ for $i, j = 1, \ldots, n_1 + n_2$ and $\lim\limits_{k\to\infty} E(v_i(k) - v_j(k))^2 = 0$ for $i, j = n_1 + 1, \ldots, n_1 + n_2$. Therefore, using Definition 1, under the proposed consensus algorithm (7), we say systems (1), (3), (4) can reach mean square consensus. □

Next, we will discuss less messages needed than one in (7) to construct consensus algorithms. Then, if we only use time delay messages of $y_j(k-\tau)$, $\widehat{y}_j(k-\tau)$ and outdated information $y_i(k-\tau)$, (7) is turned to be

$$u_i(k) = \begin{cases} k_1 \sum\limits_{j=1}^{n_1+n_2} a_{ij}[y_j(k-\tau) - y_i(k-\tau)] - \alpha \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij}\widehat{y}_j(k-\tau) - k_1\varepsilon_i y_i(k-\tau), \quad i = 1, \ldots n_1, \\ k_2 \sum\limits_{j=1}^{n_1+n_2} a_{ij}\left[y_j(k-\tau) - y_i(k-\tau)\right] + \beta \sum\limits_{j=n_1+1}^{n_1+n_2} a_{ij}\left[\widehat{y}_j(k-\tau) - \widehat{y}_i(k-\tau)\right], i = n_1 + 1 \ldots n_1 + n_2. \end{cases} \tag{18}$$

Moreover, consensus algorithms in [16] is a specific one of (18). In addition, if we only use output information related with position and current messages, (7) is changed to be

$$u_i(k) = \begin{cases} k_1 \sum\limits_{j=1}^{n_1+n_2} a_{ij}[y_j(k-\tau) - y_i(k-\tau)] - k_1\varepsilon_i y_i(k-\tau), \quad i = 1, \ldots n_1, \\ k_2 \sum\limits_{j=1}^{n_1+n_2} a_{ij}\left[y_j(k-\tau) - y_i(k-\tau)\right] - k_3\widehat{y}_i(k), i = n_1 + 1 \ldots n_1 + n_2. \end{cases} \tag{19}$$

Then, we can see that (19) is more general consensus algorithm than one in [11]. Moreover, algorithms (18) and (19) are specific cases of (7). However, (7) needs more output information than algorithms (18) and (19), which leads to complicated controller. And Section 4 gives a convergence comparison of (7), (18), (19) by simulation examples. Thus, we obtain that consensus convergence in (7) is faster than ones in (18) and (19). Thus, there is a trade-off between faster convergence of (7) and simpler controllers of (18) and (19).

Remark 3. *Theorem 1 extends the results in [5,25] to the one with communication time delays under Markovian switching topologies.*

3.2. H_∞ Consensus of Heterogeneous Multi-Agent Systems with Disturbance

In the section, we will discuss H_∞ consensus with disturbance $w(k) \neq 0$. Under switching topologies driven by a Markov chain, (9) can be rewritten

$$\begin{aligned} z(k) &= \overline{D}_{1,\sigma(k)}\xi_{\sigma(k)}(k) + \overline{D}_{2,\sigma(k)}v_{\sigma(k)}(k) \\ &= \begin{bmatrix} \overline{D}_{1,\sigma(k)} & \overline{D}_{2,\sigma(k)} \end{bmatrix}\eta_{\sigma(k)}(k) \end{aligned} \tag{20}$$

where $\overline{D}_2 = D_2\widehat{D}$, $\sigma(k)$ is identical to the one in (14).

Theorem 2. *Assume that each communication graph $\mathcal{G}_i$ contains a directed spanning tree. If there exist constants $k_1 > 0, k_2 > 0, k_3 > 0, g_1 > 0, g_2 > 0, g_3 > 0, \tau > 0, T > 0, \alpha > 0, \beta > 0, \gamma > 0$ and $\varepsilon_1 > 0$, constant matrices $\overline{D}_{1i}$, $\overline{D}_{2i}$, positive definite matrices P_i, Q, such the following linear matrix inequalities hold*

$$\Sigma_i = \begin{bmatrix} \Omega_i & R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i & R_i T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i \\ * & B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i - \tau Q & B_i T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i \\ * & * & \overline{C}_i T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i - \gamma^2 I \end{bmatrix} < 0, \; i = 1, \ldots, s_1 \tag{21}$$

where $\Omega_i = R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i + \tau Q + \begin{bmatrix} \overline{D}_{1i}^T \\ \overline{D}_{2i}^T \end{bmatrix} \begin{bmatrix} \overline{D}_{1i} & \overline{D}_{2i} \end{bmatrix} - P_i$. Thus, under the consensus algorithm (7), systems (1), (3), (4) can solve H_∞ consensus problem.

Proof. Choose the Lyapunov function as the one in (16). Then, calculating difference of (16) with $w(k) \neq 0$, we have

$$\begin{aligned} E(\Delta V(k)) &= \left[\eta^T(k) R_i^T + \eta^T(k-\tau) B_i^T + w^T(k) \overline{C}_i^T\right] \sum_{j=1}^{s_1} \pi_{ij} P_j \left[R_i \eta(k) + B_i \eta(k-\tau) + \overline{C}_i w(k)\right] - \eta^T(k) P_i \eta(k) \\ &+ \tau \eta^T(k) Q \eta(k) - \tau \eta^T(k-\tau) Q \eta(k-\tau) \\ &= \eta^T(k) R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j R_i \eta(k) + 2\eta^T(k-\tau) B_i^T \sum_{j=1}^{s1} \pi_{ij} P_j R_i \eta(k) + 2\eta^T(k) R_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i w(k) - \eta^T(k) P_i \eta(k) \\ &+ 2\eta^T(k-\tau) B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i w(k) + w^T(k) \overline{C}_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j \overline{C}_i w(k) + \eta^T(k-\tau) B_i^T \sum_{j=1}^{s_1} \pi_{ij} P_j B_i \eta^T(k-\tau) \\ &+ \tau \eta^T(k) Q \eta(k) - \tau \eta^T(k-\tau) Q \eta(k-\tau) \end{aligned}$$

Observing (20) and condition (21), we have

$$E(z^T(k) z(k) - \gamma^2 w^T(k) w(k) + \Delta V(k)) = \psi^T(k) \Sigma_i \psi(k) < 0$$

where $\psi(k) = \begin{bmatrix} \eta^T(k) & \eta^T(k-\tau) & w^T(k) \end{bmatrix}^T$.

Recalling the zero initial condition of $\eta(0) = 0$ and following a similar method in [23], we have

$$\begin{aligned} & E\left(\sum_{k=0}^{\infty} z^T(k) z(k) - \gamma^2 \sum_{k=0}^{\infty} w^T(k) w(k)\right) \\ &= -E\left(\sum_{k=0}^{\infty} \Delta V(k)\right) + \sum_{k=0}^{\infty} \psi^T(k) \Sigma_i \psi(k) \\ &\leq E(V(0)) + \sum_{k=0}^{\infty} \psi^T(k) \Sigma_i \psi(k) \leq 0 \end{aligned}$$

From above analysis, we derive $E\left(\sum_{k=0}^{\infty} z^T(k) z(k)\right) \leq \gamma^2 \sum_{k=0}^{\infty} w^T(k) w(k)$. According to Definition 2, under the algorithm (7), systems (1), (3), (4) solve H_∞ consensus problem. □

Remark 4. *In [11], H_∞ consensus problems of continuous-time multi-agent systems were investigated. However, we focus on the case of discrete-time multi-agent systems.*

4. Simulations

In this section, consider a multi-agent system consisting of four multi-agents, where agents 1–2 have first-order integrators and agents 3–4 are second-order integrators. The communication information flows jumped between graphs $\{\mathcal{G}_1, \quad \mathcal{G}_2\}$ associated with switching topologies. The graph is shown in Figure 1. And assume that the transition probability matrix is as follows

$$\pi = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}.$$

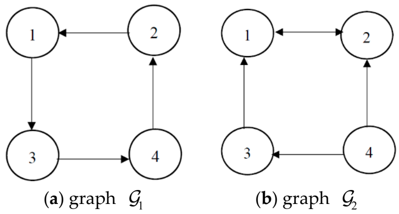

Figure 1. Two directed graphs, (**a**) graph $\mathcal{G}_1$ and (**b**) graph $\mathcal{G}_2$.

If there exists the information flow from the ith agent to the jth agent, we set $a_{ij} = 1$; otherwise it is zero. Choose $m = 1$, $\tau = 0.1$, $T = 0.1$, $k_1 = k_2 = 1.5$, $\beta = 1.8$, $k_3 = 1$, $\varepsilon_1 = 0.9$, $\alpha = 0.2$, $c_i = 1$, $d_i = 1$ for $i = 1, \ldots, 4$. When $w(k) = 0$, linear matrix inequality in (15) is feasible checked by Matlab tools. According to Theorem 1, systems (1), (3), (4) achieve mean square consensus. Figures 2–4 show trajectories of position and velocity in systems (1), (3), (4) by applying consensus algorithms (7), (18), (19), respectively. From these pictures, we can see that consensus convergence of (7) is faster than ones in (18) and (19).

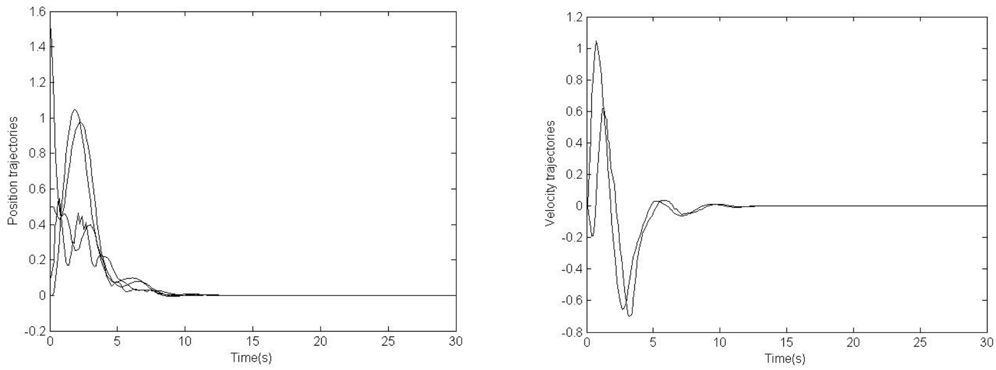

Figure 2. Trajectories of systems (1), (3), (4) with consensus algorithm (7).

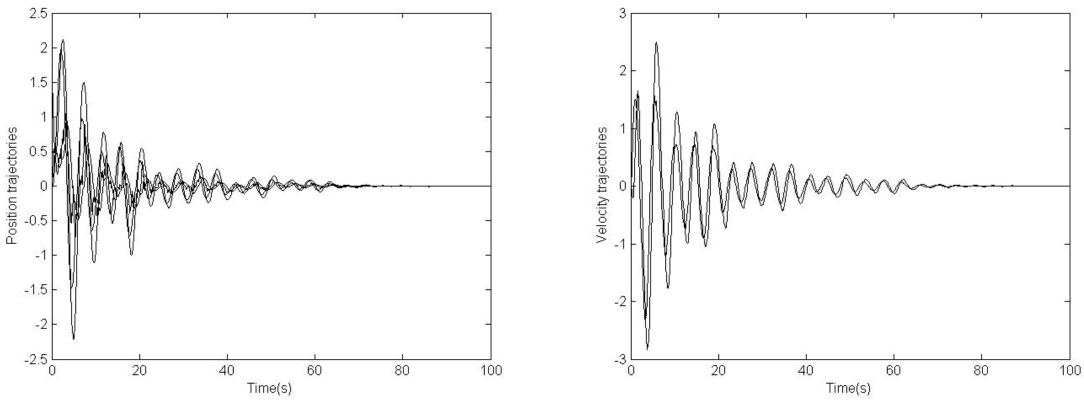

Figure 3. Trajectories of systems (1), (3), (4) with consensus algorithm (18).

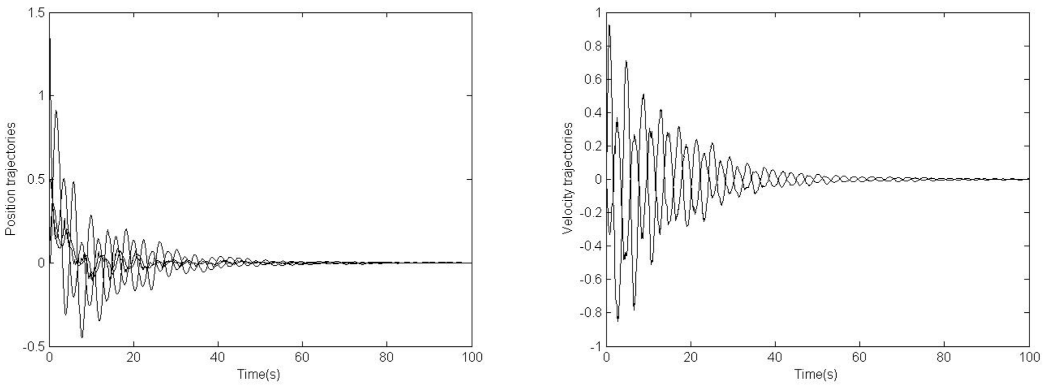

Figure 4. Trajectories of systems (1), (3), (4) with consensus algorithm (19).

For the case of $w(k) \neq 0$, we select $w(k) = \frac{1}{1+k^2}$, $\gamma = 5.2$, $g_{11} = g_{12} = 0.1$, $g_{23} = g_{24} = 0.1$, $g_{33} = 0.2$, $g_{34} = 0.2$, $\overline{D}_1 = \begin{bmatrix} 0.1 & 0.1 & 0.5 & 0.1 \end{bmatrix}$, $\overline{D}_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$. By verification of Matlab tools, linear matrix inequality in (21) is feasible. According to Theorem 2, systems (1), (3), (4) solve H_∞ consensus problem. Figure 5 gives the picture of the estimated signal $z(k)$.

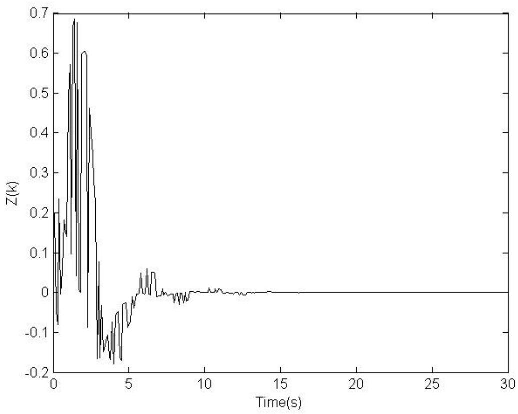

Figure 5. Picture of the estimated signal $Z(k)$.

5. Conclusions

In this paper, H_∞ consensus problem of heterogeneous multi-agent system including the first- and second-order agents under Markov switching topologies with external interference. And consensus algorithms with communication time delay via output are proposed. By using stochastic stability theory, linear matrix inequality technique and graph theory, sufficient H_∞ consensus conditions are derived. Finally, we give simulation examples to demonstrate the effectiveness of proposed results in this paper.

In addition, if we choose inappropriate parameters k_1, k_2, k_3, α, β and so on, consensus convergence of proposed algorithms (7) may not outperform compared with other algorithms. Thus, our future work is to find range of appropriate parameters in consensus algorithms by theoretical analysis of convergence.

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