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Decentralized Fuzzy Fault Estimation Observer Design for Discrete-Time Nonlinear Interconnected Systems

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Abstract: In this paper, a fault estimation technique is proposed for discrete-time nonlinear interconnected systems with uncertain interconnections. To achieve the fault estimation, the decentralized fuzzy observer is adopted based on the Takagi–Sugeno fuzzy model. Based on the estimation error model with the subsystems of the interconnected system and its decentralized fuzzy observer, the fault estimation condition with H_∞ performance is presented. The main idea of this paper is that a novel inequality condition for H_∞ performance is used, and the sufficient condition is presented to guarantee the good fault estimation performance. Also, the decentralized fuzzy observer design condition for the fault estimation is converted into linear matrix inequality formats. Finally, a simulation example is provided, and the effectiveness of the proposed fault estimation technique is verified by comparison of the fault estimation performance.

Keywords: fault estimation; discrete-time nonlinear interconnected system; uncertain interconnections; decentralized fuzzy observer; Takagi–Sugeno fuzzy model; linear matrix inequality

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1. Introduction

As society and technology advance and become more complex, the structure of systems is also becoming increasingly complexity compared to traditional system models. Among the complex systems, there are interconnected systems that have characteristics of interconnections between multiple systems. The interconnection problems of systems have attracted considerable attention in various fields such as physics [1], communication networks [2], biological systems modeling [3], climate forecasting [4], and financial markets analysis [5] and so on. As characteristics of nonlinear large-scale systems are emerging in various control issues such as automotive control, energy management, and robotics, the significance of nonlinear interconnected systems in engineering and control system fields has been steadily increasing in recent years [6]. In particular, by the widespread adoption of computer and network-based management and control for systems, more attention is being paid to discrete-time nonlinear interconnected systems [7–9]. The complex structures and interdependencies that characterize interconnected systems make the application of traditional control techniques difficult and pose challenges for the novel technique as a decentralized approach [10,11]. Accordingly, many decentralized techniques for interconnected systems have been studied to date [12–15]. However, compared to many remarkable research studies on decentralized control, not much research on decentralized estimation has been presented yet [16–19]. Especially, there is little research on fault estimation, which has been in the spotlight recently.

Apart from the interconnected system issue, enhancing the safety and reliability of systems has become a key challenge in modern control systems. Specifically, accurately estimating faults plays a crucial role in improving the safety and reliability of the system. As a result, there is a growing interest in the research field of Fault Detection and Diagnosis (FDD) [20,21], and interest is focused on fault estimation techniques. Accordingly, numerous studies have been conducted on fault estimation techniques up to the present [22–24]. Among them, fuzzy fault estimation using Takagi–Sugeno (T–S) fuzzy model [25] has the advantage of being able to easily achieve fault estimation for nonlinear systems [26]. Thus, many remarkable studies on fuzzy fault estimation are being presented as follows [27–36]: In [28,29], the continuous- and discrete-time fault estimation techniques have been presented for T–S fuzzy systems, respectively. In [27,33–35], the various problems have been solved for the fuzzy fault estimation for IT2 fuzzy systems. In [31], the fault estimation problems have been addressed for nonlinear fractional-order systems with unknown inputs. While previous studies have only focused on cases of measurable premise variables, ref. [30] specifically addressed problems considering non-measurable premise variables. Furthermore, a novel fault estimation technique has been proposed to solve the sampled-data output problem in [36].

Despite the many fuzzy fault estimation studies, fault estimation research for interconnected systems has not been sufficiently conducted, because the interconnections of interconnected systems is a great obstacle to designing the observer of the filter of the system. In fact, research on fault estimation for interconnected systems to date has been conducted under the assumption that all information about the subsystems including interconnections is known [37–41]. However, accurately knowing all interconnection information is very difficult in real engineering systems. In addition, the nonlinearity problem has not been considered in [37–40]. In [41], the fuzzy fault estimation has been addressed for nonlinear interconnected systems, but it still assumes that the all interconnections are known. To solve the unknown or uncertain interconnection problem, some techniques such as fuzzy observer [42] and fuzzy filter [43–45] have been presented for nonlinear interconnected systems with uncertain interconnection, but fault estimation issues have not been considered. To the best of the author's knowledge, the fault estimation technique has not been studied for discrete-time nonlinear interconnected systems with uncertain interconnections so far.

Motivated by the above analysis for the previous studies, the fault estimation technique is proposed for the decentralized fuzzy observer of discrete-time nonlinear interconnected systems with uncertain interconnections in this paper. To develop the technique of the fault estimation, the following research difficulties need to be addressed.

1. The general fault estimation techniques have been analyzed based on the Lyapunov stability by estimation errors alone. However, in this paper, due to interconnections of state variables, the previous approach is insufficient to address the decentralized fault estimation problem.
2. To design a fault estimation observer, the following two conditions have to be satisfied: (i) the error system should be asymptotically stable when there are no actuator fault inputs and disturbances; (ii) the estimation errors of the state variables and faults relative to faults and disturbances should be minimized.
3. It is very difficult to accurately apprehend information of all interconnections in real interconnected systems due to a complex structure. Consequently, when designing a decentralized fault estimation observer, it is imperative to solve some problems by uncertain interconnections.

To address the above research problems, we firstly consider that the subsystems of the interconnected system can be represented by a T–S fuzzy model, and the uncertain interconnection is assumed to satisfy the quadratic inequality with the given maximum interconnection bound. Also, based on the decentralized fuzzy observer and its estimation error models, the fault estimation is addressed by satisfying the H_∞ performance. H_∞ performance theory is a control theory that aims to construct controllers or observers that minimize the impact of disturbances by minimizing the norm of measured output or error signals relative to disturbances. H_∞ performance is a widely used research approach in control for tracking and filter design studies to guarantee the performance. In particular, two different inequalities for H_∞ performance are considered for the fault estimation in this paper. Then, by using the Lyapunov functional, we have developed decentralized fuzzy observers capable of addressing the fault estimation problem with H_∞ performance, and we formulated the performance inequality into linear matrix inequalities (LMIs) formats. Finally, a simulation example and analysis and comparison results are provided to demonstrate the validity of the proposed ideas, techniques and procedures.

This paper is organized as follows: Section 2 describes the fault estimation problem based on the discrete-time nonlinear interconnected systems and the decentralized fuzzy observer. The LMI conditions to design the decentralized fuzzy observer for fault estimation are proposed by using Lyapunov functional and H_∞ performance inequality in Section 3. The simulation example is given for illustration and comparison in Section 4. Finally, Sections 5 and 6 provide the concluding remarks.

Notation: The symbols $(\cdot)^T$, He$\{\cdot\}$, and $*$, respectively, represent the transpose of the given element, the summation of the element with its transpose, and the symmetrically positioned transposed element. The subscripts k and l denote the subsystem indices, and subscripts i and j denote fuzzy rule indices. Additionally, $\mathcal{I}_N$ denotes a set of integers ranging from $1, 2, \cdots, n$.

2. Preliminaries

In this section, we first consider our main research object, which is nonlinear interconnected systems. Nonlinear interconnected systems represent mathematical models of various systems [1–9], as mentioned in the Introduction. Especially, in this paper, we address the issue of discrete-time nonlinear interconnected systems with the actuator fault input. The dynamic equation is represented for a discrete-time nonlinear interconnected system composed of n subsystems, which is described as follows:

$$\begin{aligned} x_k(t+1) =& \mathcal{F}_k(x_k(t)) + \mathcal{G}_k(x_k(t))\omega_k(t) + \mathcal{H}_k(x_k(t))f_k(t) + h_k(x(t)) \\ y_k(t) =& \mathcal{C}_k(x_k(t)) \end{aligned} \tag{1}$$

where $\mathcal{F}_k(\cdot)$, $\mathcal{G}_k(\cdot)$, $\mathcal{H}_k(\cdot)$ and $\mathcal{C}_k(\cdot)$ are the nonlinear vector functions of the kth subsystem, $x_k(t) \in \mathbb{R}^{p_k}$ is the state variable, $\omega_k(t) \in \mathbb{R}^{q_k}$ is the disturbance and $y_k(t) \in \mathbb{R}^{r_k}$ is the measurement output of the kth subsystem, respectively, and $f_k(t) \in \mathbb{R}^{s_k}$ is the actuator fault input, where it is assumed that the derivative of $f_k(t)$ is norm bounded, and $h_k(x(t))$ is a nonlinear vector function representing the interconnection of interconnected systems and is assumed to satisfy the following Assumption:

Assumption 1. *The vector function $h_k(x(t))$ is unknown but satisfies the following quadratic inequality:*

$$\big(h_k(x(t))\big)^T h_k(x(t)) \leq \alpha_k^2 x(t)^T H_k^T H_k x(t) \tag{2}$$

where $\alpha_k > 0$ is a bound scalar of the interconnection term, and H_k is a given constant matrix with appropriate dimensions.

Then, considering the premise variable $z_{kp}(t) \in \mathbb{R}^{u_k}$, $k \in \mathcal{I}_{kq}$, the discrete-time nonlinear interconnected system (1) can be equivalently described by a T–S fuzzy model defined by the following IF–THEN fuzzy rules:

Fuzzy rule i of the kth subsystem:
IF $z_{k1}(t)$ is Γ_{ki1}, $\cdots$, and $z_{kq}(t)$ is Γ_{kiq},

$$\text{THEN} \begin{cases} x_k(t+1) = A_{ki}x_k(t) + B_{ki}\omega_k(t) + E_{ki}f_k(t) + h_k(x(t)) \\ y_k(t) = C_{ki}x_k(t) \end{cases} \tag{3}$$

where Γ_{kip}, $(k, i, p) \in \mathcal{I}_n \times \mathcal{I}_m \times \mathcal{I}_q$, is a fuzzy set for $z_{kp}(t)$ and A_{ki}, B_{ki}, E_{ki} and C_{ki} denote nominal system matrices with appropriate dimensions for the ith fuzzy rule of the kth subsystem.

Remark 1. *Since nonlinear functions are considered based on the state variables in the interconnected system* (1)*, the premise variables* $z_{kp}(t)$ *in the fuzzy rule* (3) *are often selected as part of the state variables.*

By employing center-average defuzzification, product inference, and a singleton fuzzifier, an IF–THEN rule (3) can be interpreted in the subsequent fuzzy subsystem as follows:

$$\begin{aligned} x_k(t+1) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t))\big(A_{ki}x_k(t) + B_{ki}\omega_k(t) + E_{ki}f_k(t)\big) + h_k(x(t)) \\ y_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t))C_{ki}x(t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} \xi_{ki}(z_k(t)) =& \eta_{ki}(z_k(t)) \Big/ \sum_{i=1}^{m} \eta_{ki}(z_k(t)), \\ \eta_{ki}(z_k(t)) =& \prod_{p=1}^{q} \Gamma_{kip}(z_{kp}(t)) \end{aligned}$$

in which $\Gamma_{kip} = \mathcal{U}_{z_{kp}(t)} \subset \mathbb{R} \to \mathbb{R}_{[0,1]}$ is the membership function of $z_{kp}(t)$ on compact set $\mathcal{U}_{z_{kp}(t)}$.

Remark 2. *The fuzzy rule* (3) *and the fuzzy system* (4) *inferred from the fuzzy IF–THEN rule by using center-average defuzzification, product inference and a singleton fuzzifier are based on the T–S fuzzy model [46]. In the T–S fuzzy model, the nonlinear models are represented by a set of fuzzy rules that denote nonlinear characteristics of the system through linear submodels. Thus, the T–S fuzzy model can present the complex nonlinear problems into the intuitive forms by using formal linguistic variables and fuzzy sets of the T–S fuzzy model. Also, the nonlinear systems also can be mathematically decomposed into linear subsystems and nonlinear weighting functions by the T–S fuzzy model. For these reasons, the T–S fuzzy system approach is one of the most widely used techniques in the field of nonlinear control, because various linear control techniques are easily applied to nonlinear systems.*

To establish the fault estimation, we consider the following decentralized fuzzy observer model:

$$\begin{aligned} \hat{x}_k(t+1) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t))\big(A_{ki}\hat{x}_k(t) + E_{ki}\hat{f}_k(t) + L_{ki}(y_k(t) - \hat{y}_k(t))\big) \\ \hat{y}_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t))C_{ki}\hat{x}_k(t) \\ \hat{f}_k(t+1) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t))\big(\hat{f}_k(t) + F_{ki}(y_k(t) - \hat{y}_k(t))\big) \end{aligned} \tag{5}$$

where $\hat{x}(t) \in \mathbb{R}^{p_k}$, $\hat{y}(t) \in \mathbb{R}^{r_k}$, and $\hat{f}(t) \in \mathbb{R}^{s_k}$ represent the estimation state, estimation output, and estimation fault, respectively. Additionally, L_{ki} and F_{ki} denote the gain matrices of the decentralized fuzzy observer, each with suitable dimensions.

Then, by defining the state estimation error as $e_{x_k}(t) = x_k(t) - \hat{x}k(t)$ and the fault estimation error as $e_{f_k}(t) = f_k(t) - \hat{f}k(t)$ and considering the first forward difference of the actuator fault input as $\Delta f_k(t) = f_k(t+1) - f_k(t)$, the error model between the (4) and (5) is derived as follows:

$$e_k(t+1) = \big(\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t)\big)e_k(t) + \mathcal{B}_k(t)v_k(t) + \tilde{h}_k(x(t)) \tag{6}$$

where

$$\begin{aligned}
\mathcal{A}_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t)) \begin{bmatrix} A_{ki} & E_{ki} \\ 0 & I \end{bmatrix}, \\
\mathcal{L}_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t)) \begin{bmatrix} L_{ki} \\ F_{ki} \end{bmatrix}, \\
\mathcal{C}_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t)) \begin{bmatrix} C_{ki} & 0 \end{bmatrix}, \\
\mathcal{B}_k(t) &= \sum_{i=1}^{m} \xi_{ki}(z_k(t)) \begin{bmatrix} B_{ki} & 0 \\ 0 & I \end{bmatrix},
\end{aligned}$$

and $e_k(t) = \mathrm{col}\{e_{x_k}(t), e_{f_k}(t)\}$, $v_k(t) = \mathrm{col}\{\omega_k(t), \Delta f_k(t)\}$, and $\tilde{h}_k(x(t)) = \mathrm{col}\{h_k(x(t)), 0\}$.

Based on the error model (6), the objective of the fault estimation problem can be presented as follows:

Problem 1. *Find the observer gain matrices L_{ki} and F_{ki} and minimize a scalar $\gamma > 0$ such that the following H_∞ performance is guaranteed:*

1. *The equilibrium point of the interconnected system based on error subsystem (6) is asymptotically stable when $v_k(t) = 0$ and $h_k(x(t)) = 0$.*
2. *To guarantee H_∞ performance, the following inequality is satisfied for a scalar γ under zero initial condition:*

$$\sum_{k=1}^{n} \sum_{t=0}^{\infty} \|e_k(t)\|^2 dt \leq \gamma^2 \sum_{k=1}^{n} \sum_{t=0}^{\infty} \left(\|v_k(t)\|^2 + \|x_k(t)\|^2 \right). \tag{7}$$

Remark 3. *In the previous fault estimation studies, the inequality for guaranteeing H_∞ performance has not included the terms of the state variable. However, in cases where uncertain interconnections are considered, as in this paper, a term of $x_k(t)$ has to be contained in the inequality for H_∞ performance for obtaining a sufficient condition that efficiently guarantees fault estimation performance. In fact, some research studies for the interconnected system with uncertain interconnections, like this paper, have included the term $x_k(t)$ in the H_∞ performance inequality, as in (7). In addition, the sufficient condition of the fault estimation has been also developed for the case where the state variable is not included in the H_∞ performance inequality in this paper. Furthermore, the performance difference between the fault estimation techniques based on two H_∞ performance inequalities is shown in the simulation.*

3. Main Results

In this section, the decentralized fuzzy observer design techniques for fault estimation have been proposed for nonlinear interconnected system (4) based on error model (6) considering Problem 1. Before presenting the proposed theorems, we need to consider the following lemma for the proof of the theorems.

Lemma 1 ([47])**.** *If X and Y are any matrices with appropriate dimensions, then for any constant $\alpha > 0$, the following property holds:*

$$X^T Y + Y^T X \leq \alpha X^T X + \alpha^{-1} Y^T Y.$$

Now, based on the above lemma, the following theorem addresses the sufficient condition for the design of decentralized fuzzy observers for fault estimation of the error model (6).

Theorem 1. *If there exist some positive definite matrices $P_k = P_k^T > 0$, some matrices N_{ki} and some scalars $\beta, \sigma > 0$ and ϱ such that the following LMIs are satisfied:*

$$\min \varrho \quad \textit{subject to}$$

$$P_k - \beta I < 0, \qquad k \in \mathcal{I}_n \tag{8}$$

$$\begin{bmatrix} -P_k + I & * & * & * \\ 0 & -\varrho I & * & * \\ P_k\mathcal{A}_{ki} - N_{ki}\mathcal{C}_{kj} & P_k\mathcal{B}_{ki} & -P_k & * \\ P_k\mathcal{A}_{ki} - N_{ki}\mathcal{C}_{kj} & P_k\mathcal{B}_{ki} & 0 & -\sigma I \end{bmatrix} < 0, \qquad (k,i,j) \in \mathcal{I}_n \times \mathcal{I}_m \times \mathcal{I}_m \tag{9}$$

$$-\varrho I + n(\beta+\sigma)\alpha_l^2 H_{lk}^T H_{lk} < 0, \qquad (k,l) \in \mathcal{I}_n \times \mathcal{I}_n \tag{10}$$

where

$$\mathcal{A}_{ki} = \begin{bmatrix} A_{ki} & E_{ki} \\ 0 & I \end{bmatrix}, \qquad \mathcal{B}_{ki} = \begin{bmatrix} B_{ki} & 0 \\ 0 & I \end{bmatrix},$$
$$\mathcal{C}_{ki} = \begin{bmatrix} C_{ki} & 0 \end{bmatrix}$$

and α_k is a given positive scaler for interconnection bound, and H_{lk} is the submatrix which has p_k columns of H_l from the ν_kth column vector where $\nu_k = p_1 + p_2 + \cdots + p_{k-1} + 1$, then the decentralized fuzzy observer (5) guarantees the H_∞ fault estimation performance for a discrete-time interconnected system based on the fuzzy subsystem (4), and γ is a minimum H_∞ performance value of fault estimation. In addition, the decentralized fuzzy observer gain matrices are given by $\mathrm{col}\{L_{ki}, F_{ki}\} = P_k^{-1}N_{ki}$, and $\gamma = \sqrt{\varrho}$ is a minimum value of the H_∞ fault estimation performance.

Proof. First, for the first condition of Problem 1, we consider a Lyapunov functional with assuming $v_k(t) = 0$ and $h_k(x(t)) = 0$ to demonstrate the stability condition of the error system based on the subsystem model (6) as follows:

$$V(t) = \sum_{k=1}^{n} V_k(e_k(t)) = \sum_{k=1}^{n} e_k(t)^T P_k e_k(t)$$

where $P_k = P_k^T > 0$. Then, we have the first forward difference of the Lyapunov functional candidate as

$$\begin{aligned} \Delta V(t) =& V(t+1) - V(t) \\ =& \sum_{k=1}^{n} \big(e_k(t+1)^T P_k e_k(t+1) - e_k(t)^T P_k e_k(t)\big). \end{aligned} \tag{11}$$

Then, the following equation can be obtained by substituting (4) and (6) into (11) with considering $v_k(t) = 0$ and $h_k(x(t)) = 0$:

$$\begin{aligned} \Delta V(t) = \sum_{k=1}^{n} \Big(& \Big(\big(\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t)\big)e_k(t)\Big)^T P_k \Big(\big(\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t)\big)e_k(t)\Big) \\ & - e_k(t)^T P_k e_k(t)\Big). \end{aligned} \tag{12}$$

Thus, if the following inequality

$$\big(\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t)\big)^T P_k \big(\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t)\big) < 0 \tag{13}$$

is satisfied, then $\Delta V(t) < 0$ is also satisfied. Furthermore, by using the Schur complement, applying the congruence transformation with $\mathrm{diag}\{I, P_k\}$ and denoting $P_k\mathcal{L}_k(t) = N_k(t)$, inequality (13) is represented as the following inequality:

$$\begin{bmatrix} -P_k & * \\ P_k\mathcal{A}_k(t) - N_k(t)\mathcal{C}_k(t) & -P_k \end{bmatrix} < 0$$

Because the above inequality is a sufficient condition of LMIs (8) and (9), we can guarantee the stability condition for the error systems (6) with $v_k(t) = 0$ and $h_k(x(t)) = 0$.

Now, to guarantee the second condition of Problem 1, we establish the H_∞ performance criteria based on the error system (6) with a zero initial condition by using the following inequality:

$$\mathcal{J} = \Delta V(t) + \sum_{k=1}^{n} \left(e_k(t)^T e_k(t) - \gamma^2 v_k(t)^T v_k(t) - \gamma^2 x_k(t)^T x_k(t)\right). \tag{14}$$

Then, by using the equation (11) and substituting (6) into (14), we have

$$\begin{aligned} \mathcal{J} = \sum_{k=1}^{n} \Big(&\Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t) + \tilde{h}_k(x(t))\Big)^T \\ &\times P_k\Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t) + \tilde{h}_k(x(t))\Big) \\ &- e_k(t)^T P_k e_k\Big) + \sum_{k=1}^{n} \left(e_k(t)^T e_k(t) - \gamma^2 v_k(t)^T v_k(t) - \gamma^2 x_k(t)^T x_k(t)\right) \end{aligned} \tag{15}$$

By using Lemma 1, the inequality (15) can be represented as follows:

$$\begin{aligned} \mathcal{J} \leq \sum_{k=1}^{n} \Big(&\Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t)\Big)^T \\ &\times (P_k + \sigma^{-1}P_k^2)\Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t)\Big) \\ &- e_k(t)^T P_k e_k(t) + \sum_{k=1}^{n} \tilde{h}_k(x(t))^T (P_k + \sigma I)\tilde{h}_k(x(t)) \\ &+ \sum_{k=1}^{n} \left(e_k(t)^T e_k(t) - \gamma^2 v_k(t)^T v_k(t) - \gamma^2 x_k(t)^T x_k(t)\right) \end{aligned} \tag{16}$$

where σ is a given positive scalar. Also, from (2) of Assumption 1, we know that the following inequality is satisfied:

$$\begin{aligned} \sum_{k=1}^{n} \tilde{h}_k(x(t))^T \tilde{h}_k(x(t)) &\leq \sum_{k=1}^{n} \alpha_k^2 x(t)^T H_k^T H_k x(t) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_k^2 x_l(t)^T H_{kl}^T H_{kl} x_l(t) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_l^2 x_k(t)^T H_{lk}^T H_{lk} x_k(t) \end{aligned} \tag{17}$$

where $H_k = [H_{k1} \;\; H_{k2} \;\; \cdots \;\; H_{kn}]$ and the matrix H_{kl} has p_l columns.

Thus, if there exists some scalar β such that $P_k - \beta I < 0$, then the inequality (16) is derived by applying the inequality (17) as follows:

$$
\begin{aligned}
\mathcal{J} \leq \sum_{k=1}^{n} \Big(& \Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t) \Big)^T \\
& \times \big(P_k + \sigma^{-1}P_k^2\big)\Big((\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t))e_k(t) + \mathcal{B}_k(t)v_k(t) \Big) \\
& - e_k(t)^T(P_k - I)e_k(t) - \gamma^2 v_k(t)^T v_k(t) \Big) \\
& + \sum_{k=1}^{n}\sum_{l=1}^{n} \Big(-\frac{1}{n}\gamma^2 x_k(t)^T x_k(t) + (\beta+\sigma)\alpha_l^2 x_k(t)^T H_{lk}^T H_{lk} x_k(t) \Big) \\
= \sum_{k=1}^{n} & \begin{bmatrix} e_k(t) \\ v_k(t) \end{bmatrix}^T \Big(\big[\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t) \quad \mathcal{B}_k(t)\big]^T \big(P_k + \sigma^{-1}P_k^2\big) \\
& \times \big[\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t) \quad \mathcal{B}_k(t)\big] + \begin{bmatrix} -P_k + I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \Big) \begin{bmatrix} e_k(t) \\ v_k(t) \end{bmatrix} \\
& + \frac{1}{n}\sum_{k=1}^{n}\sum_{l=1}^{n} x_k(t)^T \big(-\gamma^2 I + n(\beta+\sigma)\alpha_l^2 H_{lk}^T H_{lk} \big) x_k(t).
\end{aligned} \tag{18}
$$

From the result of (18), if the following inequalities are satisfied

$$
\begin{aligned}
& \big[\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t) \quad \mathcal{B}_k(t)\big]^T \big(P_k + \sigma^{-1}P_k^2\big) \\
& \times \big[\mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t) \quad \mathcal{B}_k(t)\big] + \begin{bmatrix} -P_k + I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0,
\end{aligned} \tag{19}
$$

$$
-\gamma^2 I + n(\beta+\sigma)\alpha_l^2 H_{lk}^T H_{lk} < 0, \tag{20}
$$

then $\mathcal{J} < 0$ is satisfied. Furthermore, by using the Schur complement, applying the congruence transformation with $\mathrm{diag}\{I, I, P_k, P_k\}$ and denoting $P_k \times \mathrm{col}\{L_{ki}, F_{ki}\} = N_{ki}$, the inequality (19) is majorized as LMI (9). Also, by denoting $\gamma^2 = \varrho$, the inequality (20) is also represented as LMI (10).

Now, by summing (14) for t from zero to infinity, the following can be obtained:

$$
\begin{aligned}
\sum_{t=0}^{\infty} \mathcal{J} = V(\infty) - V(0) + & \sum_{k=1}^{n}\sum_{t=0}^{\infty} e_k^T(t)e_k(t) \\
& - \gamma^2 \sum_{k=1}^{n}\sum_{t=0}^{\infty} \big(v_k(t)^T(t)v_k(t) + x_k(t)^T x_k(t)\big) \\
< 0. &
\end{aligned} \tag{21}
$$

From $V(0) = 0$, due to the consideration of the zero initial condition and $V(\infty) \geq 0$, if inequality (19) is satisfied, then H_∞ performance (7) is guaranteed from the result of (21). Finally, by summarizing the above proof, through LMIs (8) and (9), we can develop the fault estimation technique to achieve the conditions of Problem 1 from the decentralized fuzzy observer (5) for interconnected systems of (4). □

Remark 4. *From Theorem 1, for a given maximum interconnection bound α_k, satisfying the linear matrix inequality means the following: If the gain matrices L_{ki} and F_{ki} for the value of the maximum interconnection bound are obtained, it is possible for the decentralized fuzzy observer constructed by the obtained gain matrices to guarantee the fault estimation performance of Problem 1, even if the interconnection bound is considered as any value smaller than the given value α_k for the maximum interconnection bound.*

Remark 5. *The inequalities (8)–(10) obtained in Theorem 1 are represented into the LMI structure. LMI provides a remarkable mathematical framework for representing and solving various control* *and optimization problems. The main advantage of LMI is that it facilitates the use of convex optimization techniques, which can be easily solved by many efficient numerical methods, to solve various control problems. Therefore, if the control design problem can be formulated as an LMI problem, anyone can easily design a controller through simple numerical algorithms, such as the FEASP algorithm of MATLAB.*

To conquer the limitation of including the term of $x_k(t)$ in the inequality (7) for the H_∞ performance of Problem 1, a novel decentralized fuzzy observer design technique for fault estimation is introduced in the following corollary:

Corollary 1. *If there exist some matrices $P_k = P_k^T > 0$, $Q_k = Q_k^T > 0$ and N_{ki} and some scalar ϱ such that the following LMIs are satisfied:*

$$\min \varrho \quad \textit{subject to}$$

$$\begin{bmatrix} Q_k - \beta I & * \\ 0 & P_k - \beta I \end{bmatrix} < 0, \qquad k \in \mathcal{I}_n \tag{22}$$

$$\begin{bmatrix} -Q_k & * & * & * & * & * & * & * & * \\ 0 & -P_k + I & * & * & * & * & * & * & * \\ 0 & 0 & -\varrho I & * & * & * & * & * & * \\ 0 & 0 & 0 & -\varrho I & * & * & * & * & * \\ Q_k A_{ki} & 0 & Q_k \hat{B}_{ki} & Q_k E_{ki} & -Q_k & * & * & * & * \\ 0 & P_k \mathcal{A}_{ki} - N_{ki}\mathcal{C}_{kj} & P_k \mathcal{B}_{ki} & 0 & 0 & -P_k & * & * & * \\ Q_k A_{ki} & 0 & Q_k \hat{B}_{ki} & Q_k E_{ki} & 0 & 0 & -\sigma I & * & * \\ 0 & P_k \mathcal{A}_{ki} - N_{ki}\mathcal{C}_{kj} & P_k \mathcal{B}_{ki} & 0 & 0 & 0 & 0 & -\sigma I & * \\ H_{lk} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2n(\beta+\sigma)\alpha_l^2} I \end{bmatrix} < 0,$$

$$(k, l, i, j) \in \mathcal{I}_n \times \mathcal{I}_n \times \mathcal{I}_m \times \mathcal{I}_m, \tag{23}$$

where

$$\hat{B}_{ki} = \begin{bmatrix} B_{ki} & 0 \end{bmatrix}$$

and β, σ and α_k are given positive scalars, then the decentralized fuzzy observer (5) satisfies the H_∞ fault estimation performance for the interconnected system based on the fuzzy subsystem (4). In addition, the decentralized fuzzy observer gain can be obtained by $\mathrm{col}\{L_{ki}, F_{ki}\} = P_k^{-1} N_{ki}$, and $\gamma = \sqrt{\varrho}$ is a minimum value of the fault estimation performance for H_∞.

Proof. From fuzzy systems based on (4) and error model (6), we consider

$$\dot{\chi}_k(t) = \Xi_k(t)\chi_k(t) + M_k(t)v_k(t) + \hat{h}_k(x(t)) \tag{24}$$

where

$$\Xi_k(t) = \begin{bmatrix} A_k(t) & 0 \\ 0 & \mathcal{A}_k(t) - \mathcal{L}_k(t)\mathcal{C}_k(t) \end{bmatrix},$$

$$M_k(t) = \begin{bmatrix} \hat{B}_k(t) & E_k(t) \\ \mathcal{B}_k(t) & 0 \end{bmatrix},$$

$$A_k(t) = \sum_{i=1}^{r} \xi_{ki}(z_k(t)) A_{ki},$$

$$\hat{B}_k(t) = \sum_{i=1}^{r} \xi_{ki}(z_k(t)) \hat{B}_{ki},$$

$$E_k(t) = \sum_{i=1}^{r} \xi_{ki}(z_k(t)) E_{ki}$$

and $\chi_k(t) = \mathrm{col}\{x_k(t), e_k(t)\}$, $v_k(t) = \mathrm{col}\{v_k(t), f_k(t)\}$, and $\hat{h}_k(x(t)) = \mathrm{col}\{h_k(x(t)), \tilde{h}_k(x(t))\}$.

Then, based on the model (24), we firstly consider a Lyapunov function candidate with $v_k(t) = 0$ and $h_k(x(t)) = 0$ as follows:

$$V(t) = \sum_{k=1}^{n} V_k(\chi_k(t)) = \sum_{k=1}^{n} \chi_k(t)^T \mathcal{P}_k \chi_k(t)$$

where $\mathcal{P}_k = \mathcal{P}_k^T > 0$. Then, the first forward difference of the Lyapunov functional candidate becomes

$$\Delta V(t) = \sum_{k=1}^{n} \Big(\big(\Xi_k(t)\chi_k(t)\big)^T \mathcal{P}_k \big(\Xi_k(t)\chi_k(t)\big)\big) - \chi_k(t)^T \mathcal{P}_k \chi_k(t) \Big) \tag{25}$$

Thus, if the inequality $\Xi_k(t)^T \mathcal{P}_k \Xi_k(t) - \mathcal{P}_k < 0$ is satisfied, which is guaranteed by (23) by defining $\mathcal{P}_k = \mathrm{diag}\{Q_k, P_k\}$ with the positive definite matrix P_k and Q_k without the loss of generality, then $\Delta V(t) < 0$ is guaranteed.

Also, we establish the condition of H_∞ fault estimation performance with a zero initial condition by using the following inequality:

$$\mathcal{J} = \Delta V(t) + \sum_{k=1}^{n} \big(e_k(t)^T e_k(t) - \gamma^2 v_k(t)^T v_k(t)\big). \tag{26}$$

Then, through a similar procedure of Theorem 1 with a given positive scalar β, we obtain the following inequality:

$$\begin{aligned}
\mathcal{J} \leq & \sum_{k=1}^{n} \Big(\big(\Xi_k(t)\chi_k(t) + M_k(t)v_k(t)\big)^T (\mathcal{P}_k + \sigma^{-1}\mathcal{P}_k^2) \\
& \times \big(\Xi_k(t)\chi_k(t) + M_k(t)v_k(t)\big) \\
& - \chi_k(t)^T (\mathcal{P}_k - \hat{I})\chi_k(t) - \gamma^2 v_k(t)^T v_k(t) \Big) \\
& + \sum_{k=1}^{n}\sum_{l=1}^{n} 2(\beta+\sigma)\alpha_l^2 \chi_k(t)^T \hat{H}_{lk}^T \hat{H}_{lk} \chi_k(t) \\
= & \frac{1}{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \begin{bmatrix} \chi_k(t) \\ v_k(t) \end{bmatrix}^T \Big(\begin{bmatrix} \Xi_k(t) & M_k(t) \end{bmatrix}^T (\mathcal{P}_k + \sigma^{-1}\mathcal{P}_k^2) \begin{bmatrix} \Xi_k(t) & M_k(t) \end{bmatrix} \\
& + \begin{bmatrix} -\mathcal{P}_k + \hat{I} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + 2n(\beta+\sigma)\alpha_l^2 \hat{H}_{lk}^T \hat{H}_{lk} \Big) \begin{bmatrix} \chi_k(t) \\ v_k(t) \end{bmatrix}
\end{aligned} \tag{27}$$

where $\hat{H}_{lk} = \begin{bmatrix} H_{lk} & 0 \end{bmatrix}$ and $\hat{I} = \mathrm{diag}\{0, I\}$.

Thus, if the following inequalities are satisfied

$$\begin{aligned}
& \begin{bmatrix} \Xi_k(t) & M_k(t) \end{bmatrix}^T (\mathcal{P}_k + \sigma^{-1}\mathcal{P}_k^2) \begin{bmatrix} \Xi_k(t) & M_k(t) \end{bmatrix} \\
& + \begin{bmatrix} -\mathcal{P}_k + \hat{I} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + 2n(\beta+\sigma)\alpha_l^2 \hat{H}_{lk}^T \hat{H}_{lk} \\
& < 0
\end{aligned} \tag{28}$$

then $\mathcal{J} < 0$ is satisfied. In addition, by defining $\mathcal{P}_k = \mathrm{diag}\{Q_k, P_k\}$, using the Schur complement to inequality (28), applying the congruence transformation with $\mathrm{diag}\{I, I, \mathcal{P}_k, \mathcal{P}_k, I\}$ and denoting $P_k \times \mathrm{col}\{L_{ki}, F_{ki}\} = N_{ki}$ and $\gamma^2 = \varrho$, the condition of $\mathcal{J} < 0$ is majorized as inequalities (8) and (9).

Finally, by summing (26) for t from 0 to ∞, we obtain the following condition from zero initial condition

$$\sum_{k=1}^{n}\sum_{t=0}^{\infty} e_k^T(t)e_k(t)dt \leq \gamma^2 \sum_{k=1}^{n}\sum_{t=0}^{\infty} v_k(t)^T(t)v_k(t). \tag{29}$$

Thus, through the proposed LMIs, we can ensure both the stability of the interconnected system without fault, disturbance and interconnection and H_∞ performance for fault estimation. □

Remark 6. *In Corollary 1, the fault estimation is guaranteed for H performance with inequality* $\sum_{k=1}^{n}\sum_{0}^{\infty} e_k^T(t)e_k(t) \leq \gamma^2 \sum_{k=1}^{n}\sum_{0}^{\infty} v_k(t)^T(t)v_k(t)$*, which does not contain the terms of* $x_k(t)$*. It means that the fault estimation performance is not affected by state variables* $x_k(t)$*. However, Corollary 1 has the following constraints compared to Theorem 1.*

- *To satisfy the LMI conditions of Corollary 1, all subsystems of the interconnected system have to satisfy the asymptotic stability because of the term of* $A_{ki}^T Q_k A_{ki} - Q_k < 0$ *of LMI (23).*
- *Unlike Theorem 1, the scalar* β*,* σ *has to be given in Corollary 1. Thus, the LMI condition of Corollary 1 is inevitably more conservative than the LMI condition of Theorem 1.*

The performance difference between Theorem 1 and Corollary 1 is clearly shown in Section 4.

Remark 7. *The algorithm's procedure is summarized to design the decentralized fuzzy observer (5) for the fault estimation of the discrete-time nonlinear interconnected system (1):*

1. *Present T–S fuzzy subsystems (4) for the discrete-time nonlinear interconnected system while considering Assumption 1.*
2. *Construct the observer model (5) by using the presented T–S fuzzy subsystems.*
3. *Solve LMIs (8)–(10) of Theorem 1 or (22) and (23) of Corollary 1 with minimizing* ϱ *by using any numerical calculation tools, such as the FEASP algorithm on MATLAB, to solve the convex optimization problem.*
4. *Calculate the decentralized fuzzy observer gain matrices* L_{ki} *and* F_{ki} *using the results of LMIs and* $\mathrm{col}\{L_{ki}, F_{ki}\} = P_k^{-1}N_{ki}$*.*
5. *Synthesize the decentralized fuzzy fault estimation observer by using the obtained observer gain matrices.*

Remark 8. *The primary contribution of this paper is outlined in the following:*

- *The fault estimation technique is designed for discrete-time nonlinear interconnected systems based on the decentralized fuzzy observer. This technique has not been studied so far to the best of the author's knowledge.*
- *The uncertain interconnection problem is solved in the fault estimation technique development.*
- *Two fault estimation techniques are presented based on differently defined* H_∞ *performance inequalities. Also, the performance of two fault estimation techniques is compared in the simulation.*

Remark 9. *The following are noted:*

- *The decentralized fuzzy fault estimation observer proposed in this paper can be practically applied in various fields such as industrial automation, smart grids, transportation systems, monitoring systems, communication networks and so on.*
- *The methodologies employed in this paper can be adapted to develop various fault estimation techniques applicable to continuous-time systems, sampled-data systems, and beyond. Additionally, the proposed fault estimation techniques have the potential to be extended to address control problems, including fault accommodation control.*
- *Future work: In this paper, the fault estimation techniques have been presented by limiting the discrete-time systems. However, in real engineering systems, analog plants and digital observers/controllers are often combined. In this case, the sampled-data problem has to be solved in the decentralized observer design to establish the fault estimation of interconnected systems.*

4. Numerical Example

To show the effectiveness of the proposed fault estimation technique by using the decentralized fuzzy observer, the simulation results are provided in this section. We consider the discrete-time interconnected system based on the interconnected mass-spring-damper mechanical system composed of two subsystems and connected by a spring [43]. The subsystem of the interconnected mass-spring-damper mechanical system can be represented as follows:

$$m_k\ddot{\theta}_k(t) + d_k(\dot{\theta}_k(t))\dot{\theta}_k(t) + \kappa_k\theta_k(t) + h_{kl}(\theta(t)) = \omega_k(t) + f_k(t)$$

where $(k,l) \in \mathcal{I}_2 \times \mathcal{I}_2$. In addition, $\theta_k(t)$ is the relative position of the mass in the kth subsystem, $\theta(t) = [\theta_1(t)^T \quad \theta_2(t)^T]^T$ and $y_k(t)$ is the measured output. Also, m_k is the mass, κ_k is the stiffness of the springs, $d_k(\dot{\theta}_k(t))$ is the damping coefficients of the nonlinear damper as $d_k(\dot{\theta}_k(t)) = d_{k_1} + d_{k_2}\dot{\theta}_k(t)^2$, $h_{kl}(\theta(t)) = \kappa(\theta_k(t) - \theta_l(t))$ is the interconnection function for two subsystems connecting by uncertain spring constant κ, and $\omega_k(t)$ and $f_k(t)$ represent the disturbance and actuator fault input of the kth subsystem, respectively.

Then, by considering $\dot{\theta}_k(t) \in [-\Omega_k \quad \Omega_k]$, defining $\theta_k(t) = x_{k_1}(t)$ and $\dot{\theta}_k(t) = x_{k_2}(t)$ and applying the approximate discretization approach with the sampling period $T = 0.01$, the discrete-time interconnected fuzzy system composed of two subsystems can be addressed as follows:

$$\begin{aligned} x_k(tT+T) &= \sum_{i=1}^{2}\xi_{ki}(x_{k_2}(t))\big(A_{ki}x_k(t) + B_{ki}\omega_k(t) + E_{ki}f_k(t)\big) + \kappa H_k x(t) \\ y_k(t) &= \sum_{i=1}^{2}\xi_{ki}(x_{k_2}(t))\big(C_{ki}x_k(t)\big) \end{aligned}$$

where

$$\begin{aligned} A_{k1} &= \exp\left(\begin{bmatrix} 0 & 1 \\ -\kappa_k/m_k & -d_{k1}/m_k \end{bmatrix} T\right), \\ A_{k2} &= \exp\left(\begin{bmatrix} 0 & 1 \\ -\kappa_k/m_k & -d_{k1}/m_k - d_{k2}(\Omega_k)^2 \end{bmatrix} T\right), \\ B_{ki} &= \left(\int_0^T \exp(A_{ki}\tau)d\tau\right) \times \begin{bmatrix} 0 \\ 1/m_k \end{bmatrix}, \\ E_{ki} &= \left(\int_0^T \exp(A_{ki}\tau)d\tau\right) \times \begin{bmatrix} 0 \\ 1/m_k \end{bmatrix}, \\ C_{ki} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ H_1 &= \left\{\sum_{i=1}^{2}\xi_{1i}(x_{1_2}(t))\left(\int_0^T \exp(A_{1i}\tau)d\tau\right)\right\} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/m_1 & 0 & 1/m_1 \end{bmatrix}, \\ H_2 &= \left\{\sum_{i=1}^{2}\xi_{2i}(x_{2_2}(t))\left(\int_0^T \exp(A_{2i}\tau)d\tau\right)\right\} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/m_2 & 0 & -1/m_2 \end{bmatrix}, \\ \xi_{k1}(x_{k_2}(t)) &= 1 - \frac{(x_{k_2}(t))^2}{(\Omega_k)^2}, \\ \xi_{k2}(x_{k_2}(t)) &= 1 - \xi_{k1}(x_{k_2}(t)) \end{aligned}$$

for $(k,i) \in \mathcal{I}_2 \times \mathcal{I}_2$ and the maximum interconnection bound is considered as $\alpha_k = \kappa = 0.05$, which means that the decentralized fuzzy observer satisfies the H_∞ performance for fault estimation, even if the interconnection bound κ is considered any value smaller than 0.05 N/m. In addition, the parameter values of the interconnected mass-spring-damper mechanical system are shown in Table 1. The parameter values are determined from the previous study [43]. Additionally, the value of Ω_k is set to 1 as it represents a constant indicating the maximum range of state variable $x_{k_2}(t)$. Since the magnitude range of disturbances or fault signals does not exceed 1, the magnitude range of $x_{k_2}(t)$ also does not exceed 1. Thus, its upper bound is set to 1.

Table 1. Parameter values.

Parameters	Subsystem 1	Subsystem 2
m_k (kg)	1	1
κ_k (N/m)	0.2	0.3
d_{k_1} (N·s/m)	0.6	0.5
d_{k_2} (N·s/m)	0.8	0.7
Ω_k	1	1

Now, by using the MATLAB LMI Control Toolbox with the FEASP algorithm, the decentralized fuzzy observer gains are obtained from (8)–(10) of Theorem 1 as follows:

$$\begin{aligned} L_{11}^{Thm.1} &= \begin{bmatrix} 2.6591 \\ 202.6906 \end{bmatrix}, & L_{12}^{Thm.1} &= \begin{bmatrix} 2.6533 \\ 201.4591 \end{bmatrix}, \\ L_{21}^{Thm.1} &= \begin{bmatrix} 2.6600 \\ 202.8242 \end{bmatrix}, & L_{22}^{Thm.1} &= \begin{bmatrix} 2.6551 \\ 201.7870 \end{bmatrix}, \\ F_{11}^{Thm.1} &= 7.4480 \times 10^3, & F_{12}^{Thm.1} &= 7.4506 \times 10^3, \\ F_{21}^{Thm.1} &= 7.4408 \times 10^3, & F_{22}^{Thm.1} &= 7.4446 \times 10^3 \end{aligned}$$

and the minimum H_∞ performance value γ is obtained as 9. Then, from LMIs (22) and (23) of Corollary 1, the gain matrices of the decentralized fuzzy observer (5) are also presented with supposing $\beta = 800$ and $\sigma = 6 \times 10^4$ as follows:

$$\begin{aligned} L_{11}^{Cor.1} &= \begin{bmatrix} 1.0273 \\ 2.7548 \end{bmatrix}, & L_{12}^{Cor.1} &= \begin{bmatrix} 1.0272 \\ 2.7331 \end{bmatrix}, \\ L_{21}^{Cor.1} &= \begin{bmatrix} 1.0334 \\ 4.4832 \end{bmatrix}, & L_{22}^{Cor.1} &= \begin{bmatrix} 1.0332 \\ 2.4660 \end{bmatrix}, \\ F_{11}^{Cor.1} &= 3.9886, & F_{12}^{Cor.1} &= 3.9887, \\ F_{21}^{Cor.1} &= 3.2579, & F_{22}^{Cor.1} &= 3.2577. \end{aligned}$$

In addition, the disturbance signal and the actuator fault are, respectively, supposed as follows:

$$\begin{aligned} \omega_k(t) &= 0.02\sin(10t), \\ f_1(t) &= \begin{cases} 0, & t \leq 1 \ \& \ t > 0 \\ \sin(2(t-1)), & 1 < t \leq 10 \end{cases} \\ f_2(t) &= \begin{cases} 0, & t \leq 1 \ \& \ t > 0 \\ \sin(3(t-1)), & 1 < t \leq 10 \end{cases} \end{aligned}$$

for $k \in \mathcal{I}_2$. Then, the time responses of the state variables of each subsystem and the estimated state variable of the proposed observers are depicted in Figures 1–4. Additionally, the time responses of the fault signal and estimated fault of each subsystem are illustrated in Figures 5 and 6. In Figures 1 and 3, the first state variables of each subsystem are represented for the relative position of the mass–spring–damper system. Also, the second state variables are represented for velocity in Figures 2 and 4. As shown in the figures, both proposed decentralized fuzzy observers have well estimated the state variables. Even when different fault signals are applied to each subsystem, the proposed fault estimation observer continues to operate correctly, as evidenced by the results. However, compared to the outstanding fault estimation results of Theorem 1, Corollary 1 does not have good fault estimation performance. Thus, it can be confirmed that the result of Theorem 1 is better than the result of Corollary 1 in terms of the fault estimation performance. To show the performance difference in more detail, the estimation errors are represented for each subsystem in Figures 7 and 8, respectively. In addition, to emphasize the performance differences according to the maximum interconnection bound α_k, the following performance measure function is considered:

$$\mathcal{P} = \sqrt{\sum_{k=1}^{2}\sum_{t=0}^{20} e_k(t)^T e_k(t) \Big/ \sum_{k=1}^{2}\sum_{t=0}^{20} v_k(t)^T v_k(t)}$$

and comparison results of the performance measure function are presented in Table 2. In addition, to emphasize the superiority of the proposed techniques, the results of the performance measure function for the fault estimation techniques without considering the interconnection problem based on [28,36] are added in Table 2.

Table 2. Results of the performance measure function for the fault estimation techniques.

Maximum Interconnection Bound	Theorem 1	Corollary 1	[28]	[36]
0.01	0.322	3.631	divergent	3.7935
0.02	0.324	3.632	divergent	3.7941
0.05	0.333	3.633	divergent	3.7961

From Table 2, the outstanding performance of the proposed decentralized fuzzy observer for fault estimation can be confirmed once again by comparing with the results of the previous technique. In particular, not only Theorem 1, which is the main technique of this paper, but also Corollary 1, which has relatively poor performance, show better performance results than the previous techniques without considering the interconnection problem.

In addition, to check the performance for observing the states when the fault occurs in the interconnected system, we compared the performance for the fuzzy observer [48], which is one of the most used observing techniques, with respect to the observation performance of the state variables. By comparing the results of the observing performance for the state variables based on the performance measure functions provided above, the results of the proposed Theorem 1 and Corollary 1 are 2.423×10^{-3} and 0.8752, respectively, whereas the result of the fuzzy observer is 1.2954, when the maximum interconnection bound is considered as 0.05. Thus, we can confirm that the proposed fault estimation techniques based on the decentralized fuzzy observer are beneficial for interconnected systems with the actuator fault input, and Theorem 1 is particularly effective for discrete-time nonlinear interconnected systems which have uncertain interconnections.

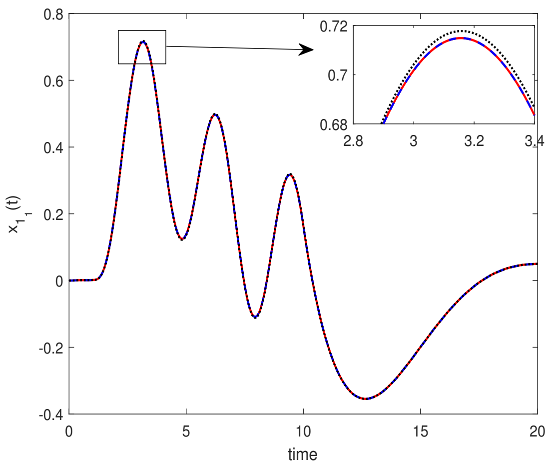

Figure 1. Results of the first state for the first subsystem: $x_{1_1}(t)$ (solid), $\hat{x}_{1_1}(t)$ from Theorem 1 (dashed) and $\hat{x}_{1_1}(t)$ from Corollary 1 (dotted).

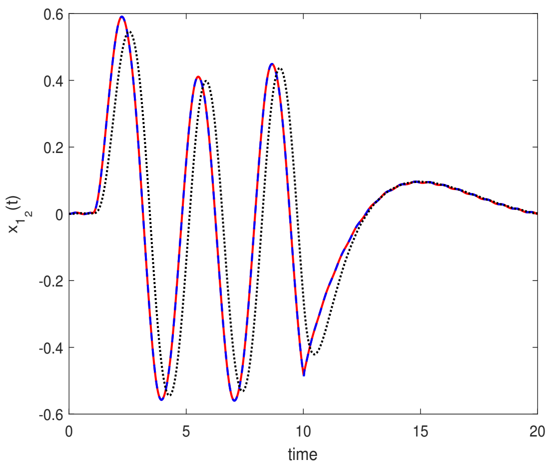

Figure 2. Results of the second state for the first subsystem: $x_{1_2}(t)$ (solid), $\hat{x}_{1_2}(t)$ from Theorem 1 (dashed) and $\hat{x}_{1_2}(t)$ from Corollary 1 (dotted).

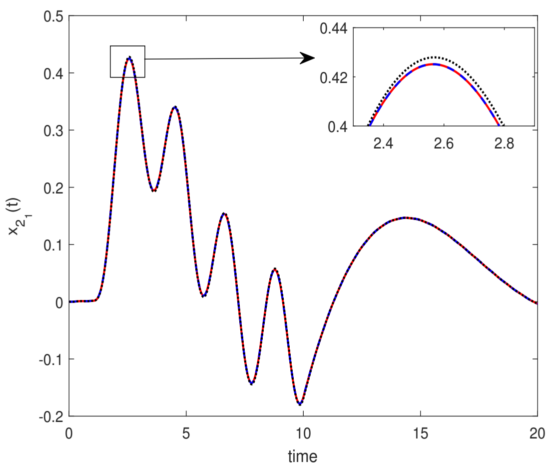

Figure 3. Results of the first state for the second subsystem: $x_{2_1}(t)$ (solid), $\hat{x}_{2_1}(t)$ from Theorem 1 (dashed) and $\hat{x}_{2_1}(t)$ from Corollary 1 (dotted).

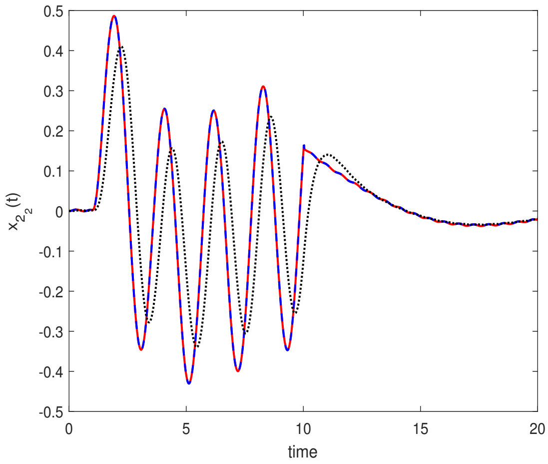

Figure 4. Results of the second state for the second subsystem: $x_{2_2}(t)$ (solid), $\hat{x}_{2_2}(t)$ from Theorem 1 (dashed) and $\hat{x}_{2_2}(t)$ from Corollary 1 (dotted).

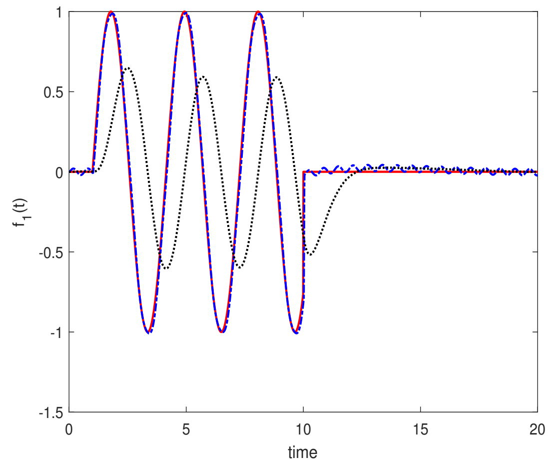

Figure 5. Results of the fault for the first subsystem: $f_1(t)$ (solid), $\hat{f}_1(t)$ from Theorem 1 (dashed) and $\hat{f}_1(t)$ from Corollary 1 (dotted).

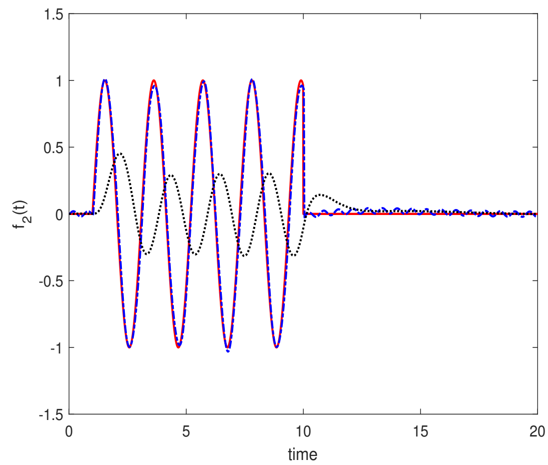

Figure 6. Results of the fault for the second subsystem: $f_2(t)$ (solid), $\hat{f}_2(t)$ from Theorem 1 (dashed) and $\hat{f}_2(t)$ from Corollary 1 (dotted).

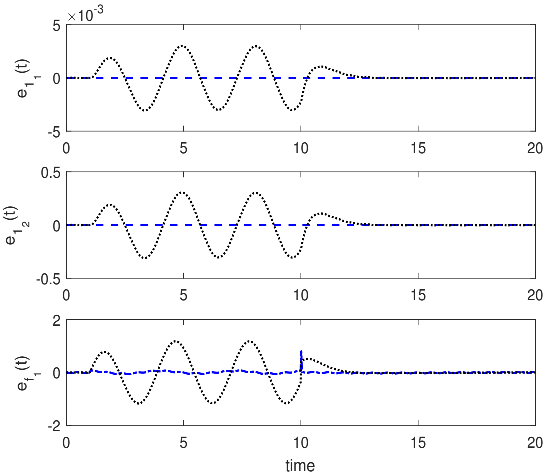

Figure 7. Results of the estimated error for the first subsystem: Theorem 1 (dashed) and Corollary 1 (dotted).

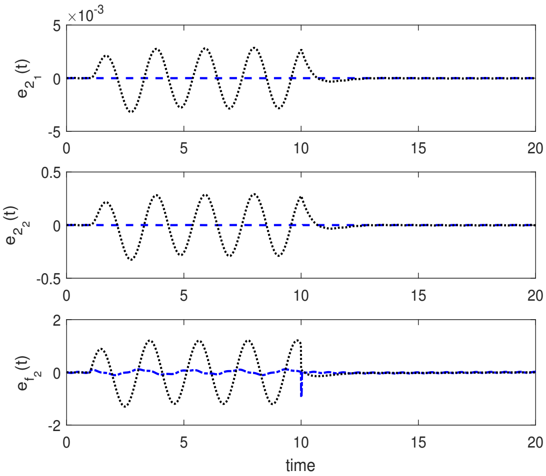

Figure 8. Results of the estimated error of the second subsystem: Theorem 1 (dashed) and Corollary 1 (dotted).

5. Discussion

The proposed techniques for the decentralized fuzzy fault estimation observer design of discrete-time nonlinear interconnected systems presents several noteworthy points. Firstly, the T–S fuzzy model is one of the best approaches to analyze nonlinear interconnected systems, because the nonlinear systems are mathematically decomposed into linear subsystems and nonlinear weighting functions by a T–S fuzzy model. The advantages of the T–S fuzzy model also apply to the discrete-time nonlinear interconnected system considered in this paper. Moreover, the decentralized observer model is suitable for the fault estimation of interconnected systems, as it can solve the problems of interconnected systems such as high dimensionality, structural constraints of the controller, and uncertain or unknown information of interconnection. In addition, the fault estimation technique effectively estimates the result for the unknown actuator fault input, which directly affects the systems.

Motivated by the above analysis, the decentralized fuzzy fault estimation observer design techniques are proposed for discrete-time nonlinear interconnected systems. There are several noteworthy points when developing fault estimation techniques. Firstly, when proposing a fault estimation technique, the scope of research has expanded by suggesting two different observer design approaches. Also, by providing the observer design algorithm into LMI format, it allows many researchers to easily implement the techniques proposed in this paper.

Lastly, to evaluate the performance of the proposed techniques, we compared various existing research results. Firstly, to highlight the importance of considering interconnection problems, the fault estimation performance has been compared with the previous fault estimation techniques that do not consider interconnection issues. Additionally, to verify the basic observing performance of the proposed fault estimation observer, a comparison has been presented for the state variable observing the performance of the fuzzy observer, which is one of the most used observing techniques. Through the comparison results, it has been guaranteed that the proposed observer design techniques for fault estimation are highly efficient in estimating the actuator fault input in discrete-time nonlinear interconnected systems.

6. Conclusions

This paper established the decentralized fuzzy fault estimation observer design for discrete-time nonlinear interconnected systems with uncertain interconnections. The main novelty or contributions of the proposed techniques in this paper are as follows:

1. This paper presents novel decentralized fuzzy observer-based fault estimation techniques for discrete-time nonlinear interconnected systems, which has not been previously studied. Especially, proposed techniques have been developed that are applicable to interconnected systems, which have uncertain information about interconnections.
2. Two approaches have been proposed to solve the observer design problem of including state variables by uncertainties. Also, the proposed approaches are algorithmized to solve the observer design problem using LMIs.
3. To demonstrate the performance of the proposed techniques, comparison results are presented for the previous fault estimation technique in the numerical example. Through performance comparison results, it validates the superiority of the proposed techniques.

To develop the decentralized fuzzy observer, it is considered that the nonlinear interconnected systems can be represented by the T–S fuzzy model, and the uncertain interconnection satisfied a quadratic inequality assumption. By defining the state and fault errors, the estimation error model was represented and the fault estimation problem was addressed. Based on two different H_∞ performance inequalities, the decentralized fuzzy observer design techniques were proposed to achieve the fault estimating performance, and observer design conditions were derived into LMI formats. Finally, the simulation and comparison results were provided to demonstrate the effectiveness of the proposed fault estimation techniques based on the decentralized fuzzy observer.

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References

1. Peddada, S.R.; Zeidner, L.E.; Ilies, H.T.; James, K.A.; Allison, J.T. Toward holistic design of spatial packaging of interconnected systems with physical interactions (spi2). *J. Mech. Des.* **2022**, *144*, 120801. [CrossRef]
2. Kumar, N.; Aryan, P.; Raja, G.L.; Muduli, U.R. Robust Frequency-Shifting Based Control Amid False Data Injection Attacks for Interconnected Power Systems with Communication Delay. *IEEE Trans. Ind. Appl.* **2024**, *60*, 3710–3723. [CrossRef]
3. Berben, L.; Floris, G.; Wildiers, H.; Hatse, S. Cancer and aging: Two tightly interconnected biological processes. *Cancers* **2021**, *13*, 1400. [CrossRef] [PubMed]
4. Perera, A.T.D.; Javanroodi, K.; Nik, V.M. Climate resilient interconnected infrastructure: Co-optimization of energy systems and urban morphology. *Appl. Energy* **2021**, *285*, 116430. [CrossRef]

5. Giudici, P.; Sarlin, P.; Spelta, A. The interconnected nature of financial systems: Direct and common exposures. *J. Bank. Financ.* **2020**, *112*, 105149. [CrossRef]
6. Lunze, J. *Feedback Control of Large-Scale Systems*; Prentice Hall: New York, NY, USA, 1992.
7. Zhang, H.; Feng, G. Stability Analysis and H_∞ Controller Design of Discrete-Time Fuzzy Large-Scale Systems Based on Piecewise Lyapunov Functions. *IEEE Trans. Syst. Man Cybern. B Cybern.* **2008**, *38*, 1390–1401. [CrossRef] [PubMed]
8. Mao, W.J.; Chu, J. Robust decentralised stabilisation of interval discrete-time singular large-scale systems. *IET Control Theory Appl.* **2010**, *4*, 244–252. [CrossRef]
9. Gielen, R.H.; Lazar, M. On stability analysis methods for large-scale discrete-time systems. *Automatica* **2015**, *55*, 66–72. [CrossRef]
10. Šiljak, D.D.; Zečević, A.I. Control of large-scale systems: Beyond decentralized feedback. *Annu. Rev. Control* **2005**, *29*, 169–179.
11. Bakule, L. Decentralized control: An overview. *Annu. Rev. Control* **2008**, *32*, 87–98. [CrossRef]
12. Peng, C.; Han, Q.L.; Yue, D. Communication-delay-distribution-dependent decentralized control for large-scale systems with IP-based communication networks. *IEEE Trans. Control Syst. Technol.* **2013**, *21*, 820–830. [CrossRef]
13. Singh, A.K.; Pal, B.C. Decentralized control of oscillatory dynamics in power systems using an extended LQR. *IEEE Trans. Power Syst.* **2016**, *31*, 1715–1728. [CrossRef]
14. Jang, Y.H.; Han, T.H.; Kim, H.S. Decentralized sampled-data fuzzy tracking control for a quadrotor UAV with communication delay. *Drones* **2022**, *6*, 280. [CrossRef]
15. Feng, Z.; Li, R.B.; Wu, L. Adaptive decentralized control for constrained strong interconnected nonlinear systems and its application to inverted pendulum. *IEEE Trans. Neural Netw. Learn. Syst.* 2023, *in press*. [CrossRef] [PubMed]
16. Zhang, H.; Dang, C.; Li, C. Decentralized H_∞ filter design for discrete-time interconnection fuzzy systems. *IEEE Trans. Fuzzy Syst.* **2009**, *17*, 1428–1440. [CrossRef]
17. Zhang, H.; Yu, G.; Zhou, C.; Dang, C. Delay-dependent decentralized H_∞ filtering for fuzzy interconnected systems with time-varying delay based on Takagi–Sugeno fuzzy model. *IET Control Theory Appl.* **2013**, *7*, 720–729. [CrossRef]
18. Kim, T.; Lee, C.; Shim, H. Completely decentralized design of distributed observer for linear systems. *IEEE Trans. Autom. Control* **2019**, *65*, 4664–4678. [CrossRef]
19. Humaloja, J.P.; Koivumäki, J.; Paunonen, L.; Mattila, J. Decentralized observer design for virtual decomposition control. *IEEE Trans. Autom. Control* **2021**, *67*, 2529–2536. [CrossRef]
20. Chen, J.; Patton, R. *Robust Model-Based Fault Diagnosis for Dynamic Systems*; Kluwer: Boston, MA, USA, 1999.
21. Ding, S.X. *Model-Based Fault Diagnosis Techniques: Design Schemes, Algorithms and Tools*; Springer : London, UK, 2008.
22. Zhu, J.W.; Yang, G.H.; Wang, H.; Wang, F.L. Fault estimation for a class of nonlinear systems based on intermediate estimator. *IEEE Trans. Autom. Control* **2016**, *61*, 2518–2524. [CrossRef]
23. Liu, Y.; Wang, Z.; Zou, L.; Zhou, D.; Chen, W.H. Joint state and fault estimation of complex networks under measurement saturations and stochastic nonlinearities. *IEEE Trans. Signal Inf. Process. Netw.* **2022**, *8*, 173–186. [CrossRef]
24. Chen, L.; Zhu, Y.; Wu, F.; Zhao, Y. Fault estimation observer design for Markovian jump systems with nondifferentiable actuator and sensor failures. *IEEE Trans. Cybern.* **2023**, *53*, 3844–3858. [CrossRef] [PubMed]
25. Tanaka, K.; Wang, H.O. *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*; Wiley: New York, NY, USA, 2001.
26. Gao, Z.; Shi, X.; Ding, X. Fuzzy state/disturbance observer design for T–S fuzzy systems with application to sensor fault estimation. *IEEE Trans. Syst. Man Cybern. B Cybern.* **2008**, *38*, 875–880. [PubMed]
27. Sakthivel, R.; Kavikumar, R.; Mohammadzadeh, A.; Kwon, O.M.; Kaviarasan, B. Fault estimation for mode-dependent IT2 fuzzy systems with quantized output signal. *IEEE Trans. Fuzzy Syst.* **2021**, *29*, 298–309. [CrossRef]
28. Fan, C.; Lam, J.; Xie, X. Fault estimation for periodic piecewise T–S fuzzy systems. *Int. J. Robust Nonlinear Control* **2021**, *31*, 8055–8074. [CrossRef]
29. Liu, Y.; Wang, Y. Actuator and sensor fault estimation for discrete-time switched T–S fuzzy systems with time delay. *J. Frankl. Inst.* **2021**, *358*, 1619–1634. [CrossRef]
30. Han, J.; Liu, X.; Wei, X.; Hu, X. Adaptive adjustable dimension observer based fault estimation for switched fuzzy systems with unmeasurable premise variables. *Fuzzy Sets Syst.* **2023**, *452*, 149–167. [CrossRef]
31. Mu, Y.; Zhang, H.; Gao, Z.; Zhang, J. A fuzzy Lyapunov function approach for fault estimation of T–S fuzzy fractional-order systems based on unknown input observer. *IEEE Trans. Syst. Man Cybern. Syst.* **2023**, *53*, 1246–1255. [CrossRef]
32. Li, Y.; Xu, W.; Chang, L. Fuzzy model based fault estimation and fault tolerant control for flexible spacecraft with unmeasurable vibration modes. *IET Control Theory Appl.* **2023**, *17*, 19–38. [CrossRef]
33. Zhou, H.; Lam, H.K.; Xiao, B. Fault estimation and fault tolerant control for interval type-2 Takagi–Sugeno fuzzy systems via membership-function-dependent approach. *Nonlinear Dyn.* **2023**, *111*, 1441–1454. [CrossRef]
34. Ding, J.; Liu, Y.; Yu, J.; Yang, X. Dissipativity-based integrated fault estimation and fault tolerant control for IT2 polynomial fuzzy systems with sensor and actuator faults. *IEEE Trans. Fuzzy Syst.* **2023**, *31*, 2956–2965. [CrossRef]
35. Zhang, X.; Zhang, J.X.; Huang, W.; Shi, P. Non-fragile sliding mode observer based fault estimation for interval type-2 fuzzy singular fractional order systems. *Int. J. Syst. Sci.* **2023**, *54*, 1451–1470. [CrossRef]
36. Koo, G.B. Sampled-data fuzzy fault estimation observer design for nonlinear systems. *IEEE Access* **2023**, *11*, 145612–145624. [CrossRef]

37. Zhang, K.; Jiang, B.; Shi, P. Distributed fault estimation observer design with adjustable parameters for a class of nonlinear interconnected systems. *IEEE Trans. Cybern.* **2019**, *49*, 4219–4228. [CrossRef] [PubMed]
38. Zhang, K.; Jiang, B.; Chen, M.; Yan, X.G. Distributed fault estimation and fault-tolerant control of interconnected systems. *IEEE Trans. Cybern.* **2021**, *51*, 1230–1240. [CrossRef] [PubMed]
39. Mu, Y.; Zhang, H.; Yan, Y.; Wang, Y. A novel design approach to state and fault estimation for interconnected systems using distributed observer. *Appl. Math. Comput.* **2023**, *449*, 127966. [CrossRef]
40. Mu, Y.; Zhang, H.; Yan, Y.; Xie, X. Distributed observer-based robust fault estimation design for discrete-time interconnected systems with disturbances. *IEEE Trans. Cybern. 2023*, **53**, 6737–6747. [CrossRef] [PubMed]
41. Li, X.J.; Yan, J.J.; Yang, G.H. Adaptive fault estimation for T–S fuzzy interconnected systems based on persistent excitation condition via reference signals. *IEEE Trans. Cybern.* **2019**, *49*, 2822–2834. [CrossRef] [PubMed]
42. Koo, G.B.; Park, J.B.; Joo, Y.H. Decentralized sampled-data fuzzy observer design for nonlinear interconnected systems. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 661–674. [CrossRef]
43. Kim, H.J.; Park, J.B.; Joo, Y.H. Decentralized H_∞ fuzzy filter for nonlinear large-scale sampled-data systems with uncertain interconnections. *Fuzzy Sets Syst.* **2018**, *344*, 145–162. [CrossRef]
44. Kim, H.J.; Park, J.B.; Joo, Y.H. Decentralized H_∞ sampled-data fuzzy filter for nonlinear interconnected oscillating systems with uncertain interconnections. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 487–498. [CrossRef]
45. Jang, Y.H.; Kim, H.S.; Kim, E.; Joo, Y.H. Decentralized sampled-data H_∞ fuzzy filtering with exponential time-varying gains for nonlinear interconnected systems. *Inf. Sci.* **2022**, *609*, 1518–1538. [CrossRef]
46. Takagi, T.; Sugeno, M. Fuzzy identification of systems and its application to modeling and control. *IEEE Trans. Syst. Man Cybern.* **1985**, *15*, 116–132. [CrossRef]
47. Petersen, I.R. A stabilization algorithm for a class of uncertain linear systems. *Syst. Control Lett.* **1987**, *8*, 351–357. [CrossRef]
48. Ma, X.J.; Sun, Z.Q.; He, Y.Y. Analysis and design of fuzzy controller and fuzzy observerl. *IEEE Trans. Fuzzy Syst.* **1998**, *6*, 41–51.