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Dark and Singular Highly Dispersive Optical Solitons with Kudryashov's Sextic Power-Law of Nonlinear Refractive Index in the Absence of Inter-Modal Dispersion

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Abstract: The current paper studies highly dispersive optical solitons with the aid of Kudryashov's integration algorithm. The governing model employs Kudryashov's sextic power law of nonlinear refractive index. The inter-modal dispersion term is absent from the model. The integration scheme retrieves dark and singular solitons to the model.

Keywords: solitons; highly dispersive; Kudryashov

1. Introduction

The dynamics of optical soliton propagation across inter-continental distances is governed by the nonlinear Schrödinger's equation (NLSE) together with the Kerr law of self-phase modulation (SPM) [1–7]. It is well-known from the fundamentals of electromagnetic theory that NLSE is derived from the basic alphabets of this theory, namely Maxwell's equation. A multiple-scale analysis leads to the formation of NLSE. This model has been extensively studied in quantum optics and from a wide range of perspectives. These include the formation of soliton radiation, birefringent fibers, DWDM topology, collision-induced timing jitter, four-wave mixing, dispersion-managed solitons, quasi-linear pulse stochasticity, supercontinuum generation, and many others. Subsequently, several new concepts from NLSE have emerged. These are non-Kerr law solitons, pure-cubic optical solitons, pure-quartic optical solitons, cubic-quartic optical solitons, and finally, the highly dispersive (HD) optical solitons, which gave way to the big influx of research results that are visible today. The details of such solitons and their implementation have been drafted and penned here.

One of the most fascinating concepts introduced in nonlinear optics is that of HD optical solitons. This was conceived when the much-needed chromatic dispersion (CD) was running critically low. Therefore, to maintain the delicate balance between nonlinearity and dispersion that propagate solitons to sustain, the concept of HD solitons was addressed. The model is represented by NLSE that contains sixth-order dispersion (6OD), fifth-order dispersion (5OD), fourth-order dispersion (4OD), and third-order dispersion (3OD) terms in addition to CD. The current model that is addressed in this paper will have the inter-modal dispersion (IMD) discarded. The self-phase modulation (SPM) effect stems from six power law nonlinear terms-a structure that was first proposed by Kudryashov and later produced several results, which have been reported in other works [8–15]. The integration algorithm that is addressed in the present paper was also developed by Kudryashov. This would give way to singular and dark solitons that are exhibited and enumerated with the use of the model. The details are sketched after an overview of the model is given and after a revisitation of Kudryashov's integration algorithm.

Governing Equation

The governing model for HD-NLSE in the absence of IMD is written as follows [8]:

$$\begin{aligned} & iq_t + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} \\ + & \left(b_1|q|^n + b_2|q|^{2n} + b_3|q|^{3n} + b_4|q|^{4n} + b_5|q|^{5n} + b_6|q|^{6n}\right) q = 0, \end{aligned} \tag{1}$$

where the SPM comes from the six nonlinear terms whose coefficients are b_j for $j = 1, \cdots, 6$. The coefficient $i = \sqrt{-1}$ is that of the linear temporal evolution of the solitons. The coefficients a_j account for the CD, 3OD, 4OD, 5OD, and 6OD effects in sequence. Lastly, $q(x,t)$ represents the optical soliton profile, with the variables being x and t, which account for spatial and temporal coordinates in sequence.

In order to obtain a detailed perspective of the model, one must understand the basic aspect and necessity of the dynamics of soliton formation and its sustainability. Solitons are the outcome of a delicate balance between CD and self-phase modulation (SPM). The dispersion terms are the coefficients of a_j ($2 \leq j \leq 6$), while the coefficients of SPM are b_j ($1 \leq j \leq 6$). Occasionally, CD alone fails to provide the necessary balance for the solitons to survive long-distance communication across the globe. The parameters a_j stem from the Taylor series expansion of the operator in Kramer–Kronig relations up until the sixth order. Therefore, these additional dispersion terms guarantee the provision of the necessary balance.

The current form of SPM was first proposed by Kudryashov in 2022 [1]. This form of SPM is one of the latest forms of the non-Kerr law of nonlinear refractive index that is studied in this area of quantum optics. Again, the contribution from IMD has been dropped since this effect is for short-distance communication, and the goal of this paper is to address soliton communications across trans-continental and trans-oceanic distances. It is worth mentioning that the proposed model is very new and is less than a year old [1]. This is definitely a theoretical model, and its experimental components are yet to be revealed. To the best of our knowledge, the model has not yet been studied and used to recover eye diagrams in laboratories anywhere in the world. Therefore, no visible eye diagrams for the model are reported as yet.

Lastly, this paper recovers singular and dark solitons; however, bright solitons are not recoverable by means of the adopted integration algorithm. It must be noted that dark solitons are not visible on an oscilloscope without the presence of a background wave. Singular solitons, on the other hand, are viable candidates for rogue waves. Furthermore, bright solitons are the ones that are visible on an oscilloscope when these are experimentally demonstrated.

2. Kudryashov's Procedure-An Overview

Consider the following governing model [16–19]:

$$F(u, u_x, u_t, u_{xt}, u_{xx}, \dots) = 0, \tag{2}$$

where $u = u(x,t)$ describes the soliton profile, while t and x are temporal and spatial variables in sequence.

Step 1: With the aid of the wave variable below:

$$u(x,t) = U(\xi), \quad \xi = k(x - vt), \tag{3}$$

Equation (2) collapses to

$$P(U, -kvU', kU', k^2U'', \dots) = 0, \tag{4}$$

where v and k are the velocity and width of the soliton in sequence.

Step 2: Equation (4) holds the following solution form:

$$U(\xi) = \sum_{l=0}^{N} \alpha_i Q^i(\xi), \ \alpha_i \neq 0, \tag{5}$$

along with the auxiliary equation

$$Q'(\xi) = Q(\xi)^2 - Q(\xi), \tag{6}$$

which satisfies the soliton structure

$$Q(\xi) = \frac{1}{1 + \eta e^{\xi}}, \tag{7}$$

where N comes from the balancing method, while α_i $(i = 0, 1, \dots, N)$ and η are constants.

Step 3: Substituting (5) along with (6) into (4) paves the way for a system of equations, which provides us with the constants in (3), (5) and (7). By inserting these constants into (5), we obtain dark and singular solitons.

3. Mathematical Analysis

The resulting soliton profile is as follows:

$$q(x,t) = U(\xi) e^{i\phi(x,t)}, \tag{8}$$

where the wave variable evolves as

$$\xi = k(x - vt), \tag{9}$$

and the soliton phase component emerges as

$$\phi(x,t) = -\kappa x + \omega t + \theta. \tag{10}$$

Here, v, κ, ω, θ, and $U(\xi)$ stand for the velocity, frequency, wave number, phase constant, and amplitude component of the soliton in sequence. Substituting (8) into (1) yields the real part:

$$a_6k^6U^{(6)} + k^4U^{(4)}\left(-15a_6\kappa^2 + 5a_5\kappa + a_4\right) + k^2\left(15a_6\kappa^4 - 10a_5\kappa^3 - 6a_4\kappa^2 + 3a_3\kappa + a_2\right)U'' + U\left(-a_6\kappa^6 - a_2\kappa^2 + a_5\kappa^5 - a_3\kappa^3 + a_4\kappa^4 - \omega\right) + b_1U^{n+1} + b_2U^{2n+1} + b_3U^{3n+1} + b_4U^{4n+1} + b_5U^{5n+1} + b_6U^{6n+1} = 0, \tag{11}$$

and the imaginary part:

$$U^{(5)}\left(a_5k^5 - 6a_6\kappa k^5\right) + U^{(3)}\left(20a_6\kappa^3k^3 - 10a_5\kappa^2k^3 - 4a_4\kappa k^3 + a_3k^3\right) - kU'\left(6a_6\kappa^5 - 5a_5\kappa^4 - 4a_4\kappa^3 + 3a_3\kappa^2 + 2a_2\kappa + v\right) = 0. \tag{12}$$

The velocity v derived from (12) evolves as follows:

$$v = -6a_6\kappa^5 + 5a_5\kappa^4 + 4a_4\kappa^3 - 3a_3\kappa^2 - 2a_2\kappa, \tag{13}$$

with the parameter constraints

$$a_5 = 6a_6\kappa, \quad 20a_6\kappa^3 - 10a_5\kappa^2 - 4a_4\kappa + a_3 = 0. \tag{14}$$

Equation (11) can be written as follows:

$$H_7U^{4n+1} + H_6U^{3n+1} + H_5U^{2n+1} + H_4U^{n+1} + H_8U^{5n+1} + H_9U^{6n+1} + H_3U^{(4)} + H_2U'' + H_1U + k^2U^{(6)} = 0, \tag{15}$$

where

$$\begin{cases} H_1 = \frac{-a_6\kappa^6 + a_5\kappa^5 + a_4\kappa^4 - a_3\kappa^3 - a_2\kappa^2 - \omega}{a_6k^4}, \\ H_2 = \frac{15a_6\kappa^4 - 10a_5\kappa^3 - 6a_4\kappa^2 + 3a_3\kappa + a_2}{a_6k^2}, \\ H_3 = \frac{-15a_6\kappa^2 + 5a_5\kappa + a_4}{a_6}, \\ H_4 = \frac{b_1}{a_6k^4}, H_5 = \frac{b_2}{a_6k^4}, H_6 = \frac{b_3}{a_6k^4}, \\ H_7 = \frac{b_4}{a_6k^4}, H_8 = \frac{b_5}{a_6k^4}, H_9 = \frac{b_6}{a_6k^4}. \end{cases} \tag{16}$$

Then, set the following restriction:

$$U(\xi) = V(\xi)^{\frac{1}{n}}. \tag{17}$$

Thus, Equation (15) transforms into:

$$n^5V^5(H_1V^{(4)} + H_2V''^2V^{(6)}) + V^4(V'(-4H_1n^4\left(H_1^4n - 1\right)V^{(3)} - 6k^2(n-1)n^4V^{(5)}) - 3H_1(n-1)n^4V''^2 + H_2(-(n-1))n^4V'^2 - 10k^2(n-1)n^4V^{(3)^2} - 15k^2(n-1)n^4V^{(4)}V'') + V^2(H_1n^2(-6n^3 + 11n^2 - 6n + 1)V'^4 - 20k^2n^2(6n^3 - 11n^2 + 6n - 1)V^{(3)}V'^3 - 45k^2n^2(6n^3 - 11n^2 + 6n - 1)V'^2V''^2) + V^3(V'^2(6H_1\left(2n^2 - 3n + 1\right)n^3V''^2\left(2n^2 - 3n + 1\right)n^3V^{(4)}) + 15k^2\left(2n^2 - 3n + 1\right)n^3V''^3 + 60k^2\left(2n^2 - 3n + 1\right)n^3V^{(3)}V'V'') + H_3n^6V^{12} + H_4n^6V^{11} + H_5n^6V^{10} + H_6n^6V^9 + H_7n^6V^8 + H_8n^6V^7 + H_9n^6V^6 + 15k^2n\left(24n^4 - 50n^3 + 35n^2 - 10n + 1\right)VV'^4V'' + k^2\left(-120n^5 + 274n^4 - 225n^3 + 85n^2 - 15n + 1\right)V'^6 = 0. \tag{18}$$

Balancing $V^5V^{(6)}$ with V^{12} in (18) collapses (5) to

$$V(\xi) = \alpha_0 + \alpha_1Q(\xi). \tag{19}$$

Substituting (19) along with (6) into (18) provides us with the following results:

Result 1:

$$k=\pm\sqrt{-H_9n^6-H_2n^4},\quad \alpha_1=-\frac{(n+2)\left(H_2(n+1)^2\left(n^2+2n+2\right)+H_9\left(n^4+4n^3+7n^2+6n+3\right)n^2\right)}{\alpha_0H_8n^2},$$
$$H_1=0,\quad H_3=\frac{H_8^6n^{10}\left(120n^5+274n^4+225n^3+85n^2+15n+1\right)\left(H_9n^2+H_2\right)}{(n+2)^6(H_2(n+1)^2(n^2+2n+2)+H_9(n^4+4n^3+7n^2+6n+3)n^2)^6},$$
$$H_4=\frac{3H_8^5n^8(5n+2)\left(24n^4+50n^3+35n^2+10n+1\right)\left(H_9n^2+H_2\right)}{(n+2)^5(H_2(n+1)^2(n^2+2n+2)+H_9(n^4+4n^3+7n^2+6n+3)n^2)^5},$$
$$H_5=\frac{5H_8^4n^6\left(78n^5+215n^4+228n^3+118n^2+30n+3\right)\left(H_9n^2+H_2\right)}{(n+2)^4(H_2(n+1)^2(n^2+2n+2)+H_9(n^4+4n^3+7n^2+6n+3)n^2)^4},$$
$$H_7=\frac{H_8^2n^2(n+1)\left(H_9\left(31n^4+90n^3+105n^2+60n+15\right)n^2+H_2\left(31n^4+90n^3+105n^2+60n+14\right)\right)}{(n+2)^2(H_2(n+1)^2(n^2+2n+2)+H_9(n^4+4n^3+7n^2+6n+3)n^2)^2},$$
$$H_6=\frac{10H_8^3n^4\left(2n^2+3n+1\right)\left(9n^3+15n^2+9n+2\right)\left(H_9n^2+H_2\right)}{(n+2)^3(H_2(n+1)^2(n^2+2n+2)+H_9(n^4+4n^3+7n^2+6n+3)n^2)^3}. \tag{20}$$

By inserting (20) along with $H_9n^2+H_2<0$ and $\eta=\pm1$ into (19), the singular and dark solitons emerge as follows:

$$q(x,t)=\left\{\alpha_0-\frac{(n+2)\left(H_2(n+1)^2\left(n^2+2n+2\right)+H_9\left(n^4+4n^3+7n^2+6n+3\right)n^2\right)}{2\alpha_0H_8n^2}\right.$$
$$\left.\times\left\{1\pm\coth\left[\frac{1}{2}\left(\sqrt{-H_9n^6-H_2n^4}\right)(x-vt)\right]\right\}\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}, \tag{21}$$

and

$$q(x,t)=\left\{\alpha_0-\frac{(n+2)\left(H_2(n+1)^2\left(n^2+2n+2\right)+H_9\left(n^4+4n^3+7n^2+6n+3\right)n^2\right)}{2\alpha_0H_8n^2}\right.$$
$$\left.\times\left\{1\pm\tanh\left[\frac{1}{2}\left(\sqrt{-H_9n^6-H_2n^4}\right)(x-vt)\right]\right\}\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{22}$$

Figures 1–3 exhibit the amplitudes of dark HD optical solitons that can be measured by viewing the waveform on an oscilloscope.

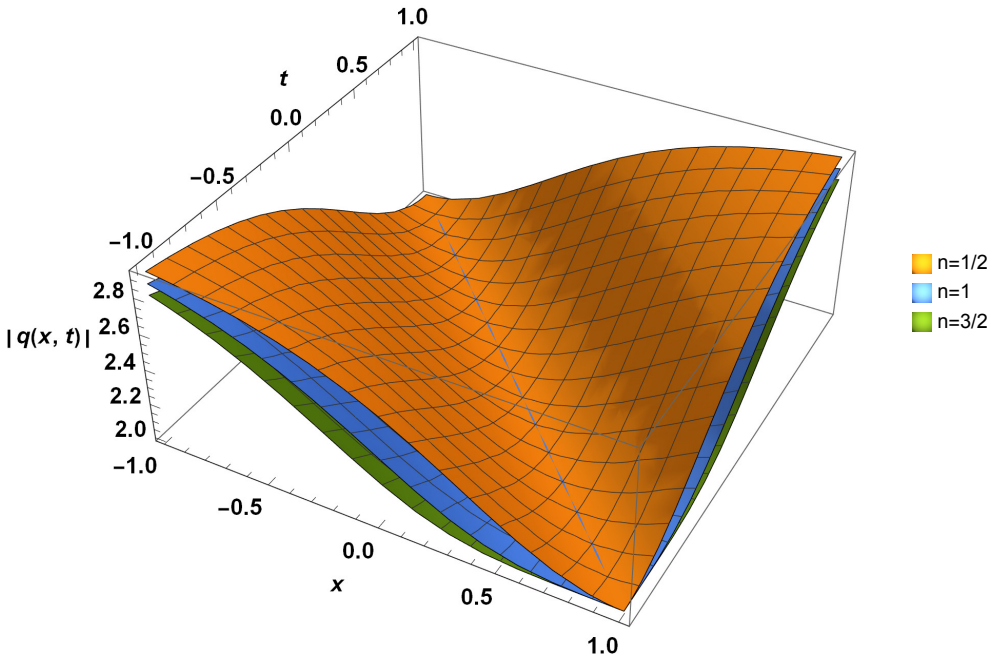


Figure 1. The amplitudes of a dark HD optical soliton (22), with $\alpha_0 = 1$, $b_6 = -1$, $a_6 = 1$, $k = 1$, $a_2 = 1$, $a_3 = 1$, $\kappa = 1$, $a_4 = 1$, and $b_5 = 1$.

Result 2:

$$k = \pm n^3\sqrt{\frac{H_9}{n^4+4n^3+7n^2+6n+2}}, \quad \alpha_1 = \pm\frac{1}{\alpha_0}\sqrt[4]{-\frac{5H_9(78n^4+137n^3+91n^2+27n+3)}{n^3+3n^2+4n+2}},$$
$$H_1 = H_8 = 0, \quad H_2 = -\frac{H_9n^2\left(n^4+4n^3+7n^2+6n+3\right)}{(n+1)^2(n^2+2n+2)},$$
$$H_3 = -\frac{20n^2+9n+1}{(6n^2+5n+1)(13n^2+12n+3)^2}\sqrt{-\frac{(78n^4+137n^3+91n^2+27n+3)(n^3+3n^2+4n+2)}{125H_9}},$$
$$H_4 = -\frac{(3-3i)\left(20n^2+13n+2\right)}{5(6n^2+5n+1)}\sqrt[4]{\frac{(78n^4+137n^3+91n^2+27n+3)^3(n^3+3n^2+4n+2)}{20H_9(13n^2+12n+3)^8}},$$
$$H_5 = 1, \quad H_6 = -\frac{2\left(9n^3+15n^2+9n+2\right)}{39n^3+49n^2+21n+3}\sqrt[4]{-\frac{5H_9(78n^4+137n^3+91n^2+27n+3)}{n^3+3n^2+4n+2}},$$
$$H_7 = \sqrt{-\frac{4H_9(15n^4+43n^3+49n^2+27n+6)^2}{5(78n^4+137n^3+91n^2+27n+3)(n^3+3n^2+4n+2)}}. \tag{23}$$

By plugging (23) along with $H_9 > 0$ and $\eta = \pm 1$ into (19), the singular and dark solitons produced are as follows:

$$q(x,t) = \left\{\alpha_0 \pm \frac{1}{2\alpha_0}\sqrt[4]{-\frac{5H_9(78n^4+137n^3+91n^2+27n+3)}{n^3+3n^2+4n+2}}\right.$$
$$\left.\times\left\{1 \pm \coth\left[\frac{n^3}{2}\sqrt{\frac{H_9}{n^4+4n^3+7n^2+6n+2}}(x-vt)\right]\right\}\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{24}$$

and

$$q(x,t)=\left\{\alpha_0\pm\frac{1}{2\alpha_0}\sqrt[4]{-\frac{5H_9(78n^4+137n^3+91n^2+27n+3)}{n^3+3n^2+4n+2}}\right.$$
$$\left.\times\left\{1\pm\tanh\left[\frac{n^3}{2}\sqrt{\frac{H_9}{n^4+4n^3+7n^2+6n+2}}(x-vt)\right]\right\}\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \quad (25)$$

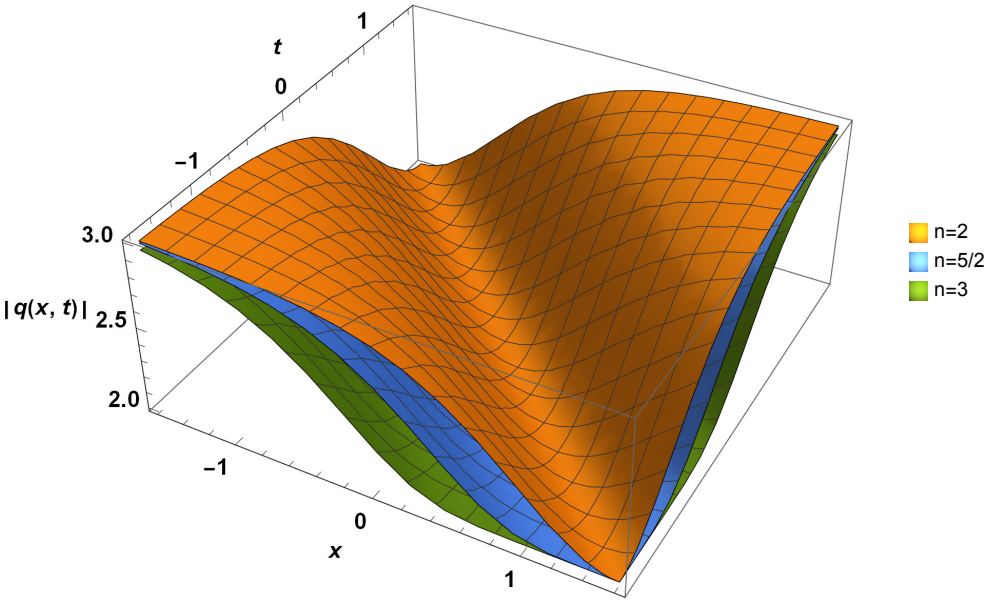


Figure 2. The amplitudes of a dark HD optical soliton (25), with $\alpha_0=1$, $b_6=1$, $a_6=1$, $k=1$, $\kappa=1$, $a_4=1$, $a_3=1$, and $a_2=1$.

Result 3:

$$\alpha_1=\pm\frac{1}{\alpha_0}\sqrt{-\frac{12(n+1)^2(30n^3+47n^2+24n+4)}{5H_7n^4(13n^2+12n+3)}},\quad k=\pm\frac{n}{\sqrt{-5(13n^2+12n+3)}},$$
$$H_1=1,\quad H_2=-\frac{64n^4+186n^3+258n^2+144n+27}{5n^2(13n^2+12n+3)},$$
$$H_4=-\frac{3(5n+2)(24n^4+50n^3+35n^2+10n+1)}{5n^4(13n^2+12n+3)}\left(\pm\sqrt{-\frac{12(n+1)^2(30n^3+47n^2+24n+4)}{5H_7n^4(13n^2+12n+3)}}\right)^{-5},$$
$$H_5=H_8=0,\quad H_6=\frac{2(n+1)(2n+1)^2(15n^2+16n+4)}{n^4(13n^2+12n+3)}\left(\pm\sqrt{-\frac{12(n+1)^2(30n^3+47n^2+24n+4)}{5H_7n^4(13n^2+12n+3)}}\right)^{-3},$$
$$H_9=\frac{(n+1)^2(64n^2+58n+13)}{5n^4(13n^2+12n+3)},\ H_3=-\frac{25H_7^3n^8(13n^2+12n+3)^2(60n^3+47n^2+12n+1)}{1728(n+1)^5(2n+1)^2(15n^2+16n+4)^3}. \quad (26)$$

By putting (26) along with $\eta=\pm1$ into (19), the singular and dark solitons evolve as follows:

$$q(x,t)=\left\{\alpha_0\pm\frac{1}{\alpha_0}\sqrt{-\frac{3(n+1)^2(30n^3+47n^2+24n+4)}{5H_7n^4(13n^2+12n+3)}}\right.$$
$$\left.\times\left\{1\pm\coth\left[\frac{n}{2\sqrt{-5(13n^2+12n+3)}}(x-vt)\right]\right\}\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \quad (27)$$

and

$$q(x,t)=\left\{\alpha_0\pm\frac{1}{\alpha_0}\sqrt{-\frac{3(n+1)^2(30n^3+47n^2+24n+4)}{5H_7n^4(13n^2+12n+3)}}\right.$$
$$\left.\times\left\{1\pm\tanh\left[\frac{n}{2\sqrt{-5(13n^2+12n+3)}}(x-vt)\right]\right\}\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{28}$$

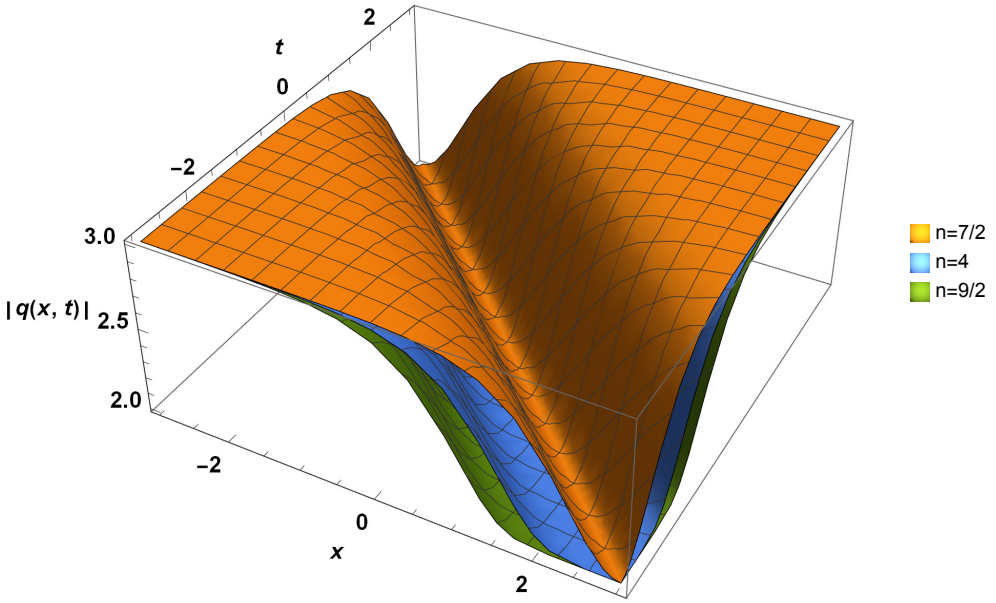


Figure 3. The amplitudes of a dark HD optical soliton (28), with $\alpha_0=1$, $b_4=-1$, $a_6=1$, $\kappa=1$, $a_4=1$, $a_3=1$, and $a_2=1$.

Result 4:

$$\alpha_1=-\frac{3(20n^2+13n+2)}{5\alpha_0H_4(13n^2+12n+3)},\quad k=\pm\frac{9n^3(20n^2+13n+2)^2}{25H_4^2(13n^2+12n+3)^2\sqrt{-5(78n^5+215n^4+228n^3+118n^2+30n+3)}},$$
$$H_2=-\frac{3n^2(20n^2+13n+2)\left(\begin{array}{c}625H_4^3(13n^2+12n+3)^4(6n^3+11n^2+6n+1)\\-27(20n^2+13n+2)^3(n^5+6n^4+15n^3+20n^2+15n+6)\end{array}\right)}{3125H_4^4(13n^2+12n+3)^5(6n^4+23n^3+28n^2+13n+2)},$$
$$H_9=\frac{3(20n^2+13n+2)\left(\begin{array}{c}625H_4^3(6n^2+5n+1)(13n^2+12n+3)^4\\-27(20n^2+13n+2)^3(n^4+5n^3+10n^2+10n+4)\end{array}\right)}{3125H_4^4(13n^2+12n+3)^5(6n^3+17n^2+11n+2)}$$
$$H_5=H_8=1,\ H_7=\frac{(n+1)\left(\begin{array}{c}625H_4^3(6n^2+5n+1)(13n^2+12n+3)^4\\+54(20n^2+13n+2)^3(15n^4+58n^3+77n^2+48n+12)\end{array}\right)}{375H_4^2(n+2)(6n^2+5n+1)(13n^2+12n+3)^3(20n^2+13n+2)},$$
$$H_1=0,\quad H_3=\frac{5H_4^2(5n+1)(13n^2+12n+3)}{9(4n+1)(5n+2)^2},\quad H_6=-\frac{6(20n^2+13n+2)(9n^3+15n^2+9n+2)}{5H_4(3n+1)(13n^2+12n+3)^2}. \tag{29}$$

By substituting (29) along with $\eta=\pm1$ into (19), the singular and dark solitons emerge as follows:

$$q(x,t) = \left\{ \alpha_0 - \frac{3\left(20n^2+13n+2\right)}{10\alpha_0 H_4(13n^2+12n+3)} \right.$$
$$\left. \times \left(1 - \coth\left[\pm \frac{9n^3\left(20n^2+13n+2\right)^2}{50H_4^2(13n^2+12n+3)^2\sqrt{-5(78n^5+215n^4+228n^3+118n^2+30n+3)}}(x-vt) \right] \right) \right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}, \tag{30}$$

and

$$q(x,t) = \left\{ \alpha_0 - \frac{3\left(20n^2+13n+2\right)}{10\alpha_0 H_4(13n^2+12n+3)} \right.$$
$$\left. \times \left(1 - \tanh\left[\pm \frac{9n^3\left(20n^2+13n+2\right)^2}{50H_4^2(13n^2+12n+3)^2\sqrt{-5(78n^5+215n^4+228n^3+118n^2+30n+3)}}(x-vt) \right] \right) \right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}. \tag{31}$$

4. Conclusions

This paper recovered singular and dark highly dispersive optical solitons with the use of a governing NLSE that is studied in the absence of IMD. The SPM effect, which was first proposed by Kudryashov, consists of six power law nonlinear effects. The integration algorithm that was implemented in this paper was also proposed by Kudryashov. This yielded dark and singular solitons. It must be noted that dark solitons are classified as being a part of topological solitons with the corresponding attached phonons that come with it, especially with the presence of higher-order dispersion terms. These topological solitons are presented from the core soliton regime since the current integration scheme fails to retrieve phonons, an inherent shortcoming of the scheme. Another issue is that the current scheme is confined to present only 1-soliton solutions, and unfortunately, multiple soliton solutions are not recoverable from such a scheme.

This scheme is therefore incomplete, in the sense that a full spectrum of solitons is not retrievable with the use of this algorithm. This leaves a gaping hole in the study. While several advances can be made with these limited results, it is imperative to secure bright soliton solutions to the model for its applicability in addressing soliton communications across inter-continental distances. One of the tacit drawbacks of this model is the shedding of energy and the slowing down of solitons that have been recovered. The higher-order dispersion terms trigger the formation of soliton radiation, which leads to the loss of energy of such solitons. A consequence of this is the slowing down of the solitons. The results of the paper, however, focus on the core soliton regime and thus have limited applicability. The singular solitons that emerged from the integration scheme could serve as a viable model for rogue waves, which could be used as the mathematical model for rogue waves. After carrying out a possible infinite-series expansion of the singular solitons, it would presumably be possible to connect to the known expression for rogue waves, which have been reported in several papers. While the results of the paper are impressive, they are still incomplete as there is an absence of bright soliton solutions and perhaps straddled solitons. The aggressive hunt now involves the securing of bright soliton solutions to the model in order to achieve a complete spectrum of solutions to the model from an optics perspective. The findings of this research would be disseminated in time.

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