electronics

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Robust Asynchronous H_∞ Observer-Based Control Design for Discrete-Time Switched Singular Systems with Time-Varying Delay and Sensor Saturation: An Average Dwell Time Approach

Mohamed Amin Regaieg [1], Mourad Kchaou [2,*], Houssem Jerbi [2], Attia Boudjemline [2] and Ahmed Hafaifa [3,4]

1 MIS Laboratory, University of Picardie Jules Verne, UFR of Sciences, 33 Rue St Leu, 80000 Amiens, France; mohamed.amin.regaieg@u-picardie.fr
2 College of Engineering, University of Hail, Hail P.O. Box 2440, Saudi Arabia; houssem.jerbi@gmail.com (H.J.); a.boudjemline@uoh.edu.sa (A.B.)
3 Applied Automation and Industrial Diagnostics Laboratory, Faculty of Science and Technology, University of Djelfa, Djelfa 17000, DZ, Algeria; hafaifa.ahmed.dz@ieee.org
4 Department of Electrical and Electronics Engineering, Nisantasi University, Sarıyer, İstanbul 34398, Turkey; ahmed.hafaifa@nisantasi.edu.tr
* Correspondence: mouradkchaou@gmail.com

Citation: Regaieg, M.A.; Kchaou, M.; Jerbi, H.; Boudjemline, A.; Hafaifa, A. Robust Asynchronous H_∞ Observer-Based Control Design for Discrete-Time Switched Singular Systems with Time-Varying Delay and Sensor Saturation: An Average Dwell Time Approach. *Electronics* **2021**, *10*, 2334. https://doi.org/10.3390/electronics10192334

Academic Editors: Dumitru Popescu, Haoping Wang, Severus C. Olteanu and Ciprian Lupu

Received: 9 August 2021
Accepted: 17 September 2021
Published: 23 September 2021

Abstract: This work discuss the robust stabilization problem for discrete-time switched singular systems with simultaneous presence of time-varying delay and sensor nonlinearity. To this end, an observer-based controller was synthesized that works under asynchronous switching signals. Investigating the average dwell time approach and using a Lyapunov–Krasovskii functional with triple sum terms, sufficient conditions were derived for achieving the existence of such asynchronous controller and guaranteeing the resulting closed-loop system to be exponentially admissible with H_∞ performance level. Subsequently, the effectiveness of the proposed control scheme was verified through two numerical examples.

Keywords: switched systems; hybrid systems; time-varying delay; observer control; average dwell time; H_∞ performance

1. Introduction

Lately, great interest has been devoted to the study of switched singular systems on both theoretical and application fronts ([1–3], and the references therein). From a mathematical point of view, switched singular systems are typically each composed of a finite number of subsystems and a switching law that specifies the active subsystems at each instant of time. Each subsystem is defined by ordinary differential equations that describe the dynamical part in the system and algebraic equations that represent the interrelationships between different components in the system. Moreover, the switching law plays a crucial role in determining the dynamic behavior of switched singular systems [2,4,5]. All the montioned works are concerned with arbitrary switching signal to study switched singular systems. Therefore, many switched singular systems fail to preserve stability under switching signals of this kind, but may be stable under some prescribed switching signals. Thus, we devote our attention in this work on the ADT approach, which means that the number of switches in a finite interval is bounded and the average time between consecutive switching is not less than a specific constant [6,7].

It should be noted that time-delay occurrence represents, usually, a source of instability and poor performance of dynamic systems. As a result, the study of switched singular delayed systems has aroused considerable attention [8–12].

Furthermore, study of robustness against external disturbances is significant [13]. For this purpose, different techniques have been investigated to ensure robust stability. Thus, the H_∞ technique has been used to study the problem of control [9,14] and filtering [4,15] for switched singular systems with time delay.

Nevertheless, all the previous works cited investigated only linear switched singular systems with synchronous switching modes between the respective controllers and systems. Generally, in practice the switching instants of the controllers exceed or lag behind those of the subsystems. Accordingly, it is necessary and more realistic to consider this phenomenon when dealing with control problems for switched singular systems. The asynchronous switched control problem covers many fields of application, such as Markovian jump systems [16], networked control systems [17] and neural systems [18,19]. To date, some appreciable studies have been reported for switched systems under asynchronous switching [20–24]. To the best of our knowledge, the stabilization problem for discrete-time switched singular time-delay systems under asynchronous switching has not been fully investigated, except in some works where the state feedback stabilization has been investigated for discrete-time singular systems [3,25], and for continuous singular systems [12,26]. That was the **primary motivation** for this work.

We note that all the methods suggested in the previous references suppose that the system state variables are available for measurements. However, the state variables for many real plants are mostly not fully accessible for many reasons, such as the non-existence of correct sensors to measure some states, or an increased number of sensors making the whole system more complex. Thus, the design of observers to estimate the system states is more reasonable. Accordingly, considerable attention has been paid to the observer-based control problem for switched singular systems. For example, in [27,28] the problem of observers design for a class of discrete-time switched singular systems subject to constant delay, unknown inputs and arbitrary switching sequences was considered. In addition, the robust H_∞ non-fragile observer-based control issue for switched discrete singular systems with time-varying delays under arbitrary switching was treated in [29]. The authors in [30] studied the observer-based asynchronous H_∞ control problem for switched singular systems with quadratically inner-bounded nonlinearity. A two step method was investigated in [31] to solve the problem of observer-based output-feedback asynchronous control design for a class of switched continuous-time Takagi–Sugeno fuzzy systems. Based on the singular value decomposition (SVD) technique and cone complementarity linearization (CCL) algorithm, the problem of asynchronous observer-based control for a class of discrete-time Markov jump systems has been treated in [32]. However, the SVD technique is difficult to use when the disturbance affects the measurements and the use of an iterative algorithm such the CCL can complicate the resolution of LMI conditions. In this regard, how to deal with dynamic systems with unavailable states for measurement by using an observer-based controller for the considered system under asynchronous switching was the **second motivation** for this work.

Due to many environmental circumstances, the actuator or sensor saturation can be interpreted as additive nonlinear exogenous disturbances. Thus, if such non-linearities are not considered in the controller design, the stability of system can be affected. Hence, in the analysis and implantation of the controller, the effect of non-linearity cannot be neglected [33–36]. Due to its theoretical and practical importance, some representative results regarding sensor saturation for switched systems have been considered [37–39]. However, the sensor saturation effect has not been investigated when dealing with the problem of asynchronous ADT observer-based control for discrete-time switched singular systems with time-varying delay. This represents the **third motivation** for this paper.

To study a general switched singular system from a practical point of view, it was assumed that the system under consideration consisted of unmeasured states, time-varying delay, and sensor saturation. The main contributions can be summarized as follows:

(i) The switched singular systems were employed to cope with the problem of asynchronous observer-based control design using the ADT approach. Compared to [14,26,40,41], the design is considered to characterize dynamic systems with unavailable states for measurement, which closely reflects the reality with a more general structure.

(ii) The exponential admissibility and H_∞ performances of the switched singular systems were established by using an appropriate Lyapunov–Krasovskii functional with a triple-sum term. Delay-dependent LMI conditions were derived using the ADT approach.
(iii) In contrast to [22,23,31], a one step method was developed to deal with the problem of asynchronous H_∞ observer-based control design without considering any appropriate algorithm.
(iv) Numerical examples were used to demonstrate the effectiveness of the proposed study.

An outline of this paper is given as follows. The system description and preliminaries are presented in Section 2. The main results, including the admissibility analysis and the asynchronous observer-based controller synthesis, are given in Sections 3 and 4. A simulation example is illustrated in Section 5. Section 6 concludes the paper.

Notations. Throughout the paper, a real symmetric matrix $Y > 0$ $(Y \geq 0)$ denotes Y being a positive definite (or positive semi-definite) matrix. $sym(Y)$ stands for $Y + Y^T$. I and 0 symbolize the identity matrix and a zero matrix with appropriate dimensions, respectively. $Y \in \mathbb{R}^s$ denotes the $s-$dimensional Euclidean space, and $Y \in \mathbb{R}^{s\times n}$ refers to the set of all $s \times n$ real matrices. $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote the minimum and maximum eigenvalues of P. In symmetric block matrices or long matrix expressions, we use a star $*$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $\|.\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. $col\{Y, X\}$ denotes a column matrix.

2. System Description and Preliminaries

Consider a class of switched singular systems with time-varying delay described by

$$\begin{cases} Ex(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d(k)) + B_{1\sigma(k)}u(k) + B_{2\sigma(k)}w(k) \\ \qquad y(k) = \varphi(C_{\sigma(k)}x(k)) \\ \qquad z(k) = C_{2\sigma(k)}x(k) + D_{\sigma(k)}w(k) \\ \qquad x(k) = \phi(k),\ k \in [-d_M, 0] \end{cases} \tag{1}$$

where $u(k) \in \mathbb{R}^{m_u}$ is the control input vector, $z(k) \in \mathbb{R}^q$ is the controlled output vector, $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^p$ is the measured output, E is a singular matrix with $rank(E) < n$, and $\phi(k)$ is a given initial condition sequence. The disturbance input vector, $w(k) \in \mathbb{R}^n$, is supposed to belong to $l_2[0,\ \infty)$. That is, $\sum_{k=0}^{\infty} w^T(k)w(k) < \infty$. $\sigma(k) : [0,\ +\infty) \rightarrow \mathfrak{I} = \{1, 2 \cdots, N\}$ is a piecewise constant switching signal, with N being the number of subsystems.

Time-varying delay, $d(k)$, is defined as

$$0 < d_m \leq d(k) \leq d_M \tag{2}$$

where d_M and d_m, positive integers, represent the bounds of the delay.

Matrices E, $A_{\sigma(k)}$, $A_{d\sigma(k)}$, $B_{1\sigma(k)}$, $C_{\sigma(k)}$, $B_{2\sigma(k)}$, $C_{2\sigma(k)}$, and $D_{\sigma(k)}$ are constants with appropriate dimensions.

The saturation function $\varphi(C_{\sigma(k)}x(k))$ is an unknown nonlinear real-valued function which represents the sensor nonlinearity and satisfies:

$$(\varphi(\omega) - M_1\omega)^T(\varphi(\omega) - M_2\omega) \leq\ 0 \tag{3}$$

where $M_1 \geq 0$ and $M_2 \geq 0$ are diagonal matrices with $M_2 > M_1$.

According to [42], the nonlinear function $\varphi(C_{\sigma(k)}x(k))$ can be decomposed into the following form:

$$\varphi(C_{\sigma(k)}x(k)) = \varphi_n(C_{\sigma(k)}x(k)) + M_1C_{\sigma(k)}x(k) \tag{4}$$

where the nonlinearity $\varphi_n(C_{\sigma(k)}x(k))$ satisfies

$$\varphi_n^T(C_{\sigma(k)}x(k))\Big[\varphi_n(C_{\sigma(k)}x(k)) - MC_{\sigma(k)}x(k)\Big] \leq 0 \tag{5}$$

where $M = M_2 - M_1 > 0$

By considering the decomposition in (4), discrete-time switched singular systems (1) is formulated as:

$$\begin{cases} Ex(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d(k)) + B_{1\sigma(k)}u(k) + B_{2\sigma(k)}w(k) \\ \quad y(k) = \varphi_n(C_{\sigma(k)}x(k)) + M_1C_{\sigma(k)}x(k) \\ \quad z(k) = C_{2\sigma(k)}x(k) + D_{\sigma(k)}w(k) \\ \quad x(k) = \phi(k),\ k \in [-d_M, 0] \end{cases} \tag{6}$$

Consider the following autonomous switched singular systems:

$$Ex(k+1) = A_ix(k) + A_{di}x(k-d(k)) \tag{7}$$

Definition 1 (Ref. [41]).

1. *For a given $i \in \mathfrak{I}$ and a complex number z, the pair (E, A_i) is said to be regular if* $\det(zE - A_i) \neq 0$.
2. *For a given $i \in \mathfrak{I}$, the pair (E, A_i) is said to be causal, if it is regular and $deg\Big(det(zE - A_i)\Big) = rank(E)$.*
3. *System (7) is said to be admissible if it is regular, causal, and stable.*

Definition 2 (Ref. [43]). *Switched system (7) with $w(k) = 0$ is said to be exponentially stable, if the solution $x(k)$ satisfies $\| x(k) \| \leqslant \theta\epsilon^{k-k_0} \| x(k_0) \|_l$, $\forall k > k_0$, for constant $\theta > 0$ and $0 < \epsilon < 1$, where $\| x(k_0) \|_l = \sup\limits_{k-d_M \leq s \leq k_0} \{\| x(s) \|\}$.*

Definition 3 (Ref. [44]). *For switching signal $\sigma(k)$ and any $k_s > k_a > k_0$, let $N_\sigma(k_a, k_s)$ be the switching number over the interval $[k_a, k_s)$. If for a given $N_0 \geq 0$ and $\tau_a \geq 0$, we have $N_\sigma(k_a, k_s) \leq N_0 + (k_s - k_a)/\tau_a$, where τ_a and N_0 are, respectively, called the average dwell time and the chatter bound.*

Lemma 1 (Ref. [6]). *Let $\varsigma(k)$ be a vector valued function. The following inequality*

$$\begin{aligned} -\sum_{s=k-d}^{k-1} \varpi(s)^T E^T V E \varpi(s) \leq & \varsigma^T(k)\begin{bmatrix} T_1^T E + E^T T_1 & -T_1^T E + E^T\ T_2 \\ * & -T_2^T E - E^T\ T_2 \end{bmatrix}\varsigma(k) \\ & + d\varsigma^T(k)\begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix} V^{-1}\begin{bmatrix} T_1 & T_2 \end{bmatrix}\varsigma(k) \end{aligned} \tag{8}$$

holds for any $V > 0$, T_1, and T_2; and an integer $d > 0$, where $\varpi(k) = x(k+1) - x(k)$ and $\varsigma(k) = \begin{bmatrix} x^T(k) & x^T(k-d) \end{bmatrix}^T$.

Lemma 2. *For any matrix $V > 0$, G_1, and G_2; and an integer $d > 0$, the following inequality holds:*

$$\begin{aligned} -\sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \varpi(s)^T E^T V E \varpi(s) \leq & \zeta_1^T(k)\begin{bmatrix} dG_1^T E + dE^T G_1 & -G_1^T + dE^T G_2 \\ * & -G_2^T - G_2 \end{bmatrix}\zeta_1(k) \\ & + \frac{d(d+1)}{2}\zeta_1^T(k)\begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} V^{-1}\begin{bmatrix} G_1 & G_2 \end{bmatrix}\zeta_1(k) \end{aligned} \tag{9}$$

where $\varpi(k) = x(k+1) - x(k)$ *and* $\zeta_1(k) = \begin{bmatrix} x^T(k) & (\sum_{s=k-d}^{k-1} Ex(s))^T \end{bmatrix}^T$.

Proof. Let $\mathbb{G} = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, $\mathbb{V} = \begin{bmatrix} V^{\frac{1}{2}} & V^{-\frac{1}{2}}\mathbb{G} \\ 0 & 0 \end{bmatrix}$; then

$$\mathbb{V}^T\mathbb{V} = \begin{bmatrix} V^{\frac{1}{2}} & V^{-\frac{1}{2}}\mathbb{G} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} V^{\frac{1}{2}} & V^{-\frac{1}{2}}\mathbb{G} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} V & \mathbb{G} \\ \mathbb{G}^T & \mathbb{G}^T V^{-1}\mathbb{G} \end{bmatrix} \geq 0 \tag{10}$$

$$\begin{aligned}
\Phi_v &= \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \begin{bmatrix} E\eta(s) \\ \zeta_1(k) \end{bmatrix}^T \mathbb{V}^T\mathbb{V} \begin{bmatrix} E\eta(s) \\ \zeta_1(k) \end{bmatrix} \\
&= \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \begin{bmatrix} E\eta(s) \\ \zeta_1(k) \end{bmatrix}^T \begin{bmatrix} V & \mathbb{G} \\ \mathbb{G}^T & \mathbb{G}^T V^{-1}\mathbb{G} \end{bmatrix} \begin{bmatrix} E\eta(s) \\ \zeta_1(k) \end{bmatrix} \\
&= \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \eta(s)^T E^T V E\eta(s) + 2\zeta_1(k)\mathbb{G}^T \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} E\eta(s) + \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \zeta_1^T(k)\mathbb{G}^T V^{-1}\mathbb{G}\zeta_1(k) \\
&= \sum_{n=-d}^{-1}\sum_{s=k+n}^{k-1} \eta(s)^T E^T V E\eta(s) + \zeta_1^T(k) \begin{bmatrix} dG_1^T E + dE^T G_1 & -G_1^T + dE^T G_2 \\ * & -G_2^T - G_2 \end{bmatrix} \zeta_1(k) \\
&+ \frac{d(d+1)}{2}\zeta_1^T(k) \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} V^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix} \zeta_1(k)
\end{aligned} \tag{11}$$

From (10), we have $0 \leq \Phi_v$, which verifies inequality (9).
This completes the proof. □

Lemma 3. *For given real matrices Y, L, and V, the following statements are equivalent:*

1.
$$\begin{bmatrix} Y & L \\ L^T & 0 \end{bmatrix} + \text{sym}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} V^T & -I \end{bmatrix} \right\} < 0 \tag{12}$$
is feasible in variable N and M
2. *Y, L, and V satisfy*
$$Y + \text{sym}(LV^T) < 0 \tag{13}$$

Proof. Let

$$\Psi = \begin{bmatrix} Y & L \\ L^T & 0 \end{bmatrix} + \text{sym}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} V^T & -I \end{bmatrix} \right\} = \begin{bmatrix} Y + sym(MV^T) & L - M + VN^T \\ L^T - M^T + NV^T & -sym(N) \end{bmatrix} \tag{14}$$

From (12), we have

$$\Psi = \begin{bmatrix} Y + sym(MV^T) & L - M + VN^T \\ L^T - M^T + NV^T & -sym(N) \end{bmatrix} < 0 \tag{15}$$

Let $\mathcal{V} = \begin{bmatrix} I & V \end{bmatrix}^T$. Pre and post-multiplying inequality (15) by $\mathcal{V}$ and $\mathcal{V}^T$, respectively, inequality (13) holds.
This completes the proof. □

3. Stability Analysis

In this section, a sufficient LMI criterion is developed to ensure the admissibility of system (7).

Theorem 1. *Take tunable scalars $0 < \alpha < 1$ and $\mu > 1$ and positive integers d_m and d_M. For any switching signal $\sigma(k)$ with ADT satisfying $\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln \alpha}$, switched singular systems (6) are exponentially admissible, if there exist symmetric definite positive matrices $P_i > 0$, $Z_{si} > 0$, and $Q_{si} > 0$; and matrices T_{1i}, T_{2i}, G_{1i}, G_{2i}, X_i, Y_i, $\mathbb{S}_i$, F_s, and $s = 1, 2, 3$, such that the following inequalities hold for all $(i, j) \in \mathfrak{I} \times : \mathfrak{I}$*

$$\begin{aligned}
\Upsilon_X(\Upsilon_i, H_T, H_G, H_{zi}) &= \begin{bmatrix} \Upsilon_i & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i H_T & -\alpha^{d_M} Z_{1i} & * & * & * \\ \sqrt{d_r}X_i^T & 0 & -\alpha^{d_M} Z_{2i} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i H_G & 0 & 0 & -\alpha^{d_M} Z_{3i} & * \\ H_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0 \\
\Upsilon_Y(\Upsilon_i, H_T, H_G, H_{zi}) &= \begin{bmatrix} \Upsilon_i & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i H_T & -\alpha^{d_M} Z_{1i} & * & * & * \\ \sqrt{d_r}Y_i^T & 0 & -\alpha^{d_M} Z_{2i} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i H_G & 0 & 0 & -\alpha^{d_M} Z_{3i} & * \\ H_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&P_i - \mu P_j < 0, \quad Q_{0i} - \mu Q_{0j} < 0, \quad Q_{1i} - \mu Q_{1j} < 0, \quad Q_{2i} - \mu Q_{2j} < 0, \\
&Q_{3i} - \mu Q_{3j} < 0, \quad Z_{1i} - \mu Z_{1j} < 0, \quad Z_{2i} - \mu Z_{2j} < 0, \quad Z_{3i} - \mu Z_{3j} < 0
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\Upsilon_i =& \Gamma_i + sym(\Gamma_{1i}) + H_{1i}^T P_i H_{1i} - \alpha H_{2i}^T P_i H_{2i} + H_3^T(d_m Z_{1i} + d_r Z_{2i} + \tilde{d}_M Z_{3i})H_3 \\
&+ sym(H_4 \mathbb{S}_i R^T H_3) + sym(\mathbb{F}\mathbb{A}_i) + H_T^T \Pi_{Ti} H_T + H_G^T \Pi_{Gi} H_G \\
\Gamma_i =& diag\big(Q_{1i} + (d_r+1)Q_{3i};\ \alpha^{d_m}(-Q_{1i} + Q_{2i});\ -\alpha^{d_M} Q_{3i};\ -\alpha^{d_M} Q_{2i}\ ; 0;\ 0;\ -\gamma I\big), \\
\Gamma_{1i} =& \begin{bmatrix} 0 & 0 & Y_i E & X_i E - Y_i E & -X_i E & 0 & 0 & 0 \end{bmatrix}, \\
H_{1i} =& \begin{bmatrix} E & 0_{n\times 3n} & I & 0 & 0 \end{bmatrix}, H_{2i} = \begin{bmatrix} E & 0_{n\times 6n} \end{bmatrix}, H_3 = \begin{bmatrix} 0 & 0_{n\times 3n} & I & 0 & 0 \end{bmatrix}, \\
H_T =& \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, H_G = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}, H_4^T = \begin{bmatrix} I & 0_{n\times 6n} \end{bmatrix}, \\
\Pi_{Ti} =& \begin{bmatrix} T_{1i}^T E + E^T T_{1i} & -T_{1i}^T E + E^T T_{2i} \\ * & -T_{2i}^T E - E^T T_{2i} \end{bmatrix}, \mathbb{T}_i = \begin{bmatrix} T_{1i} & T_{2i} \end{bmatrix}, \mathbb{G}_i = \begin{bmatrix} G_{1i} & G_{2i} \end{bmatrix}, \\
\Pi_{Gi} =& \begin{bmatrix} d_M G_{1i}^T E + d_M E^T G_{1i} & -G_{1i}^T + d_M E^T G_{2i} \\ * & -G_{2i}^T - G_{2i} \end{bmatrix}, \\
\mathbb{F}_w^T =& \begin{bmatrix} F_1 & 0 & F_2 & 0 & F_3 & 0 & 0 \end{bmatrix}, \mathbb{A}_i = \begin{bmatrix} A_i - E & 0 & A_{di} & 0 & -I & 0 & B_{2i} \end{bmatrix}, \\
H_{zi} =& \begin{bmatrix} C_{2i} & 0 & 0 & 0 & 0 & 0 & D_i \end{bmatrix},\ d_r = d_M - d_m,\ \tilde{d}_M = \frac{d_M(d_M+1)}{2}
\end{aligned} \tag{18}$$

R is any matrix with full column rank satisfying $R^T E = 0$.

Proof. The first part of this proof treats the regularity and the causality of the pair (E, A_i). For $k \in [k_r, k_{r+1})$, along the processing, switched rule $\sigma(k)$ is fixed to $i \in \mathfrak{I}$.

Since $rank(E) = r \le n$, there exist two nonsingular matrices $\mathbb{N}$ and $\mathbb{L} \in \mathbb{R}^{n\times n}$ such that

$$\tilde{E} = \mathbb{N}E\mathbb{L} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

and R can be described as $R = \mathbb{N}^T \begin{bmatrix} 0 \\ \Theta \end{bmatrix}$, where $\Theta \in \mathbb{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix.

Define

$$\tilde{A}_i = \mathbb{N}A_i\mathbb{L} = \begin{bmatrix} \tilde{A}^i_{11} & \tilde{A}^i_{12} \\ \tilde{A}^i_{21} & \tilde{A}^i_{22} \end{bmatrix}, \tilde{S}_i = \mathbb{L}^T\mathbb{S}_i = \begin{bmatrix} \tilde{S}^i_{11} \\ \tilde{S}^i_{21} \end{bmatrix}, \tilde{A}_{di} = \mathbb{N}A_{di}\mathbb{L} = \begin{bmatrix} \tilde{A}^i_{d11} & \tilde{A}^i_{d12} \\ \tilde{A}^i_{d21} & \tilde{A}^i_{d22} \end{bmatrix}, \tag{20}$$

From (16), we can easily verify that

$$\begin{aligned} &\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ * & \Lambda_{22} \end{bmatrix} < 0, \\ &\Lambda_{11} = (1-\alpha)E^T P_i E + sym(F_1^T(A_i - E) + E^T T_{1i} + d_M E^T G_{1i}) \\ &\Lambda_{12} = E^T P_i + S_i R^T + (A_i - E)^T F_3 - F_1^T \\ &\Lambda_{22} = P_i - sym(F_3) \end{aligned} \tag{21}$$

Let $\mathcal{A} = \begin{bmatrix} I & A_i^T \end{bmatrix}^T$. Pre and post-multiplying inequality (21) by $\mathcal{A}$ and $\mathcal{A}^T$, respectively, yields

$$(1-\alpha)E^T P_i E + A_i^T P_i A_i + sym(E^T(P_i - F_3)A_i + E^T(-F_1 + T_{1i} + d_M G_{1i}) + \mathbb{S}_i R^T A_i) < 0 \tag{22}$$

Checking a congruent transformation to (22) by $\mathbb{L}$, and using (19)–(20), we get

$$sym(\tilde{S}^i_{21}\Theta^T\tilde{A}^i_{22}) < 0 \tag{23}$$

Thus, $\tilde{A}^i_{22}$ is nonsingular. If we suppose that the matrix $\tilde{A}^i_{22}$ is singular, then there exists a non-zero vector ϑ_i ensuring $\tilde{A}^i_{22}\vartheta_i = 0$. Consequently, we can deduce that $\vartheta_i^T sym(\tilde{S}^i_{21}\Theta^T\tilde{A}^i_{22})\vartheta_i = 0$, which contradicts (23). Then, pair (E, A_i) is regular and causal.

Next, the exponential stability of systems (7) is demonstrated. We use the following switched Lyapunov–Krasovskii functional candidate:

$$\begin{aligned} V_i(k) &= \sum_{s=1}^{7} V_{is}(k) \\ V_{i1}(k) &= x^T(k)E^T P_i E x(k) \\ V_{i2}(k) &= \sum_{s=k-d_m}^{k-1} x^T(s)\alpha^{k-1-s}Q_{1i}x(s) \\ V_{i3}(k) &= \sum_{s=k-d_M}^{k-1-d_m} x^T(s)\alpha^{k-1-s}Q_{2i}x(s) \\ V_{i4}(k) &= \sum_{s=k-d(k)}^{k-1} x^T(s)\alpha^{k-1-s}Q_{3i}x(s) \\ V_{i5}(k) &= \sum_{n=-d_M+1}^{-d_m}\sum_{s=k+n}^{k-1} x^T(s)\alpha^{k-1-s}Q_{3i}x(s) \\ V_{i6}(k) &= \sum_{n=-d_M}^{-1}\sum_{s=k+n}^{k-1} \eta^T(s)E^T\alpha^{k-1-s}Z_{1i}E\eta(s) + \sum_{n=-d_M}^{-d_m-1}\sum_{s=k+n}^{k-1} \eta^T(s)E^T\alpha^{k-1-s}Z_{2i}E\eta(s) \end{aligned} \tag{24}$$

$$V_{i7}(k) = \sum_{s=-d_M}^{-1}\sum_{n=s}^{-1}\sum_{r=k+n}^{k-1} \eta^T(r)E^T\alpha^{k-1-r}Z_{3i}E\eta(r) \tag{25}$$

Define $\eta(k) = x(k+1) - x(k)$, and

$$\zeta(k) = col\left\{x(k),\ x(k-d_m),\ x(k-d(k)),\ x(k-d_M),\ E\eta(k),\ (\sum_{s=k-d_M}^{k-1} Ex(s))\right\} \tag{26}$$

Taking the forward difference of $V_i(k)$ in the solution of system (7) as $\Delta_\alpha V(k) = V_i(x(k+1)) - \alpha V_i(x(k))$, we get

$$\begin{aligned} \Delta_\alpha V_{i1}(k) &= x^T(k+1)E^T P_i E x(k+1) - \alpha x^T(k) E^T P_i E x(k) \\ &= \zeta^T(k) H_{1i}^T P_i H_{1i} \zeta(k) - \alpha \zeta^T(k) H_{2i}^T P_i H_{2i} \zeta(k) \end{aligned} \tag{27}$$

$$\begin{aligned} \Delta_\alpha V_{i2}(k) =& \sum_{s=k+1-d_m}^{k} x^T(s)\alpha^{k-s} Q_{1i} x(s) - \alpha \sum_{s=k-d_m}^{k-1} x^T(s) \alpha^{k-1-s} Q_{1i} x(s) \\ =& x^T(k) Q_{1i} x(k) - x^T(k-d_m)\alpha^{k-k+d_m} Q_{1i} x(k-d_m) + \\ & \sum_{s=k+1-d_m}^{k-1} x^T(s)\alpha^{k-s} Q_{1i} x(s) - \sum_{s=k+1-d_m}^{k-1} x^T(s)\alpha^{k-s} Q_{1i} x(s) \\ =& x^T(k) Q_{1i} x(k) - x^T(k-d_m)\alpha^{d_m} Q_{1i} x(k-d_m) \end{aligned}$$

$$\begin{aligned} \Delta_\alpha V_{i3}(k) =& x^T(k-d_m)\alpha^{d_m} Q_{2i} x(k-d_m) - x^T(k-d_M)\alpha^{d_M} Q_{2i} x(k-d_M) \\ \Delta_\alpha V_{i4}(k) =& x^T(k) Q_{3i} x(k) - x^T(k-d(k))\alpha^{d(k)} Q_{3i} x(k-d(k)) \\ & + \sum_{s=k+1-d(k+1)}^{k-1} x^T(s)\alpha^{k-s} Q_{3i} x(s) - \sum_{s=k+1-d(k)}^{k-1} x^T(s)\alpha^{k-s} Q_{3i} x(s) \\ \leq& x^T(k) Q_{3i} x(k) - x^T(k-d(k))\alpha^{d_M} Q_{3i} x(k-d(k)) + \sum_{s=k+1-d_M}^{k-d_m} x^T(s)\alpha^{k-s} Q_{3i} x(s) \\ \Delta_\alpha V_{i5}(k) =& d_r x^T(k) Q_{3i} x(k) - \sum_{s=k+1-d_M}^{k-d_m} x^T(s)\alpha^{k-s} Q_{3i} x(s) \end{aligned}$$

$$\begin{aligned} \Delta_\alpha V_{i6}(k) =& \sum_{n=-d_M}^{-1} \sum_{s=k+1+n}^{k} \eta^T(s) E^T \alpha^{k-s} Z_{1i} E \eta(s) + \sum_{n=-d_M}^{-d_m-1} \sum_{s=k+1+n}^{k} \eta^T(s) E^T \alpha^{k-s} Z_{2i} E \eta(s) \\ & - \sum_{n=-d_M}^{-1} \sum_{s=k+n}^{k-1} \eta^T(s) E^T \alpha^{k-s} Z_{1i} E \eta(s) + \sum_{n=-d_M}^{-d_m-1} \sum_{s=k+n}^{k-1} \eta^T(s) E^T \alpha^{k-s} Z_{2i} E \eta(s) \end{aligned} \tag{28}$$

which implies

$$\begin{aligned} \Delta_\alpha V_{i6}(k) =& \eta^T(k) E^T (d_M Z_{1i} + d_r Z_{2i}) E \eta(k) + \sum_{n=-d_M}^{-1} \eta^T(k+n) E^T (-\alpha^{-n}) Z_{1i} E \eta(k+n) \\ & - \sum_{n=-d_M}^{-d_m-1} \eta^T(k+n) E^T (-\alpha^{-n}) Z_{2i} E \eta(k+n) \end{aligned} \tag{29}$$

Since $-d_M \leq n \leq -1$, $-d_M \leq n \leq -d_m - 1$ and $0 < \alpha < 1$, we obtain, respectively,

$$\begin{aligned} \alpha^{d_M} \leq \alpha^{-n} \leq \alpha \quad &\Longrightarrow \quad -\alpha \leq -\alpha^{-n} \leq -\alpha^{d_M} \\ \alpha^{d_M} \leq \alpha^{-n} \leq \alpha^{d_m+1} \quad &\Longrightarrow \quad -\alpha^{d_m+1} \leq -\alpha^{-n} \leq -\alpha^{d_M} \end{aligned} \tag{30}$$

From (30), we get

$$\begin{aligned}\Delta_\alpha V_{i6}(k) \leq &\eta^T(k)E^T(d_M Z_{1i} + d_r Z_{2i})E\eta(k) + \sum_{n=-d_M}^{-1} \eta^T(k+n)E^T(-\alpha^{d_M})Z_{1i}E\eta(k+n) \\ &+ \sum_{n=-d_M}^{-d_m-1} \eta^T(k+n)E^T(-\alpha^{d_M})Z_{2i}E\eta(k+n) \\ =&\eta^T(k)E^T(d_M Z_{1i} + d_r Z_{2i})E\eta(k) - \sum_{s=k-d_M}^{k-1} \eta^T(s)E^T\alpha^{d_M}Z_{1i}E\eta(s) \\ &- \sum_{s=k-d(k)}^{k-d_m-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s) - \sum_{s=k-d_M}^{k-d(k)-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s)\end{aligned} \tag{31}$$

$$\begin{aligned}\Delta_\alpha V_{i7}(k) =& \sum_{s=-d_M}^{-1}\sum_{n=s}^{-1}\sum_{r=k+1+n}^{k} \alpha^{k-r}\eta^T(r)E^T Z_{3i}E\eta(r) - \sum_{s=-d_M}^{-1}\sum_{n=s}^{-1}\sum_{r=k+n}^{k-1} \alpha^{k-r}\eta^T(r)E^T Z_{3i}E\eta(r) \\ \leq& \frac{d_M(d_M+1)}{2}\eta^T(k)E^T Z_{3i}E\eta(k) - \sum_{s=-d_M}^{-1}\sum_{n=k+s}^{k-1} \eta^T(n)E^T\alpha^{d_M}Z_{3i}E\eta(n)\end{aligned} \tag{32}$$

Using Lemma 1 and defining $\zeta_0 = \begin{bmatrix} x^T(k) & x^T(k-d_M) \end{bmatrix}^T$, we can estimate

$$\begin{aligned}\Delta_\alpha V_{i6}(k) \leq &\eta^T(k)E^T(d_M Z_{1i} + d_r Z_{2i})E\eta(k) + \zeta_0^T(k)\begin{bmatrix} T_{1i}^T E + E^T T_{1i} & -T_{1i}^T E + E^T T_{2i} \\ * & -T_{2i}^T E - E^T T_{2i} \end{bmatrix}\zeta_0(k) \\ &+ d_M\zeta_0^T(k)\begin{bmatrix} T_{1i}^T \\ T_{2i}^T \end{bmatrix}(\alpha^{d_M}Z_{1i})^{-1}\begin{bmatrix} T_{1i} & T_{2i} \end{bmatrix}\zeta_0(k) \\ &- \sum_{s=k-d(k)}^{k-d_m-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s) - \sum_{s=k-d_M}^{k-d(k)-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s)\end{aligned} \tag{33}$$

For any nonsingular matrices X_i, we introduce

$$\sum_{s=k-d_M}^{k-d(k)-1} \begin{bmatrix} \zeta(k) \\ E\eta(s) \end{bmatrix}^T \begin{bmatrix} X_i\alpha^{-d_M}Z_{2i}{}^{-1}X_i^T & X_i \\ X_i^T & \alpha^{d_M}Z_{2i} \end{bmatrix} \begin{bmatrix} \zeta(k) \\ E\eta(s) \end{bmatrix} \geq 0 \tag{34}$$

Then, we can write

$$\begin{aligned}-\sum_{s=k-d_M}^{k-d(k)-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s) \leq &(d_M - d(k))\zeta^T(k)X_i\alpha^{-d_M}Z_{2i}{}^{-1}X_i^T\zeta(k) \\ &+ 2\zeta^T(k)X_iE[x(k-d(k)) - x(k-d_M)]\end{aligned} \tag{35}$$

For any nonsingular matrices Y_i, we obtain

$$\begin{aligned}-\sum_{s=k-d(k)}^{k-d_m-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s) \leq &(d(k) - d_m)\zeta^T(k)Y_i\alpha^{-d_M}Z_{2i}{}^{-1}Y_i^T\zeta(k) \\ &+ 2\zeta^T(k)Y_iE[x(k-d_m) - x(k-d(k))]\end{aligned} \tag{36}$$

From (35) and (36) and allowing $\hat{d}(k) = \frac{d_M - d(k)}{d_r}$, we get

$$\begin{aligned}&-\sum_{s=k-d_M}^{k-d_m-1} \eta^T(s)E^T\alpha^{d_M}Z_{2i}E\eta(s) \leq \zeta^T(k)\Big(d_r\hat{d}(k)X_i\alpha^{-d_M}Z_{2i}{}^{-1}X_i^T \\ &+ d_r(1-\hat{d}(k))Y_i\alpha^{-d_M}Z_{2i}{}^{-1}Y_i^T + 2\begin{bmatrix} 0 & Y_iE & X_iE - Y_iE & -X_iE & 0 & 0 \end{bmatrix}\Big)\zeta(k)\end{aligned} \tag{37}$$

Based on Lemma 2 and defining $\zeta_1(k) = \begin{bmatrix} x^T(k) & (\sum_{s=k-d_M}^{k-1} Ex(s))^T \end{bmatrix}^T$, the following inequality holds:

$$\begin{aligned} \Delta_\alpha V_{i7}(k) \leq & \frac{d_M(d_M+1)}{2}\eta^T(k)E^T Z_{3i} E\eta(k) \\ & + \zeta_1^T(k)\begin{bmatrix} d_M G_{1i}^T E + d_M E^T G_{1i} & -G_{1i}^T + d_M E^T G_{2i} \\ * & -G_{2i}^T - G_{2i} \end{bmatrix}\zeta_1(k) \\ & + \frac{d_M(d_M+1)}{2}\zeta_1^T(k)\begin{bmatrix} G_{1i}^T \\ G_{2i}^T \end{bmatrix}(\alpha^{d_M} Z_{3i})^{-1}\begin{bmatrix} G_{1i} & G_{2i} \end{bmatrix}\zeta_1(k) \end{aligned} \tag{38}$$

Moreover, for any free-weighting matrices F_s, $s = 1$, 2, and 3 satisfying $\mathbb{F} = \begin{bmatrix} F_1 & 0_{n\times(m-1)n} & 0_n & F_2 & 0_n & F_3 & 0_n \end{bmatrix}^T$, we have

$$2\,\zeta^T(k)\mathbb{F} \times \left[(A_i - E)x(k) + A_{di}x(k-d(k)) - E\eta(k)\right] = 0 \tag{39}$$

From (26) and (18), we can verify that

$$R^T H_3 \zeta(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & \eta^T(k)E^T R & 0 \end{bmatrix}^T \tag{40}$$

Since $R^T E = 0$, it is clear that

$$2\,\zeta^T(k)H_4 \mathbb{S}_i R^T H_3 \zeta(k) = 0 \tag{41}$$

Thereby, from (27) to (41), we obtain

$$\Delta_\alpha V(k) \leq \zeta^T(k)\left(\hat{d}(k)\Xi_{1i} + (1-\hat{d}(k))\Xi_{2i}\right)\zeta(k) \tag{42}$$

where

$$\begin{aligned} \Xi_{1i} =& \Upsilon_i + d_M(\mathbb{T}_i H_T)^T(\alpha^{d_M} Z_{1i})^{-1}(\mathbb{T}_i H_T) + \tilde{d}_M(\mathbb{G}_i H_G)^T(\alpha^{d_M} Z_{3i})^{-1}(\mathbb{G}_i H_G) \\ & + d_r X_i^T(\alpha^{d_M} Z_{2i})^{-1} X_i \\ \Xi_{2i} =& \Upsilon_i + d_M(\mathbb{T}_i H_T)^T(\alpha^{d_M} Z_{1i})^{-1}(\mathbb{T}_i H_T) + \tilde{d}_M(\mathbb{G}_i H_G)^T(\alpha^{d_M} Z_{3i})^{-1}(\mathbb{G}_i H_G) \\ & + d_r Y_i^T(\alpha^{d_M} Z_{2i})^{-1} Y_i \end{aligned} \tag{43}$$

We have $0 \leq \hat{d}(k) \leq 1$, which means that $(\hat{d}(k)\Xi_{1i} + (1-\hat{d}(k))\Xi_{2i})$ is a convex combination of Ξ_{1i} and Ξ_{2i}. If the inequalities in (16) are justified, then by checking the Schur complement, $(\hat{d}(k)\Xi_{1i} + (1-\hat{d}(k))\Xi_{2i}) < 0$ is proved. Thus, we get $\Delta_\alpha V(k) < 0$.

Considering (17), $\forall\, k \in [k_r, k_{r+1})$; we have

$$\begin{aligned} V_{\sigma(k)}(k) &\leq \alpha^{k-k_r} V_{\sigma(k_r)}(k_r), \\ &\leq \alpha^{k-k_r}\mu V_{\sigma(k_r-1)}(k_r) \\ &\leq \ldots \leq \alpha^{k-k_0}\mu^{(k-k_0)/\tau_a} V_{\sigma(k_0)}(k_0) \end{aligned} \tag{44}$$

Then, it becomes

$$V_{\sigma(k)}(k) \leq (\alpha\mu^{1/\tau_a})^{k-k_0} V_{\sigma(k_0)}(k_0) \tag{45}$$

Besides, there exist two positive scalars ρ_1 and ρ_2 such that

$$\rho_1 \parallel x(k) \parallel^2 \leq V_{\sigma(k)}(k); \quad V_{\sigma(k_0)}(k_0) \leq \rho_2 \parallel x(k) \parallel_l^2 \tag{46}$$

Let $\epsilon = \sqrt{\alpha\mu^{1/\tau_a}}$. Using the above inequality in (45), we get

$$\begin{aligned} \rho_1 \parallel x(k) \parallel^2 \leq V_{\sigma(k)}(k) \leq \rho_2 \epsilon^{2(k-k_0)} \parallel x(k) \parallel_l^2 \\ \parallel x(k) \parallel^2 \leq \frac{\rho_2}{\rho_1} \epsilon^{2(k-k_0)} \parallel x(k) \parallel_l^2 \end{aligned} \tag{47}$$

which means

$$\parallel x(k) \parallel \leq \sqrt{\frac{\rho_2}{\rho_1}} \epsilon^{(k-k_0)} \parallel x(k) \parallel_l \tag{48}$$

By considering the definition of τ_a, we obtain $\epsilon < 1$. Then, from Definition 2, systems (7) is exponentially stable.

To develop the H_∞ performance for systems (6) with $u(k) = 0$, we propose the following performance index:

$$J_{wz} = \sum_{k=0}^{T} \left(z^T(k)z(k) - \gamma^2 w^T(k)w(k) \right) \tag{49}$$

Define $\xi(k) = \left[\zeta^T(k)\ w^T(k)\right]^T$. The following equation holds:

$$2\, \xi^T(k)\mathbb{F}_w \times \left[(A_i - E)x(k) + A_{di}x(k-d(k)) + B_{2i}w(k) - E\eta(k) \right] = 0 \tag{50}$$

where $\mathbb{F}_w = \left[\mathbb{F}^T\ \ 0\right]^T$.

According to (16) and the Schur complement, we can obtain

$$\Delta_\alpha V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0 \tag{51}$$

Summing (51) over the range $[0,\ T]$ with initial condition $V(0) = 0$ yields

$$\sum_{k=0}^{T} \Delta_\alpha V(k) + J_{wz} \leq V(T+1) + J_{wz} < 0 \tag{52}$$

Letting $T \longrightarrow \infty$, we get

$$\sum_{k=0}^{\infty} \left(z^T(k)z(k) - \gamma^2 w^T(k)w(k) \right) < 0 \tag{53}$$

□

4. Asynchronous Controller Design

This section is reserved to studying the control design problem for switched singular systems under asynchronous switching. Based on the previous results, we developed an LMI method for designing the observer-based controller, which guarantees for the exponential admissibility of the closed-loop system.

Since it was assumed that all of the system states are not available, an observer was designed for estimating the unmeasured states as follows:

$$\begin{cases} E\hat{x}(k+1) = A_{\sigma(k)}\hat{x}(k) + A_{d\sigma(k)}\hat{x}(k-d(k)) + B_{1\sigma(k)}u(k) + L_{\sigma(k)}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C_{\sigma(k)}\hat{x}(k) \\ \hat{x}(k) = \phi_{ob}(k),\ k \in [-d_M, 0] \end{cases} \tag{54}$$

where $L_{\sigma(k)}$ are the observer gain matrices to be determined, $\hat{y}(k)$ is the observer output, and $\hat{x}(k)$ is the state estimation of $x(k)$.

It is known that, in many practical processes, a delay, called as a lag time, can occur between system modes and controller. Therefore, a mismatch between controller and switching subsystems instances, called asynchronous switching, is present. Thus, the following controller was considered:

$$u(k) = K_{\bar{\sigma}(k)}\hat{x}(k), \forall k \in [k_r, k_{r+1}) \tag{55}$$

where $\Delta_r < k_{r+1} - k_r$ is the delayed period and $\bar{\sigma}(k) = \sigma(k - \Delta_r)$ represents the switching signal of the controller with $\Delta_0 = 0$.

Let the ith subsystem be switched on at the instant k_r, and the jth subsystem be switched on at the instant k_{r+1}. In this case, corresponding controllers are switched on at the instants $k_r + \Delta_r$ and $k_{r+1} + \Delta_r$, respectively (see Figure 1).

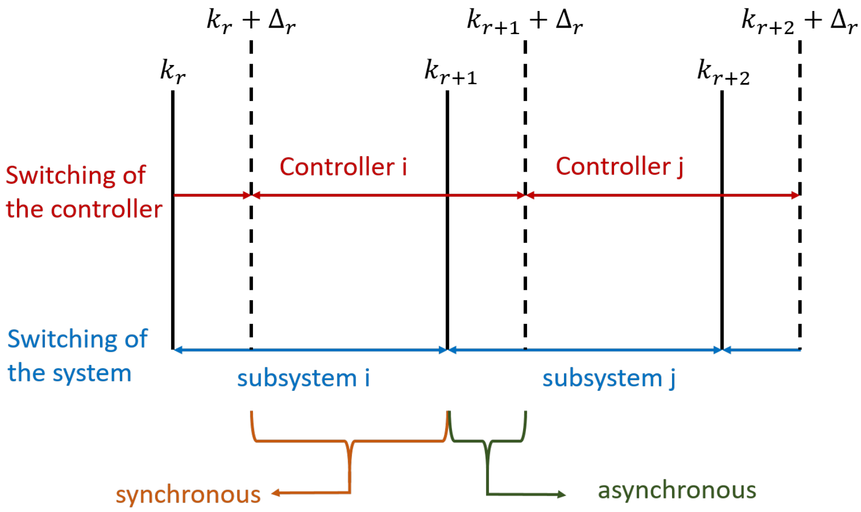

Figure 1. Diagram of asynchronous switching.

Combining (6) and (54) with (55), the augmented closed-loop system is written as

$$\begin{cases} \tilde{E}\tilde{x}(k+1) = \tilde{A}_i\tilde{x}(k) + \tilde{A}_{di}\tilde{x}(k-d(k)) + \tilde{B}_{2i}w(k) + G_{\varphi i}\varphi_n(C_i x(k)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in [k_r + \Delta_r, k_{r+1}) \\ \tilde{E}\tilde{x}(k+1) = \tilde{A}_{ij}\tilde{x}(k) + \tilde{A}_{di}\tilde{x}(k-d(k)) + \tilde{B}_{2i}w(k) + G_{\varphi i}\varphi_n(C_i x(k)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in [k_r, k_r + \Delta_r) \\ \quad z(k) = \tilde{C}_{2i}\tilde{x}(k) + \tilde{D}_i w(k) \end{cases} \tag{56}$$

where $e(k) = x(k) - \hat{x}(k)$, $\tilde{x}(k) = \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$, $\tilde{A}_i = A_{K_i} + A_{Li}, \tilde{A}_{ij} = A_{Kij} + A_{Li}$,

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \tilde{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ 0 & A_{di} \end{bmatrix}, G_{\varphi i} = \begin{bmatrix} 0 \\ -L_i \end{bmatrix}, \tilde{B}_{2i} = \begin{bmatrix} B_{2i} \\ B_{2i} \end{bmatrix}, \tilde{C}_{2i} = \begin{bmatrix} C_{2i} & 0 \end{bmatrix},$$

$$\tilde{D}_i = D_i, A_{Ki} = \begin{bmatrix} A_i + B_{1i}K_i & -B_{1i}K_i \\ 0 & A_i \end{bmatrix}, A_{Li} = \begin{bmatrix} 0 & 0 \\ L_iC_i - L_iM_1C_i & -L_iC_i \end{bmatrix},$$

$$A_{Kij} = \begin{bmatrix} A_i + B_{1i}K_j & -B_{1i}K_j \\ 0 & A_i \end{bmatrix}.$$

The corresponding controller design method is introduced by the following theorem.

Theorem 2. *Take tunable scalars* $0 < \alpha < 1$, $\beta \geq 1$, $\mu_1 > 1$, *and* $\mu_2 > 1$ *and positive integers* d_m *and* d_M. *Switched singular system* (56) *is exponentially admissible, if there exist symmetric definite positive matrices* $\tilde{P}_i$, $\tilde{P}_{ij}$, $\tilde{Z}_{si}$, $\tilde{Z}_{sij}$, $\tilde{Q}_{si}$, *and* $\tilde{Q}_{sij}$; *matrices* $\tilde{T}_{1i}$, $\tilde{T}_{2i}$, $\tilde{G}_{1i}$, $\tilde{G}_{2i}$, $\tilde{X}_i$, $\tilde{X}_{ij}$, $\tilde{Y}_i$, $\tilde{Y}_{ij}$, $\mathbb{S}_i$, $\tilde{F}_s$,

$s = 1, 2, 3$, K_i, and L_i; and positive scalar ϵ_{2i} such that for all $(i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, the following inequalities hold.

$$
\begin{aligned}
\Sigma_{Xi}(\bar{\mathbb{F}}\bar{\mathbb{A}}_i, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) &= \begin{bmatrix} \bar{\Upsilon}_i & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i\bar{H}_T & -\alpha^{d_M}\tilde{Z}_{1i} & * & * & * \\ \sqrt{d_r}\tilde{X}_i^T & 0 & -\alpha^{d_M}\tilde{Z}_{2i} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i\bar{H}_G & 0 & 0 & -\alpha^{d_M}\tilde{Z}_{3i} & * \\ \bar{H}_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0 \\
\Sigma_{Yi}(\bar{\mathbb{F}}\bar{\mathbb{A}}_i, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) &= \begin{bmatrix} \bar{\Upsilon}_i & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i\bar{H}_T & -\alpha^{d_M}\tilde{Z}_{1i} & * & * & * \\ \sqrt{d_r}\tilde{Y}_i^T & 0 & -\alpha^{d_M}\tilde{Z}_{2i} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i\bar{H}_G & 0 & 0 & -\alpha^{d_M}\tilde{Z}_{3i} & * \\ \bar{H}_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
\Sigma_{Xij}(\bar{\mathbb{F}}\bar{\mathbb{A}}_{ij}, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) &= \begin{bmatrix} \bar{\Upsilon}_{ij} & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i\bar{H}_T & -\alpha^{d_M}\tilde{Z}_{1ij} & * & * & * \\ \sqrt{d_r}\tilde{X}_{ij}^T & 0 & -\alpha^{d_M}\tilde{Z}_{2ij} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i\bar{H}_G & 0 & 0 & -\alpha^{d_M}\tilde{Z}_{3ij} & * \\ \bar{H}_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0 \\
\Sigma_{Yij}(\bar{\mathbb{F}}\bar{\mathbb{A}}_{ij}, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) &= \begin{bmatrix} \bar{\Upsilon}_{ij} & * & * & * & * \\ \sqrt{d_M}\mathbb{T}_i\bar{H}_T & -\alpha^{d_M}\tilde{Z}_{1ij} & * & * & * \\ \sqrt{d_r}\tilde{Y}_{ij}^T & 0 & -\alpha^{d_M}\tilde{Z}_{2ij} & * & * \\ \sqrt{\tilde{d}_M}\mathbb{G}_i\bar{H}_G & 0 & 0 & -\alpha^{d_M}\tilde{Z}_{3ij} & * \\ \bar{H}_{zi} & 0 & 0 & 0 & -I \end{bmatrix} < 0
\end{aligned} \tag{58}
$$

The switching rule is characterized by the following ADT condition.

$$
\tau_a > \tau_a^* = \frac{\ln(\mu_1\mu_2) + \Delta_m ln(\frac{\beta}{\alpha})}{-\ln\alpha} \tag{59}
$$

where Δ_m denotes the maximum delay period in which the switching of the controller of the ith lags behind that of the subsystem, $\mu_0 = (\frac{\alpha}{\beta})^{d_M-2}$, and $\mu_1\mu_2 \geq 1$ satisfies

$$
\begin{aligned}
&\tilde{P}_i - \mu_1\tilde{P}_{ij} < 0,\ \tilde{Q}_{0i} - \mu_1\tilde{Q}_{0ij} < 0,\ \tilde{Q}_{1i} - \mu_1\tilde{Q}_{1ij} < 0,\ \tilde{Q}_{2i} - \mu_1\tilde{Q}_{2ij} < 0, \\
&\tilde{Q}_{3i} - \mu_1\tilde{Q}_{3ij} < 0,\ \tilde{Z}_{1i} - \mu_1\tilde{Z}_{1ij} < 0,\ \tilde{Z}_{2i} - \mu_1\tilde{Z}_{2ij} < 0,\ \tilde{Z}_{3i} - \mu_1\tilde{Z}_{3ij} < 0, \\
&\beta\tilde{Q}_{0ij} - \mu_2\mu_0\tilde{Q}_{0j} < 0,\ \beta\tilde{Q}_{1ij} - \mu_2\mu_0\tilde{Q}_{1j} < 0,\ \beta\tilde{Q}_{2ij} - \mu_2\mu_0\tilde{Q}_{2j} < 0, \\
&\beta\tilde{Q}_{3ij} - \mu_2\mu_0\tilde{Q}_{3j} < 0,\ \beta\tilde{Z}_{1ij} - \mu_2\mu_0\tilde{Z}_{1j} < 0,\ \beta\tilde{Z}_{2ij} - \mu_2\mu_0\tilde{Z}_{2j} < 0, \\
&\beta\tilde{Z}_{3ij} - \mu_2\mu_0\tilde{Z}_{3j} < 0
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
\bar{\Upsilon}_{ij} =& \bar{\Gamma}_{ij} + sym(\bar{\Gamma}_{1ij}) + \bar{H}_{1i}^T\tilde{P}_{ij}\bar{H}_{1i} - \alpha\bar{H}_{2i}^T\tilde{P}_{ij}\bar{H}_{2i} + \bar{H}_3^T(d_M\tilde{Z}_{1ij} + d_r\tilde{Z}_{2ij} + \tilde{d}_M\tilde{Z}_{3ij})\bar{H}_3 \\
&+ sym(\bar{H}_4\mathbb{S}_iR^T\bar{H}_3) + sym(\bar{\mathbb{F}}\bar{\mathbb{A}}_{ij}) + sym(\bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) + \bar{H}_T^T\Pi_T\bar{H}_T + \bar{H}_G^T\Pi_G\bar{H}_G \\
&- \epsilon_{2i}sym(H_\varphi^T\bar{H}_\varphi) \\
\bar{\Gamma}_{ij} =& diag\big(\tilde{Q}_{1ij} + (d_r+1)\tilde{Q}_{3ij};\ \alpha^{d_m}(-\tilde{Q}_{1ij} + \tilde{Q}_{2ij});\ -\alpha^{d_M}\tilde{Q}_{3ij};\ -\alpha^{d_M}\tilde{Q}_{2ij}\ ;\ 0;\ 0; -\gamma I; 0\big),
\end{aligned}
$$

$$
\begin{aligned}
\bar{H}_{1i} =& \begin{bmatrix} \tilde{E} & 0_{2n\times 6n} & I & 0 & 0 & 0 \end{bmatrix}, \bar{H}_{2i} = \begin{bmatrix} \tilde{E} & 0_{2n\times 10n} & 0 & 0 \end{bmatrix},\\
\bar{H}_{3} =& \begin{bmatrix} 0 & 0_{2n\times 6n} & I & 0 & 0 & 0 \end{bmatrix}, \bar{H}_{4}^{T} = \begin{bmatrix} I & 0_{2n\times 10n} & 0 & 0 \end{bmatrix},\\
\bar{H}_{T} =& \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{H}_{G} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix},\\
H_{\varphi} =& \begin{bmatrix} 0 & 0_{2n\times 12n} & I \end{bmatrix}, \bar{H}_{\varphi} = \begin{bmatrix} -MC_i I_c & 0_{2n\times 12n} & I \end{bmatrix},\\
\Pi_T =& \begin{bmatrix} \tilde{T}_{1i}^T\tilde{E} + \tilde{E}^T\tilde{T}_{1i} & -\tilde{T}_{1i}^T\tilde{E} + \tilde{E}^T\tilde{T}_{2i} \\ * & -\tilde{T}_{2i}^T\tilde{E} - \tilde{E}^T\tilde{T}_{2i} \end{bmatrix},\\
\Pi_G =& \begin{bmatrix} d_M\tilde{G}_{1i}^T\tilde{E} + d_M\tilde{E}^T\tilde{G}_{1i} & -\tilde{G}_{1i}^T + d_M\tilde{E}^T\tilde{G}_{2i} \\ * & -\tilde{G}_{2i}^T - \tilde{G}_{2i} \end{bmatrix},\\
\bar{\Gamma}_{1ij} =& \begin{bmatrix} 0 & \tilde{Y}_{ij}\tilde{E} & \tilde{X}_{ij}\tilde{E} - \tilde{Y}_{ij}\tilde{E} & -\tilde{X}_{ij}\tilde{E} & 0 & 0 & 0 & 0 \end{bmatrix},\\
\bar{\mathbb{F}}^T =& \begin{bmatrix} \tilde{F}_1 & 0 & \tilde{F}_2 & 0 & \tilde{F}_3 & 0 & 0 & 0 \end{bmatrix},\\
\bar{\mathbb{A}}_i =& \begin{bmatrix} A_{Ki} - \tilde{E} & 0 & \tilde{A}_{di} & 0 & -I & 0 & \tilde{B}_{2i} & 0 \end{bmatrix},\\
\bar{\mathbb{A}}_{ij} =& \begin{bmatrix} A_{Kij} - \tilde{E} & 0 & \tilde{A}_{di} & 0 & -I & 0 & \tilde{B}_{2i} & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathbb{A}}_{Li} =& \begin{bmatrix} A_{Li} & 0 & 0 & 0 & 0 & 0 & 0 & G_{\varphi i} \end{bmatrix},\\
\mathbb{G}_i =& \begin{bmatrix} \tilde{G}_{1i} & \tilde{G}_{2i} \end{bmatrix},\ \mathbb{T}_i = \begin{bmatrix} \tilde{T}_{1i} & \tilde{T}_{2i} \end{bmatrix},\\
\bar{H}_{zi} =& \begin{bmatrix} \tilde{C}_{2i} & 0 & 0 & 0 & 0 & 0 & \tilde{D}_i & 0 \end{bmatrix},\\
d_r =& d_M - d_m,\ \tilde{d}_M = \frac{d_M(d_M+1)}{2},\ I_c = \begin{bmatrix} I & 0 \end{bmatrix}.
\end{aligned}
$$

R is any matrix with full column rank satisfying $R^T\tilde{E} = 0$.

Proof. Let the ith subsystem be switched on at k_r and jth one be switched on at k_{r+1}. Select the following switched Lyapunov–Krasovskii functional candidate:

$$
V(k) = \begin{cases} \bar{V}_i(k) = \sum_{s=1}^{7} \bar{V}_{is}(k), & k \in [k_r + \Delta_r, k_{r+1}) \\ \tilde{V}_{ij}(k) = \sum_{s=1}^{7} \tilde{V}_{ijs}(k), & k \in [k_r, k_r + \Delta_r) \end{cases} \tag{61}
$$

with

$$
\begin{aligned}
\bar{V}_{i1}(k) =& \tilde{x}^T(k)\tilde{E}^T\tilde{P}_i\tilde{E}\tilde{x}(k) \\
\bar{V}_{i2}(k) =& \sum_{s=k-d_m}^{k-1} \tilde{x}^T(s)\alpha^{k-1-s}\tilde{Q}_{1i}\tilde{x}(s) \\
\bar{V}_{i3}(k) =& \sum_{s=k-d_M}^{k-1-d_m} \tilde{x}^T(s)\alpha^{k-1-s}\tilde{Q}_{2i}\tilde{x}(s) \\
\bar{V}_{i4}(k) =& \sum_{s=k-d(k)}^{k-1} \tilde{x}^T(s)\alpha^{k-1-s}\tilde{Q}_{3i}\tilde{x}(s) \\
\bar{V}_{i5}(k) =& \sum_{n=-d_M+1}^{-d_m} \sum_{s=k+n}^{k-1} \tilde{x}^T(s)\alpha^{k-1-s}\tilde{Q}_{3i}\tilde{x}(s) \\
\bar{V}_{i6}(k) =& \sum_{n=-d_M}^{-1} \sum_{s=k+n}^{k-1} \tilde{\eta}^T(s)\tilde{E}^T\alpha^{k-1-s}\tilde{Z}_{1i}\tilde{E}\tilde{\eta}(s) \\
&+ \sum_{n=-d_M}^{-d_m-1} \sum_{s=k+n}^{k-1} \tilde{\eta}^T(s)\tilde{E}^T\alpha^{k-1-s}\tilde{Z}_{2i}\tilde{E}\tilde{\eta}(s) \\
\bar{V}_{i7}(k) =& \sum_{s=-d_M}^{-1} \sum_{n=s}^{-1} \sum_{r=k+n}^{k-1} \eta^T(r)\tilde{E}^T\alpha^{k-1-r}\tilde{Z}_{3i}\tilde{E}\eta(r)
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
\tilde{V}_{ij1}(k) =& \tilde{x}^T(k)\tilde{E}^T\tilde{P}_{ij}\tilde{E}\tilde{x}(k) \\
\tilde{V}_{ij2}(k) =& \sum_{s=k-d_m}^{k-1} \tilde{x}^T(s)\beta^{k-1-s}\tilde{Q}_{1ij}\tilde{x}(s) \\
\tilde{V}_{ij3}(k) =& \sum_{s=k-d_M}^{k-1-d_m} \tilde{x}^T(s)\beta^{k-1-s}\tilde{Q}_{2ij}\tilde{x}(s) \\
\tilde{V}_{ij4}(k) =& \sum_{s=k-d(k)}^{k-1} \tilde{x}^T(s)\beta^{k-1-s}\tilde{Q}_{3ij}\tilde{x}(s) \\
\tilde{V}_{ij5}(k) =& \sum_{n=-d_M+1}^{-d_m} \sum_{s=k+n}^{k-1} \tilde{x}^T(s)\beta^{k-1-s}\tilde{Q}_{3ij}\tilde{x}(s) \\
\tilde{V}_{ij6}(k) =& \sum_{n=-d_M}^{-1} \sum_{s=k+n}^{k-1} \tilde{\eta}^T(s)\tilde{E}^T\beta^{k-1-s}\tilde{Z}_{1ij}\tilde{E}\tilde{\eta}(s) \\
&+ \sum_{n=-d_M}^{-d_m-1} \sum_{s=k+n}^{k-1} \tilde{\eta}^T(s)\tilde{E}^T\beta^{k-1-s}\tilde{Z}_{2ij}\tilde{E}\tilde{\eta}(s) \\
\tilde{V}_{ij7}(k) =& \sum_{s=-d_M}^{-1} \sum_{n=s}^{-1} \sum_{r=k+n}^{k-1} \tilde{\eta}^T(r)\tilde{E}^T\beta^{k-1-r}\tilde{Z}_{3ij}\tilde{E}\tilde{\eta}(r)
\end{aligned}
\tag{63}
$$

Define $\tilde{\eta}(k) = \tilde{x}(k+1) - \tilde{x}(k)$ and

$$
\begin{aligned}
\bar{\zeta}(k) = \big[x^T(k) \quad & \tilde{x}^T(k-d_m) \quad \tilde{x}^T(k-d(k)) \quad \tilde{x}^T(k-d_M) \\
& \tilde{\eta}^T(k)\tilde{E}^T \quad (\textstyle\sum_{s=k-d_M}^{k-1} \tilde{E}\tilde{x}(s))^T \quad w^T(k) \quad \varphi_n^T(C_i x(k)) \big]^T
\end{aligned}
$$

For $k \in [k_r + \Delta_r, k_{r+1})$, the system modes and controller are switched on simultaneously.

It follows from (56) that

$$2\,\bar{\zeta}^T(k)\bar{\mathbb{F}} \times \left[(\tilde{A}_i - \tilde{E})\tilde{x}(k) + \tilde{A}_{di}\tilde{x}(k-d(k)) + \tilde{B}_{2i}w(k) + G_{\varphi i}\varphi_n(C_i x(k))\right] \\ = 2\bar{\zeta}^T(k)(\bar{\mathbb{F}}\bar{\mathbb{A}}_i)\bar{\zeta}(k) + 2\bar{\zeta}^T(k)(\bar{\mathbb{F}}\bar{\mathbb{A}}_{Li})\bar{\zeta}(k) = 0 \tag{64}$$

Furthermore, from (5) we have for any scalar $\epsilon_{2i} > 0$

$$\tilde{\varphi}_n(k) = 2\epsilon_{2i}\varphi_n^T(C_i x(k))\left[\varphi_n(C_i x(k)) - MC_i x(k)\right] \leq 0 \tag{65}$$

Since $-\tilde{\varphi}_n(k) \geq 0$, by applying the same strategy in Theorem 1 with the closed loop system (56) for the matched period, the following conditions hold by considering (64) and (65):

$$\Sigma_{Xi}(\bar{\mathbb{F}}\bar{\mathbb{A}}_i, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) = \Upsilon_X(\bar{\Upsilon}_i, \bar{H}_T, \bar{H}_G, \bar{H}_{zi}) \\ \Sigma_{Yi}(\bar{\mathbb{F}}\bar{\mathbb{A}}_i, \bar{\mathbb{F}}\bar{\mathbb{A}}_{Li}) = \Upsilon_Y(\bar{\Upsilon}_i, \bar{H}_T, \bar{H}_G, \bar{H}_{zi}) \tag{66}$$

From (57) it can be verified that $\mathcal{E}\left\{\Delta_\alpha V(k)\right\} < 0$.

Now, when $k \in [k_r, k_r + \Delta_r)$, the subsystem and the controller are switched on asynchronously. Pursuing the same proof line of the synchronous switching, we get $\mathcal{E}\left\{\Delta_\beta V(k)\right\} = \mathcal{E}\left\{\tilde{V}_{ij}(\tilde{x}(k+1)) - \beta\tilde{V}_{ij}(\tilde{x}(k))\right\} < 0$. Therefore, the conditions in (58) hold.

This completes the proof. □

Remark 1. *Theorem 2 provides sufficient conditions for the admissibility of switched singular nonlinear systems with the simultaneous presence of time-varying delay, sensor nonlinearities, disturbance, and unmeasurable states. To obtain less conservative conditions, the proposed conditions were developed by using an appropriate Lyapunov–Krasovskii functional with a triple sum term and adding some free matrices. It clear that the computational burden is the main drawback of this technique.*

Moreover, the bilinear matrix inequalities (BMIs) conditions proposed in Theorem 2 cannot be solved by using the standard numerical software. Next, LMI conditions will be proposed in the following theorem using Lemma 3.

Theorem 3. *Take tunable scalars $0 < \alpha < 1$, $\beta \geq 1$, $\mu_1 > 1$, and $\mu_2 > 1$; chosen matrices K_{ci}; and positive integers d_m and d_M. Switched singular system (56) is exponentially admissible for any switching rule satisfying (59) and (60), if there exist symmetric definite positive matrices $\tilde{P}_i$, $\tilde{P}_{ij}$, $\tilde{Z}_{si}$, $\tilde{Z}_{sij}$, $\tilde{Q}_{si}$, $\tilde{Q}_{sij}\mathbb{R}^{2n\times 2n}$, and $s = 1,2,3$; matrices $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{G}_{1i}$, $\tilde{G}_{2i}$, $\tilde{X}_i$, $\tilde{X}_{ij}$, $\tilde{Y}_i$, $\tilde{Y}_{ij}$, $\tilde{S}_i$, $\tilde{F} = \begin{bmatrix}\tilde{F}_{11} & \tilde{F}_{12}\\ 0 & \tilde{F}_{22}\end{bmatrix}$, W_{Li}, W_{ki}, and X_K; and positive scalars ϵ_{1i} and ϵ_{2i} such that for all $(i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, the following inequalities hold.*

$$\begin{bmatrix}\Sigma_{Xi}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{ci}, \mathbb{I}_\lambda\mathbb{H}_{Li}) & \mathbb{F}_{Bi}\\ * & 0\end{bmatrix} + sym\left\{\mathbb{I}\,\mathbb{W}_{ki}\right\} < 0 \\ \begin{bmatrix}\Sigma_{Yi}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{ci}, \mathbb{I}_\lambda\mathbb{H}_{Li}) & \mathbb{F}_{Bi}\\ * & 0\end{bmatrix} + sym\left\{\mathbb{I}\,\mathbb{W}_{ki}\right\} < 0 \tag{67}$$

$$\begin{bmatrix}\Sigma_{Xij}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{cij}, \mathbb{I}_\lambda\mathbb{H}_{Li}) & \mathbb{F}_{Bi}\\ * & 0\end{bmatrix} + sym\left\{\mathbb{I}\,\mathbb{W}_{kj}\right\} < 0 \\ \begin{bmatrix}\Sigma_{Yij}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{cij}, \mathbb{I}_\lambda\mathbb{H}_{Li}) & \mathbb{F}_{Bi}\\ * & 0\end{bmatrix} + sym\left\{\mathbb{I}\,\mathbb{W}_{kj}\right\} < 0 \tag{68}$$

where

$$
\begin{aligned}
\tilde{\mathbb{F}}^T &= \begin{bmatrix}\lambda_1\tilde{F} & 0 & \lambda_2\tilde{F} & 0 & \lambda_3\tilde{F} & 0 & 0 & 0\end{bmatrix}, \\
\bar{\mathbb{A}}_{ci} &= \begin{bmatrix}\bar{A}_{ci}-\tilde{E}_i & 0 & \tilde{A}_{di} & 0 & -I & 0 & \tilde{B}_{2i} & 0\end{bmatrix}, \\
\bar{\mathbb{A}}_{cij} &= \begin{bmatrix}\bar{A}_{cij}-\tilde{E}_i & 0 & \tilde{A}_{di} & 0 & -I & 0 & \tilde{B}_{2i} & 0\end{bmatrix}, \\
\mathbb{I}_\lambda{}^T &= \begin{bmatrix}\lambda_1 I & 0 & \lambda_2 I & 0 & \lambda_3 I & 0 & 0 & 0\end{bmatrix}, \\
\mathbb{H}_{Li} &= \begin{bmatrix}\tilde{A}_{Li} & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{G}_{\varphi i}\end{bmatrix}, \\
\mathbb{F}_{Bi}{}^T &= \begin{bmatrix}\lambda_1\bar{B}_{1i}^T\tilde{F} & 0 & \lambda_2\bar{B}_{1i}^T\tilde{F} & 0 & \lambda_3\bar{B}_{1i}^T\tilde{F} & \mathbb{O}\end{bmatrix}, \\
\bar{A}_{ci} &= \begin{bmatrix}A_i+B_{1i}K_{ci} & B_{1i}K_{ci} \\ 0 & A_i\end{bmatrix}, \tilde{A}_{Li} = \begin{bmatrix}0 & 0 \\ W_{Li}C_i - W_{Li}M_1C_i & -W_{Li}C_i\end{bmatrix}, \\
\bar{A}_{cij} &= \begin{bmatrix}A_i+B_{1i}K_{cj} & B_{1i}K_{cj} \\ 0 & A_i\end{bmatrix}, \bar{B}_{1i} = \begin{bmatrix}B_{1i} \\ 0\end{bmatrix},\ \tilde{G}_{\varphi i} = \begin{bmatrix}0 \\ -W_{Li}\end{bmatrix}, \\
\mathbb{W}_{ki} &= \begin{bmatrix}\mathbb{K}_{ci} & 0 & 0 & 0 & 0 & 0 & \mathbb{O} & -X_k\end{bmatrix},\ \mathbb{K}_{ci} = \begin{bmatrix}W_{ki}-X_kK_{ci} & -W_{ki}-X_kK_{ci}\end{bmatrix}, \\
\mathbb{O} &= \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix},\ \mathbb{I} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{O} & I\end{bmatrix}^T.
\end{aligned}
\tag{69}
$$

Controller and observer gains matrices K_i and L_i are given by:

$$
K_i = X_k^{-1}W_{ki}, \quad L_i = \tilde{F}_{22}^{-T}W_{Li}. \tag{70}
$$

Proof. Using the proposed conditions in Theorem 3, a feasible solution verifies $-sym(X_k) < 0$. Thus, X_k is non-singular. For the controller synthesis purpose, we introduce some auxiliary variables K_{ci} in systems (56). Thus, we obtain

$$
\begin{cases}
\tilde{E}\tilde{x}(k+1) = (\tilde{A}_i + \bar{B}_{1i}\mathcal{K}_{ci} - \bar{B}_{1i}\mathcal{K}_{ci})\tilde{x}(k) + \tilde{A}_{di}\tilde{x}(k-d(k)) + \tilde{B}_{2i}\varpi(k) \\
\qquad\qquad + G_{\varphi i}\varphi_n(C_i x(k)),\ k \in [k_r + \Delta_r, k_{r+1}) \\
\tilde{E}\tilde{x}(k+1) = (\tilde{A}_{ij} + \bar{B}_{1i}\mathcal{K}_{cj} - \bar{B}_{1i}\mathcal{K}_{cj})\tilde{x}(k) + \tilde{A}_{di}\tilde{x}(k-d(k)) + \tilde{B}_{2i}\varpi(k) \\
\qquad\qquad + G_{\varphi i}\varphi_n(C_i x(k)),\ k \in [k_r, k_r + \Delta_r)
\end{cases}
\tag{71}
$$

where $\mathcal{K}_{ci} = \begin{bmatrix}K_{ci} & K_{ci}\end{bmatrix}$.

Taking $\tilde{F}_s = \lambda_s\tilde{F}$ with $s = 1,2,3$ and $\tilde{F} = \begin{bmatrix}\tilde{F}_{11} & \tilde{F}_{12} \\ 0 & \tilde{F}_{22}\end{bmatrix}$, the following equation holds for $W_{Li} = F_{22}^T L_i$:

$$
\tilde{\mathbb{F}}\bar{\mathbb{A}}_{Li} = \mathbb{I}_\lambda\mathbb{H}_{Li} \tag{72}
$$

Applying Theorem 2 to system (71) and considering (72) yields

$$
\begin{aligned}
&\Sigma_{Xi}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{ci}, \mathbb{I}_\lambda\mathbb{H}_{Li}) + sym(\mathbb{F}_{Bi}\mathbb{K}_i) < 0, \\
&\Sigma_{Yi}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{ci}, \mathbb{I}_\lambda\mathbb{H}_{Li}) + sym(\mathbb{F}_{Bi}\mathbb{K}_i) < 0 \\
&\Sigma_{Xij}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{cij}, \mathbb{I}_\lambda\mathbb{H}_{Li}) + sym(\mathbb{F}_{Bi}\mathbb{K}_j) < 0, \\
&\Sigma_{Yij}(\tilde{\mathbb{F}}\bar{\mathbb{A}}_{cij}, \mathbb{I}_\lambda\mathbb{H}_{Li}) + sym(\mathbb{F}_{Bi}\mathbb{K}_j) < 0
\end{aligned}
\tag{73}
$$

where $\mathbb{K}_i = \begin{bmatrix}\begin{bmatrix}K_i - K_{ci} & -K_i - K_{ci}\end{bmatrix} & \mathbb{O}\end{bmatrix}$.

By applying Lemma 3 to (73), conditions in (67) and (68) hold for $W_{ki} = X_kK_i$.

□

Remark 2. *Note that inequalities (67) and (68) are linear on γ, which can be minimized as follows:*

$$
\gamma_m = \min_{\tilde{P}_i,\tilde{P}_{ij},\tilde{Q}_{si},\tilde{Q}_{sij},\tilde{Z}_{si},\tilde{Z}_{sij},\tilde{T}_1,\tilde{T}_2,\tilde{G}_{1i},\tilde{G}_{2i},\tilde{X}_i,\tilde{X}_{ij},\tilde{Y}_i,\tilde{Y}_{ij},\tilde{S}_i,\tilde{F},W_{Li},W_{ki},X_K,i\in\mathcal{I},s=1,2,3} \gamma \tag{74}
$$

Remark 3. *In [30], the study of observer-based asynchronous H_∞ control for switched singular systems with state nonlinearity was considered. To solve the problem of observer-based asynchronous control design, system transformation and Finsler's lemma involving some scalars, p_{mi}, q_{mi}, and r_{mi}, have been considered. However, this problem is solved only if the values of the used scalars are given. Otherwise, some global optimization algorithms should be used to solve the problem of bilinear matrix inequalities (BMIs), which can complicate the solvability of the LMIs. Moreover, in contrast to [23,45], the LMI conditions in Theorem 3 can be solved in one step without resorting to an iterative algorithm.*

5. Numerical Examples

Example 1. *Consider a switched singular delayed system with two modes and the following parameters:*

$$E = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1.2 & -0.3 \\ 1.2 & 1.38 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.5 & -0.3 \\ -1 & -0.1 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0.01 & 0.1 \end{bmatrix}, \; B_{11} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}, \; B_{12} = \begin{bmatrix} -1.9 & -1 \\ -1 & 1.3 \end{bmatrix}, C_1 = \begin{bmatrix} 0.5 & -1.2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.6 & -1.1 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 25 \\ 10 \end{bmatrix}, \; C_{21} = \begin{bmatrix} 0.001 & 0 \end{bmatrix}, \; C_{22} = \begin{bmatrix} 0.0001 & 0 \end{bmatrix},$$
$$D_1 = 0.6, \; D_2 = 0.7.$$

Since $deg(det(zE - A_2)) = deg(-0.25) = 0 < rank(E) = 1$, pair (E, A_2) is non causal. In this case, the unforced part of the considered subsystem is not admissible.

Let $d_m = 2$, $d_M = 3$, $\lambda_1 = 0.0001$, $\lambda_2 = -0.0004$, $\lambda_3 = -0.0002$, $K_{c1} = \begin{bmatrix} -0.9 & -0.01 \\ -1 & -3 \end{bmatrix}$, $K_{c2} = \begin{bmatrix} -0.1 & -1 \\ -0.1 & -1 \end{bmatrix}$; and take the ADT parameters $\alpha = 0.6$, $\beta = 1.03$, $\mu_1 = 1.2$, and $\mu_2 = 1.5$. Solving LMI conditions in Theorem 3, we get the minimum allowed $\gamma_m = 0.6392$ and the following controller gains:

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.6334 & -0.0375 \\ -0.9743 & -0.3784 \end{bmatrix}, \; K_2 = \begin{bmatrix} -0.1141 & -0.1352 \\ -0.1413 & -0.1637 \end{bmatrix} \\ L_1 &= \begin{bmatrix} 0.1394 \\ -0.9710 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0.2519 \\ -0.5930 \end{bmatrix} \end{aligned} \tag{75}$$

For simulation purposes, the nonlinear function and the exogenous disturbance are given, respectively, as follows:

$$\begin{aligned} \varphi(\varrho) &= 2(M_2 + M_1)\varrho + 3(M_2 - M_1)tan(\varrho). \\ w(k) &= sin(2k)e^{-0.9k}. \end{aligned}$$

where $M_1 = 0.1$ and $M_2 = 0.4$.

Under the variation of $d(k)$ depicted in Figure 4 and the switching signals depicted in Figure 2 with $\Delta_m = 0.7$ and $\tau_a = 2.1 > \tau_a^* = 1.8912$ by respecting the relationship in (59), simulation results are shown in Figures 3–5, with initial conditions $x(0) = [670 \; -700]^T$ and $\hat{x}(0) = [0 \; 0]^T$.

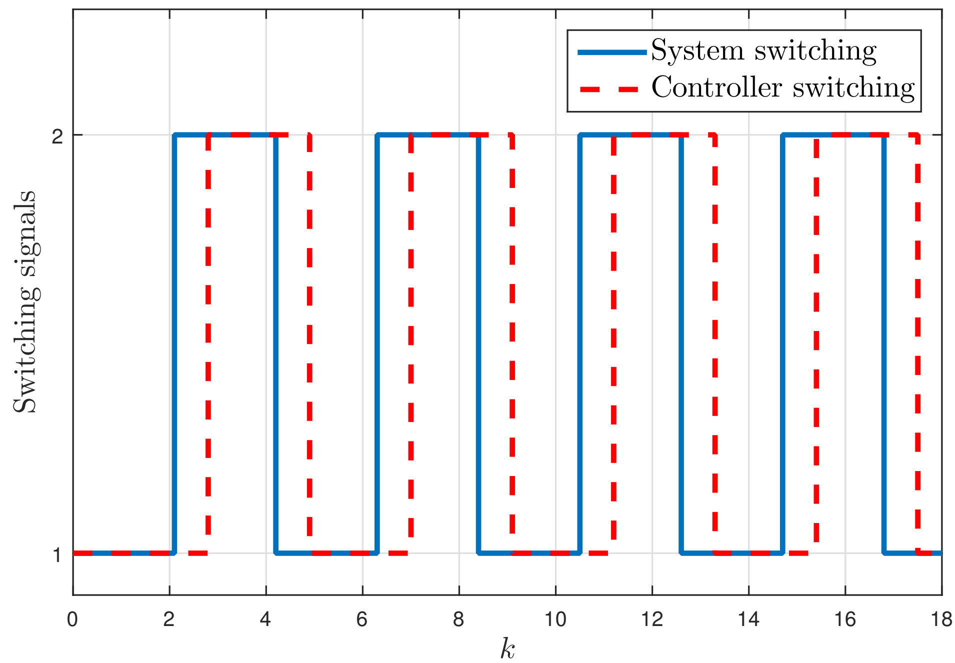

Figure 2. The system and controller switching signal.

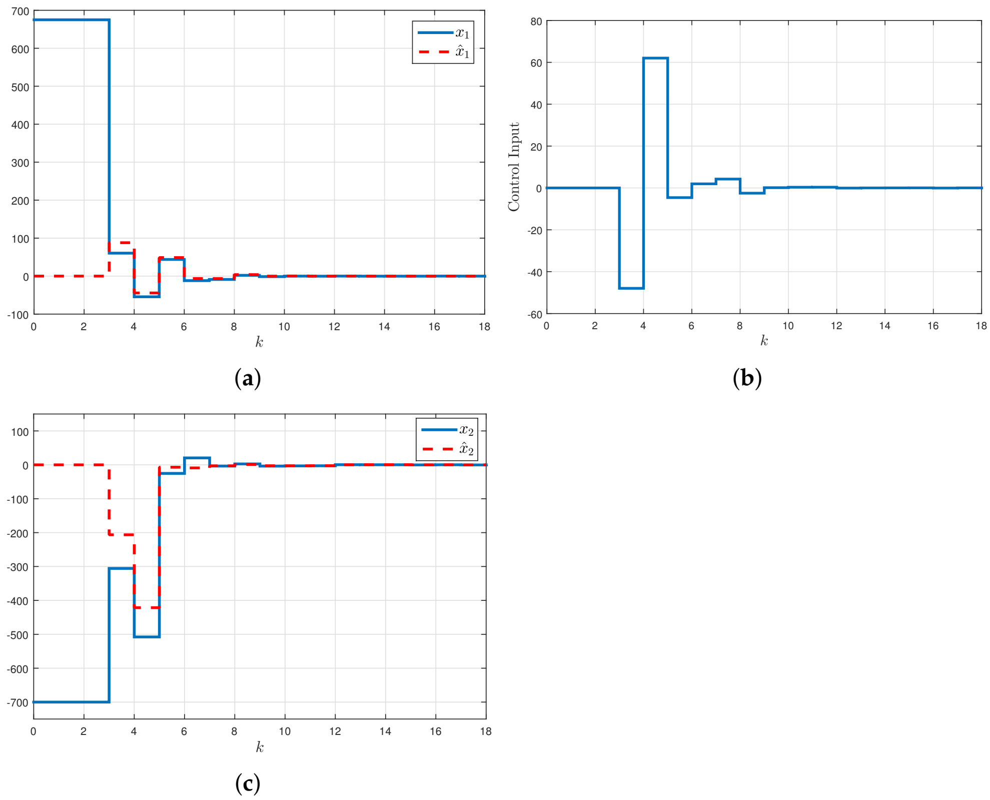

Figure 3. Simulation results for example 1. (**a**) Response and estimate trajectories of x_1. (**b**) Input trajectory. (**c**) Response and estimate trajectories of x_2.

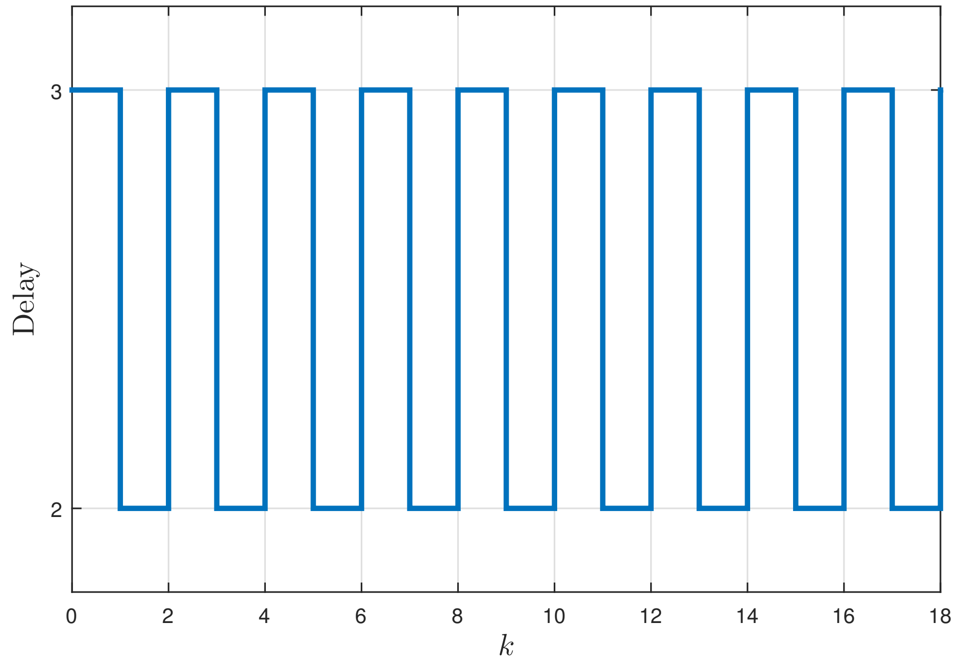

Figure 4. Variation of $d(k)$.

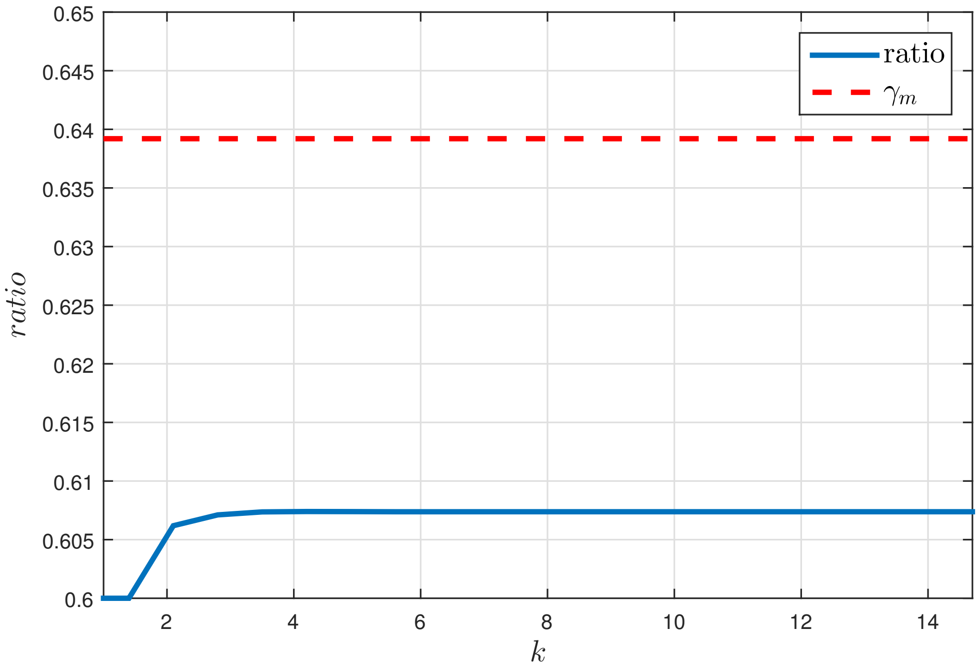

Figure 5. Ratio evolution.

In the plotted Figure 3a–c, it can be seen that the system is stabilized regardless of time-varying delay, external disturbance, and sensor saturation. Moreover, for a non-causal system, a control problem is soved when the system states are unmeasured via the proposed control scheme, which is pertinent for the analysis in this paper.

From Figure 5, we can see that the evolution of the ratio $||z||_2/||w||_2$ *under zero-initial condition tends to a constant value with a square root of about* 0.6055 *which reveals that the* H_∞ *disturbance attenuation level is less than the required minimum allowed value* $\gamma_m = 0.6392$ *of* γ.

Example 2. *To verify the merit of the controller strategy, the example of single-ended primary inductor converter (SEPIC) (Figure 6) presented in [46] is considered.*

The parameters of the converter are illustrated in Table 1.

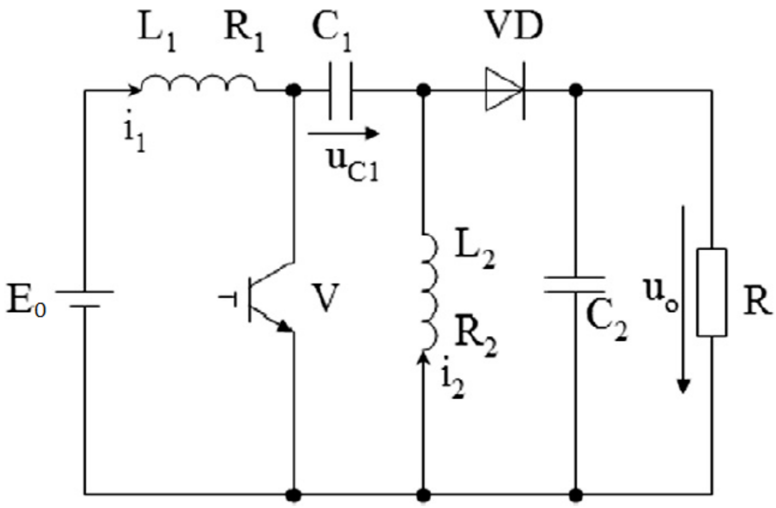

Figure 6. Single-ended primary inductor converter.

Table 1. Converter parameters.

Acronyms	Definitions	Values/Units
l_{c1}	Input inductor	$1 \times 10^{-3}H$
l_{c2}	Output inductor	$0.5 \times 10^{-3}H$
C_1	Input capacitor	$0.1 \times 10^{-3}F$
C_2	Output capacitor	$0.1 \times 10^{-3}F$
R_1	Resistor of input inductor	$2\,\Omega$
R_2	Resistor of output inductor	$0.2\,\Omega$
R	Load resistor	$2\,\Omega$

The state variables of the system are the inductive currents i_1 and i_2, the capacitor voltage u_{c1}, and the output voltage u_0. The SEPIC scan be described by the following differential equations:

- *When V is switched on:*

$$\begin{cases} \dot{i}_1 = -\dfrac{R_1}{l_{c1}} i_1 + \dfrac{1}{l_{c1}} E_0 \\ \dot{u}_{c1} = -\dfrac{1}{C_1} i_2 \\ \dot{u}_0 = -\dfrac{1}{RC_2} u_0 \\ \dot{i}_2 = -\dfrac{R_2}{l_{c2}} i_2 + \dfrac{1}{l_{c2}} u_{c1} \end{cases} \tag{76}$$

- *When V is switched off:*

$$\begin{cases} \dot{i}_1 = -\dfrac{R_1}{l_{c1}} i_1 - \dfrac{1}{l_{c1}} u_{c1} - \dfrac{1}{l_{c1}} u_0 + \dfrac{1}{l_{c1}} E_0 \\ \dot{u}_{c1} = \dfrac{1}{C_1} i_1 \\ \dot{u}_0 = \dfrac{1}{C_2} i_1 + \dfrac{1}{C_2} i_2 - \dfrac{1}{RC_2} u_0 \\ \dot{i}_2 = -\dfrac{R_2}{l_{c2}} i_2 - \dfrac{1}{l_{c2}} u_0 \end{cases} \tag{77}$$

Set the control input $u(k) = E_0$ and the state vector as $x(k) = [i_1(k)\ u_{c1}(k)\ u_0(k)\ i_2(k)]^T$.

We created the model (1) with the following data:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.96 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.2 \\ 0 & 0 & 0.99 & 0 \\ 0 & 0.04 & 0 & 0.992 \end{bmatrix}, B_{11} = B_{12} = \begin{bmatrix} 0.02 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.96 & -0.02 & -0.02 & 0 \\ 0.2 & 1 & 0 & 0 \\ 0.2 & 0 & 0.99 & 0.2 \\ 0 & 0 & -0.04 & 0.992 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A_{d1} = A_{d2} = 0, \tag{78}$$

$$B_{21} = B_{22} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, C_{21} = \begin{bmatrix} 0.0001 & 0.0001 & 0 & 0 \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} 0.0001 & 0 & 0.0001 & 0 \end{bmatrix}, D_1 = 0.1,\ D_2 = 0.01. \tag{79}$$

The nonlinear function and the exogenous disturbance are given as follows:

$$\varphi(\varrho) = \frac{M_2 + M_1}{2}\varrho + \frac{M_2 - M_1}{2}\varrho^2$$
$$`w(k) = cos(2k)e^{-0.4k}.$$

where $M_1 = 0.2$ *and* $M_2 = 0.5$.

case I: *The method in this paper: Let* $\lambda_1 = -2$, $\lambda_2 = -8$, $\lambda_3 = -8$, $K_{c1} = [0.45\ 0.28\ -0.5\ 0]$, $K_{c2} = [0.45\ 1\ 0.8\ 0]$, *and* $\Delta_m = 0.5$*; and ADT parameters* $\alpha_1 = 0.3$, $\beta = 1.03$, $\mu_1 = 1.2$, *and* $\mu_2 = 3$*; a feasible result was obtained by Theorem 3 with the minimum allowed* $\gamma_m = 0.936$ *and the following observer and controller gains:*

$$K_1 = \begin{bmatrix} 0.0112 & 0.0088 & -0.0123 & -0.0001 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.0321 & -0.0186 & -0.0337 & 0.0007 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 0.3826 & -0.0202 & 0.016 \\ -0.0294 & 0.4176 & 0.0243 \\ 0.0046 & 0.017 & 0.476 \\ -0.0005 & 0.0114 & 0.0172 \end{bmatrix}, L_2 = \begin{bmatrix} 0.1364 & -0.002 & -0.0307 \\ 0.1179 & 0.2011 & -0.0052 \\ 0.0830 & 0.0227 & 0.1435 \\ 0.0025 & -0.0051 & -0.0151 \end{bmatrix}. \tag{80}$$

case II: *The method in [47]: The observer and controller gains were:*

$$K_1 = \begin{bmatrix} -19.0129 & -3.7921 & -3.3107 & -7.1217 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -14.5563 & -0.9161 & -0.3893 & -6.6751 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 0.4081 & 0.1669 & 0.0882 \\ 0.4601 & 1.0845 & -0.0181 \\ 0.2778 & -0.0630 & 0.7716 \\ -0.3815 & -0.7171 & -0.0399 \end{bmatrix}, L_2 = \begin{bmatrix} 0.4448 & -0.0478 & 0.0512 \\ 0.1588 & 0.9432 & 0.1189 \\ -0.1689 & -0.0161 & 0.4380 \\ -0.3458 & -0.3748 & 0.1141 \end{bmatrix}. \tag{81}$$

Given the initial conditions $x(0) = \hat{x}(0) = [1\ -1\ 2\ 0.1]^T$*, the simulation results are depicted in Figures 7 and 8. The evolutions of the system states are shown in Figure 8. Figure 7 shows the evolution of the control input and the switching rule of the controller and the system in case I.*

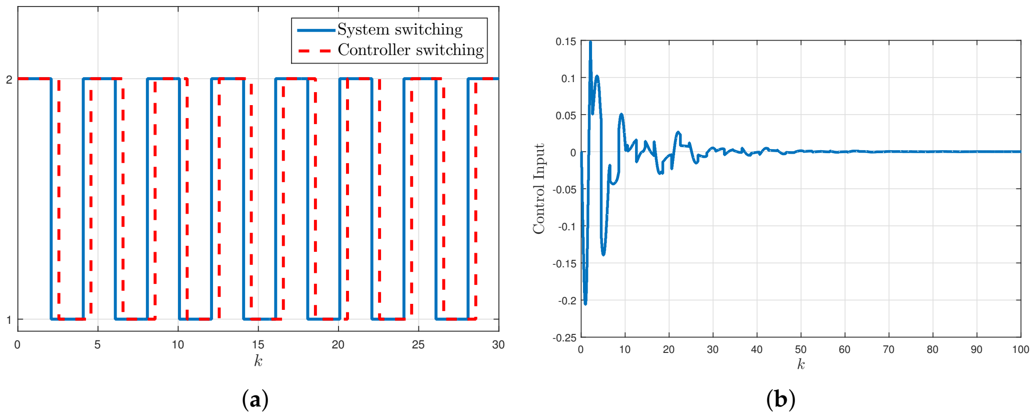

Figure 7. Switching signal and control input. (**a**) System and controller switching signal; (**b**) control input trajectory.

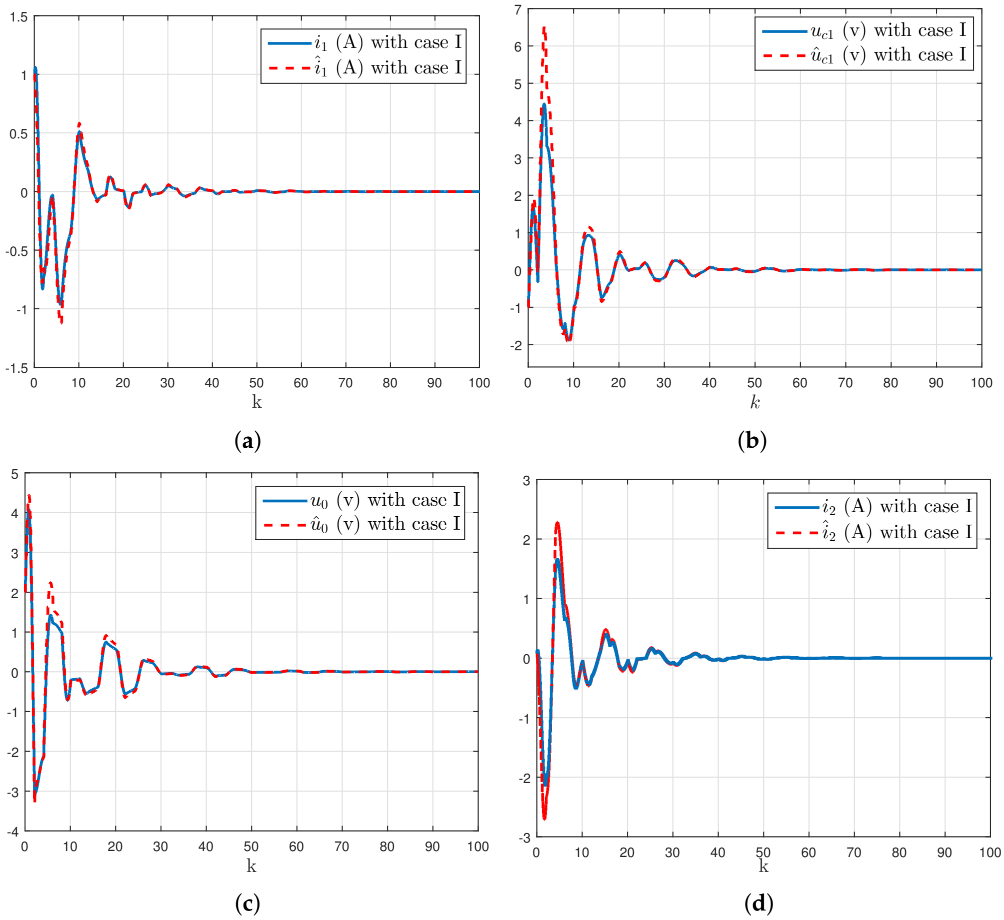

Figure 8. Simulation results for example 2. (**a**) Response and estimate trajectories of i_1. (**b**) Response and estimated trajectories of u_{c1}. (**c**) Response and estimated trajectories of u_0. (**d**) Response and estimate trajectories of i_2.

The results took into account that the converter system under asynchronous switching was stabilized even in the presence of the simultaneous sensor nonlinearity and exogenous disturbance.

Moreover, the control strategy guarantees the convergence of the system state when it is incompletely available for measurement.

To demonstrate the effectiveness of the proposed method in this work, we considered the resulting closed loop system performing by case II in Figure 9.

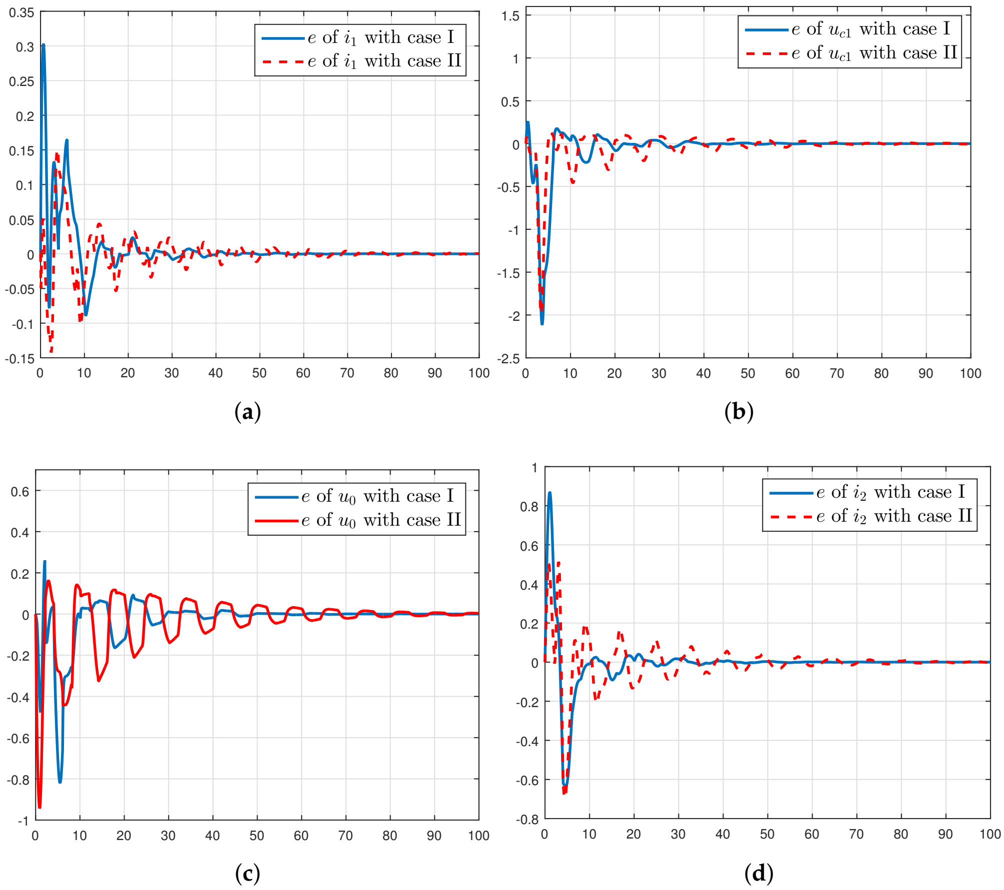

Figure 9. Estimation errors with case I and II. (**a**) Estimation error of i_1. (**b**) Estimation error of u_{c1}. (**c**) Estimation error of u_0. (**d**) Estimation error of i_2.

From Figure 9, it can be observed that all the presented controllers in case I and case II guarantee the convergence of the system. However, the evolutions of the estimated states were improved using the proposed gains in case I.

To further prove the merits of the proposed strategy, two quality criteria were considered to evaluate the states errors $e(k) = x(k) - \hat{x}(k)$: integral squared error (ISE) and integral absolute error (IAE). The comparison is provided in Table 2.

From the results in Table 2, we deduced that the total deviation of e(k) was smaller for the method developed in this paper.

Table 2. Comparison of $e(k)$ for $k \in [0, 100]$.

	ISE	IAE
Expression	$\sum_{k=0}^{100}(e(k))^2$	$\sum_{k=0}^{100}\lvert e(k)\rvert$
case I	16	0.27
case II	19	0.39

6. Conclusions

This work was a contribution to some issues in the control of switched singular non-linear time-varying systems with sensor saturation. An observer was considered to reconstruct the unmeasured system states with measurable outputs and an asynchronous control law was synthesized. Based on an appropriate Lyapunov–Krasovskii functional with a triple sum term and the ADT approach, delay-dependent sufficient conditions were proposed to ensure robust admissible of the closed loop system with H_∞ performance. The resolvability of the corresponding observer-based controller conditions has also been established using the LMI technique. The merits of the presented results were verified through two examples.

As future work, it would be interesting to extend the results for the discrete-time stochastic Markov-jump singular systems and validate the obtained results on a photovoltaic practical platform.

Author Contributions: Conceptualization, M.A.R. and M.K.; methodology, M.A.R.; software, M.K. and M.A.R.; validation, M.K. and M.A.R.; formal analysis, M.K. and M.A.R.; investigation, A.B., H.J. and A.H.; resources, M.K. and A.H.; data curation, A.B. and H.J.; writing—original draft preparation, M.A.R. and M.K.; writing—review and editing, M.A.R. and M.K.; visualization, M.A.R. and M.K.; supervision, M.A.R., M.K. and H.J.; project administration, A.B.; funding acquisition, A.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been funded by Research Deanship in University of Hail-Saudi Arabia through project number RG-20097.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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