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The Logic Structure Behind Mathematical Models: An Axiomatic Framework for System Dynamics

Fan Xia and Tangdai Xia *

Research Center of Coastal and Urban Geotechnical Engineering, Zhejiang University, Hangzhou 310058, China; xiafan.zju@gmail.com
* Correspondence: xtd@zju.edu.cn

Abstract

Modeling complex systems typically involves multiscale analysis, multiphysics coupling, and cross-domain integration. However, existing methodologies predominantly focus on constructing and solving specific equations, lacking a unified, explicit expression for underlying logical structures, which limits model comparison and combination. To address this issue, this paper proposes Axiomatic System Dynamics (ASD), a formal modeling language that decouples a model's logical structure from its mathematical form by formalizing a mathematical model as a combination of a conceptual model and its mathematical realization. Grounded in primitive concepts—state, action, and parameter—ASD introduces generalized constitutive relationships to formulate a unified representation of system evolution. On this basis, a modular "conceptual model first, mathematical model second" paradigm is established to explicitly characterize causal relationships and identify logical isomorphisms across varied mathematical models. Through case studies spanning Newtonian mechanics, ideal gases, material constitutive relationships, and soil consolidation theories, we demonstrate that classical cross-domain theories can be formalized as specific implementations of their respective underlying system dynamics "mother structure." Ultimately, ASD provides a meta-modeling framework independent of concrete mathematical forms, establishing a methodological foundation for logical expression, model comparison, and the meso-level deductive synthesis of cross-domain theories.

Keywords: complex systems; system dynamics; axiomatic modeling; logical structure; causal structure; multiphysics coupling; meta-modeling framework

1. Introduction

1.1. Complex System Modeling and Meso-Level Deductive Strategy

Traditional scientific research is predominantly established upon the classical reductionist approach, the research pathway of which is illustrated in Figure 1. When confronting complex systems, this approach first decomposes them into several relatively independent fundamental units (simple systems). Through controlled experiments and theoretical analysis, universal laws governing these units are induced (such as Newton's laws of motion, various conservation laws, and the laws of thermodynamics). Subsequently, these universal laws serve as underlying axioms or basic hypotheses, from which higher-level theories are rigorously deduced and formalized. The synthesized laws obtained thereby can describe and predict the collective behavior of the system at the macroscopic scale, ultimately achieving closed-loop verification via experimental observation.

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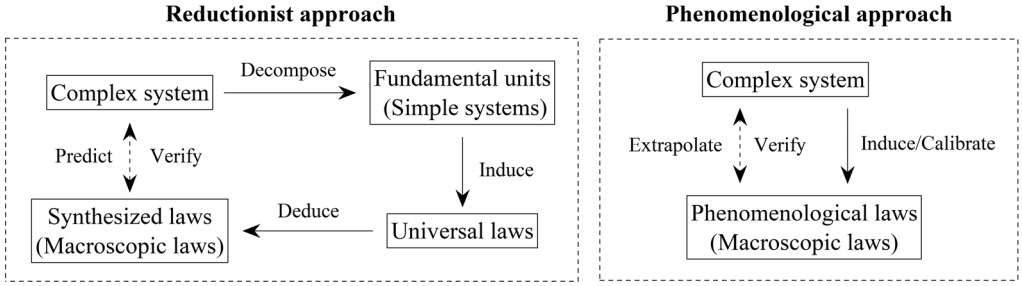


Figure 1. Research pathways of the reductionist approach and the phenomenological approach.

This "decomposition-synthesis" pathway exhibits exceptional explanatory and predictive capabilities in systems with relatively simple structures and lower hierarchical levels. For instance, combining Newtonian mechanics with statistical physics derives the macroscopic ideal gas equation of state from microscopic particle systems. Similarly, continuum mechanics constructs rigid-body mechanics, elasticity, and fluid mechanics upon material points or infinitesimal elements under fundamental physical laws and axiomatic assumptions. Overall, the reductionist approach constitutes a vital methodological foundation for modern science, yielding macroscopic theoretical models with high formal uniformity and robust cross-context extrapolation capabilities.

However, as focus shifts toward higher-level complex systems, the reductionist approach's layer-by-layer deductive link reveals its boundaries of applicability. Such systems—ranging from civil engineering objects like foundations and slopes to giant systems across social, economic, and ecological domains—inherently feature non-linear coupling of multiscale structures and multiphysics mechanisms, driving an exponential surge in computational complexity as the system scales. As the hierarchical span widens, the deductive pathway from underlying universal laws to macroscopic emergent behaviors lengthens rapidly, imposing a prohibitive computational burden. Under such conditions, direct cross-scale deduction is fundamentally limited by computational irreducibility. Taking soil systems as an example, directly deducing macroscopic constitutive relationships and foundation settlement responses from the granular scale—relying solely on fundamental physical laws—inevitably encounters a severe bottleneck due to structural complexity and multiscale coupling effects.

To overcome these challenges, the phenomenological approach serves as a crucial alternative in complex system research. By treating the system as an integral whole, this approach directly induces empirical relationships at a designated observation scale, establishing phenomenological laws and models with lower computational cost and more straightforward modeling pathways. However, phenomenological laws typically rely on specific calibration intervals and carry strong context-dependence, which weakens their cross-context extrapolation capabilities.

Organizing scientific laws by system hierarchy and complexity yields a spectral structure, as illustrated in Figure 2. Within this structure, universal laws reside at the lowest hierarchy (Region I), acting as the starting point of the logical chain and serving as underlying axioms or basic constraints for most higher-level laws.

The medium-to-high hierarchies (Regions II, III, and IV) comprise both phenomenological and synthesized laws. Among them, synthesized laws deduced from universal laws via the reductionist approach inherently possess higher consistency, interpretability, and potential for cross-context extrapolation. However, driven by escalating system complexity and the constraint of computational irreducibility, the deductive pathways originating from underlying universal laws gradually exhibit a pronounced reachability sparsity at medium-to-high hierarchies, as shown in Figure 3a. This sparsity stems not from a lack of

physical laws, but from the concurrent growth of deductive chain length and state-space complexity, which inevitably overwhelms finite resources. Conversely, the distribution density of phenomenological laws increases relatively with the hierarchy, these laws serve as an indispensable complementary path in complex system modeling.

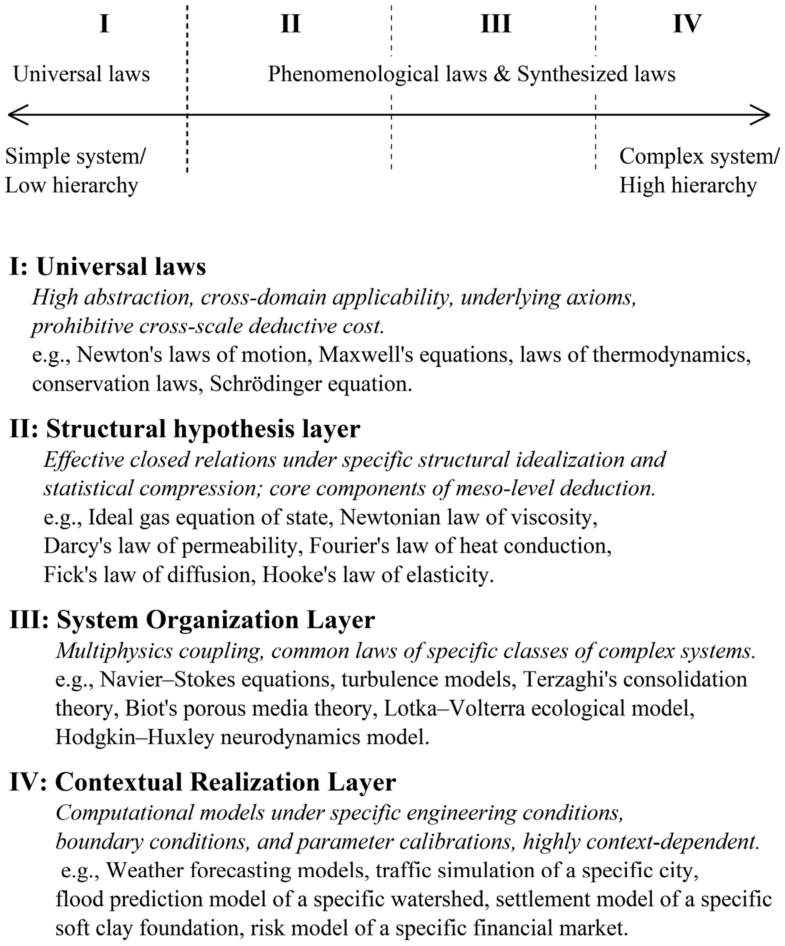


Figure 2. Spectrum of scientific laws.

Nevertheless, phenomenological laws possess an inherent limitation in their extrapolation capabilities. Characterized by significant context-dependence, their predictive reliability can no longer be assured when system states or external actions deviate substantially from their calibration ranges. For instance, in fluid mechanics, while certain empirical relationships are established under laminar conditions, they often exhibit pronounced discrepancies once the Reynolds number rises and the flow enters the turbulent regime. To enhance their applicability, increasing model parameters or expanding calibration ranges forces a trade-off between escalating calibration costs and overfitting risks. As the system hierarchy ascends, the cost of maintaining such predictive validity typically escalates further.

To reconcile the contradictions among theoretical unity, computational feasibility, and cross-context extrapolation capabilities, complex system research widely adopts a "meso-level deductive strategy." Rather than adhering to reductionism's rigid bottom-up deductive pathways, this strategy acknowledges the existence of relatively autonomous effective structures and dynamical laws across system hierarchies. Consequently, it allows lower-level phenomenological laws or existing synthesized laws to serve as new deductive foundations. By establishing stable logical fulcrums at intermediate hierarchies, researchers can construct computable deductive pathways toward higher levels, yielding synthesized

laws that characterize the macroscopic behavior of complex systems (as illustrated in Figure 3b).

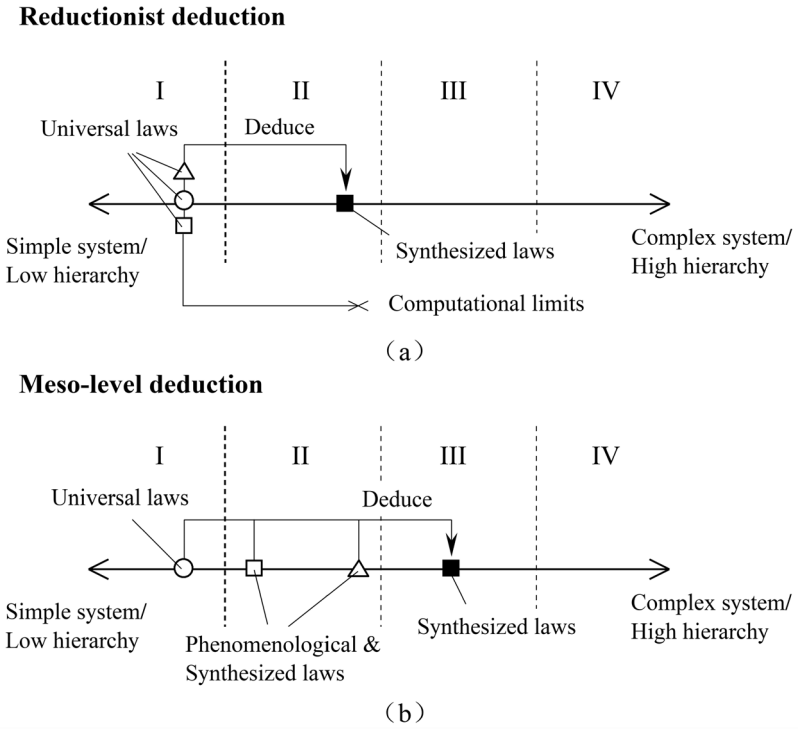


Figure 3. (**a**) Reductionist deduction and (**b**) meso-level deduction.

Compared with conventional phenomenological laws, the synthesized laws obtained via meso-level deduction inherit the extrapolation capabilities of lower-level laws to a certain extent, thereby reducing the reliance on large-scale experimental calibration. Meanwhile, compared to a strictly reductionist approach, this strategy effectively bypasses the cross-scale deductive bottlenecks caused by computational irreducibility. For instance, the Navier–Stokes equations constitute a meso-level framework grounded in the continuum hypothesis rather than molecular-scale deduction. Similarly, Terzaghi's consolidation theory blends Darcy's law, elastic constitutive relations, and mass conservation into a unified intermediate description.

The core capabilities of the meso-level deductive strategy can be summarized into three dimensions:

(1) Shortening of Logical Chains

In higher-level regions, the direct validation of macroscopic laws against underlying universal laws is severely impeded by the extreme length of reductionist deductive pathways. Meso-level deduction, however, allows lower-hierarchy laws to provide logical support for higher-hierarchy laws, thereby forming a theoretical architecture with relatively short logical chains.

(2) Diversification of Deductive Pathways

Traditional reductionism configures knowledge as a single-origin radial structure. Conversely, meso-level deduction decentralizes this architecture, permitting formalized intermediate-hierarchy laws (including phenomenological laws) to act as new deductive foundations. This pathway diversity optimizes the selection of informationally efficient starting point based on problem characteristics, facilitating cross-scale and cross-disciplinary knowledge integration.

(3) Densifying the Theoretical Spectrum

The synthesized laws generated by meso-level deduction effectively fill the reachability gaps imposed by computational constraints. This densified spectrum broadens the model selection space for engineering decisions, liberating researchers from a binary trade-off between costly reductionist approach and extrapolation-limited phenomenological approach.

1.2. The Logical Structure Behind Mathematical Models

In scientific practice, whether they are universal laws, phenomenological laws, or synthesized laws, they are typically manifested in the form of mathematical models. A mathematical model consists of a set of mathematical equations describing the quantitative relationships among system parameters (such as position, velocity, pressure, and internal energy). These include static constraint equations that characterize the geometric structure of the state space, as well as dynamic evolution equations that describe the rates of state change. Together, these equations delineate the system states and their evolution laws, thereby forming a quantitative description of system behavior.

However, mathematical models express only the quantitative relationships among parameters, whereas the underlying logical structure supporting the model's formulation is not explicitly presented in the equations. Such implicit logical structure includes: hierarchical and scale partitioning, system boundary delineation, target parameters selection, system-state quantization, the decoupling and coupling of physical mechanisms, and the causal structures and feedback loops within physical processes. Although not explicitly manifested within mathematical formulations, this information constitutes the physical semantic framework and structural constraints upon which the model's validity hinges.

In conventional engineering applications, the implicit nature of this logical structure does not pose significant difficulties. However, when employing the meso-level deductive strategy to model complex systems, this implicitness poses a critical bottleneck to theoretical consistency. Since sub-models are often grounded in distinct conceptual frameworks and structural assumptions, their logical structures may either partially overlap or exhibit intrinsic discrepancies. In the absence of a unified logical framework, directly combining sub-models at the equation level is highly problematic. Even if the mathematical derivation achieves perfect self-consistency, the synthesized model may still harbor structural inconsistencies or latent contradictions at the physical-semantic level. When sub-models span disparate sub-disciplines, this problem is acutely exacerbated by terminological and conceptual divergence. Furthermore, for sub-models situated at medium-to-high hierarchies, the opacity of their internal logical structures triggers a cascading amplification throughout the coupling process. Ultimately, the resulting synthesized model degenerates into a "black-box" that possesses only input-output mappings but lacks interpretability of internal mechanisms.

Using the one-dimensional soil consolidation detailed in Section 4 as an illustrative case, this phenomenon simultaneously involves multiple physical mechanisms, including external loading variations, fluid seepage, and soil creep. Without a logically structured analytical framework, current consolidation theories often conflate causal pathways and obscure physical mechanisms.

1.3. The Absence of Formal Modeling Languages

In summary, complex system modeling within the framework of meso-level deduction necessitates a formal language capable of explicitly articulating the logical structure underlying mathematical models. As a meta-language, it must remain independent of specific mathematical forms, focusing instead on the causal structures beneath quantitative relations; it must also exhibit a content-free nature, thereby achieving universality across heterogeneous research domains. Crucially, this language must rest on rigorous axiomatic

rules. Under these explicit logical constraints, physical mechanisms, causal pathways, and cross-scale mappings can be transparently expressed, systematically dismantled, and modularly recombined.

In pursuit of methodological breakthroughs, systems science has persistently explored universal modeling languages over recent decades. Nevertheless, regarding the core challenge of "decoupling logical structures from mathematical forms," existing theoretical frameworks still exhibit pronounced limitations:

(1) General Systems Theory [1–3] (GST): Although GST established a vital epistemological framework for complex systems through core concepts like holism, hierarchy, and openness, it remains highly abstract. Its lack of operational formal rules impedes direct utility in quantitative modeling and engineering implementation.
(2) Cybernetics [4–6]: While enhancing the mathematical rigor of system descriptions via feedback mechanisms, state spaces, and input-output mappings, Cybernetics operates under a distinct "mathematics-first" paradigm. Consequently, in complex systems where physical mechanisms remain obscure, this approach tends to rush into mathematical formalization prematurely, weakening the framework's capacity to reveal intrinsic causal pathways.
(3) System Dynamics [7–9] (SD): SD provides a meso-level expressive path, using stock-and-flow structures and causal loops to model system evolution with high intuitiveness and flexibility. However, its causal representations rely heavily on empirical, diagrammatic rules rather than strict formal syntax or an axiomatic foundation. Consequently, this methodological limitation renders SD inadequate for meso-level deduction based on mathematical models, thereby constraining its applicability within hard science domains.

In conclusion, although existing theoretical frameworks have evolved significantly, a unified modeling language that concurrently integrates logical explicitness, formal rigor, and cross-domain transferability remains elusive.

1.4. Research Content

This paper proposes Axiomatic System Dynamics (ASD), a formal modeling language for meso-level deduction. Through a rigorous axiomatic framework, ASD explicitly characterizes physical mechanisms, causal relationships, and system hierarchies, thereby serving as an "operating-system-like" supportive architecture for mathematical modeling.

This paper first outlines the fundamental axioms and concepts of ASD. Then, by modeling several simple physical systems, we demonstrate the mapping between ASD conceptual models and concrete mathematical models, reinterpreting classical theories from the perspective of logical structures to gain fresh epistemological insights and methodological enlightenment.

On this basis, taking soil consolidation as the primary case study, we construct an ASD conceptual model to evaluate its analytical capacity in complex, multiscale multiphysics coupled systems. Through degenerate derivations of classical theories, we reveal the unified logical structures underlying disparate consolidation models, while exposing the limitations of traditional theories in causal expression and mechanistic demarcation.

Finally, we summarize the theoretical characteristics and methodological significance of ASD and its potential application in complex system modeling.

2. Fundamental Axioms and Concepts

This section presents the fundamental axioms of ASD before introducing its core concepts. The axioms first establish the abstract logical relationships among the concepts, while the subsequent definitions specify their meanings within the ASD framework.

2.1. Fundamental Axioms

Axiom 1 (The First Axiom of State Change)**.** *Action is the sole factor that changes the system state. The state* $\boldsymbol{S}$ *at time* t *and the action* $\boldsymbol{F}$ *experienced during the time interval* $[t, t + \mathrm{d}t]$ *uniquely determine the state increment* $\mathrm{d}\boldsymbol{S}$ *over the interval* $[t, t + \mathrm{d}t]$.

$$\boldsymbol{S} + \boldsymbol{F} \mapsto \mathrm{d}\boldsymbol{S} \tag{1}$$

Axiom 2 (The Second Axiom of State Change)**.** *The internal action experienced by a system is uniquely determined by its current state. The state* $\boldsymbol{S}$ *at time* t *uniquely determines the internal action* $\boldsymbol{F}_\mathrm{i}$ *experienced during the time interval* $[t, t + \mathrm{d}t]$.

$$\boldsymbol{F}_\mathrm{i} = \boldsymbol{F}_\mathrm{i}(\boldsymbol{S}) \tag{2}$$

Principle 1 (The Induction Principle of External Actions)**.** *Suppose a system has a state* $\boldsymbol{S}$ *at time* t*, experiences a total action* $\boldsymbol{F}(\boldsymbol{F}_\mathrm{i}, \boldsymbol{F}_\mathrm{e})$ *comprising both internal and external actions during the interval* $[t, t + \mathrm{d}t]$*, yields a state increment* $(\mathrm{d}\boldsymbol{S})_\mathrm{B}$*. Concurrently, suppose the system has a state* $\boldsymbol{S}$ *at time* t'*, experiences zero external action (subject only to the internal action* $\boldsymbol{F}_\mathrm{i}$*) during the interval* $[t', t' + \mathrm{d}t]$*, yields a state increment* $(\mathrm{d}\boldsymbol{S})_\mathrm{A}$*. Then, the state increment of the system during the interval* $[t, t + \mathrm{d}t]$ *attributed solely to the external action* $\boldsymbol{F}_\mathrm{e}$ *is resolved into* $(\mathrm{d}\boldsymbol{S})_\mathrm{B} - (\mathrm{d}\boldsymbol{S})_\mathrm{A}$.

$$\begin{cases} \boldsymbol{S} + \boldsymbol{F}(\boldsymbol{F}_\mathrm{i}, \boldsymbol{F}_\mathrm{e}) \mapsto (\mathrm{d}\boldsymbol{S})_\mathrm{B} \\ \boldsymbol{S} + \boldsymbol{F}_\mathrm{i} \mapsto (\mathrm{d}\boldsymbol{S})_\mathrm{A} \end{cases} \Rightarrow \boldsymbol{S} + \boldsymbol{F}_\mathrm{e} \mapsto ((\mathrm{d}\boldsymbol{S})_\mathrm{B} - (\mathrm{d}\boldsymbol{S})_\mathrm{A}) \tag{3}$$

2.2. Fundamental Concepts

2.2.1. Primitive Concepts

System: A system is an abstract representation of the research object. It encompasses not only the physical entity under investigation but also the associated environmental factors that interact with the entity. The system boundary is defined by the modeling objective, which simultaneously establishes the demarcation between internal and external actions.

State: The state characterizes the condition of a system at a given moment, providing a sufficient basis for determining the system's response (i.e., the state increment) to any arbitrary action. The state is a metaphysical concept, encompassing the complete information or features relevant to the system's current condition. At any given time t, the system's state exists and is unique. In this paper, the state is denoted by the symbol $\boldsymbol{S}$.

Action: Action is the sole factor that changes the system state, and is defined as a process quantity over the time interval $[t, t + \mathrm{d}t]$. "Action" at an instantaneous moment devoid of a time interval is not considered. Action is also a metaphysical concept.

Parameter: A parameter is an observable or measurable attribute of a system, serving as a quantitative characterization of the system state. In contrast to the state, a parameter is an epistemological concept representing a subset of information or features contained within the state; it is thus a dimensionally reduced expression of the state. At any time t, the parameters of a system are entirely determined by its state at t and possess no independent history-dependence. If two systems share the same state, all their parameters must be identical. The converse, however, does not necessarily hold: any finite set of parameters does not necessarily fully characterize or constrain the state of the system.

Characteristic Time Scale: The characteristic time scale is the time interval $\mathrm{d}t$ involved in the fundamental axioms, serving to define the observation and modeling scales for system state updates. ASD characterizes state evolution in terms of discrete time; thus, the

same system may exhibit distinct evolutionary properties under different characteristic time scales. Consequently, the characteristic time scale must be pre-selected during modeling and serves as the fundamental scale parameter for the analysis.

2.2.2. Secondary Concepts

Target Parameter: A target parameter is a subset of system parameters of interest in a study, characterizing features within the system state that are relevant to the research objective. In general cases, the metaphysical state of a system is neither fully accessible nor necessary to be completely characterized. Actual research focuses on the behavior of the state under a specific projection; thus, the target parameter serves as the concentrated manifestation of this projection. The research objective is typically translated into the evolutionary law of the target parameter as the system state evolves. For instance, in the study of a moving ball, its position belongs to the target parameter; its shape and spatial orientation may also be included, whereas attributes such as its color and price are typically excluded.

Internal and **External Actions**: The action upon a system can be partitioned into internal action $\boldsymbol{F}_{\mathrm{i}}$ and external action $\boldsymbol{F}_{\mathrm{e}}$. Specifically, the internal action $\boldsymbol{F}_{\mathrm{i}}$ denotes the action of the system upon itself, while the external action $\boldsymbol{F}_{\mathrm{e}}$ represents actions originating from outside the system boundary. The total action $\boldsymbol{F}$ exerted on the system is expressed as a functional composition of the two, namely: $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e}})$.

According to Axiom 2, the system is unconditionally subject to an internal action uniquely determined by its state. It must be emphasized that the internal action does not necessarily derive solely from the physical components of the research object, but may encompass certain actions of the external environment upon these components. The partition between internal and external actions is essentially a consequence of the system boundary selection; this partition depends not only on the physical properties of the research object itself but also on the specific research objective, and must satisfy Axiom 2.

Generalized Constitutive Relationship (GCR): A generalized constitutive relationship refers to any functional relationship satisfying the form of Equation (1), which possesses an expression of "state -action-state increment." The GCR is used to characterize the abstract dynamical structure (conceptual model) of system evolution; it requires the quantification of the system's state and actions before it can be converted into a specific mathematical model. Although it can be mathematically expressed as $\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{S}(\boldsymbol{S}, \boldsymbol{F})$, adopting the expression of the form $\boldsymbol{S} + \boldsymbol{F} \mapsto \mathrm{d}\boldsymbol{S}$ in Equation (1) facilitates the highlighting of the causal relationship described by Axiom 1, and distinguishes it from other types of functional relationships. When multiple GCRs are involved, the convenience afforded by this expression becomes even more pronounced.

Coupling and **Decoupling** of Actions: The total action $\boldsymbol{F}$ exerted on a system can be decomposed into several types of sub-actions $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ according to physical mechanisms. In this case, the total action $\boldsymbol{F}$ is a coupling of these sub-actions, forming the function $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n)$. These sub-actions are said to be decouplable if the system's response under the coupled action can be (approximately) viewed as the linear superposition of the responses to these sub-actions acting individually. Specifically, assuming the system's response under the coupled action $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n)$ is:

$$\boldsymbol{S} + \boldsymbol{F}(\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n) \mapsto \mathrm{d}\boldsymbol{S} \tag{4}$$

then these sub-actions are decouplable if and only if

$$\left.\begin{array}{l} \boldsymbol{S}+\boldsymbol{F}_1 \mapsto (\mathrm{d}\boldsymbol{S})_1 \\ \boldsymbol{S}+\boldsymbol{F}_2 \mapsto (\mathrm{d}\boldsymbol{S})_2 \\ \ldots \\ \boldsymbol{S}+\boldsymbol{F}_n \mapsto (\mathrm{d}\boldsymbol{S})_n \end{array}\right\} \Rightarrow \boldsymbol{S}+\boldsymbol{F}(\boldsymbol{F}_1,\boldsymbol{F}_2,\ldots,\boldsymbol{F}_n) \mapsto (\mathrm{d}\boldsymbol{S}=(\mathrm{d}\boldsymbol{S})_1+(\mathrm{d}\boldsymbol{S})_2+\ldots+(\mathrm{d}\boldsymbol{S})_n) \quad (5)$$

Based on Principle 1, the decouplability of a system's total internal action and total external action is logically necessary.

State Creep Process: A state creep process is a state evolution process driven solely by internal action in the absence of external action, including the case where the state remains constant. This process characterizes the spontaneous evolution behavior of a system. As derived from Axiom 1 and Axiom 2, the state creep process is entirely determined by the initial state of the system; that is, given an initial state, the subsequent evolution path of the state is unique.

2.2.3. Tertiary Concepts

State Space and **State Path**: To obtain a geometric representation of a system state, the state may be viewed as a point in a state space. The state evolution process of the system then corresponds to a parametric curve in the state space with time t as the parameter; this curve is referred to as the state path of the system.

External Action Space and **External Action Path**: If there exists a function $\boldsymbol{F}_{\mathrm{e\ path}}(t)$ with time t as the independent variable, such that the external action $\boldsymbol{F}_{\mathrm{e}}$ exerted on the system over any time interval $[t, t+\mathrm{d}t]$ is entirely determined by the restriction (function segment) of this function over that interval, namely

$$\boldsymbol{F}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{e}}\Big(\Big(\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t, t+\mathrm{d}t]\Big)\Big) \quad (6)$$

then the function $\boldsymbol{F}_{\mathrm{e\ path}}(t)$ is referred to as the external action path of the external action $\boldsymbol{F}_{\mathrm{e}}$, and the image space of this function is referred to as the external action space. In common scenarios, if the external action $\boldsymbol{F}_{\mathrm{e}}$ exerted on the system can be reduced to the control of certain system parameters, the space containing the corresponding parameters can serve as the external action space, and the corresponding function of the parameters with respect to time t can serve as the external action path $\boldsymbol{F}_{\mathrm{e\ path}}(t)$. At this point, in the process of constructing specific mathematical models for the GCRs, the external action $\boldsymbol{F}_{\mathrm{e}}$ over any time interval $[t, t+\mathrm{d}t]$ can be directly equivalent to the function segment $\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t, t+\mathrm{d}t]$, namely

$$\boldsymbol{F}_{\mathrm{e}} = \Big(\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t, t+\mathrm{d}t]\Big) \quad (7)$$

thereby achieving a quantitative representation of the external action $\boldsymbol{F}_{\mathrm{e}}$.

State Parameter: State parameters are a set of parameters that can serve as an equivalent representation of the system state in a specific study. State parameters must be capable of determining or encompassing all target parameters, and serve as the decisive basis for their own changes when the system undergoes state evolution under internal and external actions. The state parameters of a system do not necessarily exist. The system state is metaphysical and may not be fully characterized or constrained by a finite set of observable parameters. The existence and specific configuration of state parameters depend on factors such as the selection of target parameters, the initial state of the system, and the set of possible external actions. When state parameters exist, the system can be deemed to satisfy the Markov property.

External Action History and **Generalized State Parameter**: If the initial state $\boldsymbol{S}_0$ of a system at a time t_0 is known or fixed, and the external action $\boldsymbol{F}_{\mathrm{e}}$ possesses an external action path $\boldsymbol{F}_{\mathrm{e\ path}}(t)$, then according to Axioms 1 and 2, the state $\boldsymbol{S}$ of the system at any time t is a functional of the initial state $\boldsymbol{S}_0$ and the function segment $\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t_0, t]$, namely

$$\boldsymbol{S} = \boldsymbol{S}\Big(\boldsymbol{S}_0, \Big(\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t_0, t]\Big)\Big) \tag{8}$$

where the function segment $\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t_0, t]$ is referred to as the external action history of the system (at time t). The external action history is deterministic with respect to the state, yet it exhibits redundancy in representation: for a given initial state, identical external action histories necessarily lead to identical states; conversely, different external action histories may correspond to the same state.

In practical modeling, it is usually sufficient to extract only those features of the external action history $\boldsymbol{F}_{\mathrm{e\ path}}(\tau), \tau \in [t_0, t]$ that exert a dominant influence on the evolution of target parameters (such as endpoint values, integral values, fractional derivatives, etc.). By combining these features with certain system parameters, a composite parameter is constructed to serve as an equivalent representation of the system state $\boldsymbol{S}$; such parameter is termed generalized state parameter.

Functionally, similar to state parameters, generalized state parameters must be capable of determining or encompassing all target parameters and serve as the decisive basis for their own changes when the system undergoes state evolution under internal and external actions.

Since generalized state parameters are no longer restricted to being functions of the state at the current moment, but may contain compressed expressions of historical information, they essentially constitute a form of history-compression operator. When a system cannot be characterized by state parameters, the introduction of generalized state parameters allows for the transformation of a non-Markovian system into a Markovian representation within an extended state space.

3. ASD Modeling Examples

This section illustrates the mapping between ASD conceptual models and concrete mathematical models via representative physical systems. The focus is not on the derivation of specific equations, but on articulating their underlying logical structures within the ASD framework to reveal inherent structural characteristics and methodological insights.

3.1. Newtonian Particle Mechanics

Consider the motion of a particle under external force in an inertial frame. The system is defined as the particle itself. The target parameter is the spatial position vector $\boldsymbol{x}$ of the particle. The ASD conceptual model of Newtonian particle mechanics is formulated as follows:

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{i}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} \tag{9}$$

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{e}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{e}} \tag{10}$$

$$\boldsymbol{S} + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e}}) \mapsto (\mathrm{d}\boldsymbol{S} = (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} + (\mathrm{d}\boldsymbol{S})_{\mathrm{e}}) \tag{11}$$

wherein, the internal action $\boldsymbol{F}_{\mathrm{i}}$ represents the inertia of the particle, and the external action $\boldsymbol{F}_{\mathrm{e}}$ represents the net external force acting on the particle.

To transform the conceptual model into a concrete mathematical model, the state $\boldsymbol{S}$, the internal action $\boldsymbol{F}_{\mathrm{i}}$, and the external action $\boldsymbol{F}_{\mathrm{e}}$ must be quantified. According to Axiom 2, the internal action $\boldsymbol{F}_{\mathrm{i}}$ is determined by the state $\boldsymbol{S}$; thus, its quantization depends inherently on the quantification scheme of the state $\boldsymbol{S}$.

In Newtonian particle mechanics, the system state $\boldsymbol{S}$ is quantified via state parameters. By selecting the position $\boldsymbol{x}$, velocity $\boldsymbol{v}$, and mass m as state parameters, we obtain $\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{v}, m)$. This set of state parameters not only contains the target parameter $\boldsymbol{x}$ but also introduces velocity $\boldsymbol{v}$ and mass m—two parameters that play a deterministic role in the dynamical evolution—thereby constituting a minimal sufficient description of the system state.

The external action $\boldsymbol{F}_\mathrm{e}$ is quantified via the external action path. Let the external action path be $\boldsymbol{F}_{\mathrm{e\ path}}(t) = \boldsymbol{F}_\mathrm{net}(t)$, where $\boldsymbol{F}_\mathrm{net}(t)$ is the function of the net external force acting on the particle with respect to time t. According to the definition of the external action path, within any time interval $[t, t + \mathrm{d}t]$, we have

$$\boldsymbol{F}_\mathrm{e} = (\boldsymbol{F}_\mathrm{net}(\tau), \tau \in [t, t + \mathrm{d}t]) \tag{12}$$

When the characteristic time scale $\mathrm{d}t$ approaches zero, the external action $\boldsymbol{F}_\mathrm{e}$ can be regarded as being determined solely by the left-endpoint value of the function segment $\boldsymbol{F}_\mathrm{net}(\tau), \tau \in [t, t + \mathrm{d}t]$, which corresponds to the current net external force $\boldsymbol{F}_\mathrm{net}(t)$, we have:

$$\boldsymbol{F}_\mathrm{e} = \boldsymbol{F}_\mathrm{net}(t) \tag{13}$$

Consequently, the mathematical model corresponding to the ASD conceptual model can be explicitly written as

$$(\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{v}, m)) + \boldsymbol{F}_\mathrm{i} \mapsto ((\mathrm{d}\boldsymbol{S})_\mathrm{i} = (\boldsymbol{v}\mathrm{d}t, 0, 0)) \tag{14}$$

$$(\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{v}, m)) + \boldsymbol{F}_\mathrm{e} \mapsto \left((\mathrm{d}\boldsymbol{S})_\mathrm{e} = \left(0, \frac{\boldsymbol{F}_\mathrm{net}(t)}{m}\mathrm{d}t, 0\right)\right) \tag{15}$$

$$(\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{v}, m)) + \boldsymbol{F}(\boldsymbol{F}_\mathrm{i}, \boldsymbol{F}_\mathrm{e}) \mapsto \left(\mathrm{d}\boldsymbol{S} = \left(\boldsymbol{v}\mathrm{d}t, \frac{\boldsymbol{F}_\mathrm{net}(t)}{m}\mathrm{d}t, 0\right)\right) \tag{16}$$

Equations (14) and (15) correspond to the ASD expressions of Newton's First and Second Laws, respectively. The iterative relationship regarding velocity $\boldsymbol{v}$ within the total state increment $\mathrm{d}\boldsymbol{S}$ is mathematically equivalent to the standard form of Newton's Second Law:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}m = \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}m = \boldsymbol{F}_\mathrm{net}(t) \tag{17}$$

From a structural perspective, the state increment $(\mathrm{d}\boldsymbol{S})_\mathrm{i}$ corresponding to the internal action depends solely on information regarding the state $\boldsymbol{S}$, which is consistent with Axiom 2. Conversely, the state increment $(\mathrm{d}\boldsymbol{S})_\mathrm{e}$ corresponding to the external action further incorporates information regarding the external action $\boldsymbol{F}_\mathrm{e} = \boldsymbol{F}_\mathrm{net}(t)$. The total state increment $\mathrm{d}\boldsymbol{S}$, obtained by the superposition of the two, constitutes an informationally complete iterative relationship of the state, thereby achieving closure from the conceptual model to the mathematical model.

From an epistemological perspective, what can be obtained directly through experiments is the evolutionary behavior of the system under internal action $\boldsymbol{F}_\mathrm{i}$ (Equation (14)), and the evolutionary behavior under coupled internal and external actions $\boldsymbol{F}(\boldsymbol{F}_\mathrm{i}, \boldsymbol{F}_\mathrm{e})$ (Equation (16)). The response behavior of the system under external action $\boldsymbol{F}_\mathrm{e}$ alone (Equation (15)) must be obtained indirectly by means of control experiments (i.e., Principle 1). This fact is naturally expressed within the ASD framework as the problem of action decomposition and response decoupling.

The ASD conceptual model corresponding to Newtonian particle mechanics represents the simplest case, comprising only one internal action and one external action, with its mathematical model forming a typical Markovian system. Here, the uniform linear motion

of a particle can be viewed as a state creep process under internal action alone, i.e., the spontaneous evolution of the system. In this formulation, inertia is explicitly modeled as an internal action, thereby incorporating motion states—traditionally deemed unconditionally valid—into a unified, action-driven framework. Consequently, Axiom 1 establishes action as the sole cause of state change, unifying the causal explanation of system evolution in formal logic and providing a foundation for a modeling methodology where logical structure precedes mathematical equations.

It must be noted that the selection of state parameters is not unique and possesses degrees of freedom for equivalent transformation. For instance, position $\boldsymbol{x}$, momentum $\boldsymbol{p} = m\boldsymbol{v}$, and mass m may be taken as state parameters, with the state being $\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{p}, m)$. It is not difficult to derive an equivalent set of dynamical equations (corresponding to Equations (14)–(16)) based on this alternative state representation. This degree of freedom reflects, to a certain extent, the relative independence between ASD conceptual models and mathematical models.

3.2. Quasi-Static Process of an Ideal Gas

Consider an ideal gas enclosed within a thermally conductive container, as illustrated in Figure 4. During a quasi-static process, the pressure P, volume V, and temperature T of the ideal gas satisfy the equation of state:

$$PV = nRT \tag{18}$$

where n is the number of moles and R is the ideal gas constant.

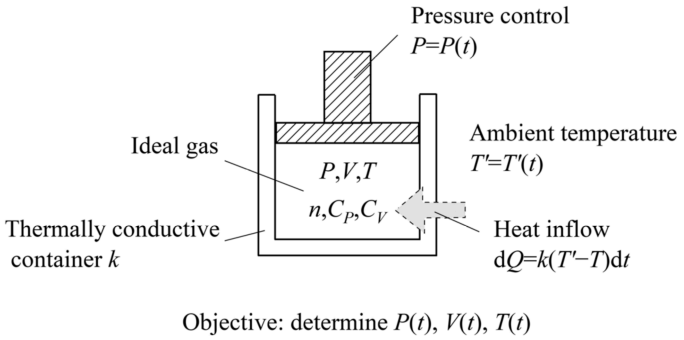


Figure 4. Ideal gas in a thermally conductive container: a quasi-static process under pressure control and prescribed ambient temperature variation.

The molar heat capacity at constant volume C_V and the molar heat capacity at constant pressure C_P are assumed to be constants. The ambient temperature of the external environment is given by $T' = T'(t)$. The container possesses a finite thermal conductivity, and the heat inflow per unit time $\mathrm{d}Q$ satisfies

$$\mathrm{d}Q = k(T' - T)\mathrm{d}t \tag{19}$$

where k is the thermal conductivity coefficient of the container (a constant). Starting from time $t = 0$, a piston is driven to prescribe the pressure path of the ideal gas $P = P(t)$, while the ambient temperature variation $T' = T'(t)$ is observed. The objective is to determine the evolution of P, V, and T during this quasi-static process.

To model this problem. the system is defined as the ideal gas along with its heat exchange with the external environment. The target parameters are the pressure P, volume

V, and temperature T of the ideal gas. The corresponding ASD conceptual model is formulated as follows:

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{i}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} \tag{20}$$

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{e1}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{e1}} \tag{21}$$

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{e2}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{e2}} \tag{22}$$

$$\boldsymbol{S} + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e1}}, \boldsymbol{F}_{\mathrm{e2}}) \mapsto (\mathrm{d}\boldsymbol{S} = (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} + (\mathrm{d}\boldsymbol{S})_{\mathrm{e1}} + (\mathrm{d}\boldsymbol{S})_{\mathrm{e2}}) \tag{23}$$

This model comprises two external actions and one internal action, all of which are decouplable. Internal action $\boldsymbol{F}_{\mathrm{i}}$ represents the instantaneous heat exchange, which is independent of variations in both the pressure P and the ambient temperature T'. External action $\boldsymbol{F}_{\mathrm{e1}}$ represents the instantaneous variation in pressure P, which is independent of both the heat exchange and variation in the ambient temperature T'. External action $\boldsymbol{F}_{\mathrm{e2}}$ represents the instantaneous variation in the ambient temperature T', which is independent of both the heat exchange and variation in the pressure P.

Selecting pressure P, volume V, temperature T, and ambient temperature T' as state parameters, the state is $\boldsymbol{S} = (P, V, T, T')$. The state parameters not only include all target parameters but also introduce the ambient temperature T', ensuring that the internal action (heat exchange) satisfies the requirements of Axiom 2.

The external actions $\boldsymbol{F}_{\mathrm{e1}}$ and $\boldsymbol{F}_{\mathrm{e2}}$ are both quantified through external action paths, defined respectively as

$$\begin{cases} \boldsymbol{F}_{\mathrm{e1\ path}}(t) = P(t) \\ \boldsymbol{F}_{\mathrm{e2\ path}}(t) = T'(t) \end{cases} \tag{24}$$

When the characteristic time scale $\mathrm{d}t$ approaches zero, these two external actions can be viewed as determined solely by the increments of their respective external action paths over the time interval $[t, t + \mathrm{d}t]$, namely

$$\begin{cases} \boldsymbol{F}_{\mathrm{e1}} = \mathrm{d}P \\ \boldsymbol{F}_{\mathrm{e2}} = \mathrm{d}T' \end{cases} \tag{25}$$

Based on the properties of an ideal gas in isobaric and adiabatic processes [10,11], the specific mathematical model can be obtained as

$$(\boldsymbol{S} = (P, V, T, T')) + \boldsymbol{F}_{\mathrm{i}} \mapsto \left((\mathrm{d}\boldsymbol{S})_{\mathrm{i}} = \left(0, \frac{Rk(T' - T)}{PC_P}\mathrm{d}t, \frac{k(T' - T)}{nC_P}\mathrm{d}t, 0\right)\right) \tag{26}$$

$$(\boldsymbol{S} = (P, V, T, T')) + \boldsymbol{F}_{\mathrm{e1}} \mapsto \left((\mathrm{d}\boldsymbol{S})_{\mathrm{e1}} = \left(\mathrm{d}P, -\frac{VC_V}{PC_P}\mathrm{d}P, \frac{V}{nC_P}\mathrm{d}P, 0\right)\right) \tag{27}$$

$$(\boldsymbol{S} = (P, V, T, T')) + \boldsymbol{F}_{\mathrm{e2}} \mapsto ((\mathrm{d}\boldsymbol{S})_{\mathrm{e2}} = (0, 0, 0, \mathrm{d}T')) \tag{28}$$

$$\begin{aligned} &(\boldsymbol{S} = (P, V, T, T')) + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e1}}, \boldsymbol{F}_{\mathrm{e2}}) \\ &\mapsto \left(\mathrm{d}\boldsymbol{S} = \left(\mathrm{d}P, -\tfrac{VC_V}{PC_P}\mathrm{d}P + \tfrac{Rk(T'-T)}{PC_P}\mathrm{d}t, \tfrac{V}{nC_P}\mathrm{d}P + \tfrac{k(T'-T)}{nC_P}\mathrm{d}t, \mathrm{d}T'\right)\right) \end{aligned} \tag{29}$$

Equation (29) provides the complete iterative relationship for the state evolution. Combined with the initial state $\boldsymbol{S}_0 = (P_0, V_0, T_0, T_0')$ at $t = 0$, the evolution of the target parameters P, V, and T can be obtained. For example, by observing the iterative relationship for temperature T in Equation (29) and utilizing the equation of state (Equation (18)), the following initial value problem can be derived:

$$\begin{cases} \frac{\mathrm{d}T}{\mathrm{d}t} = \left(\frac{R}{PC_P}\frac{\mathrm{d}P}{\mathrm{d}t} - \frac{k}{nC_P}\right)T + \frac{kT'}{nC_P} \\ T|_{t=0} = T_0 \end{cases} \tag{30}$$

This degenerates to the classical differential equation solution.

In the aforementioned problem, the external action $\boldsymbol{F}_{\mathrm{e}1}$ represents the arbitrary control over the pressure path $P = P(t)$, corresponding to a controllable input of the system. The external action $\boldsymbol{F}_{\mathrm{e}2}$ represents the variation in the ambient temperature $T' = T'(t)$, corresponding to an uncontrollable input introduced by environmental disturbances. The internal action $\boldsymbol{F}_{\mathrm{i}}$ corresponds to the heat exchange process driven by the temperature difference, which, in this specific problem, is likewise uncontrollable.

From a structural perspective, heat exchange is incorporated into the internal action $\boldsymbol{F}_{\mathrm{i}}$ within the conceptual model and is regarded as an intrinsic action of the system. To satisfy the requirement in Axiom 2 that "the internal action is determined by the state", all information governing the heat exchange process must be included in the system state $\boldsymbol{S}$ during the quantification process; this is the fundamental reason for introducing ambient temperature T' as a state parameter. The resulting state $\boldsymbol{S} = (P, V, T, T')$ thus contains not only information regarding the ideal gas itself but also the necessary external information concerning its interaction with the environment.

This case clearly demonstrates that the system boundary is not uniquely determined by the physical object, and the internal action does not necessarily originate solely from the object itself, but can include the actions exerted upon it by the external environment (such as the heat exchange here). In practical modeling, by taking "whether Axiom 2 is satisfied" as the criterion, one can complete a workable and rational partitioning of internal and external actions.

For the same problem, different modes of system partitioning and internal/external action partitioning can be adopted. For instance, if only the ideal gas itself is defined as the system, heat exchange may be reclassified as an external action $\boldsymbol{F}_{\mathrm{e}2}$ and quantified directly by the current ambient temperature, i.e., $\boldsymbol{F}_{\mathrm{e}2} = T'$. In this case, the state increment $(\mathrm{d}\boldsymbol{S})_{\mathrm{i}}$ of the system under internal action $\boldsymbol{F}_{\mathrm{i}}$ is zero, as no other intrinsic evolutionary mechanisms exist. The state parameters may be selected as $\boldsymbol{S} = (P, V, T)$, which contains only the information of the ideal gas itself. The definition of the external action $\boldsymbol{F}_{\mathrm{e}1}$ and its quantification method remain unchanged. The resulting specific mathematical model with a similar logical structure is

$$(\boldsymbol{S} = (P, V, T)) + \boldsymbol{F}_{\mathrm{i}} \mapsto ((\mathrm{d}\boldsymbol{S})_{\mathrm{i}} = (0, 0, 0)) \tag{31}$$

$$(\boldsymbol{S} = (P, V, T)) + \boldsymbol{F}_{\mathrm{e}1} \mapsto \left((\mathrm{d}\boldsymbol{S})_{\mathrm{e}1} = \left(\mathrm{d}P, -\frac{VC_V}{PC_P}\mathrm{d}P, \frac{V}{nC_P}\mathrm{d}P\right)\right) \tag{32}$$

$$(\boldsymbol{S} = (P, V, T)) + \boldsymbol{F}_{\mathrm{e}2} \mapsto \left((\mathrm{d}\boldsymbol{S})_{\mathrm{e}2} = \left(0, \frac{Rk(T' - T)}{PC_P}\mathrm{d}t, \frac{k(T' - T)}{nC_P}\mathrm{d}t\right)\right) \tag{33}$$

$$\begin{aligned}&(\boldsymbol{S} = (P, V, T)) + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e}1}, \boldsymbol{F}_{\mathrm{e}2})\\ &\mapsto \left(\mathrm{d}\boldsymbol{S} = \left(\mathrm{d}P, -\tfrac{VC_V}{PC_P}\mathrm{d}P + \tfrac{Rk(T'-T)}{PC_P}\mathrm{d}t, \tfrac{V}{nC_P}\mathrm{d}P + \tfrac{k(T'-T)}{nC_P}\mathrm{d}t\right)\right)\end{aligned} \tag{34}$$

Equation (34) equivalently yields the complete iterative relationship for all target parameters.

3.3. Stress–Strain Constitutive Relationships

Consider a material element in a triaxial test, as illustrated in Figure 5. Assume that at time $t = 0$, the element is in the initial state $\boldsymbol{S}_0$, and this initial state remains consistent across repeated tests. The research objective is to establish the stress–strain constitutive relationship [12–15] by observing the strain path $\boldsymbol{\varepsilon}(t)$ corresponding to an arbitrary human-controlled stress path $\boldsymbol{\sigma}(t)$. Neglect other minor factors such as heat conduction.

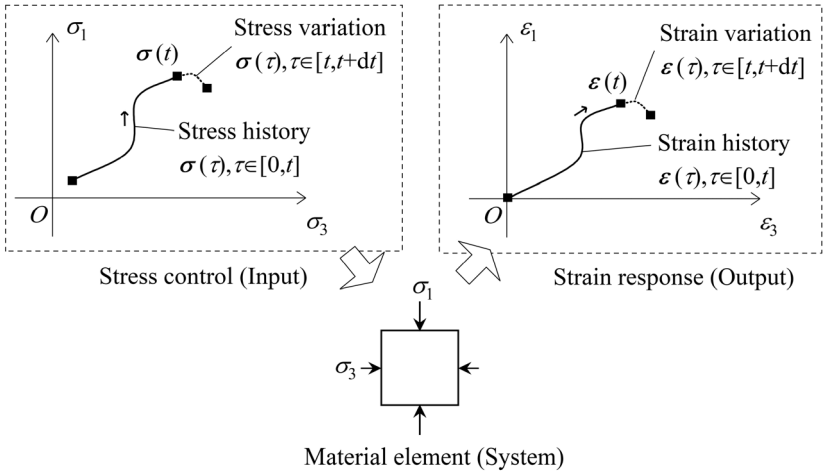


Figure 5. Stress path and strain path of a material element: establishing the stress–strain constitutive relationship under a given initial state.

To model this problem. The system is the material element. The target parameters are the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ of the element. Its ASD conceptual model is formulated as follows:

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{i}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} \tag{35}$$

$$\boldsymbol{S} + \boldsymbol{F}_{\mathrm{e}} \mapsto (\mathrm{d}\boldsymbol{S})_{\mathrm{e}} \tag{36}$$

$$\boldsymbol{S} + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{i}}, \boldsymbol{F}_{\mathrm{e}}) \mapsto (\mathrm{d}\boldsymbol{S} = (\mathrm{d}\boldsymbol{S})_{\mathrm{i}} + (\mathrm{d}\boldsymbol{S})_{\mathrm{e}}) \tag{37}$$

wherein the internal action $\boldsymbol{F}_{\mathrm{i}}$ represents the action determined by the intrinsic properties of the material (such as creep and rheology), which is independent of the variation in stress $\boldsymbol{\sigma}$. The external action $\boldsymbol{F}_{\mathrm{e}}$ represents the action induced by the variation in stress $\boldsymbol{\sigma}$. It can be seen that this problem is formally identical to Newtonian particle mechanics, yet its difficulty lies in the quantification of the state $\boldsymbol{S}$ and the external action $\boldsymbol{F}_{\mathrm{e}}$.

Due to the highly complex micro- and meso-structures of real-world materials, their stress–strain relationships typically exhibit significant path-dependency and rate-dependency, rendering the state $\boldsymbol{S}$ of the element difficult to be completely determined by a finite set of parameters. Even if two states are identical in all observable parameters, their subsequent stress–strain responses may still differ significantly. Therefore, this system is a typical non-Markovian system; without introducing additional assumptions, its state $\boldsymbol{S}$ is difficult to be fully quantified through the selection of state parameters.

The external action $\boldsymbol{F}_{\mathrm{e}}$ can still be quantified via its external action path $\boldsymbol{F}_{\mathrm{e\ path}}(t) = \boldsymbol{\sigma}(t)$, namely

$$\boldsymbol{F}_{\mathrm{e}} = (\boldsymbol{\sigma}(\tau), \tau \in [t, t + \mathrm{d}t]) \tag{38}$$

However, due to the path-dependency and rate-dependency of the material, the external action generally cannot be simplified to $\boldsymbol{F}_{\mathrm{e}} = \mathrm{d}\boldsymbol{\sigma}$.

To achieve an operational characterization of the system state $\boldsymbol{S}$, generalized state parameters must be introduced. The most direct approach is to employ the complete stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$ as the generalized state parameter, whereby the problem transforms into

$$\boldsymbol{S} = (\boldsymbol{\sigma}(\tau), \tau \in [0, t]), \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{S}) \tag{39}$$

This indicates that the current state of a material element is determined by the loading history; this conclusion can be obtained as a direct corollary within the ASD framework.

Different constitutive models in material mechanics are essentially based on different assumptions, extracting key information that dominates material behavior from the com-

plete stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$, and utilizing it to construct generalized state parameters in various forms, thereby achieving computability and varying degrees of simplification for the model.

For instance, in rate-independent models, the generalized state parameter can be chosen as the geometric trajectory L_t of the stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$ in the stress space. The problem then transforms into

$$\boldsymbol{S} = L_t, \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(L_t) \tag{40}$$

From the perspective of differential geometry, the stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$, as a regular parameterized curve in the stress space, maintains its geometric trajectory L_t invariant under any regular parameter transformation, and its corresponding current state $\boldsymbol{S}$ also remains invariant, thereby reflecting rate-independence of loading.

On this basis, classical elastic models and elasto-plastic models are the result of further compression of the history information L_t. In elastic models, the generalized state parameter degenerates to the current stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t)$ at the terminal point of the trajectory L_t, yielding

$$\boldsymbol{S} = \boldsymbol{\sigma}, \ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \tag{41}$$

At this point, the system completely loses its history-dependence; the generalized state parameter degenerates into a general state parameter, and the system degenerates into a Markovian system.

In elasto-plastic models, intermediate variables such as internal variables $\boldsymbol{\alpha}$ and plastic strain $\boldsymbol{\varepsilon}^{\mathrm{p}}$ are further introduced. These quantities generally cannot be uniquely measured at a given current state $\boldsymbol{S}$, thus they do not belong to well-defined system parameters, but are essentially methods for encoding history information. After incorporating them with the current stress $\boldsymbol{\sigma}$ into the generalized state parameters, we have

$$\boldsymbol{S} = (\boldsymbol{\alpha}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{\sigma}), \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\alpha}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{\sigma}) \tag{42}$$

This is the history compression method constructed by classical elasto-plastic models.

From a structural perspective, the introduction of generalized state parameters essentially shifts the complexity of system modeling from "determining the state" to "determining the relationship between the state and the target parameters." For instance, if the complete stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$ is taken as the generalized state parameter, then the state $\boldsymbol{S}$ of the system at any given time is logically fully determined. However, this state $\boldsymbol{S}$ itself does not directly contain information regarding the strain $\boldsymbol{\varepsilon}$, but exists merely as a quantified placeholder. To establish a valid constitutive relationship, one must further determine the functional relationship of the strain $\boldsymbol{\varepsilon}$ with respect to the state $\boldsymbol{S}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}(\tau), \tau \in [0, t]) \tag{43}$$

This relationship is typically neither obtainable in an analytical form nor exhaustible through experiments covering all possible loading histories; consequently, it lacks operability and verifiability in practice. Thus, it is evident that while generalized state parameters achieve a logically complete characterization of the state, they simultaneously introduce a high degree of complexity in functional expression.

In scenarios where the system state is difficult to quantify directly, the introduction of generalized state parameters remains fundamentally significant. This is because only after the state is quantified is it possible to systematically record and organize the evolution laws of the target parameters in experiments, and on this basis, gradually identify the historical information that plays a dominant role, remove redundancy, and thereby construct more

operable generalized state parameters. For instance, different constitutive models compress the stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$ into finite-dimensional generalized state parameters $\boldsymbol{G}$ through various history compression methods, thereby transforming the problem into

$$\boldsymbol{S} = \boldsymbol{G}, \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{G}) \tag{44}$$

At this point, the functional relationship is correspondingly simplified. However, the model complexity has not vanished; rather, it has been transferred not only to the requirement for the validity of $\boldsymbol{G}$, namely, ensuring that the material response satisfies the single-valued relationship $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{G})$, but also to the determination of $\boldsymbol{G}$ from the stress history, which typically involves complex iterative procedures.

Therefore, from a methodological perspective, the core task of material constitutive theory can be understood as finding an appropriate history compression algorithm for the system, that is, replacing the complete loading history with finite-dimensional generalized state parameters while maintaining the primary physical behaviors. Within the ASD framework, the differences between various constitutive models can be unifiedly understood as seeking a balance between the effective retention of historical information and the sufficient simplification of state expression, thereby achieving the management of system complexity.

Comparing the material element example with Newtonian particle mechanics, the fundamental difference between the two in the partition of internal and external actions can be clearly seen. If the complete stress history $\boldsymbol{\sigma}(\tau), \tau \in [0, t]$ is taken as the generalized state parameter, the strain increment $\mathrm{d}\boldsymbol{\varepsilon}$ can be decomposed into two parts generated by internal and external actions, respectively:

$$(\mathrm{d}\boldsymbol{\varepsilon})_{\mathrm{i}} = \boldsymbol{\varepsilon}\left(\left\{ \begin{array}{l} \boldsymbol{\sigma}(\tau), \tau \in [0, t] \\ \boldsymbol{\sigma}(t), \tau \in [t, t + \mathrm{d}t] \end{array} \right) - \boldsymbol{\varepsilon}(\boldsymbol{\sigma}(\tau), \tau \in [0, t]) \right. \tag{45}$$

$$(\mathrm{d}\boldsymbol{\varepsilon})_{\mathrm{e}} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}(\tau), \tau \in [0, t + \mathrm{d}t]) - \boldsymbol{\varepsilon}\left(\left\{ \begin{array}{l} \boldsymbol{\sigma}(\tau), \tau \in [0, t] \\ \boldsymbol{\sigma}(t), \tau \in [t, t + \mathrm{d}t] \end{array} \right) \right. \tag{46}$$

wherein $(\mathrm{d}\boldsymbol{\varepsilon})_{\mathrm{i}}$ represents the strain increment produced when the stress $\boldsymbol{\sigma}$ is held constant at the current value $\boldsymbol{\sigma}(t)$ during the time interval $[t, t + \mathrm{d}t]$, corresponding to the baseline response of the material in its current state (such as creep behavior), as shown in Figure 6. Conversely, $(\mathrm{d}\boldsymbol{\varepsilon})_{\mathrm{e}}$ represents the additional strain caused by the change of $\boldsymbol{\sigma}$ relative to $\boldsymbol{\sigma}(t)$ within the interval $[t, t + \mathrm{d}t]$, namely the deviation of the system from the baseline response. The latter cannot be observed directly, but needs to be identified indirectly with the aid of Principle 1 or control experiments.

It must be emphasized that for a material element, the state increment $(\mathrm{d}\boldsymbol{S})_{\mathrm{i}}$ corresponding to the internal action $\boldsymbol{F}_{\mathrm{i}}$ is not the response under the condition of zero stress ($\boldsymbol{\sigma} = 0$), but the response under the condition where "the stress $\boldsymbol{\sigma}$ is held constant at the current value $\boldsymbol{\sigma}(t)$." Although this stress originates physically from the external environment (such as the confining pressure medium and loading device in a triaxial test), its numerical information $\boldsymbol{\sigma}(t)$ has already been incorporated into the current state $\boldsymbol{S}$ at time t; therefore, it is assimilated as the internal action $\boldsymbol{F}_{\mathrm{i}}$ in constitutive modeling, thus still satisfying the requirement of Axiom 2 that "the internal action is determined by the current state."

Conversely, the state increment $(\mathrm{d}\boldsymbol{S})_{\mathrm{e}}$ corresponding to the external action $\boldsymbol{F}_{\mathrm{e}}$ originates from the change in stress $\boldsymbol{\sigma}$ relative to $\boldsymbol{\sigma}(t)$ during the time interval $[t, t + \mathrm{d}t]$, characterizing the net response of the system to additional disturbances, i.e., the deviation relative to the baseline evolution. If the stress $\boldsymbol{\sigma}$ remains constant at $\boldsymbol{\sigma}(t)$ during the time interval $[t, t + \mathrm{d}t]$, then $(\mathrm{d}\boldsymbol{S})_{\mathrm{e}}$ is zero, and the external action $\boldsymbol{F}_{\mathrm{e}}$ can be regarded as zero.

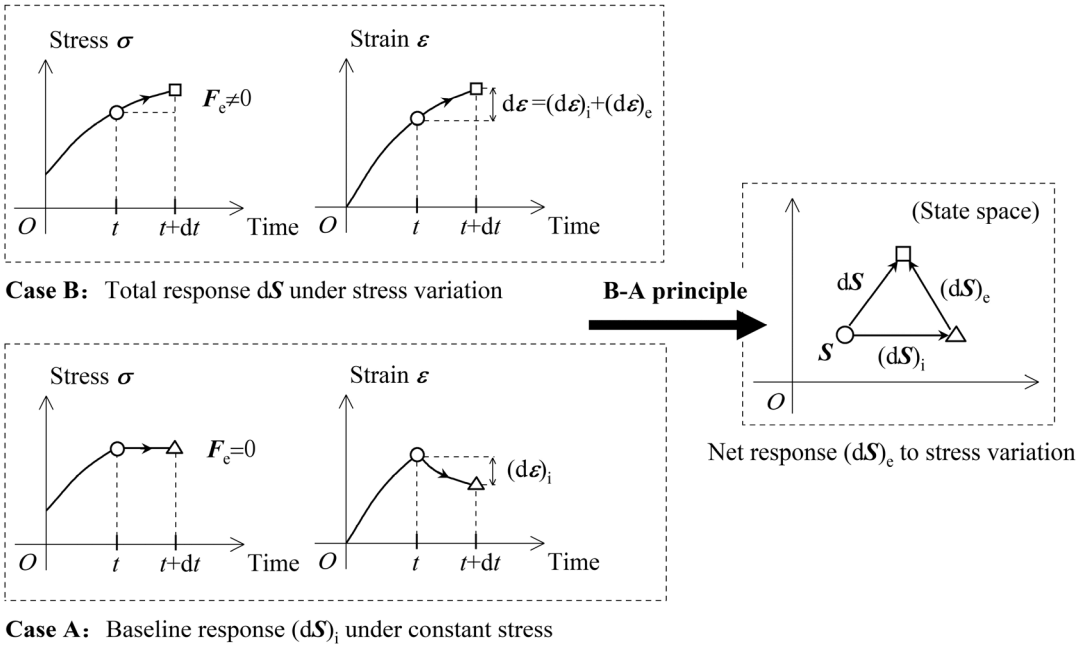


Figure 6. Partition of internal and external actions. The net response of the material element to stress variation (external action) is indirectly identified through Principle 1.

Thus, it is evident that the partition of internal and external actions is not solely determined by physical attribution, but depends on the modeling objective. In practical problems, materials are typically situated within an existing stress environment, and research interest often focuses on their response under additional loads or stress increments. Therefore, treating "constant at current stress" as the baseline for internal action or the condition of zero external action, and regarding "stress change" as the external action, is a more operable modeling choice.

3.4. Transformations in R^3 Space

To further illustrate the existence and composition of state parameters, consider an abstract system. As shown in Figure 7, taking points in the R^3 space as the system, whose coordinates (x, y, z) constitute the complete state $\boldsymbol{S}$. Define two types of transformations:

$$F_{k,l}(x, y, z) = ((k + z)x, (l + z)y, (k + l)z), k, l > 0 \tag{47}$$

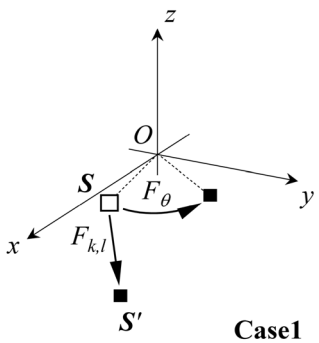
$$F_\theta(x, y, z) = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y, z),\ \theta \in \mathrm{R} \tag{48}$$

Establish a time-discrete ASD conceptual model:

$$\boldsymbol{S} + \boldsymbol{F}_e \mapsto \boldsymbol{S}' \tag{49}$$

where $\boldsymbol{S}'$ is the state at the next time step. The external action $\boldsymbol{F}_e$ represents the application of transformation $F_{k,l}$ or F_θ to the system. Internal action $\boldsymbol{F}_i$ is neglected here (i.e., the system undergoes no spontaneous evolution).

Consider the following problem: the target parameter is only a subset of x, y, z. One can repeatedly apply arbitrary external actions $\boldsymbol{F}_e$, that is, applying transformation $F_{k,l}$ or F_θ, where parameters k, l and θ can be arbitrarily specified, but their specific functional forms (Equations (47) and (48)) are unknown a priori. The research objective is to introduce appropriate state parameters through observing the evolution of the target parameter under different external actions $\boldsymbol{F}_e$, so as to construct a mathematical model characterizing the system behavior.



	Case1	Case2	Case3	Case4
Target parameters	x	x,y	x	x
Initial state $\boldsymbol{S}_0$	(1,1,0)	(1,1,0)	(1,1,1)	(1,1,0)
Set of external action $\boldsymbol{F}_e$	$F_{k,l}$	$F_{k,l}$	$F_{k,l}$	$F_{k,l}$, F_θ
Minimal state parameters	x	x,y	x,z	x,y

Figure 7. Transformations in R^3 space: modeling the evolution of coordinates driven by two types of transformations (external actions).

Figure 7 outlines four typical cases. Comparing Case 1 with Case 2–Case 4 reveals that the minimal state parameters depend on the selection of target parameters, the initial state of the system $\boldsymbol{S}_0$, and the permissible set of external actions. The minimal state parameters refer to the minimal set of parameters that, under these given conditions, achieves closure of the system dynamics and completely determines the evolution of the target parameters.

It can be discovered that, in all cases, the minimal state parameters differ from the complete state (x, y, z). This is because the description of the evolution of target parameters under permissible external actions does not require all information of the system. Through sufficient "experiments," a mathematical model that replaces the complete state with the minimal state parameters can be obtained. Corresponding to Case 1–Case 4, the respective formulations are

$$(\boldsymbol{S} = x) + (\boldsymbol{F}_e = F_{k,l}) \mapsto \left(\boldsymbol{S}' = kx\right) \tag{50}$$

$$(\boldsymbol{S} = (x,y)) + (\boldsymbol{F}_e = F_{k,l}) \mapsto \left(\boldsymbol{S}' = (kx, ly)\right) \tag{51}$$

$$(\boldsymbol{S} = (x,z)) + (\boldsymbol{F}_e = F_{k,l}) \mapsto \left(\boldsymbol{S}' = ((k+z)x, (k+l)z)\right) \tag{52}$$

$$\begin{cases} (\boldsymbol{S} = (x,y)) + (\boldsymbol{F}_e = F_{k,l}) \mapsto \left(\boldsymbol{S}' = (kx, ly)\right) \\ (\boldsymbol{S} = (x,y)) + (\boldsymbol{F}_e = F_\theta) \mapsto \left(\boldsymbol{S}' = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y)\right) \end{cases} \tag{53}$$

The aforementioned models can be understood as "partial truths" of the complete dynamics, that is, valid expressions under specific research objectives and external action constraints. Notably, in Case 3 and Case 4, the minimal state parameters contain components beyond the target parameters; if these components are unobservable, the evolution of the target parameters cannot be described in a closed form, in which case the state parameters do not exist under such conditions.

This case study demonstrates that the existence of state parameters is not a priori guaranteed; its existence and composition depend upon the selection of target parameters, the initial state of the system, and the set of external actions. The "system" is likewise not a fixed physical entity, but a dynamically closed structure constructed around the research objective.

3.5. Comparative Summary of the Examples

Although the ASD framework provides a unified description across different scientific problems, its concrete instantiations vary through different semantic interpretations of concepts and corresponding mathematical realizations. To facilitate comparison, Table 1 summarizes the examples presented in Sections 3 and 4 in terms of their target parameters, state quantification, internal and external actions, and the key ASD features illustrated by each example.

Table 1. Comparative summary of ASD instantiations across different examples.

Example		Target Parameter	State Quantification	Internal Action F_i	External Action F_e	Key ASD Feature
Newtonian particle mechanics		Position vector $\boldsymbol{x}$	State parameters $\boldsymbol{S} = (\boldsymbol{x}, \boldsymbol{v}, m)$	Inertia	Net external force	The simplest ASD conceptual model
Ideal gas (quasi-static process)		Pressure P, Volume V, Temperature T	State parameters $\boldsymbol{S} = (P, V, T, T')$	Heat exchange	Pressure control $P(t)$ & Ambient temp variation $T'(t)$	State parameters incorporate external environmental information T'
Material constitutive relationship		Stress σ, Strain ε	Generalized state parameters $\boldsymbol{S} = (\boldsymbol{\sigma}(\tau), \tau \in [0, t])$	Intrinsic evolution at constant stress	Stress variation relative to the current state	History dependent state quantification; constant stress defines the zero external action condition
Transformations in R^3 space		Coordinates x or x, y	State parameters $\boldsymbol{S} = (x)$, (x, y) or (x, z)	None	Transformations $F_{k,l}$ & F_θ	The minimal state parameter is determined by the target parameters, the initial state, and the set of external actions
1D soil consolidation	Soil element subsystem	Total stress σ, Effective stress σ', Pore water pressure u, Void ratio e	State parameters $\boldsymbol{S} = (\sigma, \sigma', u, e)$	Creep action $\boldsymbol{F}_c$	External loading action $\boldsymbol{F}_q$ & Seepage action $\boldsymbol{F}_s$	The conceptual model is structurally isomorphic to that of the ideal gas example
	Soil sample system	Spatial distribution fields $\sigma(z), \sigma'(z), u(z), e(z)$	State parameters $\boldsymbol{S}(z) = \begin{pmatrix} \sigma(z), \sigma'(z) \\ , u(z), e(z) \end{pmatrix}$	Creep action $\boldsymbol{F}_c$ & Seepage action $\boldsymbol{F}_s$	External loading action $\boldsymbol{F}_q$	The seepage action $\boldsymbol{F}_s$, which acts as an external action in the element subsystem, transforms into an internal action due to the expansion of the system boundary

4. ASD Reconstruction and Analysis of One-Dimensional Consolidation Theory

4.1. Establishment of the ASD Conceptual Model

In soil mechanics [16–19], the consolidation of soil refers to the process in which pore water is gradually expelled and the soil skeleton is gradually compressed under the action of external loads, usually accompanied by the dissipation of pore water pressure and the increase in effective stress.

Consider a soil sample with double-sided drainage boundaries, as shown in Figure 8. The research objective is to describe the spatiotemporal evolution of the total stress σ, effective stress σ', pore water pressure u, and void ratio e within the soil sample after the external load $q = q(t)$ is applied starting from time $t = 0$.

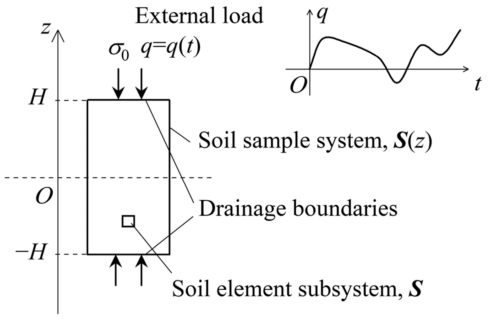


Figure 8. Schematic diagram of one-dimensional consolidation analysis.

To simplify the analysis, it is assumed that the soil is fully saturated, the soil particles and pore water are incompressible, small strain holds, and the total stress distribution $\sigma(z)$ is spatially uniform at any given time. Concurrently, the principle of effective stress is assumed to hold, namely

$$\sigma = \sigma' + u \tag{54}$$

To model this problem. The system, namely the soil sample, is composed of countless soil element subsystems. Denoting the state of an element as $\boldsymbol{S}$, the state of the soil sample can be expressed as the state distribution field $\boldsymbol{S}(z)$, which additionally implicitly contains the geometry and drainage boundary information of the soil sample. The target parameters are σ, σ', u, and e at the element level, and the spatial distribution fields $\sigma(z)$, $\sigma'(z)$, $u(z)$, and $e(z)$ of these parameters at the soil sample level. The corresponding ASD conceptual model is shown in Figure 9; A brief description of it is first provided, along with a preliminary quantification of some of the actions therein.

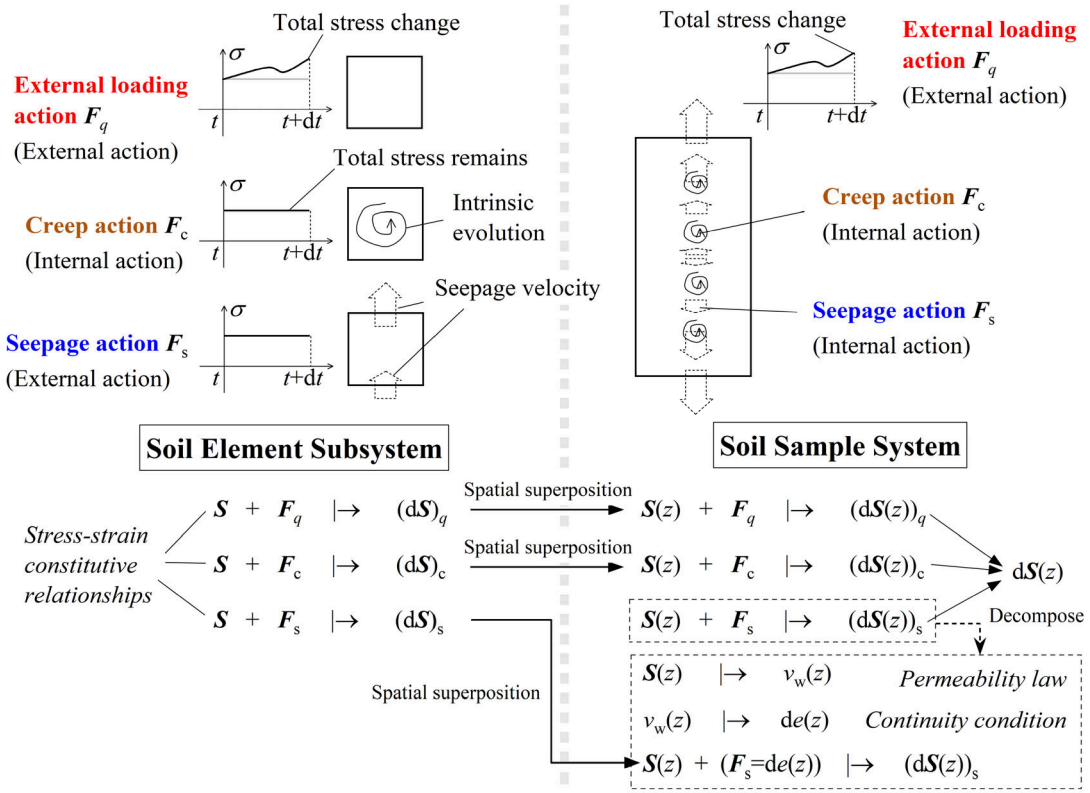


Figure 9. ASD conceptual model for one-dimensional consolidation, illustrating one internal action and two external actions at the soil element subsystem level, and two internal actions and one external action at the soil sample system level.

At the element level, the involved GCRs are as follows:

$$\boldsymbol{S} + \boldsymbol{F}_q \mapsto (\mathrm{d}\boldsymbol{S})_q \tag{55}$$

$$\boldsymbol{S} + \boldsymbol{F}_\mathrm{s} \mapsto (\mathrm{d}\boldsymbol{S})_\mathrm{s} \tag{56}$$

$$\boldsymbol{S} + \boldsymbol{F}_\mathrm{c} \mapsto (\mathrm{d}\boldsymbol{S})_\mathrm{c} \tag{57}$$

These include three decouplable actions, corresponding to three stress–strain constitutive relationships. Among them, the external loading action $\boldsymbol{F}_q$ represents the change in total stress σ, and the seepage action $\boldsymbol{F}_\mathrm{s}$ represents the effects induced by the seepage process,

both of which are external actions. The creep action $\boldsymbol{F}_{\mathrm{c}}$ is the internal action of the element, reflecting the intrinsic properties of the soil (such as creep and rheological characteristics).

The external loading action $\boldsymbol{F}_q$ possesses an external action path $\boldsymbol{F}_{q\,\mathrm{path}}(t) = q(t)$, meaning that the change in total stress σ is controlled by the external loading path $q = q(t)$.

Under the assumption that soil particles and pore water are incompressible, the direct effect of the seepage process on the element can be characterized by the change in void ratio. Therefore, taking the void ratio increment $(\mathrm{d}e)_{\mathrm{s}}$ caused by seepage within the time interval $[t, t + \mathrm{d}t]$ as the quantification method for the seepage action $\boldsymbol{F}_{\mathrm{s}}$, namely

$$\boldsymbol{F}_{\mathrm{s}} = (\mathrm{d}e)_{\mathrm{s}} \tag{58}$$

Under the current assumptions, the change in void ratio originates exclusively from the seepage process, thus $(\mathrm{d}e)_{\mathrm{s}} = \mathrm{d}e$, whereby

$$\boldsymbol{F}_{\mathrm{s}} = \mathrm{d}e \tag{59}$$

It must be pointed out that at the element level, the system dynamics cannot naturally achieve closure, because the $\mathrm{d}e$ induced by seepage cannot be directly input into the system as a control variable like the external load q.

At the soil sample level, the involved GCRs are as follows:

$$\boldsymbol{S}(z) + \boldsymbol{F}_q \mapsto (\mathrm{d}\boldsymbol{S}(z))_q \tag{60}$$

$$\boldsymbol{S}(z) + \boldsymbol{F}_{\mathrm{s}} \mapsto (\mathrm{d}\boldsymbol{S}(z))_{\mathrm{s}} \tag{61}$$

$$\boldsymbol{S}(z) + \boldsymbol{F}_{\mathrm{c}} \mapsto (\mathrm{d}\boldsymbol{S}(z))_{\mathrm{c}} \tag{62}$$

$$\boldsymbol{S}(z) + \boldsymbol{F}(\boldsymbol{F}_q, \boldsymbol{F}_{\mathrm{s}}, \boldsymbol{F}_{\mathrm{c}}) \mapsto \left(\mathrm{d}\boldsymbol{S}(z) = (\mathrm{d}\boldsymbol{S}(z))_q + (\mathrm{d}\boldsymbol{S}(z))_{\mathrm{s}} + (\mathrm{d}\boldsymbol{S}(z))_{\mathrm{c}}\right) \tag{63}$$

The GCRs of the soil sample regarding the external loading action $\boldsymbol{F}_q$ and creep action $\boldsymbol{F}_{\mathrm{c}}$ are merely the spatial superpositions of the corresponding relationships of the soil element. However, the GCR of the soil sample regarding the seepage action $\boldsymbol{F}_{\mathrm{s}}$ is further decomposed into the following three groups of causal relationships:

$$\boldsymbol{S}(z) \mapsto v_{\mathrm{w}}(z) \tag{64}$$

$$v_{\mathrm{w}}(z) \mapsto \mathrm{d}e(z) \tag{65}$$

$$\boldsymbol{S}(z) + (\boldsymbol{F}_{\mathrm{s}} = \mathrm{d}e(z)) \mapsto (\mathrm{d}\boldsymbol{S}(z))_{\mathrm{s}} \tag{66}$$

Equation (64) represents that the state distribution $\boldsymbol{S}(z)$ at time t (primarily the pore water pressure distribution $u(z)$ therein) determines the seepage velocity distribution $v_{\mathrm{w}}(z)$ at time t, which is an abstract expression of the permeability law. Equation (65) represents that the seepage velocity distribution $v_{\mathrm{w}}(z)$ at time t determines the void ratio increment distribution $\mathrm{d}e(z)$ over the time interval $[t, t + \mathrm{d}t]$, which is an abstract expression of the continuity condition (conservation of liquid phase mass). Equation (66) is the spatial superposition of the GCRs of the elements regarding the seepage action $\boldsymbol{F}_{\mathrm{s}}$.

Under the current assumptions, the continuity condition already possesses an explicit mathematical expression:

$$\mathrm{d}e(z) = -(1 + e_0)\frac{\partial v_{\mathrm{w}}}{\partial z}\mathrm{d}t \tag{67}$$

where e_0 is the initial void ratio.

Notably, for the soil sample, the external loading action $\boldsymbol{F}_q$ is the sole external action, whereas the seepage action $\boldsymbol{F}_{\mathrm{s}}$ and creep action $\boldsymbol{F}_{\mathrm{c}}$ are both internal actions. The selection of

the system boundary alters the method of partitioning internal and external actions, yet this partition still satisfies Axiom 2.

In summary, this ASD conceptual model constructs a system dynamics analysis framework that couples multiple scales (soil sample and element) and multiple physical mechanisms (external loading, seepage, and creep). Within this framework, it is only necessary to appropriately quantify the element state $\boldsymbol{S}$, select sub-mathematical models that are mutually compatible under the basic assumptions for the three types of element stress–strain constitutive relationships (Equations (55)–(57)) and the permeability law (Equation (64)), and then combine them with the continuity condition which already possesses an explicit expression (Equation (65)). Through this integration, the complete iterative relationship for the soil sample state $\boldsymbol{S}(z)$ can be further derived (Equation (63)), thereby achieving closure of the system dynamics and obtaining a specific consolidation model.

4.2. Degradation of Classical Consolidation Theories

Based on the ASD conceptual model, this section deduces several classical consolidation theories through appropriate sub-model selection and mechanism trimming, thereby revealing their isomorphism in logical structure.

Selecting all target parameters directly as the state parameters of the element, the element state is $\boldsymbol{S} = (\sigma, \sigma', u, e)$, and the soil sample state is $\boldsymbol{S}(z) = (\sigma(z), \sigma'(z), u(z), e(z))$.

First, consider exclusively the seepage action $\boldsymbol{F}_\mathrm{s}$. The permeability law (Equation (64)) is selected as Darcy's law:

$$v_\mathrm{w}(z) = -\frac{k_\mathrm{v}}{\gamma_\mathrm{w}} \frac{\partial u}{\partial z} \tag{68}$$

where the unit weight of water γ_w is constant, and the coefficient of permeability k_v is assumed to be a model constant. The element constitutive relationship under seepage action $\boldsymbol{F}_\mathrm{s}$ (Equation (56)) is selected as a linear elastic form:

$$\left(\boldsymbol{S} = \left(\sigma, \sigma', u, e\right)\right) + \left(\boldsymbol{F}_\mathrm{s} = \mathrm{d}e\right) \mapsto \left(\left(\mathrm{d}\boldsymbol{S}\right)_\mathrm{s} = \left(0, -\frac{\mathrm{d}e}{a_\mathrm{v}}, \frac{\mathrm{d}e}{a_\mathrm{v}}, \mathrm{d}e\right)\right) \tag{69}$$

and the coefficient of compressibility a_v is assumed to be a model constant. Coupling the permeability law (Equation (68)), the constitutive relationship (Equation (69)) and the continuity condition (Equation (67)) yields the Terzaghi-form GCR of the soil sample regarding the seepage action $\boldsymbol{F}_\mathrm{s}$:

$$\begin{aligned} &\left(\boldsymbol{S}(z) = \left(\sigma(z), \sigma'(z), u(z), e(z)\right)\right) + \boldsymbol{F}_\mathrm{s} \mapsto \\ &\left(\left(\mathrm{d}\boldsymbol{S}(z)\right)_\mathrm{s} = \left(0, -c_\mathrm{v} \frac{\partial^2 u}{\partial z^2} \mathrm{d}t, c_\mathrm{v} \frac{\partial^2 u}{\partial z^2} \mathrm{d}t, c_\mathrm{v} a_\mathrm{v} \frac{\partial^2 u}{\partial z^2} \mathrm{d}t\right)\right) \end{aligned} \tag{70}$$

where the coefficient of consolidation $c_\mathrm{v} = \frac{k_\mathrm{v}(1+e_0)}{\gamma_\mathrm{w} a_\mathrm{v}}$ is a constant.

In the static consolidation stage after the completion of instantaneous loading, since the external load remains constant, the external loading action $\boldsymbol{F}_q$ can be neglected. If the creep action $\boldsymbol{F}_\mathrm{c}$ is further neglected, then $\mathrm{d}\boldsymbol{S}(z) = \left(\mathrm{d}\boldsymbol{S}(z)\right)_\mathrm{s}$, at which point Equation (70) is precisely the complete iterative relationship for the soil sample state $\boldsymbol{S}(z)$. From the iterative relationship of the pore water pressure u therein, it is obtained that

$$\frac{\partial u}{\partial t} = c_\mathrm{v} \frac{\partial^2 u}{\partial z^2} \tag{71}$$

This result thus degenerates into the classical Terzaghi one-dimensional consolidation theory [20]. From the perspective of system dynamics, the core of Terzaghi's theory lies in providing the GCR of the soil sample regarding the seepage action $\boldsymbol{F}_\mathrm{s}$ (Equation (70)), which is a specific sub-model of the seepage mechanism module.

Further considering the creep action $\boldsymbol{F}_{\mathrm{c}}$ during the static consolidation process, the Yin-Graham elastic-viscoplastic model [21–23] is adopted, wherein the viscoplastic effective stress rate during the stress relaxation process is

$$\dot{\sigma}'_{\mathrm{vp}} = \dot{\sigma}'_{\mathrm{vp}}(\sigma', e) < 0 \tag{72}$$

Assuming this function is known, the Yin-Graham-form GCR of the soil sample regarding the creep action $\boldsymbol{F}_{\mathrm{c}}$ can be obtained:

$$\begin{array}{l}(\boldsymbol{S}(z) = (\sigma(z), \sigma'(z), u(z), e(z))) + \boldsymbol{F}_{\mathrm{c}} \\ \mapsto \left((\mathrm{d}\boldsymbol{S}(z))_{\mathrm{c}} = \left(0, \dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, -\dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, 0\right)\right)\end{array} \tag{73}$$

Since $\dot{\sigma}'_{\mathrm{vp}}$ is a function of (σ', e), and thereby also a function of the state $\boldsymbol{S} = (\sigma, \sigma', u, e)$, this sub-model is compatible with the selected state quantification method. Combining the Terzaghi-form GCR (Equation (70)) and the Yin-Graham-form GCR (Equation (73)) yields the complete iterative relationship for the soil sample state $\boldsymbol{S}(z)$:

$$\begin{array}{l}(\boldsymbol{S}(z) = (\sigma(z), \sigma'(z), u(z), e(z))) + \boldsymbol{F}(\boldsymbol{F}_{\mathrm{s}}, \boldsymbol{F}_{\mathrm{c}}) \mapsto \\ \left((\mathrm{d}\boldsymbol{S}(z)) = \left(0, -c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t + \dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t - \dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, c_{\mathrm{v}}a_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t\right)\right)\end{array} \tag{74}$$

From the iterative relationship of the pore water pressure u and void ratio e therein, it is obtained that

$$\begin{cases} \frac{\partial u}{\partial t} = c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2} - \dot{\sigma}'_{\mathrm{vp}} \\ \frac{\partial e}{\partial t} = c_{\mathrm{v}}a_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2} \end{cases} \tag{75}$$

This result corresponds to the static consolidation model wherein the creep mechanism is introduced on the basis of Terzaghi's theory.

Further considering the case where the external load varies with time. Taking the simplest one-dimensional strain state as an example, under this condition, the external load increment $\mathrm{d}q$ is borne entirely by the pore water pressure, and the GCR of the soil sample regarding the external loading action $\boldsymbol{F}_q$ is

$$\begin{array}{l}(\boldsymbol{S}(z) = (\sigma(z), \sigma'(z), u(z), e(z))) + (\boldsymbol{F}_q = \mathrm{d}q) \\ \mapsto \left((\mathrm{d}\boldsymbol{S}(z))_q = (\mathrm{d}q, 0, \mathrm{d}q, 0)\right)\end{array} \tag{76}$$

Coupling the Terzaghi-form GCR (Equation (70)), the Yin-Graham-form GCR (Equation (73)) and the GCR above (Equation (76)) yields the complete iterative relationship for the soil sample state $\boldsymbol{S}(z)$:

$$\begin{array}{l}(\boldsymbol{S}(z) = (\sigma(z), \sigma'(z), u(z), e(z))) + \boldsymbol{F}(\boldsymbol{F}_q, \boldsymbol{F}_{\mathrm{s}}, \boldsymbol{F}_{\mathrm{c}}) \mapsto \\ \left((\mathrm{d}\boldsymbol{S}(z)) = \left(\mathrm{d}q, -c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t + \dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, \mathrm{d}q + c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t - \dot{\sigma}'_{\mathrm{vp}}\mathrm{d}t, c_{\mathrm{v}}a_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}\mathrm{d}t\right)\right)\end{array} \tag{77}$$

Observing the iterative relationship of the pore water pressure u and void ratio e therein yields

$$\begin{cases} \frac{\partial u}{\partial t} = c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2} + \frac{\mathrm{d}q}{\mathrm{d}t} - \dot{\sigma}'_{\mathrm{vp}} \\ \frac{\partial e}{\partial t} = c_{\mathrm{v}}a_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2} \end{cases} \tag{78}$$

This model [24] is precisely the generalized form that simultaneously considers the creep mechanism and arbitrary external load variations on the basis of Terzaghi's consolidation theory.

The analysis above demonstrates that, within the ASD framework, these classical consolidation theories can be unified and understood as a family of models possessing a

common logical structure: by selecting mutually compatible state quantification methods, element constitutive relationships, and permeability laws, consolidation models with consistent structures but diverse mathematical forms can be generated. When the element constitutive relationships tend toward complexity, it is usually necessary to introduce the generalized state parameters described in Section 3.3. Although this action significantly increases the complexity of the equation forms, its underlying logical structure remains invariant.

From a methodological perspective, ASD reveals the generation mechanism of consolidation theories. Various specific models are not systems of partial differential equations independent of each other, but rather specific implementations of the same system dynamics "mother structure" under different state quantification strategies and different sub-model selections of physical mechanisms.

Furthermore, when the basic assumptions change (for instance, considering the compressibility of the soil skeleton and pore water, or extending to three-dimensional problems), ASD can still be employed to construct the corresponding conceptual model families, thereby systematically classifying, comparing, and extending more general forms of consolidation theories. This perspective provides an "operating-system-level" theoretical framework for cross-model analysis and new model construction.

4.3. Mechanistic Analysis of the Static Consolidation Process

Based on the ASD conceptual model, this section analyzes two ubiquitous problems in static consolidation theories: the inversion of causality and the mechanism confusion in the partition of primary and secondary consolidation, thereby illustrating the necessity of explicitly characterizing the logical structure.

4.3.1. Inversion of Causality in Consolidation Mechanics

Traditional consolidation theories typically posit that "the dissipation of pore water pressure u leads to the increase of effective stress σ'." However, from the perspective of system dynamics, this statement does not accurately reflect the causal relationships inherent in the consolidation process.

Based on the decomposition of the GCR regarding the seepage action $\boldsymbol{F}_{\mathrm{s}}$ (Equations (64)–(66)), the state change of the soil sample during static consolidation can be represented as the following causal chain:

$$u(z) \rightarrow v_{\mathrm{w}}(z) \rightarrow \mathrm{d}e \rightarrow \left(\mathrm{d}\sigma'\right)_{\mathrm{s}} \rightarrow \left(\mathrm{d}u\right)_{\mathrm{s}} \tag{79}$$

where

1. The non-uniform distribution of pore water pressure $u(z)$ generates the seepage velocity distribution $v_{\mathrm{w}}(z)$.
2. The non-uniform seepage velocity distribution $v_{\mathrm{w}}(z)$ leads to the reduction in element volume $\mathrm{d}e < 0$.
3. The volumetric deformation characterized by the magnitude of $\mathrm{d}e$ induces compression of the soil skeleton, leading to an increase in effective stress $\left(\mathrm{d}\sigma'\right)_{\mathrm{s}} > 0$.
4. By the principle of effective stress, the increase in effective stress $\left(\mathrm{d}\sigma'\right)_{\mathrm{s}}$ indirectly leads to an equivalent decrease in pore water pressure $\left(\mathrm{d}u\right)_{\mathrm{s}} = -\left(\mathrm{d}\sigma'\right)_{\mathrm{s}} < 0$.

The physical rationale for this conclusion lies in the fact that the conservation of liquid phase mass directly constrains the volume-related extensive variable $\mathrm{d}e$, rather than the intensive variables $\left(\mathrm{d}\sigma'\right)_{\mathrm{s}}$ and $\left(\mathrm{d}u\right)_{\mathrm{s}}$; the change in void ratio $\mathrm{d}e$ is the direct input of the seepage action $\boldsymbol{F}_{\mathrm{s}}$ into the element, independent of the specific distribution form of pore water pressure $u(z)$ that capable of generating this $\mathrm{d}e$. Concurrently, since the soil

skeleton dominates the mechanical properties of the element, the change in void ratio de first manifests as skeleton compression (analogous to spring compression), which in turn induces the change in effective stress $(\mathrm{d}\sigma')_{\mathrm{s}}$, whereas the change in pore water pressure $(\mathrm{d}u)_{\mathrm{s}}$ is passively generated solely via the principle of effective stress.

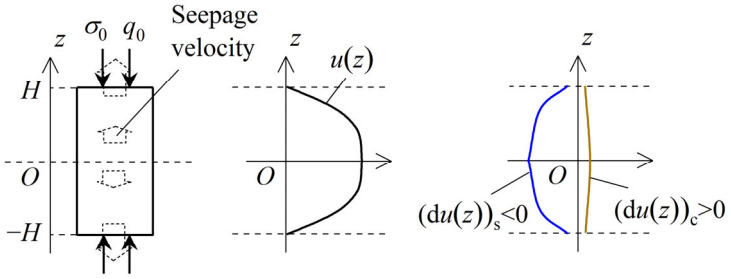
Therefore, from a mechanistic perspective, the common narrative that "the dissipation of pore water pressure leads to the increase of effective stress" contains a semantic inversion of causality.

In Terzaghi's consolidation theory, since only the seepage action $\boldsymbol{F}_{\mathrm{s}}$ is considered, the decrease in pore water pressure u and the increase in effective stress σ' are always synchronous and equivalent; hence, the aforementioned statement generally does not give rise to problems in calculation and engineering applications. However, when the problem is extended to scenarios where both the creep action $\boldsymbol{F}_{\mathrm{c}}$ and external loading action $\boldsymbol{F}_{q}$ coexist, the change in effective stress σ' is no longer induced solely by the seepage action $\boldsymbol{F}_{\mathrm{s}}$. If the simplified narrative is retained without clarifying the underlying physical processes and logical structures, the resulting framework becomes incapable of rigorously characterizing system evolution under multi-mechanism coupling. This may further introduce internal inconsistencies into theoretical generalization and weaken the explanatory capacity of the model under complex conditions.

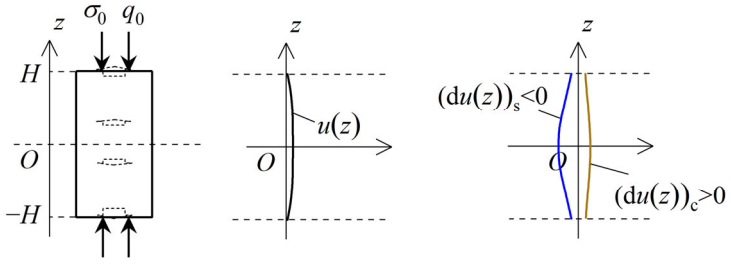
4.3.2. Mechanistic Confusion in Delineating Consolidation Phases

During the static consolidation, the soil sample is simultaneously driven by both the seepage action $\boldsymbol{F}_{\mathrm{s}}$ and the creep action $\boldsymbol{F}_{\mathrm{c}}$, as shown in Figure 10. The seepage action $\boldsymbol{F}_{\mathrm{s}}$ tends to dissipate pore water pressure ($(\mathrm{d}u(z))_{\mathrm{s}} < 0$), whereas the creep action $\boldsymbol{F}_{\mathrm{c}}$ manifested as stress relaxation tends to elevate it ($(\mathrm{d}u(z))_{\mathrm{c}} > 0$). The total pore water pressure increment satisfies

$$\mathrm{d}u(z) = (\mathrm{d}u(z))_{\mathrm{s}} + (\mathrm{d}u(z))_{\mathrm{c}} \tag{80}$$



Early stage, **primary consolidation phase**, seepage action $\boldsymbol{F}_{\mathrm{s}}$ dominates.



Later stage, **secondary consolidation phase**, Co-dominated by seepage action $\boldsymbol{F}_{\mathrm{s}}$ and creep action $\boldsymbol{F}_{\mathrm{c}}$

Figure 10. Competitive mechanisms governing pore water pressure evolution during static consolidation and the transition of dominant mechanisms across different phases.

In the early stage of consolidation, the large spatial gradient of $u(z)$ renders the seepage action $\boldsymbol{F}_{\mathrm{s}}$ highly dominant, such that

$$|(\mathrm{d}u(z))_{\mathrm{s}}| \gg |(\mathrm{d}u(z))_{\mathrm{c}}| \tag{81}$$

This means that the magnitude of the pore pressure increment induced by the seepage action $\boldsymbol{F}_{\mathrm{s}}$ is much larger than that induced by the creep action $\boldsymbol{F}_{\mathrm{c}}$. This stage features rapid pore pressure dissipation, which can be regarded as the primary consolidation phase.

In the later stage, as $u(z)$ approaches complete dissipation, the seepage action $\boldsymbol{F}_{\mathrm{s}}$ weakens significantly while the creep action $\boldsymbol{F}_{\mathrm{c}}$ becomes relatively prominent, yielding

$$|(\mathrm{d}u(z))_{\mathrm{s}}| \approx |(\mathrm{d}u(z))_{\mathrm{c}}| \tag{82}$$

This means that the magnitudes of the pore pressure increments induced by the seepage action $\boldsymbol{F}_{\mathrm{s}}$ and the creep action $\boldsymbol{F}_{\mathrm{c}}$ become comparable. The system then enters the secondary consolidation phase, where the seepage and creep mechanisms jointly govern the evolution of pore pressure.

Traditional theories typically adopt the "complete dissipation of pore water pressure" or the "inflection point of the e-lg$t$ curve" as the criteria for delineating the primary and secondary consolidation phases [25–27]. Fundamentally, this is an empirical criterion based on macroscopic response characteristics rather than on physical mechanisms. Concurrently, regarding questions such as "whether the creep mechanism exists in the primary consolidation phase" and "whether the seepage mechanism exists in the secondary consolidation phase", divergences have long existed in traditional theories, which precisely reflects that their underlying logical structures have not been explicitly depicted.

From a system dynamics perspective, the fundamental distinction between primary and secondary consolidation lies not in the presence or absence of a certain mechanism, but in the relative dominance of different mechanisms during system evolution. Within the ASD framework, a more rational partitioning method should be based upon the relative magnitude relationships of the contributions from each mechanism, such as the quantitative relationship between $|(\mathrm{d}u(z))_{\mathrm{s}}|$ and $|(\mathrm{d}u(z))_{\mathrm{c}}|$. This method of partitioning is capable of avoiding the uncertainties and controversies brought by empirical criteria based on macroscopic phenomena.

5. ASD Modeling Methodology

5.1. The Two-Layer Structure of Mathematical Models

Under the ASD framework, any mathematical model M can be decoupled into two core dimensions: a conceptual model C and a mathematical realization R, as shown in Figure 11.

The conceptual model (C) is primarily composed of GCRs and serves to expose the underlying logical structure of the mathematical model. It carries critical system-modeling information, including:

- The partitioning scheme and scale relationships between the system and its subsystems;
- The delineation and coupling of physical mechanisms (i.e., actions);
- The definition of system boundaries (i.e., the delineation between internal and external actions).

Ultimately, the conceptual model dictates which mechanisms must be modeled, at what scale they operate, and the precise causal chains and feedback loops that govern their interactions.

The mathematical realization (R) describes the mapping from the abstract conceptual model to a concrete, computable mathematical model, the core contents of which include:

- The selection of target parameters;
- The determination of the characteristic time scale dt;
- The quantification schemes for both system states and various actions (e.g., state parameters, generalized state parameters, and external action paths);
- The mathematical forms of the GCRs;
- The specific functional relationships of target parameters with respect to the system states.

Within the ASD semantic framework, a mathematical model M is formally defined as the combination of a conceptual model C and a mathematical realization R:

$$\mathrm{M} = \mathcal{M}(\mathrm{C}, \mathrm{R}) \tag{83}$$

where $\mathcal{M}$ represents the operator governing this combinative mapping.

Once conceptual model C and mathematical realization R are extracted from an existing mathematical model M, the conceptual model C constitutes the logical structure. By comparing the conceptual models of different mathematical models $\mathrm{M}_1 = \mathcal{M}(\mathrm{C}_1, \mathrm{R}_1)$ and $\mathrm{M}_2 = \mathcal{M}(\mathrm{C}_2, \mathrm{R}_2)$, it can be determined whether their differences originate from the logical structure (C-type differences) or merely the mathematical realization (R-type differences).

Consequently, ASD can be viewed as an "operating system" for organizing and managing mathematical models, whereas the expression $\mathrm{M} = \mathcal{M}(\mathrm{C}, \mathrm{R})$ constitutes the minimal logical unit of this system.

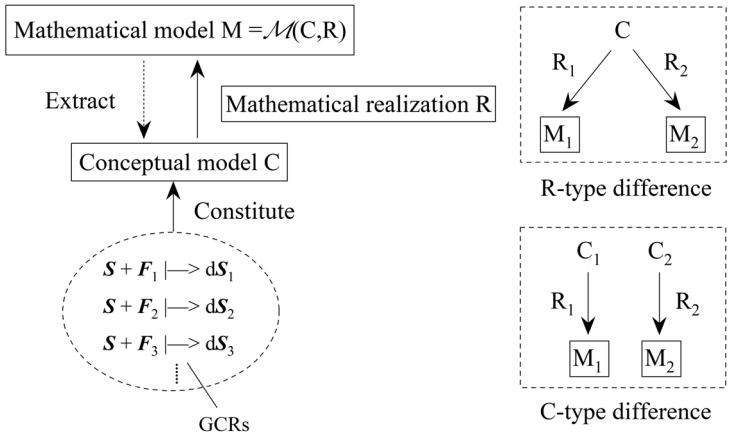


Figure 11. A mathematical model M is the combination of a conceptual model C and a mathematical realization R.

5.2. Modular Modeling Methodology

Scientific laws located at Region II of the spectrum (Figure 2) constitute the core components of meso-level deduction. In complex systems, these laws typically correspond to relatively independent physical mechanisms with straightforward logical structures, generally comprising a single internal action $\boldsymbol{F}_\mathrm{i}$ and a single external action $\boldsymbol{F}_\mathrm{e}$, as illustrated by the examples in Section 3.

For a complex system involving multiple physical mechanisms, its conceptual model C can be partitioned based on these physical mechanisms into a combination of several lower-level sub-conceptual models C_i, where each sub-conceptual model C_i corresponds to an independent physical mechanism module (Figure 12). Accordingly, each sub-mathematical

model $M_i = \mathcal{M}(C_i, R_i)$ is the instantiation of its sub-conceptual model C_i under a specific mathematical realization R_i.

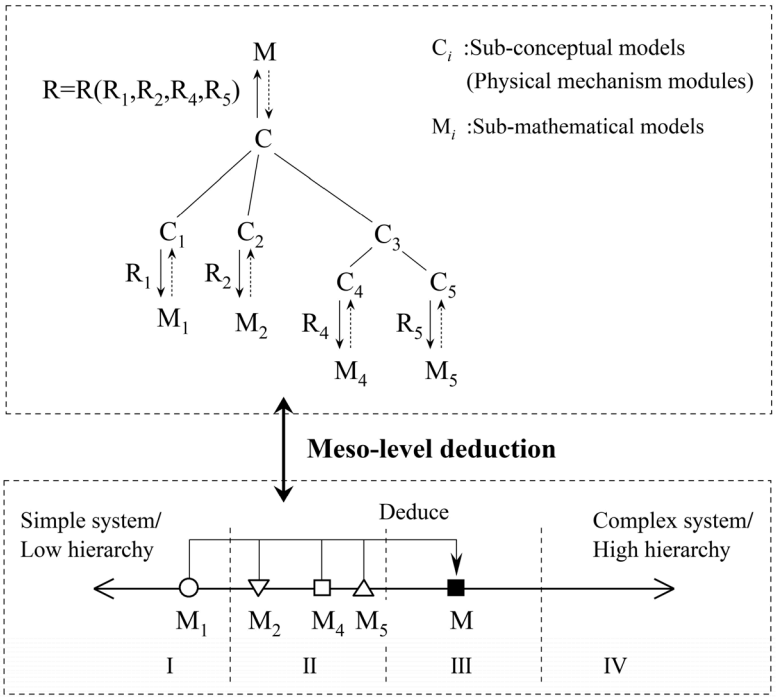


Figure 12. Modular meso-level deductive strategy of ASD, wherein the conceptual model is partitioned into multiple physical mechanism modules (sub-conceptual models).

For a given set of sub-mathematical models M_i, the necessary condition for achieving a logically consistent integration is that there exists a set of mathematical realizations R_i, such that each sub-mathematical model can be expressed as $M_i = \mathcal{M}(C_i, R_i)$, and each R_i remains compatible across key elements, including the selection of target parameters, the setting of characteristic time scales, and the quantification methods for system states and actions.

Upon satisfying the compatibility conditions, the individual sub-mathematical realizations R_i can be further synthesized into a unified mathematical realization $R = R(R_1, R_2, \ldots, R_n)$. From the synthesized conceptual model C and the synthesized mathematical realization R, the synthesized mathematical model $M = \mathcal{M}(C, R)$ of the complex system is obtained; the model is thus said to be derived from the sub-mathematical models M_i via meso-level deduction.

This modular modeling strategy of "conceptual model first, mathematical model second" establishes the combination of sub-mathematical models upon the foundation of consistent logical structures, thereby precluding systemic issues arising from structural inconsistencies. Its advantages manifest primarily in the following four aspects:

(1) Refined Management of Physical Mechanisms and Hierarchy

Modular representation renders each physical mechanism and its corresponding scale clearly discernible, thereby achieving a controllable trade-off between modeling complexity and precision. For different physical mechanism modules C_i, a mathematical model M_i matching its hierarchy can be selected, precluding the distortion of critical mechanisms or excessive complexity of the overall model.

(2) Extrapolation Capability and Model Parameter Management

All model parameters in the synthesized mathematical model M can be traced back to specific sub-models M_i, thereby clarifying their physical meanings and scope of application.

By analyzing the extrapolation capability of the sub-models M_i, the prediction confidence interval of the synthesized model M can be deduced. When predictions deviate from experimental results, module-level diagnosis and calibration can be executed, enhancing modeling efficiency.

(3) Modular Extension Strategies

R-type Extension (Evolution of Mathematical Realization): Under the premise of maintaining the conceptual model C invariant, local optimization is achieved by updating or replacing the sub-mathematical models M_i (i.e., the sub-mathematical realizations C_i) corresponding to certain physical mechanism modules C_i, featuring high transparency and controllability.

C-type Extension (Evolution of Logical Structure): On the foundation of the original conceptual model C, a new model system is constructed by introducing new physical mechanism modules or refining existing modules to alter its logical structure. This category of extension typically corresponds to substantive breakthroughs at the theoretical level.

(4) Logical Isomorphism Analysis and Theoretical Classification

By comparing the conceptual models of various classes of mathematical models, their isomorphic relationships at the level of physical mechanisms, as well as the differences in their mathematical realizations regarding shared mechanisms, can be systematically identified, thereby providing a structured methodology for model classification and theoretical analysis.

5.3. Logical Foundations of Modular Modeling

The underlying logical foundations by which ASD supports modular modeling primarily include:

(1) Decoupling of State and Parameters

ASD formally and strictly distinguishes metaphysical state from epistemological parameters, thereby achieving the separation of the conceptual model C, mathematical realization R, and mathematical model M. In the initial phase of modeling, the state $\boldsymbol{S}$ can exist as an unquantified placeholder, allowing the analysis to focus a priori on the logical structure and causal relationships rather than the quantitative relationships among parameters.

(2) Axiom 2 and the Criteria for Boundary Delineation

Axiom 2 states that the necessary and sufficient condition for system boundary delineation is that the internal action $\boldsymbol{F}_{\mathrm{i}}$ can be uniquely determined by the system state $\boldsymbol{S}$. This criterion transforms empirical judgment into a verifiable condition and provides a model diagnostic mechanism: if experiments indicate that the system still exhibits non-unique evolution when driven solely by the internal action $\boldsymbol{F}_{\mathrm{i}}$, it indicates that the quantification of the state $\boldsymbol{S}$ is incomplete or that the system is subjected to unrecognized external actions.

(3) Principle 1 and the Foundation of Modularity

Unlike the first two axioms, which describe the abstract dynamics of system evolution, Principle 1 establishes an epistemological structure for identifying the contribution of external actions. It formalizes the logic of controlled experiments: the net effect of an external action cannot be inferred directly from the system response, but must be identified through controlled comparison. By comparing the response of a system subjected to an external action (B) with that of an otherwise identical system without the action (A), the net contribution of the external action is isolated as B-A, thereby distinguishing it from the system's intrinsic evolution.

This logic is widely used across scientific disciplines. In pharmacological studies, the efficacy of a drug is identified by comparing treated and control groups rather than by observing the treated group alone. Likewise, ablation studies in large language models quantify the contribution of a component by comparing model performance before and after its removal. These examples illustrate that Principle 1 provides a rigorous logical basis for causal inference in scientific research.

Principle 1 therefore establishes a logical foundation for the decomposition and combination of physical mechanisms. It enables each physical mechanism module C_i to be logically isolated and defined through controlled comparison, thereby preventing overlap or ambiguity among different mechanisms and supporting the modular representation of complex systems.

5.4. Limitations and Future Work

ASD is a formal modeling language rather than a specific physical theory. It prescribes the organizational mode of logical structures without directly imparting physical semantics to models. Consequently, for the same mathematical model M, the extraction of its conceptual model C possesses non-uniqueness and depends upon the research objectives and application contexts.

Viewed from this sense, ASD is more akin to an open logical compiler; it provides researchers with tools to organize and reconstruct models at the structural level, while concurrently imposing higher demands on physical intuition and modeling experience.

In the modeling of complex systems, to fairly compare different mathematical models, the "greatest common divisor" principle should be followed to extract the conceptual model—that is, to identify and preserve the physical mechanism modules shared by each model to the greatest extent possible, so as to achieve maximal structural comparability. On this basis, it can be effectively determined whether model differences originate from the mathematical realization (R-type) or the logical structure (C-type).

Future research must verify the applicability of ASD through cross-disciplinary applications and continuously refine its theoretical framework. Furthermore, category theory can be introduced to reconstruct ASD, formally describing the logic of state evolution through objects and morphisms, thereby achieving a categorical representation of the axiomatic system. This direction is expected to establish structural connections between ASD and dynamical systems theory, computational mathematics, and probability theory, while promoting its application in a wider range of scientific domains.

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Abbreviations

The following abbreviations are used in this manuscript:

ASD	Axiomatic System Dynamics
GCR	Generalized Constitutive Relationship

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