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A Sociological Model of Political Regimes in the Parisi–Talagrand and Sherrington–Kirkpatrick Framework: Imposed vs. Natural Replica Symmetry in Totalitarian Systems

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Abstract

This study proposes a theoretical–empirical framework for analyzing political regimes based on a structural analogy between electoral behavior and spin-glass systems in statistical physics. Society is modeled as a system of interacting agents (voters) influenced by both interpersonal interactions and external factors such as media and institutions, formalized through a social Hamiltonian. By introducing a partition function and free energy, political regimes are interpreted as distinct macroscopic phases governed by four effective macro-parameters: external field, conformism, interaction heterogeneity, and inverse social temperature. Democratic societies correspond to a multistable regime characterized by sensitivity to initial conditions and replica symmetry breaking (RSB), reflecting the coexistence of competing social configurations. Authoritarian regimes, in contrast, arise when a strong unidirectional external field, high conformism, and low effective social temperature stabilize a single dominant macroscopic state, producing a regime analogous to replica symmetry (RS). A central result of the model is the distinction between the predictability of macroscopic outcomes and structural social multistability, as well as between natural and externally imposed homogenization of collective behavior. To illustrate the empirical relevance of the framework, the model is applied to the transition from the Weimar Republic to the National Socialist regime (1919–1933), using aggregated electoral data to construct proxy indicators for the effective parameters governing social interactions. The proposed approach enables structural identification of early signals of authoritarian transition through changes in the parameters of social dynamics.

Keywords: spin-glass model; political sociology; replica symmetry; replica symmetry breaking; totalitarianism; democracy; Sherrington–Kirkpatrick model; SK model; Parisi–Talagrand approach

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1. Introduction

Voting behavior in society emerges from the interaction of individual preferences, social influence, information flows, and institutional factors. In societies with a large number of voters, where these influences are heterogeneous and often contradictory, a clear and unanimous consensus is rarely achieved. Instead, stable groups with differing political positions typically emerge—a phenomenon that can be interpreted as a form of social frustration.

Similar features are typical of spin-glass systems in statistical physics, which exhibit disordered interactions, multiple stable states, and a complex energy landscape. The analogy between frustration in spin-glass systems and political fragmentation in society naturally raises the question of whether the mathematical tools developed in this field can be applied to the study of electoral dynamics and collective social regimes.

In this context, a central role is played by the Sherrington–Kirkpatrick (SK) model [1], which describes the thermodynamic behavior of systems with a large number of random interactions [2]. The primary objective of this model is to determine the thermodynamic limit of the average free energy as the number of degrees of freedom becomes large. The solution to this problem is provided by the Parisi variational principle [3,4], which replaces classical optimization over specific configurations with optimization over probability distributions describing the structure of overlaps between states.

1.1. Mathematical Formulation of the Sherrington–Kirkpatrick (SK) Model

In the SK model, the system consists of N binary spins $\sigma_i \in \{-1, +1\}$, where each spin interacts with every other through random couplings $J_{ij} \sim \mathcal{N}(0,\ 1/N)$. The Hamiltonian (energy function) of the system is given by [5–7]:

$$H_N(\sigma) \ = \ - \sum_{1 \le i < j \le N}^{N} J_{ij}\sigma_i\sigma_j. \tag{1}$$

The central problem is to determine the thermodynamic limit of the average free energy as $N \to \infty$, defined by

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\log Z_N(\beta)],\ \ Z_N(\beta) = \sum_{\sigma} \exp(-\beta H_N(\sigma)). \tag{2}$$

Here, N is the number of spins, $\mathbb{E}$ denotes expectation with respect to the distribution of the random couplings J_{ij}, $\beta \ = 1/T$ is the inverse temperature, and $Z_N(\beta)$ is the partition function.

The modern form of the solution is expressed through the Parisi formula:

$$\mathcal{P}(\beta) \ = \ \inf_{m\in\mathcal{M}} \{log2 \ + \ \mathcal{F}(m;\beta)\}, \tag{3}$$

where:

- $\mathcal{M}$ is the class of nonnegative, nondecreasing functions $m : [0,1] \to [0,1]$ describing the hierarchy of overlaps between states;
- $\mathcal{F}(m;\beta)$ is a functional defined via stochastic differential equations or an equivalent recursive integral representation, capturing the system's energy landscape for a given shape of the overlap distribution $m(q)$;
- β is the inverse temperature.

Within this framework, the goal is not to identify a single configuration σ, but rather to optimize over the entire statistical structure of states. In other words, instead of asking "What is the best arrangement of spins?" (as in classical optimization), the Parisi formula answers the question: "Which distribution of overlaps between states yields the lowest free energy?"

The function $m(q)$ reflects the degree of similarity between typical states. For example:

- A constant form of the function ($m(q) \ = \ 0$) describes replica symmetry (RS), where all states are equally probable.
- A stepwise or continuously increasing form of the function describes a hierarchical organization of states, as observed in spin-glass phases.

The Parisi formula provides a hierarchical description of phase space and formalizes the concept of replica symmetry breaking (RSB), associated with systems that possess multiple competing stable states [8,9]. For more than two decades, this formula remained empirically validated and widely used but lacked a rigorous mathematical proof [10]. Its validity was eventually established rigorously by Michel Talagrand for the SK model [11,12]. Talagrand formulated a new type of ultra-variational principle that is not based on finding an extremum of a function or functional, but rather on optimization over measures (distributions) of functions describing the stratification of states in the spin-glass system. Instead of seeking a single optimal solution, Talagrand proved that the very structure of the distribution of states can be described through a specific functional whose minimization yields the value predicted by Parisi.

$$\mathcal{P}(\zeta) = log2 + \Phi(0,0;\zeta) - \frac{\beta^2}{2}\int_0^1 q d\zeta(q). \tag{4}$$

Here, $\zeta \in \mathcal{M}$ is a nonnegative, nondecreasing function (measure) on the interval $[0,1]$, representing the distribution of overlaps between two independent configurations.

The function $\Phi(t,x;\zeta)$ denotes the solution of a Parisi-type equation for a fixed choice of the parameter ζ, which encodes the hierarchical structure of overlaps between states. For a fixed measure ζ, the function Φ depends on the variables (t,x), while the dependence on ζ is parametric. With this clarification, and using the shortened notation $\Phi(t,x)$, the nonlinear backward Parisi-type differential equation satisfied by Φ can be written as:

$$\frac{\partial\Phi}{\partial t} = -\frac{\beta^2}{2}\zeta(t)\left(\frac{\partial\Phi}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial^2\Phi}{\partial t^2}, \qquad \Phi(1,x) = logcoshx. \tag{5}$$

Under these formal definitions, Michel Talagrand proved that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\log Z_N(\beta)] = \inf_{\zeta}\mathcal{P}(\zeta), \tag{6}$$

and that this infimum is attained precisely for the measure ζ predicted by Parisi. In this way, the Parisi hypothesis was confirmed in a rigorous probabilistic sense for the SK model. This result transforms the paradigm into a new type of variational approach, in which the objective is to optimize over a distribution function rather than over a parameter or a specific configuration.

1.2. Models of Social Dynamics and Positioning of the SK Approach

A substantial body of literature applies methods from statistical physics to the study of opinion dynamics and collective behavior. These approaches include voter-like processes and Ising-type models on networks and bounded-confidence models, as well as various rules for opinion updating [13–17]. A classical overview of these approaches and their empirical foundations is provided in [18], while more recent syntheses of contemporary research directions and the datasets used in this field are presented in [19,20]. The development of sociophysics and computational social science demonstrates a sustained and growing interest in such interdisciplinary modeling frameworks [21,22].

The present study does not introduce a new microscopic rule for opinion updating but instead employs the mean-field framework of the Sherrington–Kirkpatrick model as an analytical reference structure. Within this framework, the variational theory of Parisi and the rigorous formalization developed by Talagrand provide macroscopic objects for describing the multiplicity of stable or metastable social configurations. Our focus is therefore not on individual updating rules (e.g., the voter rule), but rather on the Gibbs measure, the free energy, and the geometry of the state space. This perspective enables the formulation of a typology of social regimes—consensus, fragmentation, and hierarchical clustering—in terms of RS and RSB. The empirical calibration of the model parameters remains a subject for future research.

2. Social Spin-Glass Model: Voting Behavior and Disordered Interactions: Adapting the SK Model to Social Systems

Understanding voting behavior in contemporary societies requires models that account not only for individual attitudes but also for collective effects, social influence, and the nonlinear dynamics of communication networks. Classical approaches in political sociology often interpret voting as a function of individual characteristics or rational choice [23–26]. However, growing empirical evidence shows that interdependence and collective dynamics play a crucial role in shaping political preferences [27].

In this context, spin-glass theory provides a suitable formalism for describing socially complex environments. In such models, each unit is subject both to external influences and to interactions with all others through stochastic connections that may be either cooperative or conflicting. This disordered and frustrated interaction structure is particularly well suited to modeling societies in which political preferences emerge under the influence of loyalty, pressure, uncertainty, and social resonance.

The aim of this section is to adapt the SK model to social systems by introducing a formal analogy between the elements of the physical model and the processes underlying voting behavior. Within this framework, a social Hamiltonian, a partition function, a free energy, and an overlap between configurations of public opinion are defined, providing the basis for analyzing RS and its breaking in a social context.

2.1. Definition of Social Analogs

The fundamental object in the model is a binary variable that, in the social context, represents an individual voter. We introduce the following correspondences between spin-glass theory and social processes:

- voters are modeled as binary variables (spins);
- social influence is described through interactions between spins;
- external factors are modeled as fields;
- collective social tension is described by a social Hamiltonian.

2.1.1. Voters as Spins

In the classical SK model, each spin σ_i takes values ± 1, reflecting two possible orientations. In a social context, this naturally corresponds to a binary choice of the *i*-th voter between two political alternatives:

$$\sigma_i = \begin{cases} +1, & if\ voter\ i\ chooses\ party\ A \\ -1, & if\ voter\ i\ chooses\ the\ alternative\ \overline{\mathrm{A}} \end{cases}. \tag{7}$$

Here, $\overline{\mathrm{A}} = \Omega \backslash A$, where Ω denotes the set of all possible party alternatives to A, including neutral choices of voters who do not support any specific party.

This binary formulation enables a rigorous mathematical description of collective voting behavior and is fully consistent with the results of the SK model and the Parisi–Talagrand variational approach. Multiparty systems can be treated through successive binary projections, where each party is analyzed relative to its complementary alternative.

2.1.2. Social Influence as Interaction

In the SK model, the coupling J_{ij} determines the nature of the interaction between two spins. Analogously, in the social model, J_{ij} is interpreted as interpersonal influence, realized through communication, group affiliation, or social networks. Formally, we assume:

$$J_{ij} = \mathcal{D}_{soc}\left(0,\ \sigma^2\right),\ \ i,j\ =\ 1,2,\ldots,N, \tag{8}$$

where $\mathcal{D}_{soc}$ is a probability distribution describing the diversity and randomness of social ties.

The interactions are assumed to be symmetric ($J_{ij} = J_{ji}$), statistically independent for different pairs, and to have variance scaling as $1/N$, ensuring the existence of a well-defined thermodynamic limit.

In the classical SK model, interactions are of the mean-field type, meaning that each pair of agents interacts with every other pair. In a social context, this assumption should be interpreted as an effective idealization that aggregates both direct and indirect channels of influence, including social networks, media environments, and institutional structures. To account for the limited cognitive and social connectivity observed in real societies (e.g., constraints related to Dunbar-type limits), a natural extension is to introduce a contact matrix A_{ij} and define the interactions as

$$J_{ij} = \mathrm{A}_{ij}\,\widetilde{J}_{ij}, \tag{9}$$

where $\widetilde{J}_{ij}$ represents the effective influence weights, while A_{ij} specifies the existence of a social tie between individuals i and j. In the simplest case, $\mathrm{A}_{ij}\ =\ 1$ only for actual social connections (with zero diagonal), and $\mathrm{A}_{ij}\ =\ 0$ otherwise. This leads to a diluted or network-based version of the model (diluted SK/Ising on a graph), which preserves the Hamiltonian structure while introducing a more realistic topology of social influence.

The standard assumption of zero-mean interactions $\mathbb{E}\left[J_{ij}\right]\ =\ 0$ is mathematically convenient but sociologically neutral. Empirical studies of social behavior, however, indicate a systematic tendency toward conformity and alignment of opinions within groups. To incorporate this effect, we introduce a nonzero mean interaction component:

$$J_{ij} = A_{ij}\left(\frac{J_0}{N}\ +\ \frac{J}{\sqrt{N}}\varepsilon_{ij}\right),\ J_0\ >\ 0, \tag{10}$$

where J_0 represents a global tendency toward agreement, while ε_{ij} captures heterogeneous and potentially conflicting interactions arising from intergroup tensions, competing identities, or asymmetries of influence.

This formulation allows different macroscopic regimes of social organization to emerge within a unified framework. When the conformist component J_0 dominates, the system tends toward a consensus regime characterized by a single stable macrostate. Conversely, when heterogeneity dominates, the system may enter a fragmented or metastable regime with multiple competing macroscopic configurations. In this context, RSB can be interpreted as structural social multistability—the coexistence of several stable configurations of collective opinion—rather than as universal antagonism among agents. Recent developments in spin-glass theory indicate that such multistable regimes arise in a wide class of complex systems beyond physics [28,29].

2.1.3. External Factors as Fields

The parameters h_i describe individualized external influences that do not arise from interpersonal interactions, such as media environment, economic interests, or cultural identity. In the simplest case, we assume that h_i are normally distributed with zero mean and variance λ^2:

$$h_i = \mathcal{N}\left(0,\ \lambda^2\right). \tag{11}$$

This indicates that external influences vary across voters but show no systematic bias toward any specific political alternative. In more realistic applications, the parameters h_i may also be determined empirically, for example, through demographic or media-related indices. Comparable parameterizations of social influence have been adopted in recent agent-based and network models [22,30,31].

2.1.4. Social Tension as a Hamiltonian

By combining interpersonal interactions and external fields, we define the social Hamiltonian as

$$H_N(\sigma) = -\sum_{1\le i<j\le N}^{N} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} h_i\sigma_i. \tag{12}$$

The first term reflects internal systemic tensions arising from agreements and conflicts among voters, while the second represents the influence of external factors. The Hamiltonian $H_N(\sigma)$ is interpreted as a measure of the social tension or conflict associated with a given configuration of political preferences. States with low values of $H_N(\sigma)$ correspond to relatively stable social configurations in which systemic tension is minimized under the imposed constraints.

Since the Hamiltonian $H_N(\sigma)$ is defined over a discrete set of 2^N binary configurations, its energy landscape cannot be directly visualized for large values of N. Therefore, for illustrative purposes, we consider a small system ($N = 9$) and apply principal component analysis (PCA) for dimensionality reduction. The resulting geometric representation (Figure 1) illustrates local minima corresponding to stable configurations, as well as regions of high social tension.

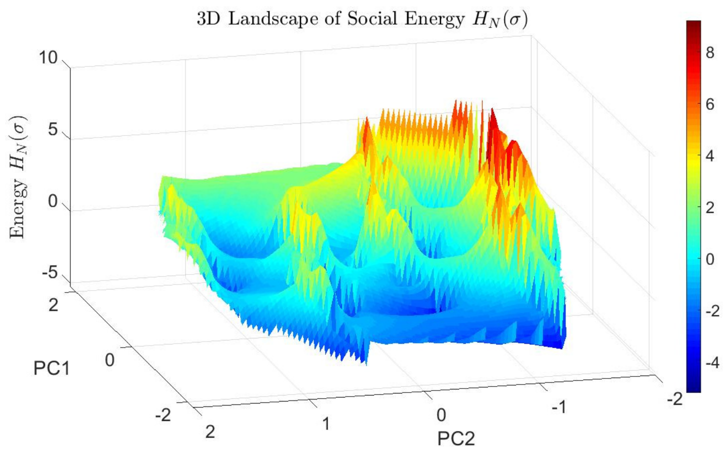

Figure 1. Undulating 3D surface illustrating an approximated energy landscape (MATLAB, 2018a).

2.2. Partition Function and Free Energy in a Social Context

In statistical mechanics, the partition function summarizes information about all possible configurations of a system and serves as the basis for deriving macroscopic quantities such as free energy and internal energy [32,33]. By analogy, in a social context it describes the probability distribution over collective electoral configurations.

We define the social partition function as

$$Z_N(\beta) = \sum_{\sigma\in\{+1,-1\}^N} \exp[-\beta H_N(\sigma)], \tag{13}$$

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where $H_N(\sigma)$ is the social Hamiltonian introduced in (12), and the parameter β is interpreted as the social interaction intensity—an analog of inverse temperature in statistical physics. Small values of β correspond to strong social fluctuations and fragmentation, whereas large values of β correspond to more ordered and socially cohesive states.

The normalized probability distribution over configurations is given by the Boltzmann–Gibbs form:

$$P_{N,\beta}(\sigma) = \frac{1}{Z_N(\beta)} \exp[-\beta H_N(\sigma)]. \quad (14)$$

To interpret distribution as the stationary measure of social dynamics, it is necessary to specify a class of micro-dynamics that realize it. A natural choice is a stochastic single-spin dynamic of the Glauber/Metropolis type (equivalent to a logit-choice rule), in which, at each discrete time step, a random individual iii is selected and a flip of their position $\sigma_i \to -\sigma_i$ is proposed.

If the configuration after the flip is denoted $\sigma^{(i)}$, the change is accepted with a probability that depends on the increment of social tension

$$\Delta H = \left[H_N\left(\sigma^{(i)}\right) - H_N(\sigma) \right], \quad (15)$$

for example,

$$\mathbb{P}(\sigma_i \to -\sigma_i \mid \sigma) = \frac{1}{1 + \exp[\beta \Delta H]}. \quad (16)$$

This dynamics satisfies the detailed balance condition with respect to the Hamiltonian, which guarantees that the stationary distribution is precisely the Boltzmann–Gibbs measure (14).

In the social interpretation, the parameter β measures the sensitivity of agents to social tension. For small values of β, fluctuations and individual deviations dominate, whereas for large values the system increasingly selects configurations with lower social tension. In this way, the distribution formalizes the idea that stable social configurations have a higher probability than configurations characterized by a high level of conflict.

We define the free energy as

$$F(\beta) = -\frac{1}{\beta N} \mathbb{E}[\log Z_N(\beta)], \quad (17)$$

where $\mathbb{E}$ denotes expectation with respect to the random interactions J_{ij} and external fields h_i.

In a social context, the free energy $F(\beta)$ serves as a macroscopic indicator of the system's stability and structure. Lower values of $F(\beta)$ correspond to more stable collective states, while higher values signal fragmentation and competing social equilibria. Through the analysis of $F(\beta)$, one can study phenomena such as polarization, consensus, and the presence of multiple stable social configurations.

Figure 2 shows the dependence of the free energy $F(\beta)$ for $\beta \in [0, 10]$ with a fixed number of voters $N = 200$.

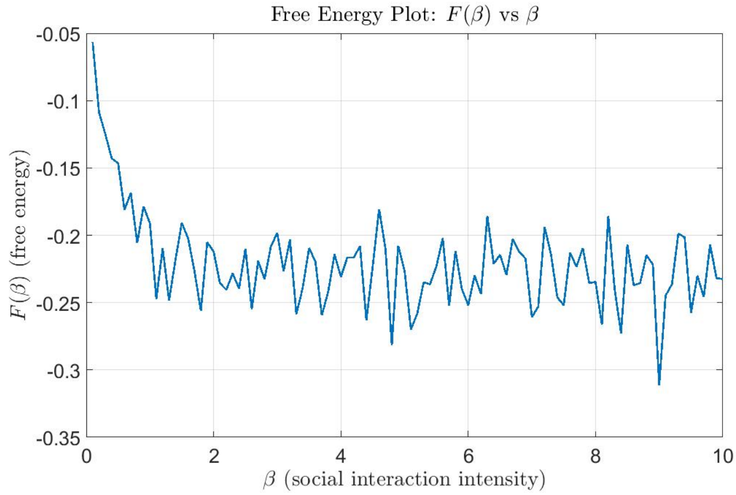

Figure 2. Free energy $F_{N,\beta}(\sigma)$ for $\beta \in [0, 10]$ and number of voters $N = 200$ (MATLAB, 2018a).

For small values of β, the free energy is higher, reflecting strong social fluctuations and the absence of a dominant macrostate. As β increases, the system transitions to a more ordered regime in which the free energy decreases and stabilizes.

As in classical statistical mechanics, the full significance of free energy emerges in the thermodynamic limit $N \to \infty$. In the social model, this corresponds to considering societies with a sufficiently large number of individuals, where $F(\beta)$ describes the global structure of voting behavior independently of individual fluctuations.

2.3. Overlap Between Configurations and RSB

To quantify the similarity between different societal configurations, we introduce the concept of overlap between voting configurations. This provides a measure of structural proximity between social states.

In spin-glass theory, this analysis is carried out using the replica method [8,9,12,34], in which n copies of the same system are considered. Formally, the average value of the logarithm of the partition function can be expressed through the identity

$$\mathbb{E}[\log Z] = \lim_{n \to \infty} \frac{\mathbb{E}[Z^n] - 1}{n}, \tag{18}$$

which allows the statistical structure of states to be studied through the mutual relationships between replicas.

Let $\sigma^{(\alpha)} = \left(\sigma_1^{(\alpha)}, \sigma_2^{(\alpha)}, \ldots, \sigma_N^{(\alpha)}\right)$ and $\sigma^{(\beta)} = \left(\sigma_1^{(\beta)}, \sigma_2^{(\beta)}, \ldots, \sigma_N^{(\beta)}\right)$ be two configurations (replicas) of a system with N voters. The overlap between them is defined as

$$q_{\alpha\beta} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(\alpha)} \sigma_i^{(\beta)}, \tag{19}$$

where $\sigma_i^{(\alpha)}, \sigma_i^{(\beta)} \in \{+1, -1\}$. The value $q_{\alpha\beta} \in [-1, 1]$ measures the degree of similarity between the two social configurations.

Under RS, all replicas are statistically equivalent and share the same typical overlap value, indicating the presence of a dominant macrostate. RSB arises when the state space becomes stratified into multiple stable but mutually distinct clusters that cannot be captured by a single average state.

The social interpretation is direct: RS ($q_{\alpha\beta} \approx 1$) corresponds to stable consensus or a clearly dominant majority, whereas RSB reflects fragmentation of society into multiple stable structures with high internal similarity and weak overlap between them. In limiting cases, this may appear as nearly independent configurations ($q_{\alpha\beta} \approx 0$) or as antagonistic social structures ($q_{\alpha\beta} \approx -1$).

To describe the global structure of overlaps, we introduce the distribution $P(q)$, which in the thermodynamic limit is defined as

$$P(q) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{n(n-1)} \sum_{\alpha\neq\beta} \delta(q - q_{\alpha\beta})\right], \quad (20)$$

where δ is the Dirac delta function and n is the number of replicas.

The function $P(q)$ provides a macroscopic characterization of the social structure.

Figure 3 illustrates social scenarios associated with characteristic forms of the overlap distribution $P(q)$.

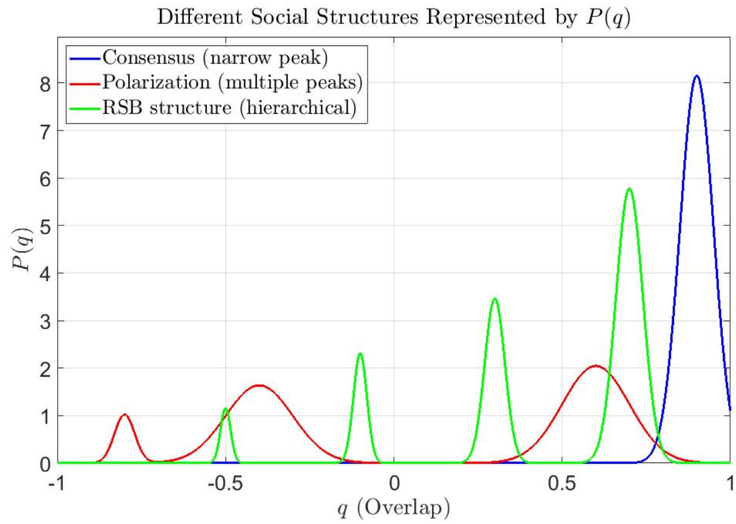

Figure 3. Social scenarios corresponding to different forms of the overlap distribution $P(q)$ for $N = 1000$ (MATLAB, 2018a).

The function $P(q)$ does not characterize the internal segmentation of a particular society, but rather the stability of the macroscopic state under a given structural disorder. It describes whether, for a fixed interaction structure, the system possesses a single typical macroscopic state or multiple alternative ones.

Under RS, the system has a single dominant macroscopic state (up to trivial global symmetry) so that different realizations of the dynamics lead to the same macroscopic outcome. In this regime, the collective result is stable with respect to variations in initial conditions and stochastic fluctuations.

Under RSB, the state space becomes stratified into multiple thermally accessible macrostates. Small differences in initial conditions or fluctuations can direct the system toward different stable configurations. Consequently, the distinction between RS and RSB should be interpreted not as homogeneity versus polarization, but rather as predictability versus structural multistability.

It is important to emphasize that polarization, understood as a stable division into social groups, may also exist in an RS regime (for example, a two-block structure with stable intra-group interactions). RSB, in contrast, is associated with the presence of multiple alternative macrostates under a fixed interaction structure, that is, with a deep structural multiplicity of possible collective outcomes.

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2.4. Possibility of RSB in a Social Context

In spin-glass theory, RS implies that all replicas of the system are statistically equivalent and share the same typical overlap value. This indicates the presence of a single dominant macrostate around which typical configurations cluster. In a social context, this corresponds to a society with stable consensus or a clearly dominant social model.

RSB arises when replicas are no longer statistically equivalent and, instead of a single characteristic overlap value, a spectrum of overlaps emerges. This implies that the space of social configurations becomes organized into multiple stable clusters with high internal similarity and significantly lower overlap between clusters.

In social terms, RSB reflects a fragmented society with multiple stable but distinct social equilibria that cannot be described by a single consensus state. Such structures are typical of highly polarized societies with a segmented public sphere and limited interaction between groups.

A formal description of this structure is given by the overlap distribution $P(q)$. A convenient equivalent representation of $P(q)$ is provided by the cumulative function:

$$x(q) \;=\; \frac{1}{n(n-1)}\sum_{\alpha<\beta} I\left[q_{\alpha\beta} \;<\; q\right], \tag{21}$$

where α and β index different replicas, and the indicator function I is defined as

$$I\left[q_{\alpha\beta} < q\right] = \begin{cases} 1, \; q_{\alpha\beta} < q \\ 0, \; q_{\alpha\beta} \;\geq\; q \end{cases}. \tag{22}$$

The function $x(q)$ measures the fraction of replica pairs with overlap less than a given threshold q and takes values in the interval $[0,\ 1]$. It is monotonically increasing and fully determines the structure of the distribution $P(q)$.

Under RS, $x(q)$ is a step function with a single jump, reflecting the presence of one dominant macrostate. In the case of one-step RSB, the function exhibits two jumps, corresponding to two main clusters of configurations. Under full RSB, as in the Parisi solution, $x(q)$ acquires a continuous step-like form that reflects a hierarchically stratified structure of states (Figure 4).

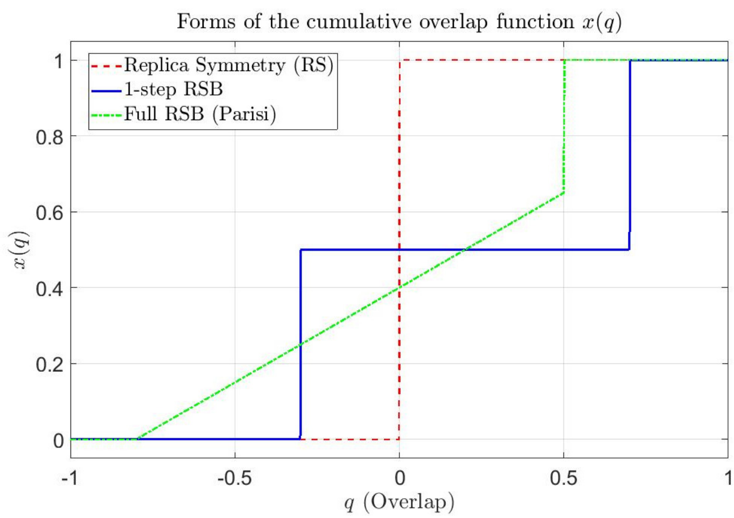

Figure 4. Cumulative function $x(q)$ under RS and RSB (MATLAB, 2018a).

In a social context, the shape of $x(q)$ provides a quantitative measure of the degree of homogeneity or fragmentation of society—from stable consensus to a highly stratified system with multiple stable groups. In this way, the function $x(q)$ offers a quantitative description both of the complexity of the social structure and of its resilience to fluctuations and external influences.

2.5. Social Interpretation of Hierarchical Clusters

The hierarchical cluster structure that emerges under full RSB has a clear social interpretation. In spin-glass theory, such stratification reflects the existence of multiple stable yet mutually distinct states. Analogously, in a social context, it corresponds to the presence of internally homogeneous but externally heterogeneous social groups.

Within the model, each replica represents a possible configuration of public opinion or political attitudes. Hierarchical grouping of configurations means that, within a given cluster, social states exhibit high overlap ($q_{\alpha\beta} \approx 1$), reflecting strong internal cohesion and consensus, whereas the overlap between different clusters decreases sharply. This indicates substantial differences in perceptions, attitudes, and value systems.

This structure allows modeling of societies that are not simply divided into two opposing camps but are instead stratified into a network of stable social groups, each with its own identity, internal logic, and degree of stability. Typical examples of such hierarchical clusters include the following:

- ideological bubbles—groups of individuals sharing comprehensive worldviews, sustained through selective information consumption and algorithmically filtered communication environments;
- electoral cores—stable supporters of political parties or movements, marked by high mobilization and resistance to external influence;
- cultural groups—deeper social formations united not only by political preferences but also by language, values, and lifestyle.

These clusters are not independent but hierarchically embedded within one another. Within a given ideological bubble, smaller factions may exist that differ in specific political or economic priorities. This nesting of opinions and identities reflects the hierarchy of overlaps in the Parisi solution: the deeper the level of clustering, the higher the internal similarity and the more specific the social identity.

The hierarchical structure of social clusters has several important consequences:

- it reduces the possibility of broad and stable consensus since no single dominant macrostate exists;
- it increases the stability of fragmentation—each group is internally robust and only weakly susceptible to external influence;
- it raises the risk of information isolation, social resonance, and political radicalization.

In this sense, hierarchical clusters, described through the function $x(q)$ and the variational structure of Parisi, provide a powerful analytical tool for studying social fragmentation, the resilience of identities, and the dynamics of group cohesion. The adaptation of symmetry-breaking mechanisms to the social sciences helps illuminate deep structural causes of political instability, social conflict, and informational segregation in contemporary societies.

3. Strong External Field and Apparent RS in Totalitarian Social Systems

In democratic environments, political preferences typically form a complex, multimodal landscape of macrostates, including different ideological groups and multiple locally stable configurations. Within the present model, this structure is described by the overlap distribution $P(q)$, which in a democratic regime is richly segmented and reflects the presence of multiple competing social models.

In authoritarian and totalitarian regimes, from the perspective of spin-glass theory, the three principal parameters—the external field h, the interpersonal interactions J_{ij}, and the social interaction intensity β—shift toward extreme values. A strong unidirectional external field emerges, interpersonal interactions become biased toward conformism, and the social temperature decreases. As a result, the system enters a single-phase regime dominated by a unique global minimum of free energy, in which the observable structure of public opinion takes on the appearance of RS.

However, this symmetry is not the result of spontaneous social consensus, but rather, it is imposed by extreme external parameters. The model allows for a distinction between different social layers, for example, between publicly observable behavior and the internal cognitive structure of individuals. Under such stratification, even when public choices (the σ-layer) are nearly fully aligned, internal beliefs (the τ-layer) may remain hierarchically structured and latently discordant.

From this perspective, totalitarian regimes exhibit a superficial form of RS that conceals the underlying structural complexity of internal beliefs. The observed homogeneity is therefore not a sign of social cohesion, but rather the consequence of a strong external field that suppresses fluctuations and eliminates competing macrostates already at the level of free energy.

This interpretation links the parameters h, J_{ij}, and β to observable social indicators such as concentration of power, media control, degree of repression, and polarization. In this way, the model becomes an analytical framework both for describing stable totalitarian structures and for identifying early signals of authoritarian drift through changes in social parameters.

3.1. Parameterization of the Totalitarian Model

Let us recall the introduced social Hamiltonian (12), analogous to the SK model:

$$H_N(\sigma) = -\sum_{1\le i<j\le N}^{N} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} h_i\sigma_i. \tag{23}$$

Here, the quantities h_i describe individualized external influences on voters, J_{ij} represents interpersonal interactions, and the parameter β (introduced through the partition function and free energy) is interpreted as social interaction intensity, or inverse social temperature. Large values of β correspond to a low social temperature and suppressed fluctuations, whereas small values correspond to more chaotic and competitive social regimes.

The aim of this section is to analyze how the parameters h_i, J_{ij}, and β manifest and combine under a totalitarian political regime.

3.1.1. External Fields h_i

In a democratic system, the external fields h_i are naturally distributed near zero mean with moderate variance, reflecting the absence of a dominant center of influence and the presence of competing informational and institutional sources.

Under a totalitarian regime, this structure changes qualitatively. A strong unidirectional component $h > 0$ emerges, acting on nearly all voters and systematically favoring support for the dominant party or leader. Alternative external influences are suppressed, leading to a highly asymmetric and concentrated distribution of $\{h_i\}$, which can be approximated as

$$h_i \approx h + \eta_i, \tag{24}$$

where $h \gg 0$ is the dominant totalitarian component, and η_i are small individual deviations.

Figure 5 illustrates the distributions of external fields under democracy and totalitarianism.

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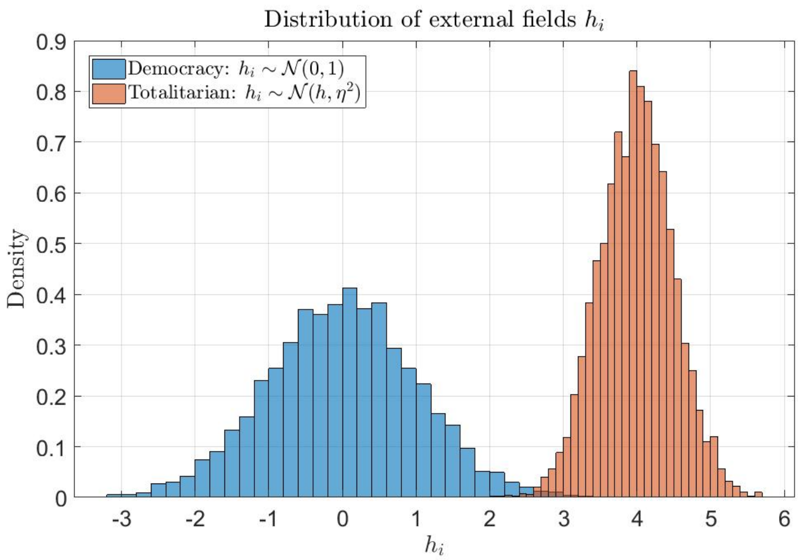

Figure 5. Distributions of external fields h_i under different political regimes. Democracy: $h_i \approx \mathcal{N}(0,1)$. Totalitarian regime: $h_i \approx \mathcal{N}(h,\eta^2)$, with $h\ = 4, \eta\ = 0.5$ (MATLAB, 2018a).

Under these conditions, the social Hamiltonian takes the simplified form

$$H_N(\sigma) = -\sum_{1 \le i < j \le N}^{N} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i. \tag{25}$$

3.1.2. Social Interactions J_{ij}

In a democratic regime, the interactions J_{ij} can be regarded as stochastic variables with mean close to zero, reflecting a mixture of agreement and conflict. In a totalitarian system, this structure becomes distorted in two main ways: conformism is systematically encouraged, while conflicting ties are suppressed through institutional pressure.

To model this effect, we represent the interactions as the sum of a deterministic component and random noise, using standard mean-field scaling. In the original SK model, the interactions are

$$J_{ij} = \frac{J}{\sqrt{N}}\varepsilon_{ij}, \tag{26}$$

where $\varepsilon_{ij} \sim \mathcal{N}(0,\ 1)$, hence

$$\mathbb{E}[J_{ij}] = 0. \tag{27}$$

Under totalitarian conditions, we introduce an additional systematic component J_0, acting equally on all pairs of individuals:

$$J_{ij} = A_{ij}\left(\frac{J_0}{N} + \frac{J}{\sqrt{N}}\varepsilon_{ij}\right), \tag{28}$$

where

- J_0 is the deterministic component measuring the average tendency toward agreement ($J_0 > 0$) or antagonism ($J_0 < 0$);
- J determines the amplitude of random fluctuations in the interactions;
- $A_{ij} \in \{0,1\}$ denotes the contact (network) matrix, where $A_{ij} = A_{ji}$ for $i \neq j$ and $A_{ii} = 0$.

The deterministic component contributes a global term of order $\mathrm{O(N)}$ to the Hamiltonian, while the noise component retains zero mean fluctuations. In a totalitarian regime, J_0

increases and the relative role of random interactions weakens, driving the system toward a conformist, low-frustration state.

It is important to emphasize that this structural alignment does not imply genuine internal unanimity among individuals but rather an organization of social interactions in which disagreement cannot amplify and stabilize as a collective alternative.

The heatmap of the interaction matrix J_{ij}, shown in Figure 6 for a system with $N = 60$, illustrates the qualitative differences between democratic and totalitarian social regimes. Under democracy, the matrix contains links of varying intensity and sign, reflecting a heterogeneous structure of social interactions. Under totalitarianism, the matrix takes on a more unidirectional and homogeneous structure, reflecting the dominance of a deterministic component and the suppression of fluctuations.

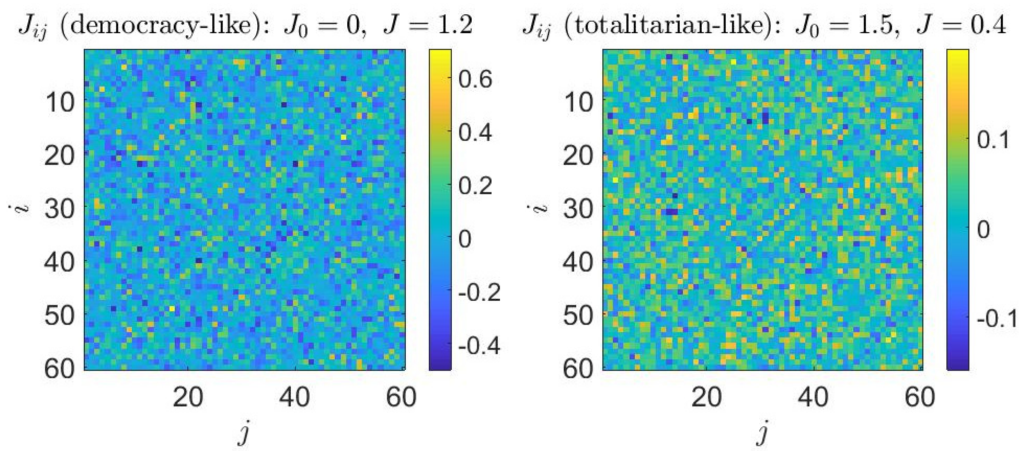

Figure 6. Heatmap of the interaction matrix J_{ij} for a system with $N = 60$. Democracy: $J_0 \approx 0$ and larger noise J. Totalitarianism: $J_0 > 0$ and smaller noise J (MATLAB, 2018a).

3.1.3. Social Interaction Intensity β

Institutional pressure in a totalitarian regime acts as a mechanism for effectively increasing social interaction intensity. Control over the informational environment and the sanctioning of deviations reduce social noise and suppress spontaneous fluctuations, leading to a decrease in the system's effective temperature and the dominance of configurations with minimal social energy.

3.1.4. Summary

As a result of the analysis, the totalitarian regime can be qualitatively parameterized by the combination

$$h \gg 0, \; J_0 > 0, \; \beta \gg 1, \tag{29}$$

together with suppressed heterogeneity of interactions. This parameter configuration leads to a single-phase regime with apparent RS, in which the observed social homogeneity results from externally imposed constraints rather than spontaneous consensus.

3.2. Energy Analysis of the Hamiltonian Under Totalitarian Parameters

In this section, a qualitative and quantitative analysis is carried out of how the totalitarian regime parameters (29) determine the energy landscape of the social system. We show how the external field and interpersonal interactions jointly align individual preferences and make deviations from the dominant configuration energetically unfavorable.

In this regime, the social Hamiltonian takes the form

$$H_N(\sigma) = -\sum_{1\le i<j\le N}^{N} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \tag{30}$$

where the first term describes interpersonal interactions and the second represents the influence of the external field.

3.2.1. Contribution of the External Field

The term describing the external influence can be expressed through the magnetization:

$$-h\sum_{i=1}^{N}\sigma_i = -hNm, \tag{31}$$

where

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma_i,\ m \in [-1,1], \tag{32}$$

The minimum of this term is achieved at full alignment $m = 1$. The energy difference between a configuration with magnetization $m < 1$ and the fully aligned state is

$$\Delta \mathrm{E}_{field} = hN(1-m). \tag{33}$$

For large values of hN, this difference grows linearly with the number of individuals, meaning that even small deviations from full alignment lead to a significant increase in social energy (Figure 7). Therefore, the external field effectively pulls the global energy minimum toward the state $m \to 1$, corresponding to the configuration

$$\sigma_i = +1,\ \ \forall\, i = 1,2,\ldots,N. \tag{34}$$

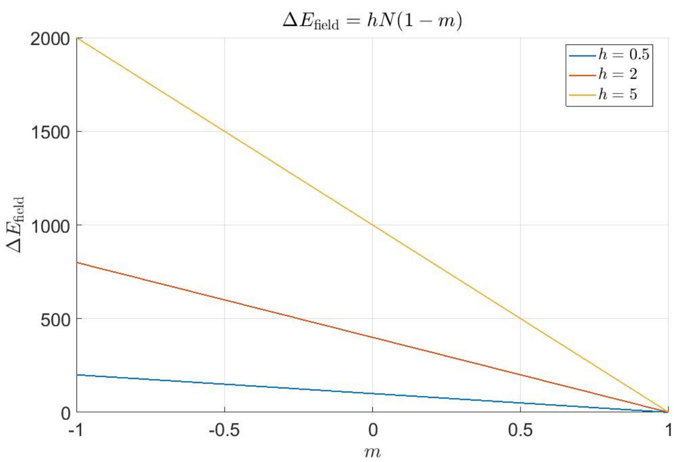

Figure 7. Energy difference $\Delta\mathrm{E}_{field}$ induced by the external field as a function of magnetization m for different values of h, with a fixed number of individuals $N = 200$ (MATLAB, 2018a).

3.2.2. Contribution of Interpersonal Interactions

For the totalitarian regime, we use

$$J_{ij} = \frac{J_0}{N} + \frac{J}{\sqrt{N}}\varepsilon_{ij}. \tag{35}$$

where $\varepsilon_{ij} \sim \mathcal{N}(0,\ 1)$.

The contribution of interpersonal interactions to the energy can be decomposed into deterministic and noise components:

$$\sum_{1\leq i<j\leq N}^{N} J_{ij}\sigma_i\sigma_j = \frac{J_0}{N}\sum_{1\leq i<j\leq N}^{N}\sigma_i\sigma_j + \frac{J}{\sqrt{N}}\sum_{1\leq i<j\leq N}^{N}\varepsilon_{ij}\sigma_i\sigma_j. \tag{36}$$

The deterministic term can be expressed in terms of m in the standard way:

$$\frac{J_0}{N}\sum_{1\leq i<j\leq N}^{N}\sigma_i\sigma_j = \frac{J_0}{2}N\left(m^2 - \frac{1}{N}\right) \approx \frac{J_0}{2}Nm^2. \tag{37}$$

Under totalitarian conditions where $m \to 1$, this term reaches its maximum value $\frac{J_0}{2}N$, meaning that interpersonal interactions themselves energetically favor highly aligned configurations.

The noise component

$$X_N = \frac{J}{\sqrt{N}}\sum_{1\leq i<j\leq N}^{N}\varepsilon_{ij}\sigma_i\sigma_j. \tag{38}$$

has zero mean and variance

$$Var(X_N) = \frac{J^2}{2}N, \tag{39}$$

so, fluctuations are of order $O(N)$.

Although the noise grows linearly with N, it does not act systematically in one direction. In contrast, the external term hN has a definite sign and, for $h \gg 0$, dominates the energy balance. When $\beta \gg 1$, these fluctuations are further suppressed and cannot sustainably compensate for the imposed field.

3.2.3. Summary

The combined effect of the strong external field and the deterministic component of interactions produces an energy landscape with a single global minimum corresponding to near-complete alignment of individuals. Random interpersonal interactions generate only local and unstable deviations that cannot stabilize into alternative macrostates. This explains why, under totalitarian parameters, the system exhibits apparent RS and high observable homogeneity despite the presence of latent individual differences.

3.3. Free Energy Analysis Under Totalitarian Conditions

In this section, we move from a local energy analysis to the global thermodynamic quantity of the system—the free energy—which summarizes all possible configurations and their probabilities. In the totalitarian regime ($\beta \gg 1$), the system effectively selects configurations with minimal social energy, allowing a simple asymptotic expression for the free energy and explaining the absence of competing macrostates.

The free energy is defined as

$$VF(\beta, h) = -\frac{1}{\beta N}\mathbb{E}[\log Z_N(\beta, h)], \tag{40}$$

where

$$Z_N(\beta, h) = \sum_{\sigma} e^{-\beta H(\sigma)}. \tag{41}$$

Under totalitarian conditions ($h \gg 0, J_0 > 0, \beta \gg 1$), configurations with minimal social energy dominate. As shown earlier, this minimum energy is achieved for the fully aligned configuration $\sigma_i = +1,\ \forall\, i = 1, 2, \ldots, N$.

In this case, the Hamiltonian takes the value

$$H_{min} = -\sum_{i<j} J_{ij} - hN. \tag{42}$$

In the low-temperature limit $\beta \to \infty$, the free energy reduces to the energy of the dominant state:

$$F(\beta, h) \underset{\beta\to\infty}{\to} \frac{1}{N} H_{min} = -\frac{1}{N}\sum_{i<j} J_{ij} - h. \tag{43}$$

Using the adopted parameterization of the interactions J_{ij}, we obtain asymptotically

$$\sum_{i<j} J_{ij} = \frac{J_0}{2} N + O\left(\sqrt{N}\right) \tag{44}$$

and therefore

$$F(\beta, h) \approx -\frac{J_0}{2} - h. \tag{45}$$

Figure 8 illustrates this linear dependence of the free energy on the external field h in the low-temperature (totalitarian) limit.

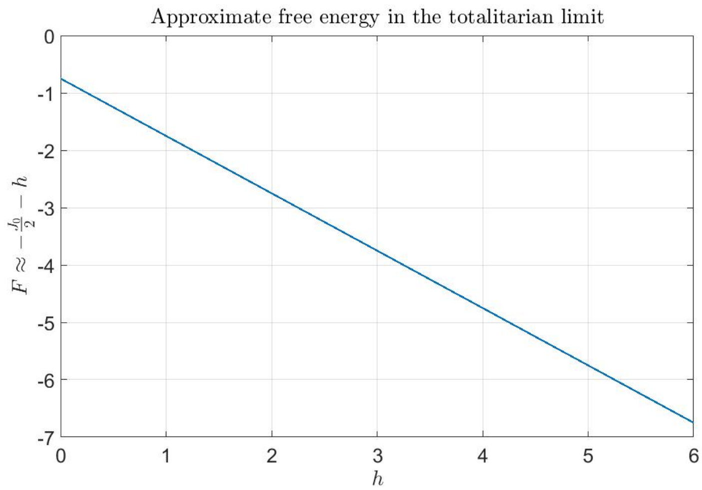

Figure 8. Approximate free energy $F(\beta, h)$ in the totalitarian limit as a function of the external field h for $J_0 = 1.5$ (MATLAB, 2018a).

Two key results follow from this analysis:

- In the thermodynamic limit, the free energy is determined entirely by the deterministic parameters J_0 and h; the noise component of the interactions J_{ij} does not contribute at leading order.
- The free energy possesses a single global minimum, meaning that the system is in a single-phase regime without competing macrostates and without spontaneous frustration.

In social terms, this indicates that under a strong external field and low social temperature, the political system inevitably organizes around a single macroscopic state. All alternative configurations are energetically unfavorable and have an exponentially small probability of realization. Pluralism is not merely institutionally suppressed but eliminated at the level of collective dynamics.

Importantly, random and conflictual interpersonal interactions, although they may exist, cannot organize into stable alternative macrostates. Social noise remains statistically invisible at the global level. The observed RS is therefore imposed rather than spontaneous—

it arises from the elimination of all competing political alternatives already at the level of free energy.

This result provides a formal explanation of why totalitarian regimes can appear extremely stable for long periods despite the presence of latent dissatisfaction. As long as the parameters h, J_{ij}, and β remain within the totalitarian range, the system lacks an internal mechanism for the spontaneous emergence of an oppositional structure. Change becomes possible only through a reconfiguration of the effective parameters—weakening the external field, increasing social noise, or breaking down conformist interactions.

3.4. Overlap of Replicas Under Totalitarian Conditions

To assess the degree of difference between various realizations of society under the same structural parameters, we introduce the overlap q between two independent configurations (replicas). It serves as a quantitative indicator of pluralism: a broad distribution of q indicates the presence of multiple competing macrostates, whereas concentration around a single value signals dominance of a single macroscopic state.

Under totalitarian parameters ($h \gg 0,\ J_0 > 0,\ \beta \gg 1$), consider two independent replicas $\sigma^{(1)}$ and $\sigma^{(2)}$, whose overlap is defined as

$$q\left(\sigma^{(1)},\, \sigma^{(2)}\right) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)}\sigma_i^{(2)}. \tag{46}$$

For configurations $\sigma^{(1)} = \sigma^{(2)} = \{+1,\, +1, \ldots, +1\}$, the overlap reaches its maximal value

$$q\left(\sigma^{(1)},\, \sigma^{(2)}\right) = \frac{1}{N}N = 1. \tag{47}$$

In the thermodynamic limit, this leads to a collapse of the overlap distribution toward

$$P(q) \;\approx\; \delta(q-1). \tag{48}$$

Figure 9 illustrates the concentration of $P(q)$ near $q = 1$, indicating that different replicas of the system are practically indistinguishable in terms of observable behavior.

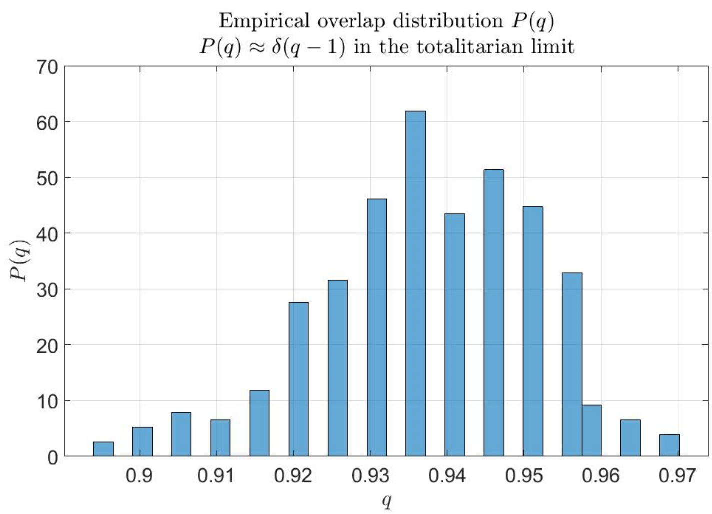

Figure 9. Concentration of the overlap distribution $P(q)$ near $q = 1$ in the totalitarian regime. Parameters: $N = 400$, $p = 0.02$ (MATLAB, 2018a).

The delta function $\delta(q - 1)$ means that the entire probability mass is concentrated at a single point in overlap space. This is the limiting case of full RS, where competing macrostates and alternative minima of free energy are absent. All observed configurations are aligned with a single dominant macroscopic regime.

It is important to emphasize that the concentration $P(q) \approx \delta(q - 1)$ characterizes the observable layer of the system, namely public behavior. It does not necessarily imply complete agreement in individuals' internal beliefs but rather reflects strongly imposed behavioral alignment. Within the one-layer model, this leads to a single-phase regime in which alternative configurations are energetically suppressed and cannot stabilize as competing macrostates.

This result naturally motivates a future extension of the model with an additional τ-layer, allowing analysis of the potential discrepancy between public RS and a latent multicluster structure of internal beliefs.

4. Illustrative Example: The Transition from the Weimar Republic to the National Socialist Regime

As an illustrative example, we consider the transition from the Weimar Republic to the National Socialist regime (1919–1933). This case is selected due to its historical completeness and the availability of well-documented electoral data, which allows a structural comparison between the theoretical model and the observed political dynamics.

4.1. Empirical Data and Ideological Structure of the German Party System (1919–1933)

The data for the Reichstag elections during the period 1919–1933 were obtained from official historical sources [35–39] and are aggregated in Table 1. The table presents the percentage of votes received by the main political formations, grouped by ideological type.

Table 1. Summary of Reichstag election results (1919–1933).

Year	Social Democrats SPD	Centre (Zentrum/BVP)	Liberals (DDP/DVP)	National Conservatives DNVP	Radical Left (USPD/KPD)	National Socialists NSDAP	Others
1919	37.90%	19.70%	18.6% (DDP)	10.30%	7.6% (USPD)	–	5.90%
1928	29.80%	15.10%	13.5% (DDP + DVP [1])	14.30%	10.6% (KPD)	2.60%	14.10%
1930	24.50%	14.80%	4.5% (DVP)	7.00%	13.1% (KPD)	18.30%	17.80%
Jul 1932	21.60%	15.70%	1.2% (DVP)	5.90%	14.3% (KPD)	37.30%	4%
Mar 1933	18.30%	14.00%	1.1% (DVP)	8.0% [2]	12.3% (KPD [3])	43.90%	2.40%

[1] DDP 4.8%, DVP 8.7%. [2] DNVP within KSWR coalition. [3] KPD was severely repressed at the time of the election.

During the considered period, the following main ideological groups can be distinguished:

- Social Democrats (SPD)—the main center-left political force;
- Centre (Zentrum/BVP)—a Catholic centrist bloc with relatively stable electoral support;
- Liberals (DDP, DVP)—moderate liberal forces with significant influence during 1919–1928 and declining support after 1930;
- National Conservatives (DNVP)—a conservative and nationalist formation present throughout the entire period;
- Radical Left (USPD/KPD)—the radical left spectrum, dominated by USPD in 1919 and by KPD in later elections;
- National Socialists (NSDAP)—a party with initially limited representation that experienced a rapid increase in support after 1930.

The general trends during the examined period can be summarized as follows:

- In the period 1919–1928, the German party system was multipolar and relatively balanced.

- After 1930, a collapse of the liberal center is observed, accompanied by simultaneous growth of both far-left and far-right political forces.
- By 1933, the system had shifted toward a concentrated structure dominated by an extreme right political pole and characterized by severely reduced political competition.

4.2. Construction of Effective Macro-Parameters

Although the model is formulated at the microscopic level through the parameters h_i, J_{ij}, and β, the historical data used in this study are available only in aggregated form. Consequently, a direct reconstruction of the microscopic parameters is not feasible. Instead, we employ proxy indicators—h_0, J_0, β—to estimate the effective macroscopic parameters of the system.

The purpose of this procedure is to examine the structural compatibility between the theoretical framework and the historical case under consideration.

The proxy indicators are computed directly from the electoral results. Let

- v_i denote the vote share of party iii (obtained by dividing the percentage of votes by 100);
- n denote the number of parties considered;
- the vote shares be normalized such that $\sum_{i=1}^{n} v_i = 1$.

The uniform distribution is then defined as

$$v_i^{(0)} = \frac{1}{n}. \tag{49}$$

4.2.1. External Field h_0

The external field measures the presence of systemic pressure toward a particular political pole. In the context of electoral results, a natural indicator of such an effect is the degree of dominance of the largest political force.

The external field can be defined as the deviation of the dominant party from the symmetric distribution:

$$h_0 = v_{max} - \frac{1}{n}, \tag{50}$$

where $v_{max} = \max_i v_i$.

Small values of h_0 correspond to a balanced multiparty system without a clearly dominant political pole. As h_0 increases, asymmetry in the distribution of political support emerges. Under empirical conditions, a strong external field may be considered to occur when the dominant party receives approximately one half or more of the total votes cast.

4.2.2. Mean Conformity J_0

The parameter J_0 describes the average tendency of agents to align with their social environment. Empirically, this quantity can be approximated through the degree of concentration of political support.

For a quantitative assessment, we use the entropy of the vote-share distribution:

$$H = -\sum_{i=1}^{n} v_i \ln(v_i). \tag{51}$$

The maximum possible entropy under a uniform distribution is

$$H_{max} = \ln(n). \tag{52}$$

The normalized conformity indicator is then defined as

$$J_0 = 1 - \frac{H}{\ln(n)}, \quad 0 \leq J_0 \leq 1. \tag{53}$$

When $J_0 \approx 0$, the distribution of votes is highly fragmented across many parties, corresponding to a low level of collective conformity. As J_0 increases, political support becomes progressively concentrated around a smaller number of political actors. When J_0 approaches unity, the distribution becomes strongly concentrated, reflecting a high degree of social alignment.

4.2.3. Social Temperature and the Parameter β

The parameter β plays the role of inverse social temperature and controls the degree of fluctuations in collective behavior.

In an empirical context, social temperature can be associated with the degree of political instability. As a proxy indicator, we use the volatility of electoral outcomes between consecutive elections, measured by the Pedersen index [40]:

$$V_t = \frac{1}{2}\sum_{i=1}^{n}\left|v_i^{(t)} - v_i^{(t-1)}\right|. \tag{54}$$

This quantity measures the overall redistribution of political support between two consecutive elections. High values of V_t indicate strong fluctuations in electoral behavior and correspond to high social temperature.

We define β as

$$\beta = \frac{1}{V_t}. \tag{55}$$

Thus, high electoral volatility corresponds to high social temperature and a small value of β, characterizing unstable political regimes. Conversely, low volatility leads to larger values of β, corresponding to lower social temperature and a more stable political system.

4.3. Regime Dynamics of the German Party System (1919–1933)

Using the proxy parameters h_0, J_0, and β defined in the previous section, the evolution of the German political system during the period 1919–1933 can be interpreted as a change in the regime of collective dynamics. In this sense, electoral outcomes allow us to trace how the system transitions from a multipolar and relatively balanced configuration to a strongly asymmetric structure dominated by a single political pole.

The quantitative estimates of the parameters are summarized in Table 2.

Table 2. Estimated model parameters for the German party system (1919–1933).

Year	v_{max}	h_0	J_0	V	β=1/V
1919	0.379	0.236	0.179	–	–
1928	0.298	0.155	0.073	0.178	5.62
1930	0.245	0.102	0.056	0.219	4.57
Jul 1932	0.373	0.23	0.169	0.211	4.74
Mar 1933	0.439	0.296	0.205	0.087	11.49

Based on the observed parameter ranges in the examined period, a qualitative classification into low, moderate, and high levels is introduced (Table 3). It should be noted that these categorical thresholds are illustrative and may be adjusted in other empirical cases or standardized within a broader comparative framework.

Table 3. Qualitative classification intervals for the effective parameters.

Parameter	Low	Moderate	High
h_0 (external field)	$h_0 < 0.15$	$0.15 \leq h_0 < 0.25$	$h_0 \geq 0.25$
J_0 (conformity)	$J_0 < 0.10$	$0.10 \leq J_0 < 0.19$	$J_0 \geq 0.19$
β (inverse social temperature)	$\beta < 5.0$	$5.0 \leq \beta < 8.0$	$\beta \geq 8.0$

These categories are used in Table 4 to provide a regime-based interpretation of the political dynamics.

Table 4. Regime interpretation of the German party system (1919–1933).

Year	Dominant Party	h_0	J_0	β	Model Regime	Historical Interpretation
1919	SPD (37.9%)	moderate	moderate	–	multistable regime (RSB-like)	early pluralistic democracy
1928	SPD (29.8%)	moderate	low	moderate	stable pluralism	stable period of the Weimar Republic
1930	SPD (24.5%)	low	low	low	deep multistable regime (RSB)	beginning of political crisis
Jul 1932	NSDAP (37.3%)	moderate	moderate	low	strongly asymmetric multistability	rapid rise of NSDAP
Mar 1933	NSDAP (43.9%)	high	high	high	RS regime (single dominant macrostate)	authoritarian consolidation

Table 4 reveals a gradual transformation of the German political system from a multistable pluralistic regime toward a configuration dominated by a single political pole. In the language of the model, this corresponds to a transition from a regime analogous to RSB to a regime characterized by a single stable minimum of free energy (RS).

To illustrate the relationship between the parameter values and the observed political dynamics, the key electoral moments of the period are briefly discussed below. The regime interpretation is derived from the proxy parameters, while historical developments serve as an external consistency check between the model and the empirical political process.

1919: Pluralistic Multistability

The elections to the National Assembly in 1919 show a strongly fragmented distribution of political support across multiple parties without a clearly dominant political pole. The largest party (SPD) receives 37.9% of the vote, which is far from a single-pole dominance. Such a configuration corresponds to a multistable political regime in which numerous coalition combinations and alternative macroscopic states of the system are possible.

1928: Stabilized Pluralism

By the late 1920s, the German party system remained multiparty but exhibited relatively greater institutional stability. Political support was distributed among several major parties, corresponding to a regime of stable pluralism. Within the model framework, the system remained multistable, but fluctuations were more limited and the political configurations were relatively stable.

1930: Beginning of Destabilization

The elections of 1930 marked a significant structural shift in the political system. Support for radical political forces increased considerably, while the liberal center weakened. In terms of the model, this manifests as increased fluctuations and a decrease in the effective parameter β, corresponding to a highly unstable multistable configuration.

July 1932: Maximum Multistability

By 1932, the political system had reached a state of polarized instability. Support for extreme political forces increased significantly, and competition among political blocs intensified. Within the model, this corresponds to a regime characterized by multiple nearby macroscopic minima of free energy and high sensitivity to small fluctuations.

March 1933: Regime Transformation

After 1933, the political system shifted to a qualitatively different configuration. The concentration of political power, institutional pressure, and repressive measures led to a substantial increase in the effective external field and in social conformity. Within the model framework, this is interpreted as a transition toward a stabilized regime with a single dominant macroscopic minimum of free energy.

The evolution of the parameters reveals a clear trajectory of the system in the parameter space (h_0, J_0, β) (Figure 10). In the initial period, the German party system was characterized by a relatively weak external field, low to moderate conformity, and high social temperature, corresponding to a multistable political configuration with multiple possible macroscopic states. Around 1930, the system reached a phase of maximal instability, reflected in low values of h_0, J_0, and β. In the following elections, the increasing asymmetry of political support led to a rapid growth of the effective external field and social conformity. The culmination of this process occurred in 1933, when the parameters reached values corresponding to a stabilized configuration with a dominant political pole. In terms of the model, this evolution can be interpreted as a transition from a multistable regime to a regime characterized by a single dominant minimum of free energy.

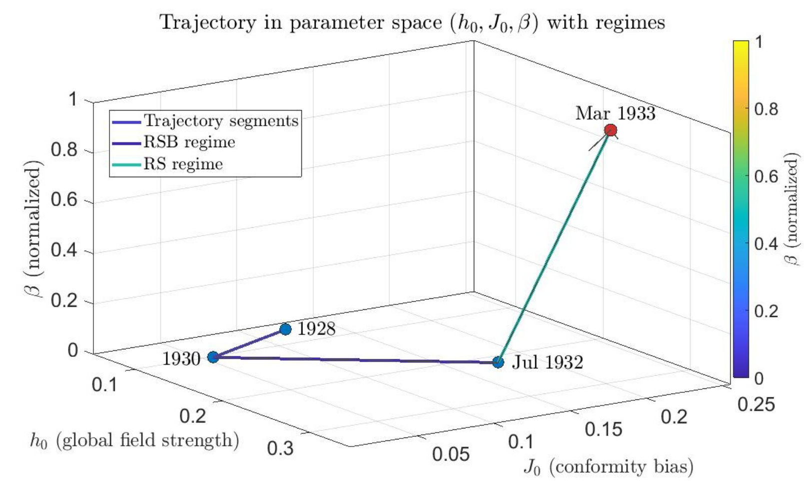

Figure 10. Trajectory of the German political system in the parameter space (h_0, J_0, β) for the period 1919–1933 (MATLAB 2018a). The parameter β is rescaled to the interval $[0, 1]$ for visualization purposes. The year 1919 is omitted due to missing data for β.

The historical example considered here illustrates how the proposed theoretical framework allows political transformations to be interpreted as regime changes in the space of effective parameters governing social interaction.

5. Discussion

The present study proposes a mathematical framework for analyzing voting behavior and social dynamics based on spin-glass theory and the Parisi–Talagrand variational principle. The aim of the model is not deterministic prediction of specific electoral outcomes but a structural description of collective social regimes, phase transitions, and stable macroscopic states through effective macro-parameters.

5.1. Social Phases and Interpretation of Macroscopic Regimes

One of the main results of the model is the possibility of interpreting different political regimes as distinct phases in the sense of statistical mechanics. Under democratic conditions—characterized by moderate external fields, high social temperature, and heterogeneous interactions—the system enters a multimodal regime with multiple local minima of the free energy. This manifests itself in a stratified structure of the overlap distribution $P(q)$, the presence of frustration, and the possibility of RSB. At the social level, this regime corresponds to pluralism, polarization, and the stable coexistence of competing groups.

Conversely, the analysis of the totalitarian regime shows that under a strong external field, a positive systematic component of interactions, and low social temperature, the system collapses into a single-phase regime. The free energy exhibits a unique global minimum, the distribution $P(q)$ concentrates near $q \approx 1$, and RS is effectively restored. This formalizes the well-known sociological phenomenon of apparent stability and homogeneity in totalitarian systems.

5.2. Apparent Stability and Imposed RS

A key conceptual contribution of the model is the distinction between natural and imposed RS. In classical physical systems, RS usually indicates the absence of frustration and a homogeneous structure of phase space. In the social context, however, the symmetry observed in totalitarian regimes does not arise from natural consensus but from the action of a strong external field and systematic suppression of fluctuations.

The model shows that observed macroscopic homogeneity in such cases is not necessarily an expression of social cohesion but may instead be the result of structural coercion. This distinction has important interpretive consequences: it explains why totalitarian systems can appear stable for long periods despite the presence of latent dissatisfaction, cognitive dissonance, and internal fragmentation. As long as the parameters (h, J_0, β) remain within the totalitarian range, the model does not allow an internal mechanism for the spontaneous emergence of an alternative macroscopic state.

5.3. Limitations and Future Extensions of the Model

The main limitation of the proposed model lies in its single-layer structure. In its current formulation, each variable σ_i simultaneously represents internal belief and publicly expressed behavior. While this assumption is reasonably acceptable in democratic systems, it becomes problematic in the analysis of authoritarian and totalitarian regimes, where divergence between private attitudes and public expression can be structurally significant.

A natural extension of the model is the introduction of a two-layer structure with separate variables for public behavior (σ) and internal beliefs (τ), coupled through a conflict or cognitive term of the form $-K\sum_i \sigma_i\tau_i$. Such an extension would allow empirical validation as well as analysis of cognitive dissonance, latent polarization, and sudden phase transitions triggered by changes in external parameters.

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5.4. Methodological Status of the Analogy

The proposed model should be understood as a structural analogy rather than a reductionist claim that societies are physical systems. The use of the spin-glass formalism does not imply a physical identity between social and material systems, but rather an isomorphism of mathematical structure: a system composed of many interacting units, characterized by competing influences, frustration, and the possibility of multiple stable macrostates.

The socio-political motivation for employing such a formalism arises from empirically observed characteristics of contemporary societies, including:

- interdependence of individual decisions;
- the presence of both cooperative and antagonistic social interactions;
- multistability of collective behavior;
- sensitivity to initial conditions and the possibility of abrupt regime transitions.

These properties are structurally analogous to the characteristics of disordered mean-field systems studied in spin-glass theory. For this reason, the model does not claim direct empirical predictability; rather, it provides a conceptual and analytical framework that enables the dynamics of social regimes to be investigated using clearly defined mathematical tools.

6. Conclusions

This study developed a theoretical–empirical framework for analyzing social and political dynamics based on a structural analogy between voting behavior and spin-glass systems in statistical physics. By introducing a social Hamiltonian, partition function, and free energy, we show that collective political regimes can be interpreted as different phases of a complex system governed by effective macro-parameters related to external influence, interpersonal interactions, and the level of social "noise".

The proposed model provides a unified mathematical scheme for describing both pluralistic and authoritarian regimes. Under pluralistic conditions, the system exhibits multistability and sensitivity to initial conditions, which can be associated with RSB-like behavior and the existence of multiple competing macroscopic configurations. Conversely, under parameter regimes characterized by a strong unidirectional external field, increased conformism, and low effective temperature (high β), the system transitions to a regime with a single dominant macroscopic minimum of the free energy, corresponding to RS-like behavior.

An important contribution of the study is the formal distinction between natural and imposed RS. We show that the observed homogenization of collective behavior in authoritarian and totalitarian regimes does not necessarily arise from spontaneous social consensus but may instead result from structural suppression of fluctuations and the elimination of competing macrostates already at the level of free energy. This provides a mathematical interpretation of the apparent stability of such regimes and allows a clearer distinction between a predictable regime (RS) and a regime characterized by multiple possible outcomes (RSB), which may emerge under closely related initial conditions.

To illustrate the connection with real socio-political processes, the model was applied to the transition from the Weimar Republic to the National Socialist regime (1919–1933). Using aggregated electoral data as an empirical basis, proxy indicators for the effective parameters (h_0, J_0, β) were defined, allowing the historical dynamics to be interpreted as a trajectory in parameter space. The obtained results suggest that the considered transition is structurally consistent with a phase-transition scenario from multistable to single-phase behavior.

Finally, it should be emphasized that the proposed framework does not aim to reconstruct the microscopic interactions J_{ij} quantitatively or to derive the distribution $P(q)$ directly from the available aggregated data. Its primary purpose is to provide a conceptual and analytical tool for studying regime dynamics and transitions between different political configurations using the methods of statistical mechanics. This approach establishes a foundation for future research in which the model could be further validated using richer empirical datasets—such as social network structures, panel surveys, or indices of media freedom and institutional pressure—as well as through comparative analyses of other historical and contemporary political transformations.

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Abbreviations

The following abbreviations are used in this manuscript:

SK	Sherrington–Kirkpatrick
RS	Replica Symmetry
RSB	Replica Symmetry Breaking

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