

Article

Prediagnosis of Disease Based on Symptoms by Generalized Dual Hesitant Hexagonal Fuzzy Multi-Criteria Decision-Making Techniques

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Abstract: Multi-criteria decision-making (MCDM) is now frequently utilized to solve difficulties in everyday life. It is challenging to rank possibilities from a set of options since this process depends on so many conflicting criteria. The current study focuses on recognizing symptoms of illness and then using an MCDM diagnosis to determine the potential disease. The following symptoms are considered in this study: fever, body aches, fatigue, chills, shortness of breath (SOB), nausea, vomiting, and diarrhea. This study shows how the generalised dual hesitant hexagonal fuzzy number ($GDHH_{\chi}FN$) is used to diagnose disease. We also introduce a new de-fuzzification method for $GDHH_{\chi}FN$. To diagnose a given condition, $GDHH_{\chi}FN$ coupled with MCDM tools, such as the fuzzy criteria importance through inter-criteria correlation (FCRITIC) method, is used for finding the weight of criteria. Furthermore, the fuzzy weighted aggregated sum product assessment (FWASPAS) method and a fuzzy combined compromise solution (FCoCoSo) are used to rank the alternatives. The alternative diseases are chosen to be malaria, influenza, typhoid, dengue, monkeypox, ebola, and pneumonia. A sensitivity analysis is carried out on three patients affected by different diseases to assess the validity and reliability of our methodologies.

Keywords: disease recognition; $GDHH_{\chi}FN$; de-fuzzification; FCRITIC; FWASPAS; FCoCoSo



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1. Introduction

The maximum value of the degree of belongingness in fuzzy sets is 1. Suppose more than one decision-maker is participating in the same context when making a qualitative judgment. This maximum degree of membership may differ from one decision-maker to the other. Thus, the generalized fuzzy number was introduced [1]. The idea behind generalized fuzzy numbers is to extend the concept of fuzzy numbers beyond their traditional triangular or trapezoidal shapes. Generalized fuzzy numbers allow for more flexible shapes, which can better capture the uncertainty or imprecision inherent in many real-world situations. Generalized fuzzy numbers have been widely used in fuzzy mathematics [2–4], decision-making [5,6], and control theory [7], among other fields [2,8,9]. They provide a useful tool for representing uncertain or vague information in a formal way and for making decisions in situations where there is a high degree of uncertainty or ambiguity. Hesitant fuzzy sets (HFS) can be used to recognize diseases by allowing the degree of membership of an element to multiple possible sets or categories to be expressed as a range of values rather than a single crisp value. This is particularly useful in medical diagnosis [10], where

there may be a degree of uncertainty or ambiguity in assigning a patient to a particular disease category based on their symptoms or test results. This method can be used to model the uncertainty and imprecision inherent in medical diagnosis by allowing the degree of membership of an element to be expressed as a range of values between 0 and 1. This makes it possible to make decisions that are more flexible and nuanced, which can help to reduce the risk of misdiagnosis. Overall, hesitant fuzzy sets can be a valuable tool for recognizing diseases, particularly in situations where there is a high degree of uncertainty or ambiguity in the diagnosis process. Liao, H. et al. [11] proposed qualitative decision-making with correlation coefficients of hesitant fuzzy linguistic term sets in medical diagnosis. Mardani, A. et al. [12] introduced a novel extended approach using hesitant fuzzy sets to design a framework for assessing the key challenges of digital health intervention adoption during the COVID-19 pandemic. Ghorui, N. et al. [13] identified the dominant risk factor involved in the spread of COVID-19 using hesitant fuzzy MCDM methodology. In addition to medical diagnosis, a great deal of study in other areas [14,15] involving hesitant fuzzy sets has been carried out. Additionally, many studies have been conducted on generalized fuzzy numbers, including generalized triangular intuitionistic fuzzy numbers (*GTIFN*) [16], generalized trapezoidal hesitant fuzzy numbers (*GTrHFN*) [2], generalized hesitant fuzzy numbers (*GHFN*) [8], and generalized hexagonal fuzzy numbers (*GH_χFN*) [17]. However, as far as we know, there is currently no research on the use of a generalized dual hesitant hexagonal fuzzy (*GDHH_χF*) MCDM technique for disease recognition. Hence, in this paper, we focus on this issue and use specific MCDM techniques, such as fuzzy criteria importance through inter-criteria correlation (FCRITIC), fuzzy weighted aggregated sum product assessment (FWASPAS), and fuzzy combined compromise solution (FCoCoSo), to recognize a patient's disease using a new fuzzy environment called a generalized dual hesitant hexagonal fuzzy number (*GDHH_χFN*).

The generalized dual hesitant hexagonal fuzzy multi-criteria decision-making (*GDHH_χF-MCDM*) technique can be used to recognize diseases by considering multiple criteria and factors that may affect the diagnosis process. The first step in using the *GDHH_χF-MCDM* technique for disease recognition is to identify the relevant criteria and factors that are associated with the disease. These criteria may include medical symptoms, patient history, laboratory results, imaging tests, and other diagnostic tools. Next, the experts or decision-maker in the field can assign weights to each criterion based on its importance in the disease recognition process. These weights can be determined using professional judgment, statistical analysis, or other methods. Then, the experts can use the *GDHH_χF-MCDM* methods to evaluate each criterion based on its linguistic variable and their degree of membership and non-membership values. This technique allows for the use of dual hesitant fuzzy numbers, which can represent uncertainty and vagueness in the evaluation process. Finally, the *GDHH_χF-MCDM* technique can be used to aggregate the evaluations of each criterion and provide an overall diagnosis of the disease. The technique can also be used to rank the potential diseases of a patient based on their likelihood of being the correct diagnosis.

Overall, the *GDHH_χF-MCDM* technique can help to improve the accuracy and efficiency of disease recognition by considering multiple criteria and sub-criteria in the decision-making process.

1.1. Motivation of the Study

There need to be more suitable decision mechanisms in the context of current studies relating to disease-recognizing issues. The multiple-criteria decision-making (MCDM) methods have used numerous successful real-world challenges to examine difficult decisions and appropriately make a proper decision. To address the vital characteristic aspects that match integrating MCDM and the diseases recognition problem, further research into the emergence of MCDM in addressing the diseases recognition problem is required. On the other hand, if uncertainties are with the whole situation, then the problem is more complex to solve. Hesitant fuzzy ideology is one of the preferable uncertainties for the

diagnosis problem. This study will provide MCDM, fuzzy sets theory, and its extensions to researchers and the broader community for tackling an accurate decision for disease diagnosis problems.

1.2. Research Outline

Based on the above introduction and especially research motivation, the primary outlines of this paper are as follows:

- For pre-diagnosis of the diseases, pick the disorders that are caused by viruses, have many overlapping symptoms, and have been scientifically identified.
- Identifying the symptoms of different diseases and comparative studies on signs with slight differences.
- Develop a comparative matrix based on criteria and alternatives for the Generalized Dual Hesitant Hexagonal fuzzy number ($GDHHxFN$) by the decision-makers (DMs) to capture all the ambiguity, hesitancy, and vagueness.
- Calculate the criteria weight using the MCDM method FCRITIC. According to decision experts, this part identifies which symptoms are more severe and dangerous than others.
- Rank the diseases by FWASPAS and FCoCoSo, two MCDM methods for pre-diagnosis and early treatment. The two approaches mentioned above are considered concurrently for robustness and to ensure the consistency of our ranking.
- We conducted sensitivity and comparative analyses to check the result's vagueness and unbiasedness. It demonstrates that how internally consistent our results are.

1.3. Structure of the Paper

Section 1 contains the introduction and motivation for this study. The literature review on methodology and applications are covered in Section 2. Then, Sections 3 and 4 include preliminaries of mathematical tools and MCDM methodology, respectively. Then the alternatives are discussed in Section 5. Section 6 described the symptoms of the disease and model setup. Next, data collection and numerical description are covered in Sections 7 and 8, respectively. Sensitivity analysis and Comparative analysis are described in Section 9. Section 10 discuss practical implication. Finally, the conclusion and future research extension are contained in Section 11.

2. Literature Review

This section introduces the necessary basic key terms and their recent uses. Disease recognition is crucial in healthcare domains, and MCDM supports decision-making. MCDM is a decision-making technique considering multiple criteria and a weighted combination of these criteria from various alternatives. The literature on disease recognition using MCDM covers numerous aspects, including the application of MCDM in medical decision-making, the development of novel MCDM models for disease recognition, and the evaluation of the effectiveness of these models. Several studies have used MCDM to develop decision support systems for various diseases in Tables 1 and 2. Moreover, Tables 3 and 4 cover recent work on dual hesitant fuzzy numbers and generalized fuzzy numbers, respectively.

Table 1. MCDM in disease control.

Authors	Citation	Years	MCDM Methods	Application Area
Ritmak, N. et al.	[18]	2023	AHP and TOPSIS	Health sustainability model
Mumtaz, A.	[19]	2022	AHP	Prioritizing and overcoming barriers to e-health
Yas, Q. M. et al.	[20]	2021	TOPSIS	Evaluation of multi diabetes mellitus symptoms
Yildirim, F. S. et al.	[21]	2021	PROMETHEE and VIKOR	Treatment of COVID-19
Nguyen, P.H. et al.	[22]	2021	SF-AHP and FWASPAS	COVID-19 Pandemic
Addy, M. et al.	[23]	2020	MCDM	Detection and severity monitoring to serve for Dengue
Mohammed, M. A. et al.	[24]	2020	Entropy and TOPSIS	Selection of optimal COVID-19 diagnostic model
Abdel-Basset, M. et al.	[25]	2020	N-MCDM	Heart disease diagnosis
Sharawat, K. et al.	[26]	2018	AHP and TOPSIS	Diet recommendation for diabetic patients
Basanta, H. et al.	[27]	2016	AHP	IoT-based H2U healthcare system
This paper		2023	FCRITIC, FWASPAS, FCoCoSo	Diagnosis of a patient

Table 2. Recent use MCDM technique in Disease.

Authors	Citation	Year	Types of Diseases	Optimization Techniques
Faith-Michael Uzoka et al.	[28]	2022	Typhoid	AHP
Jacques Fantini et al.	[29]	2022	Monkeypox	–
Jasdeep Kaler et al.	[30]	2022	Monkeypox	–
Rose Nakasi et al.	[31]	2021	Malaria	Faster R-CNN and SSD MobileNet
Abodayeh, K. et al.	[32]	2021	Pneumonia	Euler, Runge–Kutta and non-standard finite difference technique
Madhumita Addy et al.	[23]	2020	Dengue	MCDM
G. Bhuju et al.	[33]	2020	Dengue	Fuzzy SEIR-SEI
Anna Marie Nathan et al.	[34]	2020	Pneumonia	–
Umar M. Modibbo et al.	[35]	2019	Malaria	AHP
Funda Samanlioglu	[36]	2019	Influenza	AHP and VIKOR
Nuriah Abd Majid et al.	[37]	2019	Dengue	Spatial Mean Center, Standard Distant and Standard Deviation Ellipse
Nivetha Martin et al.	[38]	2019	Ebola	Fuzzy VIKOR
Gbadamosi, B. et al.	[39]	2018	Dengue	Disease-Free Equilibrium (DFE)
Jung-Yoon Kim et al.	[40]	2018	Malaria	AHP and PROMETHEE
Tilahun, G. T. et al.	[41]	2017	Typhoid	Disease-Free Equilibrium (DFE)
Athena P. Kourtis et al.	[42]	2015	Ebola	–
Ashraf M. Dewan et al.	[43]	2013	Typhoid	Spatial analysis and Cluster mapping
Ozgur Maraz	[44]	2013	Influenza	AHP
Jie Zhang et al.	[45]	2011	Influenza	One class classification model
Pavan K. Attaluri et al.	[46]	2009	Influenza	Hidden Markov Model
This paper		2023	7 diseases	FCRITIC, FWASPAS, FCoCoSo

Table 3. Recent use of a Dual Hesitant Fuzzy Set method in different applications.

Authors	Citation	Year	Application Area	Methodology
Sindhu et al.	[47]	2023	Selection of alternative based on linear programming	Fuzzy TOPSIS
S. F. Ismael et al.	[48]	2023	A Pavement Strategy Selection	DH-FWZIC, DH-FDOSM
Daekook Kang et al.	[49]	2022	determine a used PPE kit disposal	MARCOS
Verma et al.	[50]	2022	Consensus group decision-making	A genetic algorithm based
Ni, Y. et al.	[51]	2022	Product marketing	Dual hesitant MADM
Guirao et al.	[52]	2020	Multiple Criteria Decision-Making Based on Vector Similarity Measures	Linear programming
Garg et al.	[53]	2020	Quantifying gesture information in brain hemorrhage patients	Entropy
Z. Ren et al.	[54]	2019	The Strategy Selection Problem on Artificial Intelligence	VIKOR, AHP
Huiping Chen et al.	[55]	2019	Dual Hesitant Fuzzy Information Measures and Their Applications	Entropy
Narayanamoorthy, S. et al.	[56]	2019	Site selection for hydrogen underground storage	VIKOR
Liu et al.	[57]	2019	Risk Evaluation	Three phased MCGDM
Hao, Z. et al.	[58]	2017	Risk evaluation	PDHFs MCDM
This paper		2023	Prediagnosis of disease based on symptoms	FCRITIC, FWASPAS, and FCoCoSo

Table 4. Literature review on Generalized fuzzy number.

Authors	Year	Hesitant Type	Optimization Methods	Applications
Deli, I. et al.	[59] 2021	Generalized Trapezoidal Hesitant Fuzzy Numbers (GTHFN)	Ranking method of GTHFN	Best option to invest an investment company
Sultan, A. et al.	[60] 2021	Triangular hesitant fuzzy set	TOPSIS	Determine the business chain for most revenue
Büyükoğkan, G. et al.	[61] 2021	Hesitant fuzzy linguistic term set	AHP and MULTIMOORA	Supply chain analysis on logistics services
Farid, H.M.A. et al.	[62] 2021	q-rung orthopair fuzzy numbers	Gq-ROFEIWG	Determine its upcoming year's strategy on multinational corporation
Zulqarnain, R.M. et al.	[63] 2020	Generalized triangular fuzzy number (GTFN)	TOPSIS	Garments industry hire a supplier
Yue, Q. et al.	[64] 2020	Hesitant fuzzy number (HFN)	Two-sided matching (TsM) method	Smart intelligent technique transfer
Garg, H. et al.	[65] 2020	Generalized trapezoidal hesitant fuzzy number (GTHFN)	TOPSIS and Choquet integral (CI)	Artificial numerical example
Zhong, Y. et al.	[66] 2019	Pythagorean hesitant fuzzy set	MCDM method based on the PHEFMM and WPHEFMM operators	Green supply chain management
Garg, H. et al.	[67] 2018	Dual hesitant fuzzy soft set	Aggregation operator	Monsoon rains in river catchments
Faizi, S. et al.	[68] 2018	L–R-type generalized fuzzy numbers	COMET	Electrical resistance
Faizi, S. et al.	[69] 2017	L–R-type generalized fuzzy numbers	Characteristic Objects Method (COMET)	Production on mobile factory
Liang, D. et al.	[70] 2017	Hesitant Pythagorean fuzzy sets	TOPSIS	Energy development strategy
Zhiliang, R. et al.	[71] 2017	Dual hesitant fuzzy set	VIKOR	Smartphone design firms to increasing profit
Khan, M.S.A. et al.	[72] 2017	Pythagorean hesitant fuzzy set	MADM on Pythagorean hesitant fuzzy information	Choose the best option of investment company
Wang, L. et al.	[73] 2016	Dual hesitant fuzzy set	Power aggregation operator	Urban traffic route choices
This paper	2023	Generalized dual hesitant hexagonal fuzzy set	FCRITIC, FWASPAS, and FCoCoSo	Prediagnosis of disease based on symptoms

3. Preliminaries

This section appries the basic definitions relating to fuzzy sets (FS), membership function, generalization of FS, hesitation of FS, and some algebraic operations which will be needed to elucidate the scope of this paper.

3.1. Fuzzy Set Theory

Prof. Lotfi A. Zadeh presented fuzzy sets [74] from the University of California at Berkeley in 1965. A fuzzy set is a set that is characterized by a membership function (MF). Qualitative and subjective viewpoints are communicated by linguistic variables, which are quantitatively expressed by a fuzzy set in the context of discourse and the corresponding MF [75,76]. The fuzzy set represents impreciseness, fully intent on decreasing intricacy by disposing of the sharp limit isolating the members from the pair from non-members. Every component in this set with its degree of membership value [77]. The definition of the fuzzy set and its membership function is described below. There exist different MCDM problems associated with an extension of the fuzzy set, such as interval-valued fuzzy soft set [78] and the interval-valued intuitionistic fuzzy set [79].

3.1.1. Generalized Fuzzy Number

A generalized fuzzy number $\tilde{A} = (\xi_a, \xi_b, \xi_c, \xi_d, \xi_e, \xi_f; \tilde{\Omega})$ is described as any fuzzy set (especially, $\tilde{A}$ be a generalized hexagonal fuzzy number, more details are later) of the real line $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$ that possesses the following features:

1. $\mu_{\tilde{A}}(y) : \mathbb{R} \rightarrow [0, \tilde{\Omega}]$ is continuous, $\tilde{\Omega} \in (0, 1]$,
2. $\mu_{\tilde{A}}(y) = 0$ for all $y \in (-\infty, \xi_a)$,
3. $\mu_{\tilde{A}}(y)$ is monotonically increasing on $[\xi_a, \xi_b)$ and $[\xi_b, \xi_c)$,
4. $\mu_{\tilde{A}}(y) = \tilde{\omega}$ for all $y \in [\xi_c, \xi_d]$,
5. $\mu_{\tilde{A}}(y)$ is monotonically decreasing on $(\xi_d, \xi_e]$ and $(\xi_e, \xi_f]$,
6. $\mu_{\tilde{A}}(y) = 0$ for all $y \in (\xi_f, \infty)$.

3.1.2. Hesitant Fuzzy Number

Torra, V. et al. [80] put out a notion known as the hesitant fuzzy set (HFS). When a decision-maker is not entirely satisfied with a precise membership degree, HFSs can address the intrinsic ambiguity in the decision-making process by enabling the decision-maker to describe their preference with a membership function containing a greater variety of values [81]. Chen, N. et al. [82] also discussed HFS. When compromised decisions are used in group decision-making challenges, HFSs are preferred more. Torra, V. et al. [80] represented the HFS as follows:

Definition 1. Assume Y_θ be a fixed set. Hesitant fuzzy set (HFS) $\tilde{F}_{eh}$ is defined on Y_θ to $[0, 1]$. The HFS is represented as follows,

$$\tilde{F}_{eh} = \{ \langle y, \mu_{\tilde{F}_{eh}}(y) \rangle \mid y \in Y_\theta \} \quad (1)$$

where $\mu_{\tilde{F}_{eh}}(y)$ denotes the possible membership function of the set $\tilde{F}_{eh}$, bounded by $[0, 1]$. For convenience, $\mu_{\tilde{F}_{eh}} = \mu_{\tilde{F}_{eh}}(y)$ is called a hesitant fuzzy element (HFE).

3.1.3. Dual Hesitant Fuzzy Number

Zhu, B. et al. [83] defined the dual hesitant fuzzy number (DHFN). Tyagi, S.K. [84] also discusses it. DHFN are defined as follows:

Definition 2. Suppose there is a fixed set Y_θ . A DHFS $\tilde{D}_{eh}$ on the fixed set Y_θ is described as follows,

$$\tilde{D}_{eh} = \{ \langle y, \mu_{\tilde{D}_{eh}}(y), \nu_{\tilde{D}_{eh}}(y) \rangle \mid y \in Y_\theta \} \quad (2)$$

where $\mu_{\tilde{D}_{eh}}(y)$ and $\nu_{\tilde{D}_{eh}}(y)$ are the membership and non-membership degree, respectively, and satisfies the following conditions:

- (a). $0 \leq \mu_{\tilde{D}_{eh}}, \nu_{\tilde{D}_{eh}} \leq 1$ for all $y \in Y_\theta$
 (b). $0 \leq \mu_{\tilde{D}_{eh}}^+ + \nu_{\tilde{D}_{eh}}^- \leq 1$ where $\mu_{\tilde{D}_{eh}}^+ = \max_{y \in Y_\theta} \{\mu_{\tilde{D}_{eh}}(y)\}$ and $\nu_{\tilde{D}_{eh}}^- = \max_{y \in Y_\theta} \{\nu_{\tilde{D}_{eh}}(y)\}$

3.1.4. Basic Operation of DHFS:

Consider two DHFS $\tilde{\Delta}_{eh}, \tilde{\Gamma}_{eh}$ on Y_θ and described as $\tilde{\Delta}_{eh} = \{ \langle \xi, \mu_{\tilde{\Delta}_{eh}}(\xi), \nu_{\tilde{\Delta}_{eh}}(\xi) \rangle \mid \xi \in Y_\theta \}$ and $\tilde{\Gamma}_{eh} = \{ \langle \xi, \mu_{\tilde{\Gamma}_{eh}}(\xi), \nu_{\tilde{\Gamma}_{eh}}(\xi) \rangle \mid \xi \in Y_\theta \}$. Then, some basic set operations and arithmetic operations on them are defined as follows:

A. Complement of $\tilde{\Delta}_{eh}$

$$\tilde{\Delta}_{eh}^c = \{ \langle \xi, \nu_{\tilde{\Delta}_{eh}}(\xi), \mu_{\tilde{\Delta}_{eh}}(\xi) \rangle \mid \xi \in Y_\theta \} \quad (3)$$

B. Union of $\tilde{\Delta}_{eh}$ and $\tilde{\Gamma}_{eh}$

$$\tilde{\Delta}_{eh} \cup \tilde{\Gamma}_{eh} = \bigcup_{\substack{\mu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \mu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}, \\ \nu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \nu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}}} \left\{ \langle \xi, \max\{\mu_{\tilde{\Delta}_{eh}}(\xi), \mu_{\tilde{\Gamma}_{eh}}(\xi)\}, \min\{\nu_{\tilde{\Delta}_{eh}}(\xi), \nu_{\tilde{\Gamma}_{eh}}(\xi)\} \rangle \mid \xi \in Y_\theta \right\} \quad (4)$$

C. Intersection of $\tilde{\Delta}_{eh}$ and $\tilde{\Gamma}_{eh}$

$$\tilde{\Delta}_{eh} \cap \tilde{\Gamma}_{eh} = \bigcap_{\substack{\mu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \mu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}, \\ \nu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \nu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}}} \left\{ \langle \xi, \min\{\mu_{\tilde{\Delta}_{eh}}(\xi), \mu_{\tilde{\Gamma}_{eh}}(\xi)\}, \max\{\nu_{\tilde{\Delta}_{eh}}(\xi), \nu_{\tilde{\Gamma}_{eh}}(\xi)\} \rangle \mid \xi \in Y_\theta \right\} \quad (5)$$

D. Addition of $\tilde{\Delta}_{eh}$ and $\tilde{\Gamma}_{eh}$

$$\tilde{\Delta}_{eh} \oplus \tilde{\Gamma}_{eh} = \bigcup_{\substack{\mu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \mu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}, \\ \nu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \nu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}}} \left\{ \langle \xi, (\mu_{\tilde{\Delta}_{eh}}(\xi) + \mu_{\tilde{\Gamma}_{eh}}(\xi) - \mu_{\tilde{\Delta}_{eh}}(\xi)\mu_{\tilde{\Gamma}_{eh}}(\xi)), (\nu_{\tilde{\Delta}_{eh}}(\xi) + \nu_{\tilde{\Gamma}_{eh}}(\xi) - \nu_{\tilde{\Delta}_{eh}}(\xi)\nu_{\tilde{\Gamma}_{eh}}(\xi)) \rangle \mid \xi \in Y_\theta \right\} \quad (6)$$

E. Multiplication of $\tilde{\Delta}_{eh}$ and $\tilde{\Gamma}_{eh}$

$$\tilde{\Delta}_{eh} \otimes \tilde{\Gamma}_{eh} = \bigcup_{\substack{\mu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \mu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}, \\ \nu_{\tilde{\Delta}_{eh}} \in \tilde{\Delta}_{eh}, \nu_{\tilde{\Gamma}_{eh}} \in \tilde{\Gamma}_{eh}}} \left\{ \langle \xi, (\mu_{\tilde{\Delta}_{eh}}(\xi)\mu_{\tilde{\Gamma}_{eh}}(\xi)), (\nu_{\tilde{\Delta}_{eh}}(\xi) + \nu_{\tilde{\Gamma}_{eh}}(\xi) - \nu_{\tilde{\Delta}_{eh}}(\xi)\nu_{\tilde{\Gamma}_{eh}}(\xi)) \rangle \mid \xi \in Y_\theta \right\} \quad (7)$$

3.2. Hexagonal Fuzzy Number ($H_\chi FN$)

Definition 3. [85,86] Let Y_θ be the fixed set and $\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u \in \mathbb{R}$, such that $\xi_p \leq \xi_q \leq \xi_r \leq \xi_s \leq \xi_t \leq \xi_u$. Then, a hexagonal fuzzy number ψ_N on Y_θ is defined as

$$\begin{aligned} \psi_N &= \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) : \alpha, 1 \rangle \\ &= \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) : \alpha \rangle \end{aligned} \quad (8)$$

with membership function μ_{ψ_N} defined as

$$\mu_{\psi_N}(y) = \begin{cases} 0 & \text{if } y < \xi_p \\ \alpha \frac{(y - \xi_p)}{(\xi_q - \xi_p)} & \text{if } \xi_p \leq y < \xi_q \\ \alpha + (1 - \alpha) \frac{(y - \xi_q)}{(\xi_r - \xi_q)} & \text{if } \xi_q \leq y < \xi_r \\ 1 & \text{if } \xi_r \leq y \leq \xi_s \\ \alpha + (1 - \alpha) \frac{(y - \xi_t)}{(\xi_s - \xi_t)} & \text{if } \xi_s < y \leq \xi_t \\ \alpha \frac{(y - \xi_t)}{(\xi_u - \xi_t)} & \text{if } \xi_t < y \leq \xi_u \\ 0 & \text{if } \xi_u < y \end{cases} \quad (9)$$

where $y \in \mathbb{R}$ and $\alpha \in [0, 1]$.

Note 1. The range of α is $[0, 1]$ and if

- $\alpha = 0$ then hexagonal fuzzy number ψ_N becomes trapezoidal fuzzy number $\psi = \langle (\xi_q, \xi_r, \xi_s, \xi_t) \rangle$.
- $\alpha = 1$ then hexagonal fuzzy number ψ_N becomes trapezoidal fuzzy number $\psi = \langle (\xi_p, \xi_q, \xi_t, \xi_u) \rangle$.
- $\alpha \in (0, 1)$ then hexagonal fuzzy numbers ψ_N are not restructured.

Therefore, in general, always consider $\alpha \in (0, 1)$.

Figure 1 describes the structural representation of the membership function of $H_\chi FN$, where α lies in $(0, 1)$ and the highest membership value is 1.

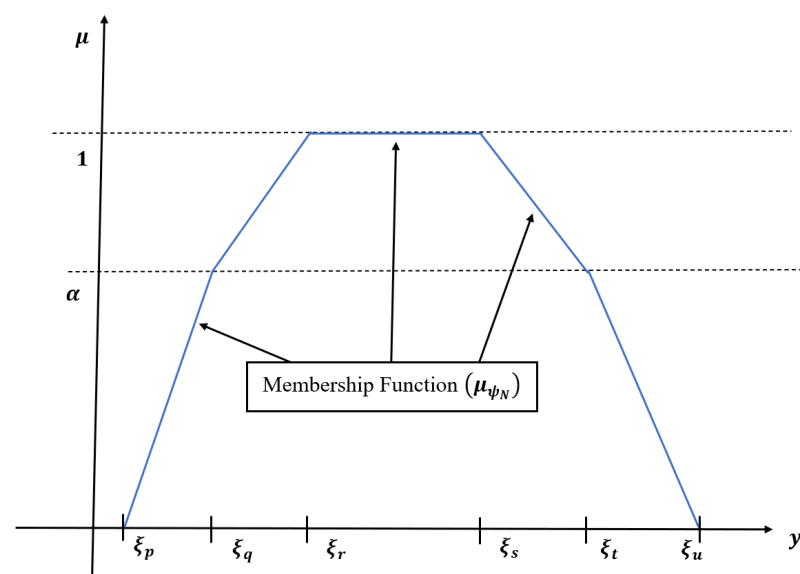


Figure 1. Membership function of $H_\chi FN$.

3.2.1. Arithmetic Operation on $H_\chi FN$

Let $U = (u_1, u_2, u_3, u_4, u_5, u_6)$ and $V = (v_1, v_2, v_3, v_4, v_5, v_6)$ be two $H_\chi FN$, then their general arithmetic operation can be defined in the following way:

1. Addition:

$$U \oplus V = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5, u_6 + v_6) \quad (10)$$

2. Subtraction:

$$U \ominus V = (u_1 - v_6, u_2 - v_5, u_3 - v_4, u_4 - v_3, u_5 - v_2, u_6 - v_1) \quad (11)$$

3. Multiplication:

$$U \otimes V = (u_1 v_1, u_2 v_2, u_3 v_3, u_4 v_4, u_5 v_5, u_6 v_6) \quad (12)$$

4. Scalar Multiplication:

$$\alpha U = (\alpha u_1, \alpha u_2, \alpha u_3, \alpha u_4, \alpha u_5, \alpha u_6) \quad (13)$$

5. Division:

$$U \oslash V = \left(\frac{U}{V} \right) = \left(\frac{u_1}{v_6}, \frac{u_2}{v_5}, \frac{u_3}{v_4}, \frac{u_4}{v_3}, \frac{u_5}{v_2}, \frac{u_6}{v_1} \right) \quad (14)$$

6. Inverse:

$$U^{-1} = \frac{1}{U} = \left(\frac{1}{u_6}, \frac{1}{u_5}, \frac{1}{u_4}, \frac{1}{u_3}, \frac{1}{u_2}, \frac{1}{u_1} \right) \quad (15)$$

3.2.2. Generalised Hexagonal Fuzzy Set

Definition 4. [87] Let Y_θ be the fixed set and $\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u \in \mathbb{R}$ such that $\xi_p \leq \xi_q \leq \xi_r \leq \xi_s \leq \xi_t \leq \xi_u$. Then, a generalised hexagonal fuzzy set ψ_g on Y_θ is defined as:

$$\psi_g = \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) : \alpha, \beta \rangle \quad (16)$$

with membership function μ_{ψ_g} defined as

$$\mu_{\psi_g}(y) = \begin{cases} 0 & \text{if } y < \xi_p \\ \alpha \frac{(y - \xi_p)}{(\xi_q - \xi_p)} & \text{if } \xi_p \leq y < \xi_q \\ \alpha + (\beta - \alpha) \frac{(y - \xi_q)}{(\xi_r - \xi_q)} & \text{if } \xi_q \leq y < \xi_r \\ \beta & \text{if } \xi_r \leq y \leq \xi_s \\ \alpha + (\beta - \alpha) \frac{(y - \xi_t)}{(\xi_s - \xi_t)} & \text{if } \xi_s < y \leq \xi_t \\ \alpha \frac{(y - \xi_u)}{(\xi_t - \xi_u)} & \text{if } \xi_t < y \leq \xi_u \\ 0 & \text{if } \xi_u < y \end{cases} \quad (17)$$

where $y \in \mathbb{R}$ and $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$.

Note 2. Depending on the range and relation between α and β , the generalised hexagonal fuzzy set (GH_χFS) ψ_g is restructured. The range is $0 \leq \alpha \leq \beta \leq 1$. Then, depending on the relation α and β , reshape the GH_χFS ψ_g in the following way:

- If $\alpha = \beta = 0$ then the GH_χFS ψ_g becomes a zero set.
- If $\alpha = 0$ and $\beta \in (0, 1)$ then the GH_χFS ψ_g becomes generalised trapezoidal fuzzy set $\psi = \langle (\xi_q, \xi_r, \xi_s, \xi_t) : \beta \rangle$.
- If $\alpha = 0$ and $\beta = 1$ then the GH_χFS ψ_g becomes trapezoidal fuzzy set $\psi = \langle (\xi_q, \xi_r, \xi_s, \xi_t) \rangle$.
- If $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha < \beta$ then the GH_χFS ψ_g unchanged.
- If $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha = \beta$ then the GH_χFS ψ_g becomes generalised trapezoidal fuzzy set $\psi = \langle (\xi_p, \xi_q, \xi_t, \xi_u) : \alpha \rangle$.
- If $\alpha \in (0, 1)$ and $\beta = 1$ then the GH_χFS ψ_g becomes hexagonal fuzzy set $\psi = \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) : \alpha \rangle$.
- If $\alpha = \beta = 1$ then the GH_χFS ψ_g becomes trapezoidal fuzzy set $\psi = \langle (\xi_p, \xi_q, \xi_t, \xi_u) \rangle$.

Figure 2 represented the generalised hexagonal fuzzy number $\psi_g = \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) : \alpha, \beta \rangle$. The highest membership value of ψ_g is β and the value of α are in between 0 and β .

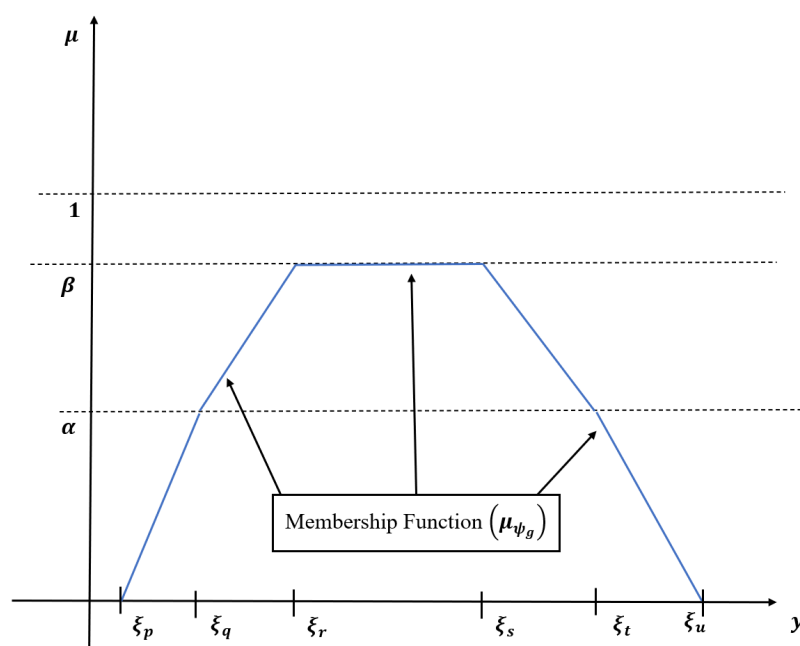


Figure 2. Membership function of $GH_{\chi}FN$.

3.3. Generalized Hexagonal Dual Hesitant Fuzzy Set

A Hexagonal Dual Hesitant Fuzzy Set ($H_{\chi}DHFS$) is a hesitant fuzzy set introduced as an extension of the dual hesitant fuzzy set.

3.3.1. Hexagonal Dual Hesitant Fuzzy Set

Definition 5. Let Y_{θ} be a fixed set. Hexagonal dual hesitant fuzzy set ($H_{\chi}DHFS$) $D_{eh_{\chi}}$ on Y_{θ} is described as,

$$D_{eh_{\chi}} = \{ \langle y, (\mu_{eh_{\chi}}(y), \nu_{eh_{\chi}}(y)) \rangle \mid y \in Y_{\theta} \} \quad (18)$$

where $\mu_{eh_{\chi}}(y)$ is the set of HFN possible membership values and $\nu_{eh_{\chi}}(y)$ is the set of HFN possible non-membership values of the element $y \in Y_{\theta}$ to the set $D_{eh_{\chi}}$.

3.3.2. Hexagonal Dual Hesitant Fuzzy Number ($H_{\chi}DHFN$)

If $H_{\chi}F_h(y) = \{(\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u) \mid \{\mu(y), \nu(y)\}\}$ then $H_{\chi}F_h(y)$ is called the Hexagonal Dual Hesitant Fuzzy Element.

3.3.3. Generalized Hesitant Fuzzy Number ($GHFN$)

It is a collection of generalized fuzzy numbers reflecting various membership functions that could be applied to the element $y \in Y_{\theta}$ in the set of real numbers $\mathbb{R}$. The generalized triangular hesitant fuzzy number ($GTHFN$), generalized trapezoidal hesitant fuzzy number ($GTrHFN$), and generalized pentagonal hesitant fuzzy numbers ($GPHFN$) are some examples of this form.

3.3.4. Generalized Hexagonal Fuzzy Number ($GH_{\chi}FN$)

Generally, the maximum membership value of any fuzzy number is 1, but it needs not be the same for $GH_{\chi}FN$. Any values from $(0, 1]$ are acceptable. The $GH_{\chi}FN$ is changed to $H_{\chi}FN$ if the maximum membership value is 1.

3.4. Generalized Dual Hesitant Hexagonal Fuzzy Number ($GDHH_{\chi}FN$)

A $GDHH_{\chi}FN$ is defined in this paper in two forms. In $GDHH_{\chi}FN$ (1st type), the number of $H_{\chi}FN$ can be more than one, but the number of membership and non-membership

functions is restricted to one. In $GDHH_\chi FN$ (2nd type), the number of the $H_\chi FN$ is restricted to one, but the membership and non-membership functions can be more than one.

3.4.1. Benefits of $DHFS$ Rather than FS

A fuzzy set variation, known as the dual hesitant fuzzy set ($DHFS$), enables a more sophisticated representation of uncertainty and ambiguity. In contrast to fuzzy sets, which only allow partial membership of an element in a set, $DHFS$ leads to a more fine-grained representation of uncertainty by allowing an element to be neither entirely in nor fully out of a set. Here are some benefits of $DHFS$ over fuzzy sets:

- (A) $DHFS$ may represent membership degrees that are positive or negative, whereas fuzzy sets can only do so for positive membership degrees. This makes it possible for $DHFS$ to represent scenarios when one element is uncertain and potentially not a set member.
- (B) The uncertainty and hesitancy of a decision-maker can be better captured by $DHFS$ than FS , producing more precise and dependable outcomes.
- (C) Decision-makers can make better decisions with access to more complete and accurate information from $DHFS$.
- (D) When handling uncertain or missing data, $DHFS$ delivers additional flexibility, which can result in more reliable and adaptable decision-making systems.

3.4.2. Benefits of $GDHH_\chi FS$ Rather than FS

Somebody can use further development of the dual hesitant fuzzy set, the generalized dual hesitant hexagonal fuzzy set ($GDHH_\chi FS$), to communicate uncertainty more elaborately than fuzzy sets. The advantages of $GDHH_\chi FS$ over fuzzy sets are as follows:

- (a) $GDHH_\chi FS$ provides a more accurate and improved representation of uncertainty and ambiguous information than fuzzy sets. It can also represent the degree of hesitation in decision-making.
- (b) For handling ambiguous or missing data, $GDHH_\chi FS$ provides greater flexibility. Under a single framework, it enables the representation of several kinds of uncertainty, including randomness, fuzziness, and incompleteness.
- (c) $GDHH_\chi FS$ provides a clear and intuitive interpretation of each element's membership and non-membership degrees, making it easy for decision-makers to understand and use and allowing for more informed and reliable decision-making.

Now we are focusing on $GDHH_\chi FN$.

Definition 6. (Generalised Dual Hesitant Hexagonal Fuzzy Number $GDHH_\chi FN$). Assume Y_θ to be a fixed set, $i \in T \subset \mathbb{N}$ be an integer and $\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i} \in \mathbb{R}$, such that the relation among them are $\xi_{p_i} \leq \xi_{q_i} \leq \xi_{r_i} \leq \xi_{s_i} \leq \xi_{t_i} \leq \xi_{u_i}$, for all $i \in T$. Then generalised dual hesitant hexagonal fuzzy number $GDHH_\chi FN$ is defined as

$$\psi = \left\{ \left\langle \bigcup_{i \in T} (\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i}); \mu_\psi, \nu_\psi \right\rangle : i \in T \right\} \quad (19)$$

where μ_ψ and ν_ψ denotes the membership function and non-membership function, respectively.

Remark 1. A generalized dual hesitant hexagonal fuzzy number is a collection of hexagonal fuzzy numbers with $i \in T \subset \mathbb{N}$. Additionally, $GDHH_\chi FN$ can be formed by generalized fuzzy numbers, hesitant fuzzy numbers, dual hesitant fuzzy numbers, and hexagonal fuzzy numbers, as depicted in Figure 3.

We construct two forms of generalized dual hesitant hexagonal fuzzy numbers from the above definitions. In the first type of $GDHH_\chi FN$, the membership and non-membership functions are fixed, but the $H_\chi FN$ can vary. In the second type, the $H_\chi FN$ are

considered to be fixed, but the membership and non-membership function for a specific $H_\chi FN$ can differ depending on the preference of DMs.

Definition 7. Let $\psi = \left\{ \left\langle y, \bigcup_{i \in T} (\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i}) : \mu_i, v_i \right\rangle : i \in T; y \in Y_\emptyset \right\}$ represents a set of different $GDHH_\chi FN$. If $\mu_i = \mu$ for all $i \in T$ then $GDHH_\chi FN$ is reduced to a single membership function for different $GDHH_\chi FN$. Similarly, for non-membership function $v_i = v$ for all $i \in T$. This form of $GDHH_\chi FN$ is said to be $GDHH_\chi FN$ of the 1st type. It is structurally expressed in the following way

$$\psi_{GDHH_\chi FN}^1 = \left\{ \left\langle y, \bigcup_{i \in T} (\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i}); \{\mu, v\} \right\rangle : i \in T, y \in Y_\emptyset \right\} \quad (20)$$

The example of $GDHH_\chi FN$ of the first type is presented in Example 1. Here, consider $i = 2$ and obtain two sets of $H_\chi FN$.

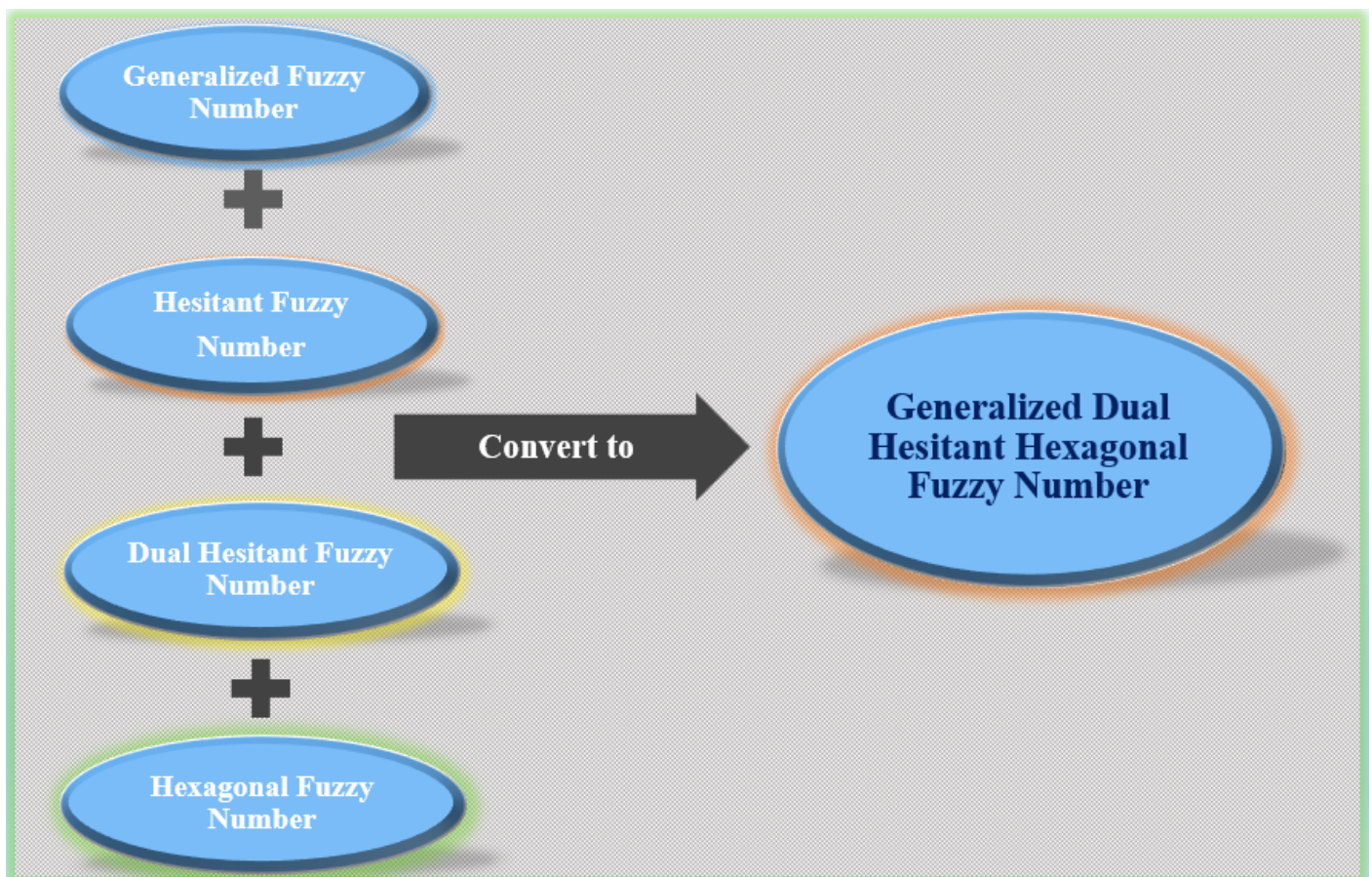


Figure 3. Visualization of diagram illustrating the structure of a $GDHH_\chi FN$.

Example 1. Consider ψ_a^1 where $a = 1, 2, \dots, 6$ are the $GDHH_\chi FN$ of first type and represented as follows:

- $\psi_1^1 = \langle (5.3, 5.4, 5.6, 5.7, 5.8, 5.9), (5.4, 5.5, 5.6, 5.8, 5.9, 6); \{0.9, 0.1\} \rangle$
- $\psi_2^1 = \langle (4.4, 4.5, 4.7, 4.8, 4.9, 5), (4.8, 4.9, 5.1, 5.2, 5.3, 5.4); \{0.8, 0.1\} \rangle$
- $\psi_3^1 = \langle (3.8, 4.9, 4, 4.1, 4.2, 4.3), (4.5, 4.6, 4.7, 4.8, 4.9, 5); \{0.7, 0.2\} \rangle$
- $\psi_4^1 = \langle (3, 3.1, 3.2, 3.4, 3.5, 3.6), (3.4, 3.5, 3.6, 3.7, 3.8, 4); \{0.5, 0.4\} \rangle$
- $\psi_5^1 = \langle (2, 2.2, 2.3, 2.4, 2.5, 2.6), (2.4, 2.5, 2.6, 2.7, 2.8, 2.9); \{0.3, 0.6\} \rangle$
- $\psi_6^1 = \langle (1, 1.2, 1.3, 1.4, 1.5, 1.6), (1.3, 1.4, 1.5, 1.6, 1.7, 1.9); \{0.2, 0.5\} \rangle$

Remark 2. $GDHH_\chi FN$ can be used to describe the hesitancy or vagueness of a DM or group of DMs in different possible ways. Unlike the Generalized Dual Hesitant Triangular Fuzzy Number (GDHTFN), Generalized Dual Hesitant Trapezoidal Fuzzy Number (GDHTTrFN), or Generalized Dual Hesitant Pentagonal Fuzzy Number (GDHPFN), which represents the imprecision in three numbers, or four numbers or five numbers, respectively, $GDHH_\chi FN$ considers six numbers and, thus, captures the uncertainty in better form.

Definition 8. Let $\Gamma = \{ \langle y, \bigcup_{i \in T} (\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i}); \mu_\Gamma^i, \nu_\Gamma^i : i \in T \rangle : y \in Y_\theta \}$ represents a set of different $GDHH_\chi FN$. If $\xi_p = \xi_{p_i}, \xi_q = \xi_{q_i}, \xi_r = \xi_{r_i}, \xi_s = \xi_{s_i}, \xi_t = \xi_{t_i}, \xi_u = \xi_{u_i}$ for all $i \in T$, $GDHH_\chi FN$ is reduced to single valued generalized dual hesitant hexagonal fuzzy number with different set of membership and non-membership value. This form of $GDHH_\chi FN$ is said to be the second type. Structurally expressed as

$$\Gamma_{GDHH_\chi FN}^2 = \left\{ \left\langle y, (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u); \{(\mu_\Gamma^i, \nu_\Gamma^i) : \mu_\Gamma^i \in \mu_\Gamma, \nu_\Gamma^i \in \nu_\Gamma; i \in T\} \right\rangle : y \in Y_\theta \right\} \quad (21)$$

where $\mu_\Gamma^i (\in [0, 1])$ represent the membership value and $\nu_\Gamma^i (\in [0, 1])$ represent the non-membership value for every $i \in T$ and $y \in Y_\theta$.

Definition 8 is a special form of dual hesitant fuzzy set on the set of real number $\mathbb{R}$. The membership and non-membership functions of $GDHH_\chi FN$ [88] are defined as follows:

$$\{\mu_\Gamma^i(y), \nu_\Gamma^i(y)\} = \begin{cases} \{0, 1\} & \text{if } y < \xi_p \\ \left\{ \frac{\mu_\Gamma}{2} \left(\frac{y - \xi_p}{\xi_q - \xi_p} \right), \frac{\nu_\Gamma}{2} + \left(1 - \frac{\nu_\Gamma}{2} \right) \left(\frac{y - \xi_q}{\xi_p - \xi_q} \right) \right\} & \text{if } \xi_p \leq y < \xi_q \\ \left\{ \frac{\mu_\Gamma}{2} + \left(1 - \frac{\mu_\Gamma}{2} \right) \left(\frac{y - \xi_q}{\xi_r - \xi_q} \right), \frac{\nu_\Gamma}{2} + \frac{\nu_\Gamma}{2} \left(\frac{y - \xi_r}{\xi_q - \xi_r} \right) \right\} & \text{if } \xi_q \leq y < \xi_r \\ \{\mu_\Gamma, \nu_\Gamma\} & \text{if } \xi_r \leq y \leq \xi_s \\ \left\{ \frac{\mu_\Gamma}{2} + \left(1 - \frac{\mu_\Gamma}{2} \right) \left(\frac{y - \xi_s}{\xi_t - \xi_s} \right), \frac{\nu_\Gamma}{2} + \frac{\nu_\Gamma}{2} \left(\frac{y - \xi_t}{\xi_s - \xi_t} \right) \right\} & \text{if } \xi_s < y \leq \xi_t \\ \left\{ \frac{\mu_\Gamma}{2} \left(\frac{y - \xi_u}{\xi_t - \xi_u} \right), \frac{\nu_\Gamma}{2} + \left(1 - \frac{\nu_\Gamma}{2} \right) \left(\frac{y - \xi_t}{\xi_u - \xi_t} \right) \right\} & \text{if } \xi_t < y \leq \xi_u \\ \{0, 1\} & \text{if } \xi_u < y \end{cases} \quad (22)$$

Geometric representation of membership and non-membership functions of $GDHH_\chi FN$ are displayed in Figure 4.

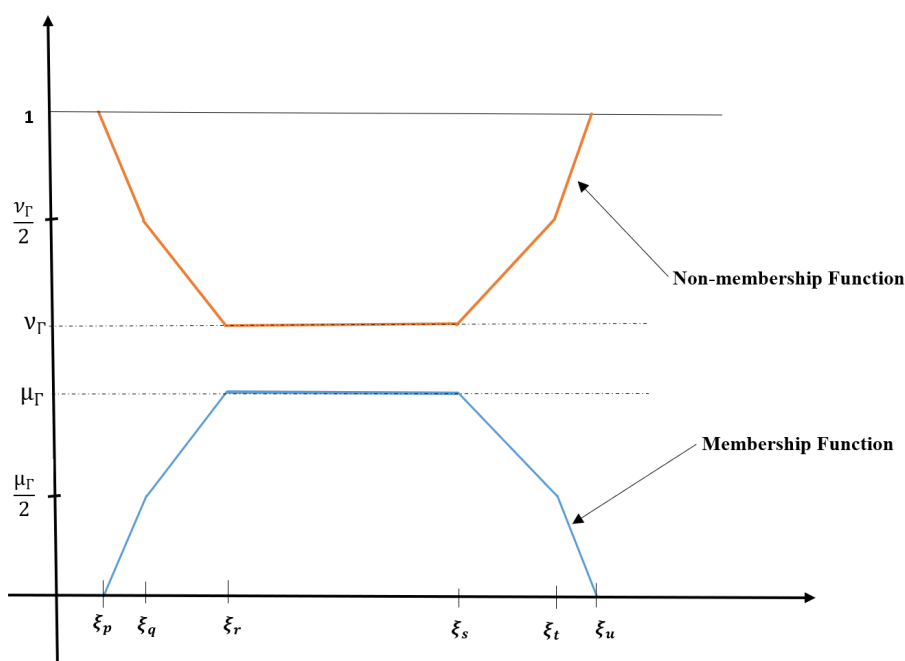


Figure 4. Membership function of $GDHH_{\chi}FN$.

Example 2. Numerically $GDHH_{\chi}FN$ (2nd type) are shown as:

- $\tilde{F}_1^2 = \langle (8, 8.5, 9, 10, 10.5, 11); \{(0.70, 0.15), (0.75, 0.20), (0.65, 0.20)\} \rangle >$
- $\tilde{F}_2^2 = \langle (7, 7.5, 8, 9, 9.5, 10); \{(0.65, 0.15), (0.85, 0.05), (0.75, 0.10)\} \rangle >$
- $\tilde{F}_3^2 = \langle (6, 6.5, 7, 8, 8.5, 9); \{(0.80, 0.10), (0.75, 0.15), (0.70, 0.15)\} \rangle >$
- $\tilde{F}_4^2 = \langle (5, 5.5, 6, 7, 7.5, 8); \{(0.85, 0.05), (0.65, 0.15), (0.75, 0.15)\} \rangle >$
- $\tilde{F}_5^2 = \langle (4, 4.5, 5, 6, 6.5, 7); \{(0.60, 0.25), (0.65, 0.20), (0.70, 0.15)\} \rangle >$
- $\tilde{F}_6^2 = \langle (3, 3.5, 4, 5, 5.5, 6); \{(0.80, 0.15), (0.85, 0.10), (0.75, 0.10)\} \rangle >$
- $\tilde{F}_7^2 = \langle (2, 2.5, 3, 4, 4.5, 5); \{(0.90, 0.05), (0.80, 0.10), (0.85, 0.10)\} \rangle >$

Here, $\tilde{F}_j^2; 1 \leq j \leq 7$ are 2nd type of $GDHH_{\chi}FN$ where hesitancy are in $(\mu_{\tilde{F}_j}^i, \nu_{\tilde{F}_j}^i)$ with $i \in T$ and number of hesitancy is 3 (i.e., cardinality of T ($|T|$) is 3). Additionally, $0 \leq \mu_{\tilde{F}_j}, \nu_{\tilde{F}_j} \leq 1$ and $0 \leq \max_{i \in T} \mu_{\tilde{F}_j}^i + \max_{i \in T} \nu_{\tilde{F}_j}^i \leq 1$.

3.5. Arithmetic Operation of $GDHH_{\chi}FN$

3.5.1. Arithmetic Operation of $GDHH_{\chi}FN$ (1st Type):

The arithmetic operations on $GDHH_{\chi}FN$ (1st type) is applicable only if the $H_{\chi}FN$ s $(\xi_{p_i}, \xi_{q_i}, \xi_{r_i}, \xi_{s_i}, \xi_{t_i}, \xi_{u_i})$ of $GDHH_{\chi}FN$ are different but $\{\mu, \nu\}$ is fixed. Consider two $GDHH_{\chi}FN$ s $\tilde{F}_{GDHH_{\chi}FN}^1$ and $\tilde{\Lambda}_{GDHH_{\chi}FN}^1$ defined as

$\tilde{F}_{GDHH_{\chi}FN}^1 = \langle (\xi_{p_1}^{\tilde{F}}, \xi_{q_1}^{\tilde{F}}, \xi_{r_1}^{\tilde{F}}, \xi_{s_1}^{\tilde{F}}, \xi_{t_1}^{\tilde{F}}, \xi_{u_1}^{\tilde{F}}), (\xi_{p_2}^{\tilde{F}}, \xi_{q_2}^{\tilde{F}}, \xi_{r_2}^{\tilde{F}}, \xi_{s_2}^{\tilde{F}}, \xi_{t_2}^{\tilde{F}}, \xi_{u_2}^{\tilde{F}}); \{\mu^{\tilde{F}}, \nu^{\tilde{F}}\} \rangle >$,
 $\tilde{\Lambda}_{GDHH_{\chi}FN}^1 = \langle (\xi_{p_1}^{\tilde{\Lambda}}, \xi_{q_1}^{\tilde{\Lambda}}, \xi_{r_1}^{\tilde{\Lambda}}, \xi_{s_1}^{\tilde{\Lambda}}, \xi_{t_1}^{\tilde{\Lambda}}, \xi_{u_1}^{\tilde{\Lambda}}), (\xi_{p_2}^{\tilde{\Lambda}}, \xi_{q_2}^{\tilde{\Lambda}}, \xi_{r_2}^{\tilde{\Lambda}}, \xi_{s_2}^{\tilde{\Lambda}}, \xi_{t_2}^{\tilde{\Lambda}}, \xi_{u_2}^{\tilde{\Lambda}}); \{\mu^{\tilde{\Lambda}}, \nu^{\tilde{\Lambda}}\} \rangle >$ and $\epsilon (\geq 0)$ be a scalar. Then arithmetic operations are defined as follows:

A. Addition of two $GDHH_{\chi}FN$ s:

$$\begin{aligned} \tilde{F}_{GDHH_{\chi}FN}^1 \oplus \tilde{\Lambda}_{GDHH_{\chi}FN}^1 = & \langle (\xi_{p_1}^{\tilde{F}} + \xi_{p_1}^{\tilde{\Lambda}}, \xi_{q_1}^{\tilde{F}} + \xi_{q_1}^{\tilde{\Lambda}}, \xi_{r_1}^{\tilde{F}} + \xi_{r_1}^{\tilde{\Lambda}}, \xi_{s_1}^{\tilde{F}} + \xi_{s_1}^{\tilde{\Lambda}}, \xi_{t_1}^{\tilde{F}} + \xi_{t_1}^{\tilde{\Lambda}}, \xi_{u_1}^{\tilde{F}} + \xi_{u_1}^{\tilde{\Lambda}}), \\ & (\xi_{p_2}^{\tilde{F}} + \xi_{p_2}^{\tilde{\Lambda}}, \xi_{q_2}^{\tilde{F}} + \xi_{q_2}^{\tilde{\Lambda}}, \xi_{r_2}^{\tilde{F}} + \xi_{r_2}^{\tilde{\Lambda}}, \xi_{s_2}^{\tilde{F}} + \xi_{s_2}^{\tilde{\Lambda}}, \xi_{t_2}^{\tilde{F}} + \xi_{t_2}^{\tilde{\Lambda}}, \xi_{u_2}^{\tilde{F}} + \xi_{u_2}^{\tilde{\Lambda}}); \\ & \{\max\{\mu^{\tilde{F}}, \mu^{\tilde{\Lambda}}\}, \min\{\nu^{\tilde{F}}, \nu^{\tilde{\Lambda}}\}\} \rangle \end{aligned} \quad (23)$$

B. Subtraction of two $GDHH_{\chi}FN$ s:

$$\begin{aligned} \tilde{\Gamma}_{GDHH_{\chi}FN}^1 \ominus \tilde{\Lambda}_{GDHH_{\chi}FN}^1 = & \left\langle (\zeta_{p_1}^{\tilde{\Gamma}} - \zeta_{p_1}^{\tilde{\Lambda}}, \zeta_{q_1}^{\tilde{\Gamma}} - \zeta_{q_1}^{\tilde{\Lambda}}, \zeta_{r_1}^{\tilde{\Gamma}} - \zeta_{r_1}^{\tilde{\Lambda}}, \zeta_{s_1}^{\tilde{\Gamma}} - \zeta_{s_1}^{\tilde{\Lambda}}, \zeta_{t_1}^{\tilde{\Gamma}} - \zeta_{t_1}^{\tilde{\Lambda}}, \zeta_{u_1}^{\tilde{\Gamma}} - \zeta_{u_1}^{\tilde{\Lambda}}), \right. \\ & (\zeta_{p_2}^{\tilde{\Gamma}} - \zeta_{p_2}^{\tilde{\Lambda}}, \zeta_{q_2}^{\tilde{\Gamma}} - \zeta_{q_2}^{\tilde{\Lambda}}, \zeta_{r_2}^{\tilde{\Gamma}} - \zeta_{r_2}^{\tilde{\Lambda}}, \zeta_{s_2}^{\tilde{\Gamma}} - \zeta_{s_2}^{\tilde{\Lambda}}, \zeta_{t_2}^{\tilde{\Gamma}} - \zeta_{t_2}^{\tilde{\Lambda}}, \zeta_{u_2}^{\tilde{\Gamma}} - \zeta_{u_2}^{\tilde{\Lambda}}); \\ & \left. \left\{ \max\{\mu^{\tilde{\Gamma}}, \mu^{\tilde{\Lambda}}\}, \min\{\nu^{\tilde{\Gamma}}, \nu^{\tilde{\Lambda}}\} \right\} \right\rangle \end{aligned} \quad (24)$$

C. Scalar multiplication of $GDHH_{\chi}FN$:

$$\begin{aligned} \epsilon \times \tilde{\Gamma}_{GDHH_{\chi}FN}^1 = \epsilon \tilde{\Gamma}_{GDHH_{\chi}FN}^1 = & \left\langle (\epsilon(\zeta_{p_1}^{\tilde{\Gamma}}), \epsilon(\zeta_{q_1}^{\tilde{\Gamma}}), \epsilon(\zeta_{r_1}^{\tilde{\Gamma}}), \epsilon(\zeta_{s_1}^{\tilde{\Gamma}}), \epsilon(\zeta_{t_1}^{\tilde{\Gamma}}), \epsilon(\zeta_{u_1}^{\tilde{\Gamma}})), \right. \\ & \left. (\epsilon(\zeta_{p_2}^{\tilde{\Gamma}}), \epsilon(\zeta_{q_2}^{\tilde{\Gamma}}), \epsilon(\zeta_{r_2}^{\tilde{\Gamma}}), \epsilon(\zeta_{s_2}^{\tilde{\Gamma}}), \epsilon(\zeta_{t_2}^{\tilde{\Gamma}}), \epsilon(\zeta_{u_2}^{\tilde{\Gamma}})); \left\{ 1 - (1 - \mu^{\tilde{\Gamma}})^{\epsilon}, (\nu^{\tilde{\Gamma}})^{\epsilon} \right\} \right\rangle \end{aligned} \quad (25)$$

where $\epsilon(\geq 0)$ be a scalar.

D. Multiplication of two $GDHH_{\chi}FN$ s:

$$\begin{aligned} \tilde{\Gamma}_{GDHH_{\chi}FN}^1 \otimes \tilde{\Lambda}_{GDHH_{\chi}FN}^1 = & \left\langle (\zeta_{p_1}^{\tilde{\Gamma}} \zeta_{p_1}^{\tilde{\Lambda}}, \zeta_{q_1}^{\tilde{\Gamma}} \zeta_{q_1}^{\tilde{\Lambda}}, \zeta_{r_1}^{\tilde{\Gamma}} \zeta_{r_1}^{\tilde{\Lambda}}, \zeta_{s_1}^{\tilde{\Gamma}} \zeta_{s_1}^{\tilde{\Lambda}}, \zeta_{t_1}^{\tilde{\Gamma}} \zeta_{t_1}^{\tilde{\Lambda}}, \zeta_{u_1}^{\tilde{\Gamma}} \zeta_{u_1}^{\tilde{\Lambda}}), \right. \\ & \left. (\zeta_{p_2}^{\tilde{\Gamma}} \zeta_{p_2}^{\tilde{\Lambda}}, \zeta_{q_2}^{\tilde{\Gamma}} \zeta_{q_2}^{\tilde{\Lambda}}, \zeta_{r_2}^{\tilde{\Gamma}} \zeta_{r_2}^{\tilde{\Lambda}}, \zeta_{s_2}^{\tilde{\Gamma}} \zeta_{s_2}^{\tilde{\Lambda}}, \zeta_{t_2}^{\tilde{\Gamma}} \zeta_{t_2}^{\tilde{\Lambda}}, \zeta_{u_2}^{\tilde{\Gamma}} \zeta_{u_2}^{\tilde{\Lambda}}); \left\{ \max\{\mu^{\tilde{\Gamma}}, \mu^{\tilde{\Lambda}}\}, \min\{\nu^{\tilde{\Gamma}}, \nu^{\tilde{\Lambda}}\} \right\} \right\rangle \end{aligned} \quad (26)$$

E. Division of two $GDHH_{\chi}FN$ s:

$$\begin{aligned} \tilde{\Gamma}_{GDHH_{\chi}FN}^1 \oslash \tilde{\Lambda}_{GDHH_{\chi}FN}^1 = & \left\langle \left(\frac{\zeta_{p_1}^{\tilde{\Gamma}}}{\zeta_{u_1}^{\tilde{\Lambda}}}, \frac{\zeta_{q_1}^{\tilde{\Gamma}}}{\zeta_{t_1}^{\tilde{\Lambda}}}, \frac{\zeta_{r_1}^{\tilde{\Gamma}}}{\zeta_{s_1}^{\tilde{\Lambda}}}, \frac{\zeta_{s_1}^{\tilde{\Gamma}}}{\zeta_{r_1}^{\tilde{\Lambda}}}, \frac{\zeta_{t_1}^{\tilde{\Gamma}}}{\zeta_{q_1}^{\tilde{\Lambda}}}, \frac{\zeta_{u_1}^{\tilde{\Gamma}}}{\zeta_{p_1}^{\tilde{\Lambda}}} \right), \right. \\ & \left. \left(\frac{\zeta_{p_2}^{\tilde{\Gamma}}}{\zeta_{u_2}^{\tilde{\Lambda}}}, \frac{\zeta_{q_2}^{\tilde{\Gamma}}}{\zeta_{t_2}^{\tilde{\Lambda}}}, \frac{\zeta_{r_2}^{\tilde{\Gamma}}}{\zeta_{s_2}^{\tilde{\Lambda}}}, \frac{\zeta_{s_2}^{\tilde{\Gamma}}}{\zeta_{r_2}^{\tilde{\Lambda}}}, \frac{\zeta_{t_2}^{\tilde{\Gamma}}}{\zeta_{q_2}^{\tilde{\Lambda}}}, \frac{\zeta_{u_2}^{\tilde{\Gamma}}}{\zeta_{p_2}^{\tilde{\Lambda}}} \right); \left\{ \max\{\mu^{\tilde{\Gamma}}, \mu^{\tilde{\Lambda}}\}, \min\{\nu^{\tilde{\Gamma}}, \nu^{\tilde{\Lambda}}\} \right\} \right\rangle \end{aligned} \quad (27)$$

F. Inverse of $GDHH_{\chi}FN$:

$$\begin{aligned} (\tilde{\Gamma}_{GDHH_{\chi}FN}^1)^{-1} = & \frac{1}{\tilde{\Gamma}_{GDHH_{\chi}FN}^1} = \left\langle \left(\frac{1}{\zeta_{u_1}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{t_1}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{s_1}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{r_1}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{q_1}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{p_1}^{\tilde{\Gamma}}} \right), \right. \\ & \left. \left(\frac{1}{\zeta_{u_2}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{t_2}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{s_2}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{r_2}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{q_2}^{\tilde{\Gamma}}}, \frac{1}{\zeta_{p_2}^{\tilde{\Gamma}}} \right); \left\{ \mu^{\tilde{\Gamma}}, \nu^{\tilde{\Gamma}} \right\} \right\rangle \end{aligned} \quad (28)$$

G. Power (scalar) of $GDHH_{\chi}FN$:

$$\begin{aligned} (\tilde{\Gamma}_{GDHH_{\chi}FN}^1)^{\epsilon} = & \left\langle ((\zeta_{p_1}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{q_1}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{r_1}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{s_1}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{t_1}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{u_1}^{\tilde{\Gamma}})^{\epsilon}), \right. \\ & \left. ((\zeta_{p_2}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{q_2}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{r_2}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{s_2}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{t_2}^{\tilde{\Gamma}})^{\epsilon}, (\zeta_{u_2}^{\tilde{\Gamma}})^{\epsilon}); \left\{ (\mu^{\tilde{\Gamma}})^{\epsilon}, 1 - (1 - \nu^{\tilde{\Gamma}})^{\epsilon} \right\} \right\rangle \end{aligned} \quad (29)$$

3.5.2. Arithmetic Operation of $GDHH_{\chi}FN$ (2nd Type)

Assume two $GDHH_{\chi}FN$ (2nd type) are defined as

$$\begin{aligned} \tilde{\Gamma}_{GDHH_{\chi}FN}^2 = & \left\langle (\zeta_p^{\tilde{\Gamma}}, \zeta_q^{\tilde{\Gamma}}, \zeta_r^{\tilde{\Gamma}}, \zeta_s^{\tilde{\Gamma}}, \zeta_t^{\tilde{\Gamma}}, \zeta_u^{\tilde{\Gamma}}); \left\{ \bigcup_{i \in T} (\mu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Gamma}}) \right\} \right\rangle, \\ \tilde{\Lambda}_{GDHH_{\chi}FN}^2 = & \left\langle (\zeta_p^{\tilde{\Lambda}}, \zeta_q^{\tilde{\Lambda}}, \zeta_r^{\tilde{\Lambda}}, \zeta_s^{\tilde{\Lambda}}, \zeta_t^{\tilde{\Lambda}}, \zeta_u^{\tilde{\Lambda}}); \left\{ \bigcup_{i \in T} (\mu_i^{\tilde{\Lambda}}, \nu_i^{\tilde{\Lambda}}) \right\} \right\rangle \text{ and } \epsilon(\geq 0) \text{ be a scalar. Then the} \\ \text{arithmetic operations on } & \tilde{\Gamma}_{GDHH_{\chi}FN}^2 \text{ and } \tilde{\Lambda}_{GDHH_{\chi}FN}^2 \text{ define as:} \end{aligned}$$

A. Addition of two $GDHH_{\chi}FN$ s:

$$\tilde{\Gamma}_{GDHH_{\chi}FN}^2 \oplus \tilde{\Lambda}_{GDHH_{\chi}FN}^2 = \left\langle \left(\tilde{\zeta}_p^{\tilde{\Gamma}} + \tilde{\zeta}_p^{\tilde{\Lambda}}, \tilde{\zeta}_q^{\tilde{\Gamma}} + \tilde{\zeta}_q^{\tilde{\Lambda}}, \tilde{\zeta}_r^{\tilde{\Gamma}} + \tilde{\zeta}_r^{\tilde{\Lambda}}, \tilde{\zeta}_s^{\tilde{\Gamma}} + \tilde{\zeta}_s^{\tilde{\Lambda}}, \tilde{\zeta}_t^{\tilde{\Gamma}} + \tilde{\zeta}_t^{\tilde{\Lambda}}, \tilde{\zeta}_u^{\tilde{\Gamma}} + \tilde{\zeta}_u^{\tilde{\Lambda}} \right); \right. \\ \left. \left\{ \bigcup_{i \in T} \left(\max\{\mu_i^{\tilde{\Gamma}}, \mu_i^{\tilde{\Lambda}}\}, \min\{\nu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Lambda}}\} \right) \right\} \right\rangle \quad (30)$$

B. Subtraction of two $GDHH_{\chi}FN$ s:

$$\tilde{\Gamma}_{GDHH_{\chi}FN}^2 \ominus \tilde{\Lambda}_{GDHH_{\chi}FN}^2 = \left\langle \left(\tilde{\zeta}_p^{\tilde{\Gamma}} - \tilde{\zeta}_p^{\tilde{\Lambda}}, \tilde{\zeta}_q^{\tilde{\Gamma}} - \tilde{\zeta}_q^{\tilde{\Lambda}}, \tilde{\zeta}_r^{\tilde{\Gamma}} - \tilde{\zeta}_r^{\tilde{\Lambda}}, \tilde{\zeta}_s^{\tilde{\Gamma}} - \tilde{\zeta}_s^{\tilde{\Lambda}}, \tilde{\zeta}_t^{\tilde{\Gamma}} - \tilde{\zeta}_t^{\tilde{\Lambda}}, \tilde{\zeta}_u^{\tilde{\Gamma}} - \tilde{\zeta}_u^{\tilde{\Lambda}} \right); \right. \\ \left. \left\{ \bigcup_{i \in T} \left(\max\{\mu_i^{\tilde{\Gamma}}, \mu_i^{\tilde{\Lambda}}\}, \min\{\nu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Lambda}}\} \right) \right\} \right\rangle \quad (31)$$

C. Scalar multiplication of $GDHH_{\chi}FN$:

$$\epsilon \times \tilde{\Gamma}_{GDHH_{\chi}FN}^2 = \epsilon \tilde{\Gamma}_{GDHH_{\chi}FN}^2 = \left\langle \left(\epsilon(\tilde{\zeta}_p^{\tilde{\Gamma}}), \epsilon(\tilde{\zeta}_q^{\tilde{\Gamma}}), \epsilon(\tilde{\zeta}_r^{\tilde{\Gamma}}), \epsilon(\tilde{\zeta}_s^{\tilde{\Gamma}}), \epsilon(\tilde{\zeta}_t^{\tilde{\Gamma}}), \epsilon(\tilde{\zeta}_u^{\tilde{\Gamma}}) \right); \right. \\ \left. \left\{ \bigcup_{i \in T} (1 - (1 - \mu_i^{\tilde{\Gamma}})^{\epsilon}, \nu_i^{\tilde{\Gamma}})^{\epsilon} \right\} \right\rangle \quad (32)$$

where $\epsilon (\geq 0)$ be a scalar.

D. Multiplication of two $GDHH_{\chi}FN$ s:

$$\tilde{\Gamma}_{GDHH_{\chi}FN}^2 \otimes \tilde{\Lambda}_{GDHH_{\chi}FN}^2 = \left\langle \left(\tilde{\zeta}_p^{\tilde{\Gamma}} \tilde{\zeta}_p^{\tilde{\Lambda}}, \tilde{\zeta}_q^{\tilde{\Gamma}} \tilde{\zeta}_q^{\tilde{\Lambda}}, \tilde{\zeta}_r^{\tilde{\Gamma}} \tilde{\zeta}_r^{\tilde{\Lambda}}, \tilde{\zeta}_s^{\tilde{\Gamma}} \tilde{\zeta}_s^{\tilde{\Lambda}}, \tilde{\zeta}_t^{\tilde{\Gamma}} \tilde{\zeta}_t^{\tilde{\Lambda}}, \tilde{\zeta}_u^{\tilde{\Gamma}} \tilde{\zeta}_u^{\tilde{\Lambda}} \right); \left\{ \bigcup_{i \in T} \left(\max\{\mu_i^{\tilde{\Gamma}}, \mu_i^{\tilde{\Lambda}}\}, \min\{\nu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Lambda}}\} \right) \right\} \right\rangle \quad (33)$$

E. Division of two $GDHH_{\chi}FN$ s:

$$\tilde{\Gamma}_{GDHH_{\chi}FN}^2 \oslash \tilde{\Lambda}_{GDHH_{\chi}FN}^2 = \left\langle \left(\frac{\tilde{\zeta}_p^{\tilde{\Gamma}}}{\tilde{\zeta}_u^{\tilde{\Lambda}}}, \frac{\tilde{\zeta}_q^{\tilde{\Gamma}}}{\tilde{\zeta}_t^{\tilde{\Lambda}}}, \frac{\tilde{\zeta}_r^{\tilde{\Gamma}}}{\tilde{\zeta}_s^{\tilde{\Lambda}}}, \frac{\tilde{\zeta}_s^{\tilde{\Gamma}}}{\tilde{\zeta}_r^{\tilde{\Lambda}}}, \frac{\tilde{\zeta}_t^{\tilde{\Gamma}}}{\tilde{\zeta}_q^{\tilde{\Lambda}}}, \frac{\tilde{\zeta}_u^{\tilde{\Gamma}}}{\tilde{\zeta}_p^{\tilde{\Lambda}}} \right); \left\{ \bigcup_{i \in T} \left(\max\{\mu_i^{\tilde{\Gamma}}, \mu_i^{\tilde{\Lambda}}\}, \min\{\nu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Lambda}}\} \right) \right\} \right\rangle \quad (34)$$

F. Inverse of $GDHH_{\chi}FN$:

$$(\tilde{\Gamma}_{GDHH_{\chi}FN}^2)^{-1} = \frac{1}{\tilde{\Gamma}_{GDHH_{\chi}FN}^2} = \left\langle \left(\frac{1}{\tilde{\zeta}_u^{\tilde{\Gamma}}}, \frac{1}{\tilde{\zeta}_t^{\tilde{\Gamma}}}, \frac{1}{\tilde{\zeta}_s^{\tilde{\Gamma}}}, \frac{1}{\tilde{\zeta}_r^{\tilde{\Gamma}}}, \frac{1}{\tilde{\zeta}_q^{\tilde{\Gamma}}}, \frac{1}{\tilde{\zeta}_p^{\tilde{\Gamma}}} \right); \left\{ \bigcup_{i \in T} (\mu_i^{\tilde{\Gamma}}, \nu_i^{\tilde{\Gamma}}) \right\} \right\rangle \quad (35)$$

G. Power (scalar) of $GDHH_{\chi}FN$:

$$(\tilde{\Gamma}_{GDHH_{\chi}FN}^2)^{\epsilon} = \left\langle \left((\tilde{\zeta}_p^{\tilde{\Gamma}})^{\epsilon}, (\tilde{\zeta}_q^{\tilde{\Gamma}})^{\epsilon}, (\tilde{\zeta}_r^{\tilde{\Gamma}})^{\epsilon}, (\tilde{\zeta}_s^{\tilde{\Gamma}})^{\epsilon}, (\tilde{\zeta}_t^{\tilde{\Gamma}})^{\epsilon}, (\tilde{\zeta}_u^{\tilde{\Gamma}})^{\epsilon} \right); \left\{ \bigcup_{i \in T} ((\mu_i^{\tilde{\Gamma}})^{\epsilon}, 1 - (1 - \nu_i^{\tilde{\Gamma}})^{\epsilon}) \right\} \right\rangle \quad (36)$$

Note 3. We take hesitancy only on membership and non-membership functions to keep the hexagonal fuzzy number fixed. So second type $GDHH_{\chi}F$ sets can be more robust to noise and outliers in data since they allow multiple membership values to be associated with each element. Hence, we take the second type $GDHH_{\chi}FN$ to solve the problem in this study.

3.6. De-Fuzzification of Generalized Dual Hesitant Hexagonal Fuzzy Number

In fuzzy logic systems, the defuzzification process transforms a fuzzy output value, which indicates the degree of membership in a fuzzy set, into a crisp output value that may be utilized as a decision or control action. The choice of de-fuzzification method depends on the application and the desired behavior of the system.

Consider $\tilde{P} = \langle (\xi_p, \xi_q, \xi_r, \xi_s, \xi_t, \xi_u); \{(\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\} \rangle$ be a generalised dual hesitant hexagonal fuzzy number. Then the proposed de-fuzzification technique of $\tilde{P}$ denoted by $\mathcal{D}(\tilde{P})$ and described as

$$\mathcal{D}(\tilde{P}) = \frac{1}{18}(\xi_p + 2 \times \xi_q + 3 \times \xi_r + 3 \times \xi_s + 2 \times \xi_t + \xi_u) \times (1 + \mu - \nu) \quad (37)$$

where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$ and $\nu = \max\{\nu_1, \nu_2, \nu_3\}$.

Remark 3. There are various possible advantages to introducing a new defuzzification technique for generalized dual hesitant hexagonal fuzzy numbers (GDHH_χFN), including:

- (I) Generalized dual hesitant hexagonal fuzzy numbers are a complex type of fuzzy number that can better represent uncertain or ambiguous information. We can obtain more accurate and precise defuzzification results by developing a new de-fuzzification method specifically for this type of fuzzy number.
- (II) With this defuzzification procedure, the de-fuzzified value is often between the six numbers of a hexagonal fuzzy number for any GDHH_χFN.
- (III) This new de-fuzzification method can be applied to various applications in different fields, such as engineering, finance, and economics, where generalized dual hesitant hexagonal fuzzy numbers are commonly used.
- (IV) The new de-fuzzification technique can be changed and adjusted to accommodate various fuzzy number types or problem domains. More versatility and adaptation in particular situations are possible because of this flexibility.
- (V) Creating a novel de-fuzzification approach can enhance the study of fuzzy sets and fuzzy logic since it can provide a unique perspective and methods for accessing ambiguous or uncertain data.

Example 3. Let consider $\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3$ and $\tilde{\Delta}_4$ are four GDHH_χFNs define as

$$\tilde{\Delta}_1 = \langle (5, 6, 7, 8, 9, 10); \{(0.7, 0.2), (0.9, 0.05), (0.85, 0.1)\} \rangle,$$

$$\tilde{\Delta}_2 = \langle (0.1, 0.2, 0.3, 0.6, 0.7, 0.8); \{(0.8, 0.15), (0.75, 0.2), (0.9, 0)\} \rangle,$$

$$\tilde{\Delta}_3 = \langle (0.60, 0.65, 0.75, 0.85, 0.95, 1); \{(0.9, 0.05), (0.75, 0.2), (0.85, 0.1)\} \rangle \text{ and}$$

$$\tilde{\Delta}_4 = \langle (9, 10, 12, 14, 16, 17); \{(1, 0), (0.9, 0.05), (0.8, 0.15)\} \rangle.$$

Table 5 describe the de-fuzzification of the above-mentioned GDHH_χFNs by various studies and proposed method.

Table 5. De-fuzzification of GDHH_χFNs by different methods.

GDHH _χ FN	Kahram-an, G. et al. [89]	Rajend-ran, C. et al. [17]	Sangee-tha K. et al. [90]	Jangid, V. et al. [91]	Proposed Model
	Equation (24)	Equation (4)	Equation (6)	Equation (40)	Equation (33)
$\tilde{\Delta}_1$	11.25	11.25	11.25	2.8125	7.5
$\tilde{\Delta}_2$	0.6975	0.6975	0.6975	0.1744	0.465
$\tilde{\Delta}_3$	1.24	1.24	1.24	0.31	0.8267
$\tilde{\Delta}_4$	20.15	20.15	20.15	5.0375	13.4333

Remark 4. The de-fuzzified value of four GDHH_χFN is described here. We can see that the de-fuzzified values in each instance do not belong to the range of hexagonal fuzzy numbers (H_χFN). Nonetheless, according to our method, any GDHH_χFN's de-fuzzified value belongs to the chosen H_χFN, so we can determine the de-fuzzified value of GDHH_χFN for categorizing those numbers.

4. Multi-Criteria Decision Making Methodology

This section give details explanations about MCDM methodology which are applied for determine the criteria weight and prioritized the alternatives for the present research work.

4.1. The Fuzzy Criteria Importance Through Inter-Criteria Correlation (FCRITIC) Method

The Criteria Importance Through Inter-criteria Correlation (CRITIC) method was proposed by Diakoulaki, D. et al. in 1995 [92]. The Fuzzy Criteria Importance Through Inter-criteria Correlation (FCRITIC) Multiple-Criteria-Decision Making (MCDM) techniques is a mathematical approach that can be used to evaluate complex decision problems by considering multiple criteria [93]. This technique is widely used in various healthcare fields to support decision-making processes. In disease recognition, the FCRITIC technique can analyze and evaluate multiple diagnostic criteria for a particular disease. By applying this technique, a healthcare professional can assess the importance of each diagnostic criteria and determine how well each criteria correlates with the disease. By considering multiple criteria simultaneously, the FCRITIC technique can provide a more comprehensive and accurate assessment of a patient's condition, leading to improved diagnosis and treatment outcomes. This method can be used when decision-makers have conflicting views on the value of weights or need help comparing different criteria.

Consider n number of criteria with m number of alternatives in this study. Additionally, K number of decision-makers give their opinion in linguistic terms and then convert it in $GDHH_{\chi}FN$ by Table 6. Based on K DMs decision, K decision matrices are formed of $m \times n$ order as input data. The FCRITIC method steps are as follows:

Table 6. Linguistic variables representing $GDHH_{\chi}FN$ (2nd type) and their de-fuzzified values.

Linguistic Terms	$GDHH_{\chi}FN$ (2nd Type)	De-Fuzzified Value
Strongly Significant (SSG)	$< (8, 8.5, 9, 10, 10.5, 11); \{(0.8, 0.15), (0.7, 0.25), (0.7, 0.15)\} >$	9.18
Highly Significant (HSG)	$< (7, 7.5, 8, 9, 9.5, 10); \{(0.8, 0.05), (0.7, 0.25), (0.7, 0.15)\} >$	8.22
More Significant (MSG)	$< (6, 6.5, 7, 8, 8.5, 9); \{(0.8, 0.1), (0.75, 0.15), (0.8, 0.15)\} >$	8.0
Significant (SG)	$< (5, 5.5, 6, 7, 7.5, 8); \{(0.85, 0.1), (0.8, 0.15), (0.75, 0.2)\} >$	6.72
Less Significant (LSG)	$< (4, 4.5, 5, 6, 6.5, 7); \{(0.75, 0.2), (0.7, 0.2), (0.8, 0.1)\} >$	5.5
Below Significant (BSG)	$< (3, 3.5, 4, 5, 5.5, 6); \{(0.8, 0.1), (0.8, 0.2), (0.75, 0.2)\} >$	4.65
Weakly Significant (WSG)	$< (2, 2.5, 3, 4, 4.5, 5); \{(0.9, 0.1), (0.8, 0.1), (0.8, 0.2)\} >$	3.73

- A: Formulate a decision matrix in terms of Generalized Dual Hesitant Hexagonal Fuzzy Number ($GDHH_{\chi}FN$):

Decision matrix D_k given by DMs are as

$$\tilde{D}_k = \begin{bmatrix} (\tilde{x}_{11})_k & (\tilde{x}_{12})_k & \dots & (\tilde{x}_{1n})_k \\ (\tilde{x}_{21})_k & (\tilde{x}_{22})_k & \dots & (\tilde{x}_{2n})_k \\ \vdots & \vdots & \ddots & \vdots \\ (\tilde{x}_{m1})_k & (\tilde{x}_{m2})_k & \dots & (\tilde{x}_{mn})_k \end{bmatrix} \quad (38)$$

i.e.,

$$D_k = [(x_{\zeta\eta})_k]_{mn} \quad (39)$$

where

$$(x_{\zeta\eta})_k = \left\{ ([\tilde{\zeta}_p]_{\zeta\eta})_k, ([\tilde{\zeta}_q]_{\zeta\eta})_k, ([\tilde{\zeta}_r]_{\zeta\eta})_k, ([\tilde{\zeta}_s]_{\zeta\eta})_k, ([\tilde{\zeta}_t]_{\zeta\eta})_k, ([\tilde{\zeta}_u]_{\zeta\eta})_k; \cup_{i=1}^N \{ < ([\mu_i]_{\zeta\eta})_k, ([v_i]_{\zeta\eta})_k > \} \right\} \quad (40)$$

are k th DMs rating in $GDHH_{\chi}FN$, given on alternatives ζ with respect to criteria η . Here, criteria $\eta = 1, 2, \dots, n$ and alternative $\zeta = 1, 2, \dots, m$. Additionally, consider, hesitant variable $N = 3$.

- B: Aggregation of decisions of DMs assigned in Equation (40) on $GDHH_{\chi}FN$ using the following equation

$$\tilde{x}_{\zeta\eta} = \begin{cases} [\bar{\zeta}_p]_{\zeta\eta} = \min_{k=1,2,\dots,K} ([\bar{\zeta}_p]_{\zeta\eta})_k \\ [\bar{\zeta}_q]_{\zeta\eta} = \min_{k=1,2,\dots,K} ([\bar{\zeta}_q]_{\zeta\eta})_k \\ [\bar{\zeta}_r]_{\zeta\eta} = \sqrt[K]{\prod_{k=1}^K ([\bar{\zeta}_r]_{\zeta\eta})_k} \\ [\bar{\zeta}_s]_{\zeta\eta} = \sqrt[K]{\prod_{k=1}^K ([\bar{\zeta}_s]_{\zeta\eta})_k} \\ [\bar{\zeta}_t]_{\zeta\eta} = \max_{k=1,2,\dots,K} ([\bar{\zeta}_t]_{\zeta\eta})_k \\ [\bar{\zeta}_u]_{\zeta\eta} = \max_{k=1,2,\dots,K} ([\bar{\zeta}_u]_{\zeta\eta})_k \\ [\mu_i]_{\zeta\eta} = \min_{k=1,2,\dots,K} ([\mu_i]_{\zeta\eta})_k \\ [\nu_i]_{\zeta\eta} = \max_{k=1,2,\dots,K} ([\nu_i]_{\zeta\eta})_k \end{cases} \quad (41)$$

where $i = 1, 2, 3$ for all μ and ν .

The aggregated decision results of the K decision makers are obtained by the Equation (41). Therefore, the aggregating decision matrix is

$$\tilde{D}^a = [\tilde{x}_{\zeta\eta}]_{mn} \quad (42)$$

where

$$\tilde{x}_{\zeta\eta} = \{([\bar{\zeta}_p]_{\zeta\eta}, [\bar{\zeta}_q]_{\zeta\eta}, [\bar{\zeta}_r]_{\zeta\eta}, [\bar{\zeta}_s]_{\zeta\eta}, [\bar{\zeta}_t]_{\zeta\eta}, [\bar{\zeta}_u]_{\zeta\eta}); \cup_{i=1}^N \{([\mu_i]_{\zeta\eta}, [\nu_i]_{\zeta\eta})\}\} \quad (43)$$

here, $\eta = 1, 2, \dots, n$ and $\zeta = 1, 2, \dots, m$ with $N = 3$.

C: De-fuzzify the decision matrix ($\tilde{D}^a$):

De-fuzzify the decision matrix $\tilde{D}^a$ by the de-fuzzified formula and convert $GDHH_{\chi}FN$ second type number to crisp number by the Equation (37). Now the de-fuzzifies decision matrix ($\tilde{D}_d$) is formed as

$$\tilde{D}_d = [(\tilde{x}_d)_{\zeta\eta}]_{mn} \quad (44)$$

where $(\tilde{x}_d)_{\zeta\eta}$ is the de-fuzzifies value of $GDHH_{\chi}FN$ number $\tilde{x}_{\zeta\eta}$.

D: Normalize the decision matrix

Then, find the normalized decision matrix from the de-fuzzifies decision matrix ($\tilde{D}_d$), using the following formula:

$$\tilde{x}'_{\zeta\eta} = \frac{\tilde{x}_{\zeta\eta} - \tilde{x}_{\eta}^{worst}}{\tilde{x}_{\eta}^{best} - \tilde{x}_{\eta}^{worst}} \quad (45)$$

where

$$\begin{cases} \tilde{x}_{\eta}^{best} = \max_{\zeta=1,2,\dots,m} \tilde{x}_{\zeta\eta} \\ \tilde{x}_{\eta}^{worst} = \min_{\zeta=1,2,\dots,m} \tilde{x}_{\zeta\eta} \end{cases} \quad (46)$$

E: Calculate the standard deviation σ_{η} for each criteria by the equation, as follows:

$$\sigma_{\eta} = \sqrt{\frac{\sum (\tilde{x}_{\eta} - \bar{\tilde{x}}_{\eta})^2}{n-1}} \quad (47)$$

where $\tilde{x}_\eta$ is the population mean, n is size of the population (i.e., number of criteria) and $\eta = 1, 2, \dots, n$.

- F: Find the linear correlation coefficient between the criteria c_η and $c_{\eta'}$. Determine the symmetric matrix of $n \times n$ with elements $\tilde{x}_{\zeta\eta}$ which is the linear correlation coefficient between the vectors $\tilde{x}_\eta$ and $\tilde{x}_{\eta'}$ and correlation coefficient criteria c_η to criteria $c_{\eta'}$ denoted by $\tilde{\theta}_{\eta\eta'}$.
- G: Measure of the conflict created by the criteria:
In this step, we have to calculate the measure of the conflict created by the criterion η with respect to the decision situation defined by the rest of criteria.

$$\tilde{\chi}_\eta = \sum_{\eta'=1}^n (1 - \tilde{\theta}_{\eta\eta'}) \quad (48)$$

- H: Determining the quantity of the information in relation to each criteria by

$$C_\eta = \sigma_\eta \times \tilde{\chi}_\eta \quad (49)$$

where criteria $\eta = 1, 2, \dots, n$.

- I: Determining the objective weights:
Calculate the weight of the criteria η denoted by C_η^w and defined as follows:

$$C_\eta^w = \frac{C_\eta}{\sum_{\eta=1}^n C_\eta} \quad (50)$$

Finally, Equation (50) gives the criteria weight C_η^w of each criteria $\eta = 1, 2, \dots, n$.

4.2. The Fuzzy Weighted Aggregated Sum Product Assessment (FWASPAS) Approach

This section uses the Weighted Aggregated Sum Product Assessment (WASPAS) method with Generalized Dual Hesitant Hexagonal fuzzy data, called Fuzzy Weighted Aggregated Sum Product Assessment (FWASPAS) [94,95] which is introduced. Consider a decision-making scenario in which a set of m alternatives $A_1, A_2, \dots, A_m$ are assessed using n criteria $C_1, C_2, \dots, C_n$. GDHFNs are used to estimate how well each alternative performs or rates concerning each criterion.

The FWASPAS MCDM technique is widely used in various fields, including healthcare, finance, engineering, and environmental management. In healthcare, it can be used to evaluate different treatment options, medical devices, and diagnostic criteria, as well as to prioritize patients for treatment based on their medical history, symptoms, and test results. Using the FWASPAS MCDM technique, decision-makers can consider the uncertainties and imprecisions associated with decision-making and make more informed decisions based on a systematic and quantitative approach. After the criteria weights have been calculated, the FWASPAS MCDM technique is used to evaluate the alternatives based on the criteria. The alternatives are compared using a decision matrix that shows the performance of each alternative on each criterion. The fuzzy numbers are multiplied by the corresponding criterion weights, and the results are aggregated using a fuzzy aggregation operator to obtain a total score for each alternative. The alternative with the highest score is selected as the best decision. Like the previous section, assume that K number of DMs assign their rating on n numbers of criteria and m numbers of alternatives. DMs are given their views in linguistic terms, and we convert them to GDHFN by Table 6. Then established K numbers of $m \times n$ order decision matrix. The FWASPAS methodology proceeding steps are as follows:

- A. Develop the decision matrix with respect to GDHFN:
Decision matrix D_k formulated by DMs are shown in Equation (38). All inputs are given in GDHFN $(x_{\zeta\eta})_k$ by DMs displayed in Equation (40).

- B. Aggregation of DMs opinion:
All ratings assigned by DMs on K decision matrices are aggregated by the Equation (41) and make a single decision matrix ($\tilde{D}^a$), shown in the previous section.
- C. Normalize the aggregated decision matrix ($\tilde{D}^a$):
The aggregated decision matrix ($\tilde{D}^a$) shown in the Equation (42) where every entry are $GDHH_{\chi}FN$ of the second type appear in Equation (43). Then, normalize the decision matrix by normalizing every element. Normalized $GDHH_{\chi}FN$ is denoted by $[\tilde{x}_{\zeta\eta}]_z$ and describe as follows:

$$\begin{cases} [\tilde{x}_{\zeta\eta}]_z = \left\{ \left(\frac{[\tilde{\zeta}_p]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+}, \frac{[\tilde{\zeta}_q]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+}, \frac{[\tilde{\zeta}_r]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+}, \frac{[\tilde{\zeta}_s]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+}, \frac{[\tilde{\zeta}_t]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+}, \frac{[\tilde{\zeta}_u]_{\zeta\eta}}{[\tilde{\zeta}_u]_{\zeta\eta}^+} \right); \cup_{i=1}^N \{([\mu_i]_{\zeta\eta}, [\nu_i]_{\zeta\eta})\} \right\} \\ \quad \text{when criteria } c_{\eta} \text{ be beneficial criteria} \\ [\tilde{x}_{\zeta\eta}]_z = \left\{ \left(\frac{[\tilde{\zeta}_p]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}}, \frac{[\tilde{\zeta}_q]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}}, \frac{[\tilde{\zeta}_r]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}}, \frac{[\tilde{\zeta}_s]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}}, \frac{[\tilde{\zeta}_t]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}}, \frac{[\tilde{\zeta}_u]_{\zeta\eta}^-}{[\tilde{\zeta}_p]_{\zeta\eta}} \right); \cup_{i=1}^N \{([\mu_i]_{\zeta\eta}, [\nu_i]_{\zeta\eta})\} \right\} \\ \quad \text{when criteria } c_{\eta} \text{ be non-beneficial criteria} \end{cases} \quad (51)$$

where

$$\begin{cases} [\tilde{\zeta}_u]_{\zeta\eta}^+ = \max_{\zeta=1,2,\dots,m} [\tilde{\zeta}_u]_{\zeta\eta} & \text{when criteria } c_{\eta} \text{ be beneficial criteria} \\ [\tilde{\zeta}_p]_{\zeta\eta}^- = \min_{\zeta=1,2,\dots,m} [\tilde{\zeta}_p]_{\zeta\eta} & \text{when criteria } c_{\eta} \text{ be non-beneficial criteria} \end{cases} \quad (52)$$

Then, the normalized decision matrix ($\tilde{D}^n$) is shown as

$$\tilde{D}^n = [[\tilde{x}_{\zeta\eta}]_z]_{mn} \quad (53)$$

- D. Weight of the criteria:
Criteria weights are calculated by the FCRITIC method in previous section by the Equation (50) and criteria weights are C_{η}^w .
- E. Weighted Sum Normalized Decision Matrix:
Find out the weighted sum normalized decision matrix $\tilde{D}^s$ by the formula

$$\tilde{D}^s = C_{\eta}^w \times [\tilde{x}_{\zeta\eta}]_z \quad (54)$$

where scalar multiplication of $GDHH_{\chi}FN$ of the second type obeys the Equation (31).

- F. De-fuzzify the weighted sum normalized decision matrix $\tilde{D}^s$:
De-fuzzify matrix $\tilde{D}^s$ by the Equation (37) and update the decision matrix as

$$\tilde{D}_d = [(\tilde{x}_d)_{\zeta\eta}]_{mn} \quad (55)$$

- G. Weighted Sum model (WSM):
Determine the weighted sum model $\tilde{Q}_{\zeta}^{(1)}$ where $\zeta = 1, 2, \dots, m$ as follows

$$\tilde{Q}_{\zeta}^{(1)} = \sum_{\eta=1}^n (\tilde{x}_d)_{\zeta\eta} \quad (56)$$

- H. Weighted Product Normalized Decision Matrix:
Find out the weighted product normalized decision matrix $\tilde{D}^p$ by the equation

$$\tilde{D}^p = ([\tilde{x}_{\zeta\eta}]_z)^{C_{\eta}^w} \quad (57)$$

Here the scalar power of $GDHH_{\chi}FN$ of the second type is obtained by the Equation (32).

- I. De-fuzzifies the weighted product normalized decision matrix $\tilde{D}^p$:
De-fuzzifies the matrix $\tilde{D}^p$ by the Equation (37) and renovate the decision matrix as

$$\tilde{D}'_d = [(\tilde{x}'_d)_{\zeta\eta}]_{mn} \quad (58)$$

- J. Weighted Product model (WPM):
Formulate the weighted product model $\tilde{Q}^{(2)}_\zeta$ where $\zeta = 1, 2, \dots, m$ as follows

$$\tilde{Q}^{(2)}_\zeta = \prod_{\eta=1}^n (\tilde{x}'_d)_{\zeta\eta} \quad (59)$$

- K. Evaluate $\tilde{Q}_\zeta$:
Now, the Weighted Aggregated Sum Product Assessment (WASPAS) model value $\tilde{Q}_\zeta$ which is a unique combination of WSM and WPM is calculated as follows

$$\tilde{Q}_\zeta = \lambda \tilde{Q}^1_\zeta + (1 - \lambda) \tilde{Q}^2_\zeta \quad (60)$$

where $\lambda \in [0, 1]$, is the coefficient of combined optimally. If the Weighted Sum Model and Weighted Product Model approaches have equal effect on the combined optimality criteria, then consider $\lambda = 0.5$.

- L. Ranking the alternative
The alternatives are ordered based on the value $\tilde{Q}_\zeta$ of the Equation (60), where $\zeta = 1, 2, \dots, m$ are alternatives numbers. The alternative with the largest $\tilde{Q}_\zeta$ value is the best alternative and ranks first.

4.3. The Fuzzy Combined Compromise Solution (FCoCoSo) Approach

The Combined Compromise Solution (CoCoSo) method [96] extends the WASPAS method, significantly depending on WSM and WPM. Therefore, initial steps of CoCoSo are similar to WASPAS, which are mentioned in Section 4.2. n numbers of criteria and m numbers of alternatives are considered. Additionally, K numbers of DMs give the decision in linguistic terms and converted into $GDHH_\chi FN$ mentioned in Table 6. K DMs express their opinion in the form of a decision matrix of $m \times n$ order. The FCoCoSo method process steps are the following:

1. Build up the decision matrix on $GDHH_\chi FN$:
Decision matrix constructed by DMs is shown in step "A" of Section 4.1. Decision matrix shown in Equation (38) with every entry are formed like Equation (40).
2. Aggregation of DMs opinion:
Aggregate the K decision matrix in a single decision matrix ($\tilde{D}^a$) by Equation (41).
3. Normalization of the decision matrix ($\tilde{D}^a$):
Normalize the decision matrix by the Equation (51) and construct normalized decision matrix ($\tilde{D}^n$) shown in Equation (53).
4. Weight of the criteria:
Obtain weight of the criteria using FCRITIC method in the Section 4.1. Criteria weights are denoted by C^w_η and described in Equation (50).
5. Weighted Sum model (WSM):
In FWASPAS method (Section 4.2), calculated WSM data $(\tilde{Q}^{(1)}_\zeta)$ using the Equation (56).
6. Weighted Product Model (WPM):
Similar way, from FWASPAS method (in Section 4.2) obtain WPM data $(\tilde{Q}^{(2)}_\zeta)$ applying Equation (59).

7. Calculated $K_{\zeta a}$, $K_{\zeta b}$ and $K_{\zeta c}$ are using Equations (61)–(63), respectively, using the data of $\tilde{Q}_{\zeta}^1$ and $\tilde{Q}_{\zeta}^2$ is taken from previous steps.

$$K_{\zeta a} = \frac{\tilde{Q}_{\zeta}^1 + \tilde{Q}_{\zeta}^2}{\sum_{\zeta=1}^m (\tilde{Q}_{\zeta}^1 + \tilde{Q}_{\zeta}^2)} \quad (61)$$

$$K_{\zeta b} = \frac{\tilde{Q}_{\zeta}^1}{\min_{1 \leq \zeta \leq m} \tilde{Q}_{\zeta}^1} + \frac{\tilde{Q}_{\zeta}^2}{\min_{1 \leq \zeta \leq m} \tilde{Q}_{\zeta}^2} \quad (62)$$

$$K_{\zeta c} = \frac{\lambda \tilde{Q}_{\zeta}^1 + (1 - \lambda) \tilde{Q}_{\zeta}^2}{\lambda \max_{1 \leq \zeta \leq m} \tilde{Q}_{\zeta}^1 + (1 - \lambda) \max_{1 \leq \zeta \leq m} \tilde{Q}_{\zeta}^2} \quad (63)$$

where $\lambda \in [0, 1]$. Usually consider $\lambda = 0.5$.

8. Finally, rank the alternatives obtained based on the value of K_{ζ}

$$K_{\zeta} = \{K_{\zeta a} \times K_{\zeta b} \times K_{\zeta c}\}^{\frac{1}{3}} + \frac{K_{\zeta a} + K_{\zeta b} + K_{\zeta c}}{3} \quad (64)$$

where $K_{\zeta a}$, $K_{\zeta b}$ and $K_{\zeta c}$ are taken from previous steps. Here, alternative numbers are $\zeta = 1, 2, \dots, m$.

Rank alternatives based on K_{ζ} values in increasing order.

Figure 5 depicts the flow chart diagram of the MCDM methodologies.

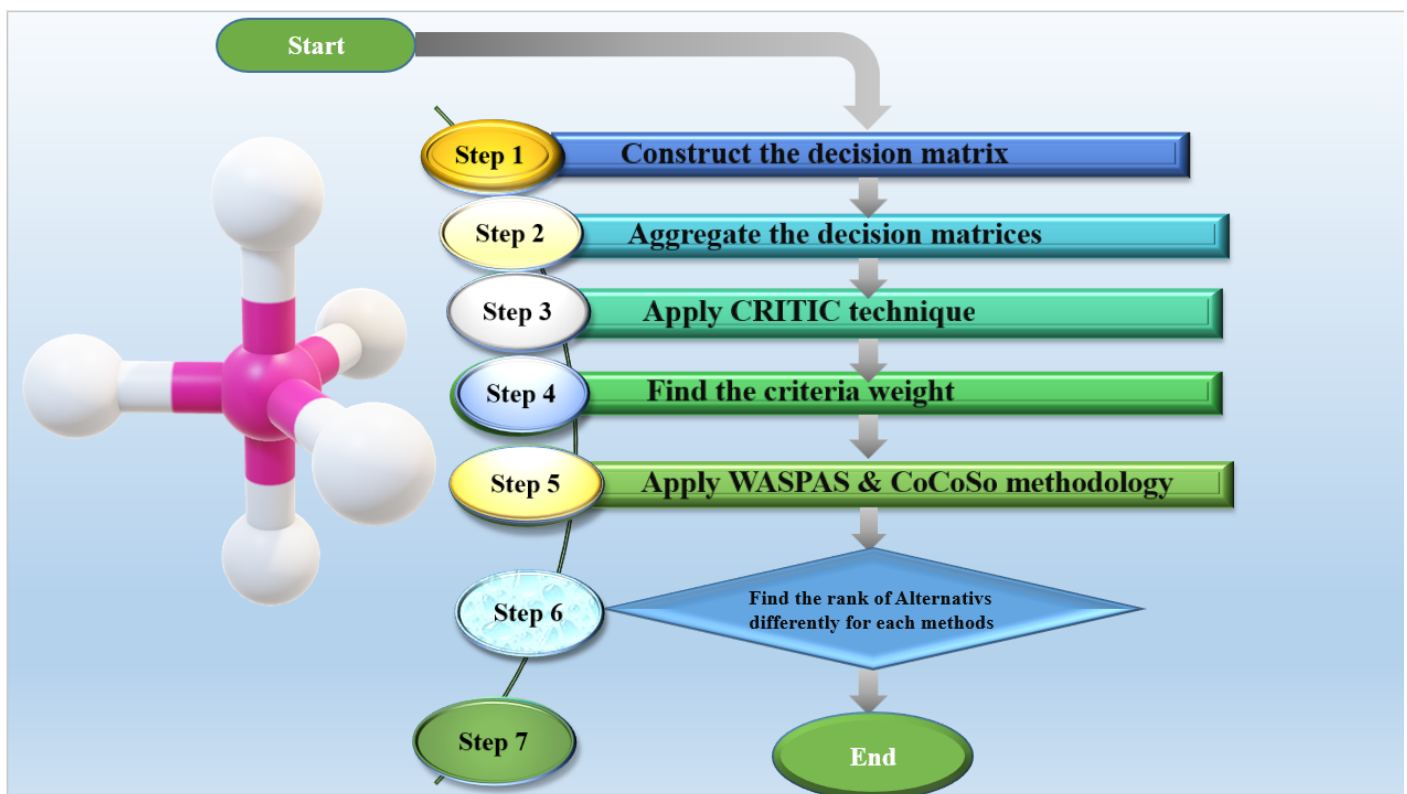


Figure 5. Flow chart of this study.

4.4. Pseudo Code Depicting the Empirical Study Application

Set up the structure with n numbers of criteria and m numbers of alternatives. K numbers of decision-makers give their overview on their knowledge in linguistic terms on the decision matrix of $m \times n$ order. Then linguistic terms are transformed into $GDH\tilde{H}_{\chi}FN$. MCDM techniques are applied to find the weight of the criteria and rank the alternatives. The K decision matrix of $m \times n$ order is used as input data.

INPUT: K numbers of decision matrix of $m \times n$ order

OUTPUT: Ranking the alternatives

COMPUTE: Weight of the criteria

INITIALIZE: $GDHH_{\chi}FN$

OPERATION: CRITIC, WASPAS and CoCoSo

```

1  MERGE merge the  $K$  numbers of DMs inputs in decision matrix
2  THEN normalize the comparison matrix
3  FOR CRITIC
4      FIND comparison matrix used to evaluate linear correlation coefficients between
        .           criteria to criteria
5      THEN compute the measure of conflict
6      FIND determine the weight of the criteria
7  END FOR
8  BEGIN WASPAS
9      COMPUTE calculation of the ranking of the alternatives using weighted
        .           normalization by using WSM and WPM from decision matrix
10 END WASPAS
11 BEGIN CoCoSo
12     COMPUTE calculation of the ranking of the alternatives on the basis of  $K_r$  value
        .           find the  $K_{ra}$ ,  $K_{rb}$  and  $K_{rc}$  using WSM and WPM from decision
        .           matrix
13 END CoCoSo

```

5. Brief Discussion of Alternative Diseases

In this section, we describe the details explanation of alternatives associated with the present model based on diseases recognition.

5.1. Malaria

Malaria [31,35,40] is often spread by mosquito bites from an infected female Anopheles mosquito. An infected person's red blood cells contain the malaria parasite, which can also be spread through blood transfusions, organ transplants, and the shared use of needles or syringes tainted with blood. Additionally, before or during delivery, a mother's unborn child may contract malaria from her. Malaria symptoms include fever and a flu-like disease with shivering chills, headache, muscular aches, and fatigue. Additionally, possible side effects include nausea, vomiting, and diarrhea. Due to the loss of red blood cells, malaria can result in anemia and jaundice (yellow skin and eyes).

5.2. Influenza

The influenza viruses [36,44–46] A, B, C, and D are the four different varieties of the virus. Influenza A virus (IAV), common in many animals, including humans and pigs, is primarily acquired through aquatic birds. The primary way influenza viruses spread among people is through respiratory droplets created when someone coughs or sneezes. Fever, sore muscles, sweats and chills, headache, shortness of breath, a persistent cough made of dryness, weakness and exhaustion, runny or congested nose, throat pain, a headache, vomiting, and diarrhea are common, but more commonly in children than in adults.

5.3. Typhoid

A bacterial infection [28,41,43] called typhoid fever can spread throughout the body and harm numerous organs. It can lead to significant problems and even be fatal without early treatment. Typically, contaminated food or water is how it is disseminated. The symptoms are high temperature, headache, nausea, weakness, and loose feces.

5.4. Dengue

Dengue [23,33,37,39] is a viral disease transmitted to humans through the bites of infected Aedes mosquitoes, primarily Aedes aegypti. The symptoms of dengue can range from mild to severe and usually appear between 3 and 14 days after infection. Dengue fever is typically characterized by a sudden onset of a high fever lasting several days. Additionally, people with dengue fever may experience severe headaches, particularly behind the eyes. High temperature, headache, muscle aches, nausea, and vomiting are the main symptoms of this disease. There is no specific treatment for dengue fever, and supportive care is often the only option.

5.5. Monkey Pox

Anyone can contract monkeypox [29,30] through close, direct, frequently skin-to-skin contact. A rash on the hands, feet, chest, face, or mouth appears in monkeypox patients. The flu, chills, enlarged lymph nodes, exhaustion, back pain, muscle aches, headache, and respiration issues (e.g., sore throat, nasal congestion, or cough) have also been observed in patients.

5.6. Ebola

The Ebola virus [38,42] causes severe, and frequently fatal, sicknesses known as Ebola virus disease (EVD), also referred to as Ebola hemorrhagic fever. It was first discovered during an outbreak in Sudan and the Democratic Republic of the Congo in 1976. Several attacks have since occurred in Africa, with the biggest being in West Africa between 2014 and 2016. People who come into contact with an infected bat or non-human primate or a sick or deceased Ebola virus-infected person directly may obtain EVD. Fever, aches and pains, such as a bad headache and painful muscles and joints, weakness, exhaustion, stiff neck, appetite loss, abdominal pain, diarrhea, vomiting, unexplained bleeding, bruising, or hemorrhaging are gastrointestinal symptoms. Red eyes, a rash, and hiccups are other symptoms that could occur (late-stage).

5.7. Pneumonia

A lung infection known as pneumonia [32,34] causes lung tissue swelling or inflammation. Viruses or bacterial infections are frequently to blame. Pneumonia symptoms may appear suddenly throughout 24 to 48 h or more gradually over several days. Chest pain, sweating, shivering, a cough, trouble breathing, a fast heartbeat, a high temperature, and an overall feeling of unwellness are symptoms.

6. Symptom of Disease and Model Setup

In this section, we describe the criteria for the proposed model. Figure 6 show how a doctor diagnoses a patient based on their symptoms and how to computationally identify and rank of disease.

In this section, let us choose the alternative disease mentioned by $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ among the above-described disease of Section 5. We do not want to reveal the identity of the disease the patient has because we only want to demonstrate how our methodology can be used to diagnose patients.

Remark 5. There are usually many factors to take to think about when diagnosing a disease, and the level of certainty for each factor can vary. Hexagonal fuzzy numbers ($H_\chi FN$) allow for a more flexible representation of uncertainty compared to traditional fuzzy numbers. This idea is expanded upon by the use of dual hesitant hexagonal fuzzy numbers ($DHH_\chi FN$), which provide the expression of both positive and negative degrees of hesitation for a disease. $GDHH_\chi FN$ s are particularly useful because they can represent not only the degree of membership but also the degree of non-membership and the degree of hesitancy in a more generalized way to capture any symptoms of the disease. This can help people make more informed judgments by accounting for a wider range of factors. Aside from this, it has been demonstrated that $GDHH_\chi FN$ s are computationally efficient

in several types of operations, including addition, subtraction, multiplication, and division. This can be beneficial for disease recognition where there are often many calculations involved in the diagnosis process. Overall, using $GDHH_{\chi}FN$ s can offer a more adaptable, precise, and effective method of identifying diseases, improving diagnosis and therapeutic outcomes.

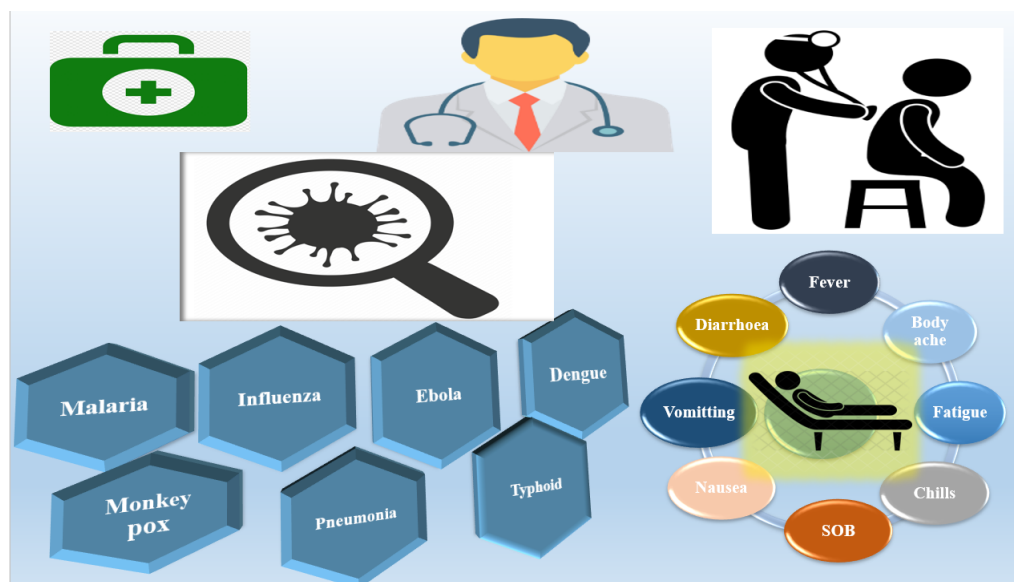


Figure 6. Diagnosing of a patient by a doctor.

7. Data Collection

In this section, how the data are collected for the numerical study is shown briefly. There are three decision-makers from whom we obtained the data about patients. The experts are given below:

- DM1: MD-Internal Medicine , MBBS General Physician
- DM2: MBBS, MD-Medicine, MD-USC, California, General Physician
- DM3: MBBS, MD-Pediatrics, MRCP (UK), Diploma in Child Health (DCH), Pediatrician

In this study, eight factors have been considered to recognize the disease in a patient. This concept has been taken from the author of [97]. Let there be the n numbers of diseases and the m numbers of symptoms associated with these diseases. If a patient V has almost all these symptoms, recognizing the disease becomes cumbersome. Hence, to capture this hesitancy or uncertainty, $GDHH_{\chi}FN$ has been used with MCDM tools WASPAS and CoCoSo to identify the specific medical problem and understand the exact diagnosis required for that patient. The eight symptoms considered are fever, body ache, fatigue, chills, shortness of breath (SOB), nausea, vomiting, and diarrhea. The symptoms have been weighted by another MCDM technique CRITIC. Seven alternatives have been taken for the study. The symptoms have been assigned linguistic terms with regard to obtaining the ranking, and another MCDM technique CoCoSo is used to verify the robustness of our result. $GDHH_{\chi}FN$ with WASPAS and CoCoSo model is applied to rank the treatment alternatives. Some steps of our methodology are shown in a later section.

Table 6 shows the linguistic term and corresponding $GDHH_{\chi}FN$ with their defuzzified values. In Table 7, we display the decision-makers (DMs) rating in linguistic terms.

Table 7. Linguistic ratings obtained for the diseases with regard to the factors.

Criteria		Fever	Body Ache	Fatigue	Chills	SOB	Nausea	Vomiting	Diarrhea
Alternative									
DM 1	D ₁	SSG	LSG	LSG	MSG	WSG	BSG	LSG	SG
	D ₂	SSG	HSG	SG	LSG	SG	SG	LSG	HSG
	D ₃	SSG	SG	WSG	MSG	WSG	HSG	LSG	BSG
	D ₄	SSG	LSG	WSG	MSG	WSG	HSG	LSG	BSG
	D ₅	SG	SG	BSG	MSG	HSG	HSG	BSG	WSG
	D ₆	SSG	HSG	LSG	LSG	BSG	BSG	HSG	SG
	D ₇	SSG	LSG	BSG	SG	HSG	SG	LSG	WSG
Criteria		Fever	Body Ache	Fatigue	Chills	SOB	Nausea	Vomiting	Diarrhea
Alternative									
DM 2	D ₁	HSG	LSG	LSG	SG	WSG	LSG	LSG	MSG
	D ₂	SSG	HSG	MSG	LSG	SG	SG	BSG	SSG
	D ₃	HSG	MSG	WSG	MSG	WSG	MSG	SG	BSG
	D ₄	HSG	SG	WSG	HSG	WSG	HSG	BSG	LSG
	D ₅	SG	SG	WSG	SG	HSG	HSG	LSG	WSG
	D ₆	SSG	MSG	LSG	LSG	BSG	SG	HSG	MSG
	D ₇	HSG	SG	BSG	SG	SSG	SG	SG	WSG
Criteria		Fever	Body Ache	Fatigue	Chills	SOB	Nausea	Vomiting	Diarrhea
Alternative									
DM 3	D ₁	HSG	LSG	LSG	HSG	WSG	BSG	BSG	SG
	D ₂	SSG	HSG	SG	LSG	MSG	SG	SG	HSG
	D ₃	SSG	SG	WSG	HSG	WSG	SSG	LSG	LSG
	D ₄	SSG	LSG	BSG	MSG	BSG	HSG	SG	BSG
	D ₅	SG	SG	LSG	MSG	HSG	HSG	BSG	WSG
	D ₆	SSG	HSG	LSG	LSG	BSG	WSG	MSG	SG
	D ₇	SSG	BSG	SG	SG	HSG	SG	SG	WSG

8. Numerical Description

The numerical simulation is performed in this section. We give the details explanation how the data are fitted to the proposed model and how the proposed method is applied for finding the numerical solution.

First, we gather information from three decision-makers in distinct ways, which are shown in Table 7. The linguistic variables are represented by $GDH\mathcal{H}_\chi FN$ second type of Table 6. Then we aggregate the three decision-makers' opinions into a single decision matrix using the Equation (41). Then using FCRITIC method mentioned in Section 4.1 evaluates the symptoms or criteria weights, which are shown in Table 8.

Table 8. Criteria weight of diagnosing using FCRITIC.

Criteria	Fever	Body Ache	Fatigue	Chills	SOB	Nausea	Vomiting	Diarrhea
Weight	0.10781	0.11548	0.10293	0.16443	0.14482	0.11271	0.09636	0.15546

Using decision matrix in Table 7 and criteria weight (from Table 8), calculate the alternative ranking by FWASPAS methodology, which mentioned in Section 4.2 are shown in Table 9.

Table 9. Ranking of disease alternatives using WASPAS technique.

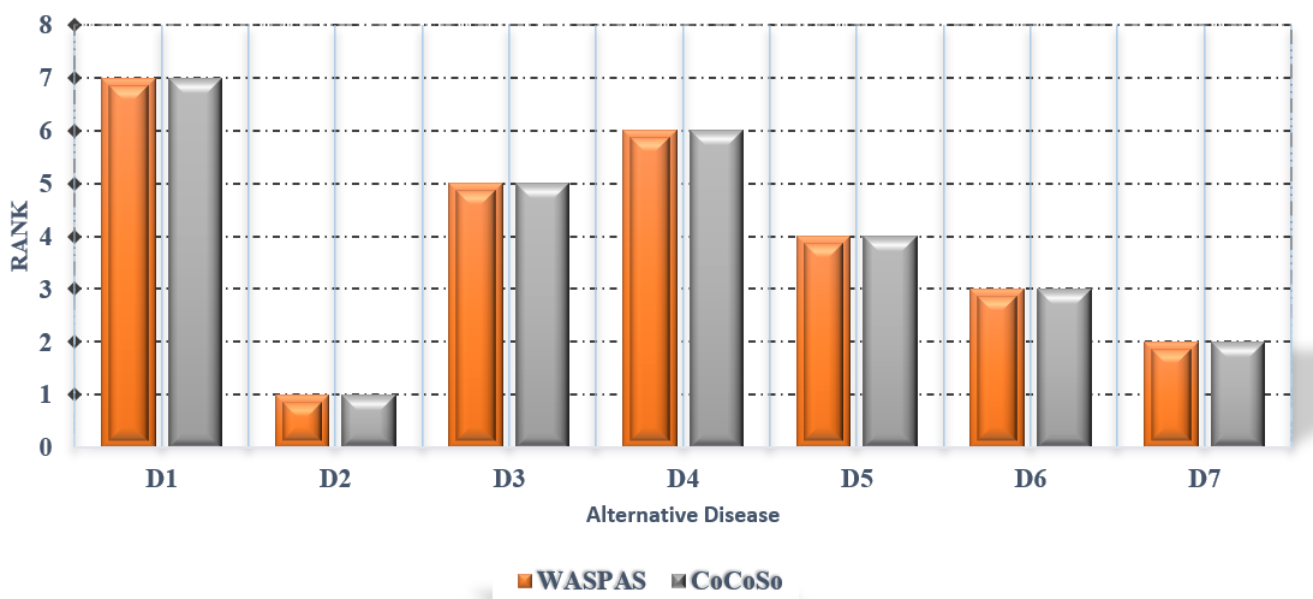
Alternative	WSM ($\tilde{Q}_i^{(1)}$)	WPM ($\tilde{Q}_i^{(2)}$)	WASPAS	Ranking
D_1	0.11443	3.84640	4.79597	7
D_2	0.14261	4.87374	4.94932	1
D_3	0.12596	4.14258	4.84629	5
D_4	0.12660	4.11709	4.84148	6
D_5	0.12949	4.22887	4.85601	4
D_6	0.13054	4.35577	4.87842	3
D_7	0.13559	4.42553	4.88686	2

Using the FWASPAS data (from Table 9) and the FCoCoSo technique mentioned in Section 4.3, calculate alternatives ranking are shown in Table 10.

Table 10. Ranking of disease alternatives using CoCoSo technique.

Alternative	K_{ra}	K_{rb}	K_{rc}	K_r	Ranking
D_1	0.12820	2.00000	0.78958	1.55978	7
D_2	0.16237	2.51340	1.00000	1.96700	1
D_3	0.13816	2.17778	0.85092	1.69061	5
D_4	0.13736	2.17675	0.84597	1.68578	6
D_5	0.14107	2.23108	0.86883	1.72940	4
D_6	0.14521	2.27323	0.89434	1.77012	3
D_7	0.14763	2.33555	0.90925	1.81015	2

Remark 6. When aggregated the three decision-makers, we obtain same rank of disease alternatives in both WASPAS and CoCoSo methods, which is demonstrated in Tables 9 and 10 and Figure 7. $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ are at the rank 7, 1, 5, 6, 4, 3, 2 in both systems, respectively.

**Figure 7.** Rank of disease alternatives of aggregating decision makers using WASPAS and CoCoSo.

Computational Complexity

This section discusses the computational complexity of this proposed fuzzy MCDM model. The idea of this numerical analysis is mentioned in several studies [98]. The total number of numerical calculations conducted to obtain the solution is described by time complexity and denoted by T_c . Here, we assume n numbers of criteria and m numbers of alternatives are performed in the evaluation process. Total K numbers of DMs give their decision in $GDHH_\chi FN$. Then, the following steps are considered to reach the solution.

1. For the FCRITIC method, the K decision matrices have $m \times n$ entries. Then, the aggregated decision matrix has $m \times n$ operations. To de-fuzzifies the decision matrix, another $m \times n$ operation are conducted. Normalizing the decision matrix $m \times n + 2n$ calculation are performed. Calculate the standard division by n number of operations. We performed n^2 operation to find the correlation coefficients. To measure the conflict created by the criteria, n^2 operations are conducted. Finally, obtain the criteria weight by conducted $2n + 1$ operations. Total operations performed for the FCRITIC method are $mn + mn + mn + 2n + n + n^2 + n^2 + 2n + 1 = 3mn + 2n^2 + 5n + 1$.
2. For the FWASPAS technique, from the FCRITIC method, we already obtain aggregated decision matrix. Then, normalize the aggregated decision matrix by conducting $m \times n + n$ operations. Another $m \times n$ calculations was performed to the weighted sum decision matrix. To de-fuzzified the weighted sum decision matrix $m \times n$ operations operated and weighted sum value calculated by m operations. Similarly, finding to weighted product decision matrix, de-fuzzification, and weighted product value $m \times n$, $m \times n$, and m operations performed, respectively. Finally, to rank the alternatives $2m$ operations are conducted. Then, the total calculations conducted for the FWASPAS method are $mn + n + mn + mn + m + mn + mn + m + 2m = 5mn + 4m + n$.
3. For the FCoCoSo method, form the FCRITIC and the FWASPAS methods, we already obtain weighted sum value and weighted product value. Then, to calculate $K_{\zeta a}$, $K_{\zeta b}$ and $K_{\zeta c}$ values $3m$ operations conducted. Finally, to rank the alternatives on the basis of $2m$ operations. Total calculation conducted for the FCoCoSo method are $3m + 2m = 5m$.

In this study, time complexity (T_c) is calculated by criteria $n = 8$, alternative $m = 7$, and decision-maker $K = 3$ as follows:

- For FCRITIC, number of calculation are $3 \times 7 \times 8 + 2 \times 8^2 + 5 \times 8 + 1 = 337$.
- For FWASPAS, number of operations are $5 \times 7 \times 8 + 4 \times 7 + 8 = 316$.
- For FCoCoSo, number of operations are $5 \times 7 = 35$.

The time complexity of this study is $337 + 316 + 35 = 688$.

9. Sensitivity Analysis and Comparative Analysis

We are interested to find some crucial results for decision-making problems in different cases. We must conduct a sensitivity analysis and comparative analysis to accomplish that. The accuracy of the final result may, therefore, be affected by initial criteria, parameters, data and measures, so doing a sensitivity analysis will aid in identifying the important factors. Additionally, the different fuzzy number gives the solutions flexibility and robustness using comparative analysis.

9.1. Sensitivity Analysis

In this section, we have considered three more patients V_1 , V_2 , and V_3 . While interacting with them regarding the ailments they are suffering, we came across varied symptoms.

Case 1: The patient V_1 informed that fever and chills, these two symptoms were quite less on an average. So, the rating of fever and chills are removed. Considering this case, the ranking indicates that the patient is suffering from D_2 .

Remark 7. The ranking of these alternative diseases is also same when we eliminate the criteria fever and chills, which is shown in Table 11 and Figure 8. The diseases D_1 , D_2 , D_3 , D_4 , D_5 , D_6 , D_7 are ranked as rank 6, 1, 5, 7, 4, 2, 3 in both WASPAS and CoCoSo, respectively.

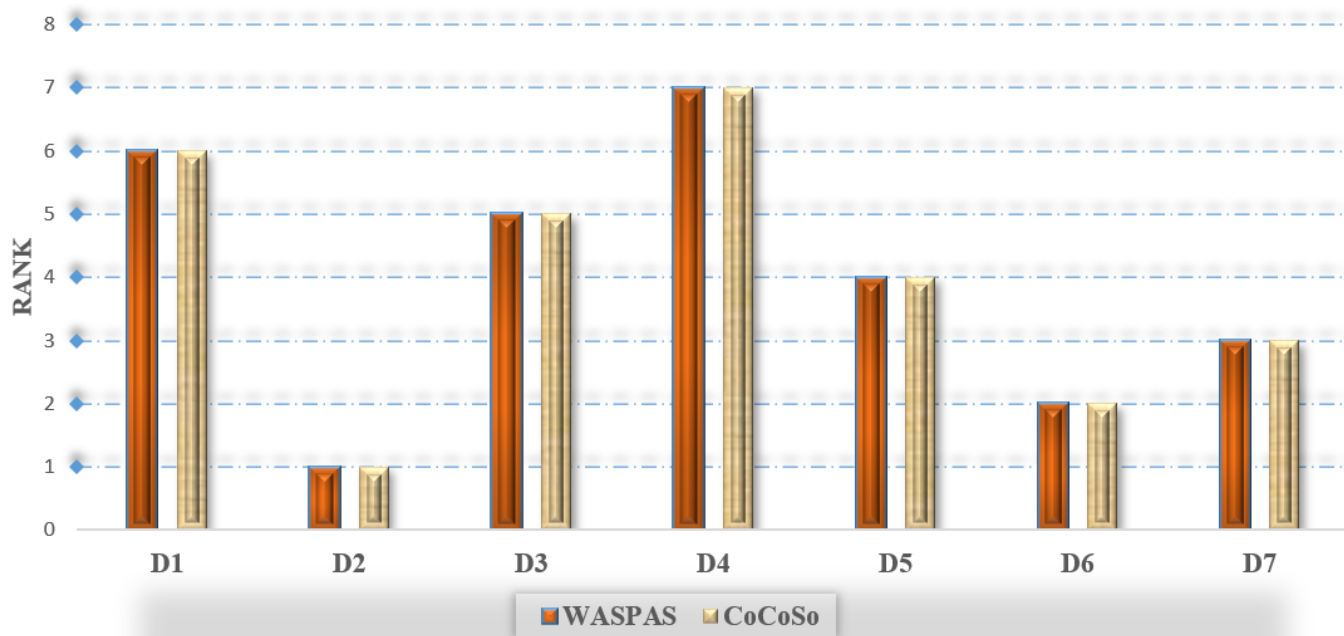


Figure 8. Rank of disease when the criteria Fever and Chills are removed.

Table 11. Comparison ranking of disease when the criteria Fever and Chills are removed.

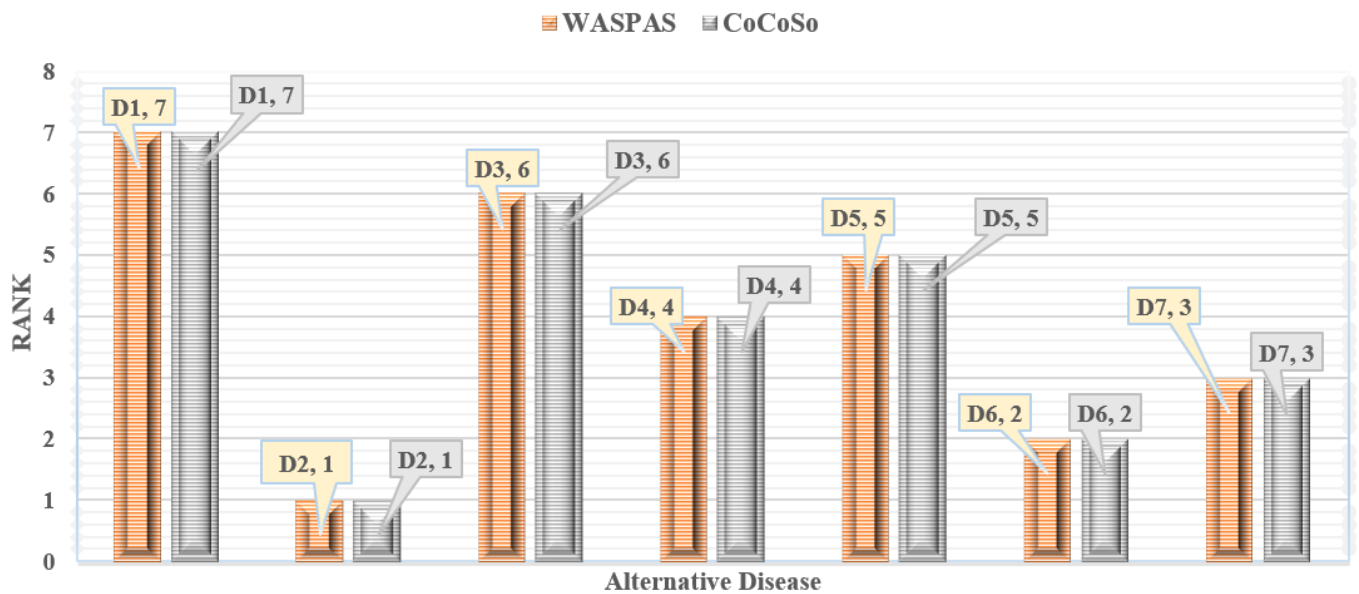
Alternative	WASPAS	CoCoSo
D_1	6	6
D_2	1	1
D_3	5	5
D_4	7	7
D_5	4	4
D_6	2	2
D_7	3	3

Case 2: The patient V_2 informed that body ache and fatigue, these two symptoms were negligible or very less on an average. The ranking so obtained considering this case indicates that the patient is also suffering from D_2 .

Remark 8. In this case the Table 12 and Figure 9 show, there are no change in rank of the alternatives, when the criteria body ache and fatigue are removed. The alternatives diseases D_1 , D_2 , D_3 , D_4 , D_5 , D_6 , and D_7 have the same rank as 7, 1, 6, 4, 5, 2, 3, respectively, in both system WASPAS and CoCoSo.

Table 12. Comparison ranking of disease when the criteria Body Ache and Fatigue are removed.

Alternative	WASPAS	CoCoSo
D_1	7	7
D_2	1	1
D_3	6	6
D_4	4	4
D_5	5	5
D_6	2	2
D_7	3	3

**Figure 9.** Rank of disease when the criteria Body Ache and Fatigue are removed.

Case 3: The patient V_3 informed that SOB and nausea, these two symptoms have no issue. The ranking so obtained reveals that the patient is suffering from D_2 or D_6 .

Remark 9. Here, the Table 13 and the Figure 10 show that the ranking of alternative diseases are completely altered for removing the criteria SOB and Nausea. The rank of the alternatives D_1 , D_2 , D_3 , D_4 , D_5 , D_6 , and D_7 are 3, 1, 5, 4, 7, 2, and 6, respectively, in WASPAS, and 4, 2, 5, 3, 7, 1, and 6, respectively, in CoCoSo.

Table 13. Comparison ranking of disease when the criteria SOB and Nausea are removed.

Alternative	WASPAS	CoCoSo
D_1	3	4
D_2	1	2
D_3	5	5
D_4	4	3
D_5	7	7
D_6	2	1
D_7	6	6

Considering these three cases of the three individual patients, transforming the linguistic rating to $GDHH_{\chi}FN$, the ranking obtained are represented in Tables 11–13 and Figures 8–10, respectively.

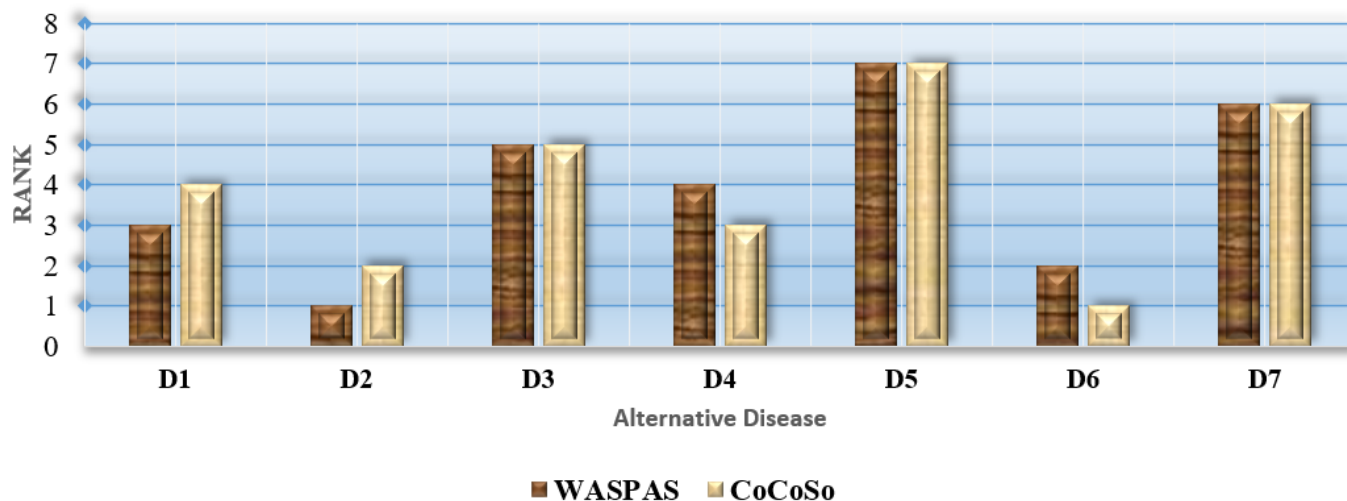


Figure 10. Rank of disease when the criteria SOB and Nausea are removed.

9.2. Comparative Analysis

We would need to examine using three different fuzzy numbers in order to check the consistency and reliability of our procedures, as well as the accuracy and specificity of our results. In addition to $GDHH_{\chi}FN$, we also used two other fuzzy numbers which are generalized dual hesitant trapezoidal fuzzy number ($GDHTrFN$) and generalized dual hesitant pentagonal fuzzy number ($GDHPFN$). Table 14 shows the rank of alternative disease by $GDHTrFN$, $GDHPFN$, $GDHH_{\chi}FN$, respectively, in WASPAS and CoCoSo methods.

Table 14. Comparison ranking of $GDHTrFN$, $GDHPFN$, and $GDHH_{\chi}FN$ using the WASPAS and CoCoSo method.

Alternatives	$GDHTrFN$		$GDHPFN$		$GDHH_{\chi}FN$	
	WASPAS	CoCoSo	WASPAS	CoCoSo	WASPAS	CoCoSo
D_1	7	7	7	7	7	7
D_2	1	1	1	1	1	1
D_3	2	3	4	4	5	5
D_4	6	6	6	6	6	6
D_5	5	5	2	2	4	4
D_6	4	4	5	5	3	3
D_7	3	2	3	3	2	2

Remark 10. Table 14 and Figures 11 and 12 depict that the disease D_2 is at rank 1 for each three fuzzy numbers in both methods WASPAS and CoCoSo, and the rank of remaining diseases are slightly changed. For $GDHTrF$ WASPAS and CoCoSo, D_1 , D_4 , D_5 , and D_6 have same rank which are 7, 6, 5, and 4, respectively, and D_3 have rank 2 and 3 and D_7 have rank 3 and 2. In the same way, for $GDHPFN$ and $GDHH_{\chi}FN$, all diseases have the same rank for both methods WASPAS and CoCoSo, respectively, which is displayed in Table 14.

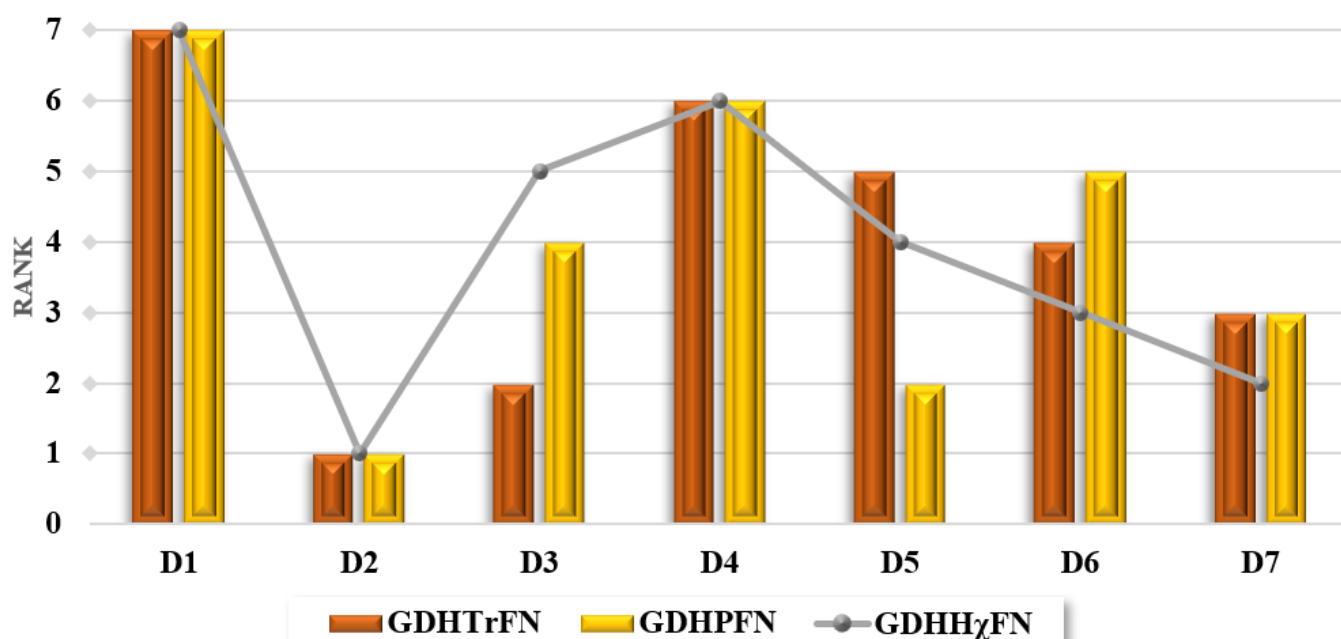


Figure 11. Ranking of disease in WASPAS of three different fuzzy environments.

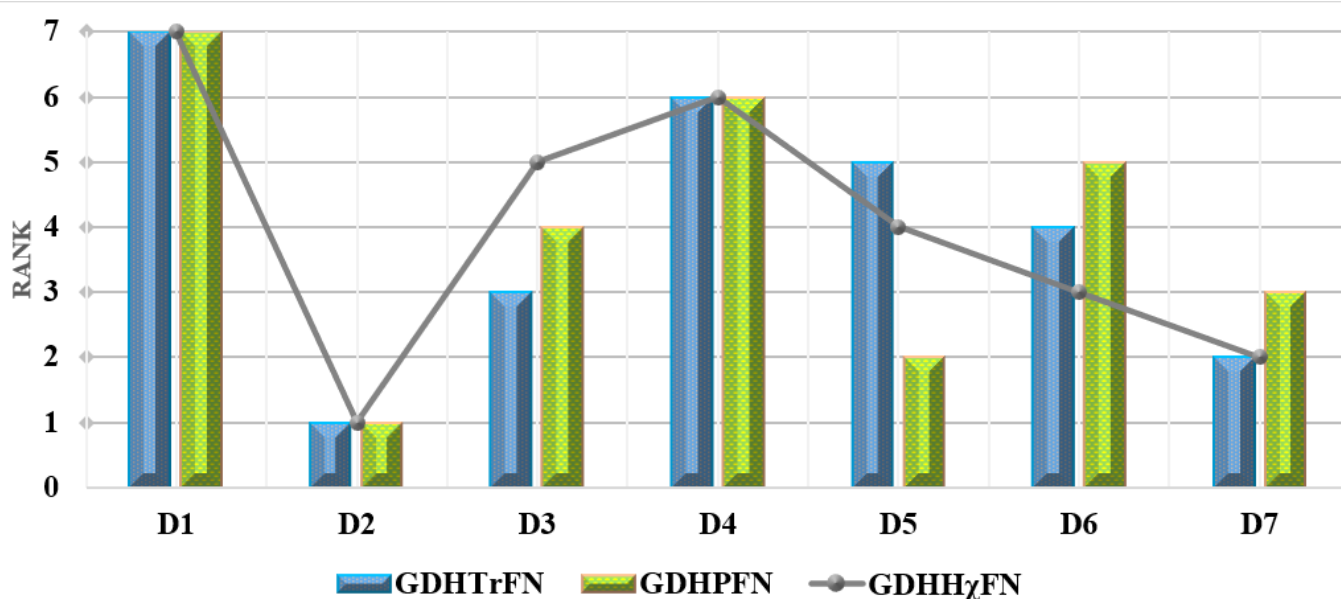


Figure 12. Ranking of disease in CoCoSo of three different fuzzy environments.

Finally, we observe that in each fuzzy environment using the provided MCDM methodologies, the disease D_2 is always at rank 1. Hence, we may conclude with certainty that the patient has the disease D_2 .

10. Practical Implications

The whole study is based on the symptomatic dataset and statistical analysis. The proposed methodology is to find the most changes among the related diseases. Therefore, this method can be used entirely for clustering the presence of associated diseases with typical symptoms. Before the lab test or consulting with a medical supervisor, anyone can easily find the related illnesses they suffer. As it is common to be reliant on technology, whether it be a laptop, smartwatch, or mobile phone, it may be easy to store data for analysis purposes. This work only shows how the decision may be taken after the mathematical

and numerical analyses. Suppose any software is made to take data about symptoms from an infected person, then uses the proposed algorithm to conclude that initial disease identification and move forward to analysis and lab tests.

11. Conclusions and Future Research Extension

In conclusion, this research study has focused on exploring the applicability of the multi-criteria decision-making (MCDM) technique using Generalized Dual Hesitant Hexagonal Fuzzy Number ($GDHH_{\chi}FN$) for the diagnosis of the patient. The study has demonstrated the effectiveness of the proposed strategy in dealing with ambiguity and the uncertainty that is prevalent in the medical diagnosis process.

11.1. Summary of Study and Contributions

The study of disease recognition by the generalized dual hesitant hexagonal fuzzy MCDM technique is an approach to evaluate and recognize diseases based on multiple criteria. The proposed method uses a combination of dual hesitant fuzzy set theory and hexagonal fuzzy numbers to model the uncertain and imprecise information associated with disease recognition. This study provides a valuable contribution to the field of medical diagnosis by introducing a new approach to decision-making that is more accurate, comprehensive, and efficient. The contributions of our study are as follows:

- (a) In order to compare each $GDHH_{\chi}FN$, we have first proposed a new defuzzification technique for a $GDHH_{\chi}FN$ and modified the arithmetic operation of $GDHH_{\chi}FN$ first type and second type.
- (b) Then, we develop an aggregation formula to combine more than one decision matrix.
- (c) Here, we have used the FCritic approach to determine the weight of the criteria (symptom).
- (d) We have applied the FWASPAS and FCoCoSo methods simultaneously to determine the rank of the alternatives (Disease).
- (e) Finally, we conducted a comparative analysis using two alternative fuzzy numbers, $GDHPFN$ and $GDHTrFN$.

11.2. The Crucial Findings

The $GDHH_{\chi}F$ MCDM approach can successfully handle the uncertainty and ambiguity involved with medical diagnosis, which is an important finding of these studies. Specifically, the technique enables the representation of decision-making criteria and alternatives using dual hesitant fuzzy sets, which are a particular kind of fuzzy set that allows decision-makers to express their degree of hesitation or uncertainty about the membership and non-membership of an element in a set. The hexagonal structure of the dual hesitant fuzzy sets used in the $GDHH_{\chi}F$ MCDM technique provides more flexibility in representing the degree of hesitancy. Overall, the use of the $GDHH_{\chi}F$ MCDM technique in disease recognition has shown promising results and can potentially be used as a tool to aid medical professionals in making more informed decisions in the diagnosis of diseases.

11.3. The Restriction and Directions for Future Research

The proposed approach has been validated through a case study on the diagnosis of patients, and the results have shown that it can produce accurate and reliable outcomes compared to other existing methods. Therefore, the use of $GDHH_{\chi}FN$ based MCDM technique can be considered a promising approach for medical diagnosis applications. We have used seven alternative diseases and eight criteria (symptoms) to diagnose a patient.

However, further research is necessary to investigate the generalization of this approach to other diseases and to explore the potential of integrating additional data sources, such as patient history, laboratory test results, and medical imaging, by another generalized fuzzy environment with a new de-fuzzification method. More criteria and alternatives can be taken to identify a patient's disease. Overall, this research paper has contributed to the growing body of literature on the application of MCDM techniques in medical diagnosis,

and it has demonstrated the potential of $GDH\mathcal{H}_\chi FN$ based MCDM for accurate and reliable disease diagnosis.

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