Article

Three-Dimensional Couette Flow of a Second-Grade Fluid Along Periodic Injection/Suction

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Abstract: This work explores the three-dimensional laminar flow of an incompressible second-grade fluid between two parallel infinite plates. The assumed suction velocity comprises a basic steady dispersal with a superimposed weak transversally fluctuating distribution. Because of variation of suction velocity in transverse direction on the wall, the problem turns out to be three-dimensional. Analytic solutions for velocity field, pressure and skin friction are presented and effects of dimensionless parameters emerging in the model are discussed. It is observed that the non-Newtonian parameter plays dynamic part to rheostat the velocity component along main flow direction.

Keywords: second-grade fluid; couette flow; three-dimensional flow; periodic injection/suction

1. Introduction

In recent years the laminar flow control (LFC) problem has attained significant importance particularly in the field of aeronautical engineering. In fact, it provides a mechanism to reduce drag and hence to improve the vehicle power by a substantial amount. The boundary layer can be controlled artificially by a number of methods. The boundary layer suction method is the one of the effective techniques of reducing the drag coefficient. According to this method the transition from laminar to turbulent flow, causing escalation of drag coefficient may be deferred or prevented by removing the slowed fluid particles in the boundary layer via holes in the wall into the interior of the body [1]. Many researchers have explored fluid flow problems with suction; however, the majority of these investigations cope with two-dimensional flows only. Gersten and Gross [2] studied the impact of transverse sinusoidal suction velocity on viscous fluid flow with heat transfer over a porous plane wall. Three dimensional fluctuating viscous fluid flow with transpiration cooling and heat transfer along a plate with suction was studied by Singh [3].

Three-dimensional viscous fluid flow through infinites parallel planes with injection/suction was studied by Chaudhary et al. [4]. Workers [5] explored three-dimensional fluctuating flow of viscous fluid through two parallel infinite plates with heat transfer. Three-dimensional viscous fluid flow between two parallel infinite planes was deliberated with injection/suction by [6]. Dominant effect of Prandtl number over temperature field as compared to injection/suction was reported. Transient heat effects on three-dimensional viscous fluid flow through two parallel infinite plates partially filled by permeable material were scrutinized by [7]. Further, several investigators [8–16] witnessed under various physical conditions the three-dimensional viscous fluid flow past a permeable plane.

Above cited investigations deal with viscous fluid. Though the Navier–Stokes equations can cope with the Newtonian fluids flows only but these are insufficient to incorporate the configurations of

non-Newtonian fluids. Researchers [17–22] considered three-dimensional flow of non-Newtonian fluids under various physical conditions. According to the authors, knowledge Couette flow of the second-grade fluid with perpendicular sinusoidal injection/suction velocity has not yet been examined. Therefore, in this study, three-dimensional Couette flow of the second-grade fluid along sinusoidal injection/suction is inspected. A constant injection or suction velocity at the plane tends to two-dimensional flow [2]; however, because of varying suction velocity in the perpendicular direction on plane, the problem becomes three-dimensional. Perturbation method is employed to handle the nonlinearity involved in the governing equations. The outline of this paper is as follows. Section 2 describes the problem, Section 3 discusses problem formulation, Section 4 estimates solutions, Section 5 includes the discussion, and Section 6 summarizes the conclusions.

2. Problem Description

Let the three-dimensional laminar flow of an incompressible second-grade fluid between two parallel porous plates be steady and fully developed. Also, assume that the separation between the plates is h . Take the x^*z^* -plane along the plates and the y^* -axis normal to the plane of the plates as shown in Figure 1. The injection/suction velocity distribution [5] takes the form of:

$$v^*(z^*) = V_0 \left(1 + \varepsilon \cos \pi \frac{z^*}{h} \right) \quad (1)$$

where V_0 is injection/suction velocity and ε is amplitude of the injection/suction velocity. The bottom plate is fixed whereas upper is moving with constant velocity U in +ve x^* direction. The crosswise periodic injection of the fluid at the fixed plate and its equivalent removal by periodic suction through the moving plate is taken. Moreover, u^* , v^* , and w^* are the velocity components along the x^* -, y^* -, and z^* -axes respectively. As the flow is expected to be laminar and fully developed so all the physical quantities do not depend on x^* . Obviously, the flow remains three-dimensional because of injection/suction velocity (Equation (1)).

Figure 1. Schematic diagram of the problem.

3. Formulation of the Problem

For a Second-grade fluid model the Cauchy's stress tensor is

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \quad (2)$$

in which p , μ , I , and α_i ($i = 1, 2$) denote the pressure, the dynamic viscosity, the identity tensor, and material constants respectively. A_1 and A_2 are the Rivlin–Ericksen tensors defined as:

$$\left. \begin{aligned} A_1 &= \text{grad}V + (\text{grad}V)^T \\ A_2 &= \frac{dA_1}{dt^*} + A_1 \text{grad}V + (\text{grad}V)^T A_1 \end{aligned} \right\} \quad (3)$$

where “ T ” represents the transpose and t^* is the time. The material parameters meets the conditions [23]:

$$\alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0, \mu \geq 0 \quad (4)$$

so that thermodynamics is compatible with the model (2) in the sense that the Clausius–Duhem inequality is satisfied by all the motions and the specific Helmholtz free energy is minimum in equilibrium. The conservation laws of mass and momentum are given by:

$$\text{div}V = 0 \quad (5)$$

$$\rho \frac{\partial V}{\partial t^*} = \text{div}T \quad (6)$$

Thus, the equations of motion are:

$$\frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (7)$$

$$\rho \left(v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} \right) = \mu \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right) + \alpha_1 \left(v^* \frac{\partial^3 u^*}{\partial y^{*3}} + w^* \frac{\partial^3 u^*}{\partial z^* \partial y^{*2}} + v^* \frac{\partial^3 u^*}{\partial y^* \partial z^{*2}} + w^* \frac{\partial^3 u^*}{\partial z^{*3}} \right) \quad (8)$$

$$\rho \left(v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} \right) = -\frac{\partial p^*}{\partial y^*} + \mu \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}} \right) + \alpha_1 \left(\begin{aligned} &v^* \frac{\partial^3 v^*}{\partial y^{*3}} + w^* \frac{\partial^3 v^*}{\partial z^* \partial y^{*2}} + \frac{\partial^3 v^*}{\partial y^* \partial z^{*2}} + w^* \frac{\partial^3 v^*}{\partial z^{*3}} + \\ &5 \frac{\partial v^*}{\partial y^*} \frac{\partial^2 v^*}{\partial y^{*2}} + 2 \frac{\partial u^*}{\partial y^*} \frac{\partial^2 u^*}{\partial y^{*2}} + 2 \frac{\partial w^*}{\partial y^*} \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial v^*}{\partial z^*} \frac{\partial^2 w^*}{\partial y^{*2}} \\ &+ \frac{\partial u^*}{\partial z^*} \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial u^*}{\partial y^*} \frac{\partial^2 u^*}{\partial z^{*2}} + \frac{\partial v^*}{\partial y^*} \frac{\partial^2 v^*}{\partial z^{*2}} + \frac{\partial w^*}{\partial z^*} \frac{\partial^2 v^*}{\partial y^{*2}} \end{aligned} \right) \quad (9)$$

$$\rho \left(v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} \right) = -\frac{\partial p^*}{\partial z^*} + \mu \left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} \right) + \alpha_1 \left(\begin{aligned} &w^* \frac{\partial^3 w^*}{\partial y^{*3}} + v^* \frac{\partial^3 w^*}{\partial z^* \partial y^{*2}} + v^* \frac{\partial^3 w^*}{\partial z^* \partial y^{*2}} + w^* \frac{\partial^3 w^*}{\partial z^{*3}} + \\ &\frac{\partial w^*}{\partial y^*} \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial u^*}{\partial y^*} \frac{\partial^2 u^*}{\partial z^{*2}} + 5 \frac{\partial v^*}{\partial z^*} \frac{\partial^2 w^*}{\partial z^{*2}} + \frac{\partial w^*}{\partial y^*} \frac{\partial^2 v^*}{\partial z^{*2}} \\ &+ 2 \frac{\partial u^*}{\partial z^*} \frac{\partial^2 u^*}{\partial z^{*2}} + 2 \frac{\partial v^*}{\partial z^*} \frac{\partial^2 v^*}{\partial z^{*2}} + \frac{\partial u^*}{\partial z^*} \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial w^*}{\partial z^*} \frac{\partial^2 w^*}{\partial y^{*2}} \end{aligned} \right) \quad (10)$$

subject to boundary conditions:

$$\left. \begin{aligned} &\text{at } y^* = 0; u^* = 0, v^*(z^*) = V_0(1 + \varepsilon \cos \pi \frac{z^*}{h}), w^* = 0 \\ &\text{at } y^* = h; u^* = U, v^*(z^*) = V_0(1 + \varepsilon \cos \pi \frac{z^*}{h}), w^* = 0 \end{aligned} \right\} \quad (11)$$

Dimensionless parameters:

$$\alpha = \frac{V_0}{U}, \text{Re} = \frac{hU}{\nu}, K = \frac{\alpha_1}{\rho h^2}, p = \frac{p^*}{\rho U^2}, y = \frac{y^*}{h}, z = \frac{z^*}{h}, u = \frac{u^*}{U}, v = \frac{v^*}{U}, w = \frac{w^*}{U} \quad (12)$$

where α , K , and Re represent injection/suction parameter, elastic parameter, and Reynolds number respectively. Substituting Equation (12) into the Equations (7)–(11) to get:

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (13)$$

$$v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + K \left(v \frac{\partial^3 u}{\partial y^3} + w \frac{\partial^3 u}{\partial z \partial y^2} + v \frac{\partial^3 u}{\partial y \partial z^2} + w \frac{\partial^3 u}{\partial z^3} \right) \quad (14)$$

$$v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}} \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + K \left(\frac{v \frac{\partial^3 v}{\partial y^3} + w \frac{\partial^3 v}{\partial z \partial y^2} + v \frac{\partial^3 v}{\partial y \partial z^2} + w \frac{\partial^3 v}{\partial z^3} + \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial y} + \frac{5 \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial z} \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial z^2} \right) \quad (15)$$

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\text{Re}} \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + K \left(\frac{w \frac{\partial^3 v}{\partial y^3} + v \frac{\partial^3 w}{\partial y^3} + v \frac{\partial^3 w}{\partial y \partial z^2} + w \frac{\partial^3 w}{\partial z^3} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial z} + \frac{5 \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z^2} + \frac{\partial w}{\partial y} \frac{\partial^2 v}{\partial z^2} + 2 \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z^2} \right) \quad (16)$$

subject to nondimensional boundary conditions

$$\left. \begin{aligned} \text{at } y = 0; \quad u = 0, v(z) = \alpha(1 + \varepsilon \cos \pi z), w = 0 \\ \text{at } y = 1; \quad u = 1, v(z) = \alpha(1 + \varepsilon \cos \pi z), w = 0 \end{aligned} \right\} \quad (17)$$

where u , v and w are velocity components along the x -, y -, and z -axes respectively.

4. Solution

In this section the perturbation solutions for the velocity profile and skin friction components are established.

4.1. Cross Flow Solution

Since ε is positive and small, hence we assume the solution of the type

$$g(y, z) = g_0(y) + \varepsilon g_1(y, z) + \varepsilon^2 g_2(y, z) + \dots \quad (18)$$

where g denotes any of the physical parameters u, v, w , and p . It is worth mentioning that the velocity components $v_1(y, z)$, $w_1(y, z)$ and pressure $p_1(y, z)$ do not depend upon the main flow velocity component $u_1(y, z)$. The first order continuity equation and momentum equations along the x - and z -axes, respectively, are

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (19)$$

$$\alpha \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{\text{Re}} \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) + K \alpha \left(\frac{\partial^3 v_1}{\partial y^3} + \frac{\partial^3 v_1}{\partial y \partial z^2} \right) \quad (20)$$

$$\alpha \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{\text{Re}} \left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right) + K \alpha \left(\frac{\partial^3 w_1}{\partial y^3} + \frac{\partial^3 w_1}{\partial y \partial z^2} \right) \quad (21)$$

the corresponding boundary conditions are

$$v_1(0, z) = \alpha \cos \pi z, \quad w_1(0, z) = 0, \quad v_1(1, z) = \alpha \cos \pi z, \quad w_1(1, z) = 0 \quad (22)$$

The injection/suction velocity comprises of basic constant distribution V_0 with a super imposed weak periodic distribution $\varepsilon V_0 \cos \pi z$, thus the velocity components $v_1(y, z)$, $w_1(y, z)$, and pressure $p_1(y, z)$ are also separated into small and main periodic components. Therefore, we consider the following:

$$p_1(y, z) = p_{11}(y) \cos \pi z \quad (23)$$

$$v_1(y, z) = v_{11}(y) \cos \pi z \quad (24)$$

$$w_1(y, z) = -\frac{1}{\pi} v'_{11} \sin \pi z \quad (25)$$

here “ $'$ ” denotes derivative with respect to “ y ”. Moreover, the velocity components (24) and (25) satisfy the continuity Equation (19) identically. Setting Equations (23)–(25) into Equations (20) and (21) to obtain

$$K \alpha (v'''_{11} - \pi^2 v'_{11}) + \frac{1}{\text{Re}} (v''_{11} - \pi^2 v_{11}) - \alpha v'_{11} = p'_{11} \quad (26)$$

$$K\alpha(v_{11}'' - \pi^2 v_{11}') + \frac{1}{\text{Re}}(v_{11}''' - \pi^2 v_{11}') - \alpha v_{11}'' = \pi^2 p_{11} \quad (27)$$

Eliminating the pressure p_{11} from Equations (26) and (27) to get

$$KR(v_{11}'' - 2\pi^2 v_{11}''' + \pi^4 v_{11}') + v_{11}'' + \pi^4 v_{11} - 2\pi^2 v_{11}'' - R(v_{11}''' - \pi^2 v_{11}') = 0 \quad (28)$$

where $\alpha \text{Re} = R$. Assuming $K \ll 1$, and taking

$$v_{11}(y) = v_{110}(y) + Kv_{111}(y) + O(K^2) \quad (29)$$

then the general solution of zeroth-order problem

$$v_{110}'' + \pi^4 v_{110} - 2\pi^2 v_{110}'' - R(v_{110}''' - \pi^2 v_{110}') = 0 \quad (30)$$

subject to the corresponding boundary conditions

$$v_{110}(0) = \alpha = v_{110}(1), v_{110}'(0) = 0 = v_{110}'(1) \quad (31)$$

is

$$v_{110}(y) = E_3 e^{-\pi y} + E_4 e^{\pi y} + E_5 e^{E_1 y} + E_6 e^{E_2 y} \quad (32)$$

Similarly, the first-order equation is

$$v_{111}'' - 2\pi^2 v_{111}'' - R(v_{111}''' - \pi^2 v_{111}') + \pi^4 v_{111} = -R(v_{110}'' - 2\pi^2 v_{110}''' + \pi^4 v_{110}') \quad (33)$$

and the corresponding boundary conditions are

$$v_{111}(0) = 0, v_{111}(1) = 0, v_{111}'(0) = 0, v_{111}'(1) = 0 \quad (34)$$

The solution of the boundary value problem (33)–(34) is

$$v_{111}(y) = E_7 e^{-\pi y} + E_8 e^{\pi y} + E_9 e^{E_1 y} + E_{10} e^{E_2 y} + y(E_{11} e^{E_1 y} + E_{12} e^{E_2 y}) \quad (35)$$

Thus, Equations (29), (24), and (25), respectively, become

$$v_{11}(y) = E_3 e^{-\pi y} + E_4 e^{\pi y} + E_5 e^{E_1 y} + E_6 e^{E_2 y} + K(E_7 e^{-\pi y} + E_8 e^{\pi y} + E_9 e^{E_1 y} + E_{10} e^{E_2 y} + y(E_{11} e^{E_1 y} + E_{12} e^{E_2 y})) \quad (36)$$

$$v_1(y, z) = \begin{pmatrix} E_3 e^{-\pi y} + E_4 e^{\pi y} + E_5 e^{E_1 y} + E_6 e^{E_2 y} + \\ K \begin{pmatrix} E_7 e^{-\pi y} + E_8 e^{\pi y} + E_9 e^{E_1 y} + \\ E_{10} e^{E_2 y} + y(E_{11} e^{E_1 y} + E_{12} e^{E_2 y}) \end{pmatrix} \end{pmatrix} \cos \pi z \quad (37)$$

$$w_1(y, z) = -\frac{1}{\pi} \begin{pmatrix} -E_3 \pi e^{-\pi y} + E_4 \pi e^{\pi y} + E_5 E_1 e^{E_1 y} + E_6 E_2 e^{E_2 y} \\ + K \begin{pmatrix} -E_7 \pi e^{-\pi y} + E_8 \pi e^{\pi y} + \\ E_9 E_1 e^{E_1 y} + E_{10} E_2 e^{E_2 y} \\ + y(E_{11} E_1 e^{E_1 y} + E_{12} E_2 e^{E_2 y}) \\ + (E_{11} e^{E_1 y} + E_{12} e^{E_2 y}) \end{pmatrix} \end{pmatrix} \sin \pi z \quad (38)$$

where constants E_i ($i = 1, 2, 3, \dots, 12$) are defined in Appendix A.

4.2. Main Flow Solution

The problem become two dimensional flow when $\varepsilon = 0$, and therefore zeroth-order problem and corresponding boundary conditions are as follows

$$KR \frac{d^3 u_0}{dy^3} + \frac{d^2 u_0}{dy^2} - R \frac{du_0}{dy} = 0 \quad (39)$$

$$u_0(0) = 0, \quad u_0(1) = 1 \quad (40)$$

Since $K \ll 1$, so assuming

$$u_0(y) = u_{00}(y) + Ku_{01}(y) + O(K^2) \quad (41)$$

Then the solution of zeroth-order boundary value problem

$$\frac{d^2 u_{00}}{dy^2} - R \frac{du_{00}}{dy} = 0 \quad (42)$$

$$u_{00}(0) = 0, \quad u_{00}(1) = 1 \quad (43)$$

is

$$u_{00}(y) = 1 + \frac{e^{Ry} - e^R}{e^R - 1} \quad (44)$$

Similarly, the first-order boundary value problem

$$\frac{d^2 u_{01}}{dy^2} - R \frac{du_{01}}{dy} = -R \frac{d^3 u_{00}}{dy^3} \quad (45)$$

and corresponding boundary conditions are

$$u_{01}(0) = 0, \quad u_{01}(1) = 0 \quad (46)$$

The solution of the problems (45) and (46) is

$$u_{01}(y) = E_{13} + E_{14}e^{Ry} + yE_{15}e^{Ry} \quad (47)$$

Thus,

$$u_0(y) = 1 + \frac{e^{Ry} - e^R}{e^R - 1} + K(E_{13} + E_{14}e^{Ry} + yE_{15}e^{Ry}) \quad (48)$$

where the constants E_{13} , E_{14} , and E_{15} are defined in Appendix A. In the case $\varepsilon \neq 0$, the equations of motion (14)–(16) and boundary conditions (17) are perturbed by taking

$$u(y, z) = u_0(y) + \varepsilon u_1(y, z) + O(\varepsilon^2) \quad (49)$$

$$v(y, z) = v_0(y) + \varepsilon v_1(y, z) + O(\varepsilon^2) \quad (50)$$

$$w(y, z) = w_0(y) + \varepsilon w_1(y, z) + O(\varepsilon^2) \quad (51)$$

Then the first-order equation

$$\alpha \frac{\partial u_1}{\partial y} + v_1 \frac{du_0}{dy} = \frac{1}{\text{Re}} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) + K \left(\alpha \frac{\partial^3 u_1}{\partial y^3} + \alpha \frac{\partial^3 u_1}{\partial y \partial z^2} + v_1 \frac{d^3 u_0}{dy^3} \right) \quad (52)$$

and corresponding boundary conditions are

$$u_1(0, z) = 0 = u_1(1, z) \quad (53)$$

The solution of Equation (52) can be expressed, such as $u_1(y, z) = u_{11}(y)\cos\pi z$. Then,

$$\frac{d^2 u_{11}}{dy^2} - R \frac{du_{11}}{dy} - \pi^2 u_{11} = \operatorname{Re} v_{11} \left(\frac{du_0}{dy} - K \frac{d^3 u_0}{dy^3} \right) + KR \left(\pi^2 \frac{du_{11}}{dy} - \frac{d^3 u_{11}}{dy^3} \right) \quad (54)$$

The corresponding boundary conditions (53) become

$$u_{11}(0) = 0 = u_{11}(1) \quad (55)$$

The order of differential Equation (54) is three while only two boundary conditions are given. Therefore, we assume the solution of the type

$$u_{11}(y) = u_{110}(y) + Ku_{111}(y) + O(K^2) \quad (56)$$

Then the solution of zeroth-order problem

$$\frac{d^2 u_{110}}{dy^2} - R \frac{du_{110}}{dy} - \pi^2 u_{110} = \operatorname{Re} v_{110} \frac{du_{00}}{dy} \quad (57)$$

$$u_{110}(0) = 0 = u_{110}(1) \quad (58)$$

is

$$u_{110}(y) = E_{16}e^{E_1 y} + E_{17}e^{E_2 y} + E_{18}e^{(R-\pi)y} + E_{19}e^{(R+\pi)y} + E_{20}e^{(R+E_1)y} + E_{21}e^{(R+E_2)y} \quad (59)$$

Similarly, the first-order problem is

$$\frac{d^2 u_{111}}{dy^2} - R \frac{du_{111}}{dy} - \pi^2 u_{111} = R \left(\pi^2 \frac{du_{110}}{dy} - \frac{d^3 u_{110}}{dy^3} \right) + \operatorname{Re} \left(v_{110} \left(\frac{du_{01}}{dy} - \frac{d^3 u_{00}}{dy^3} \right) + v_{111} \frac{du_{00}}{dy} \right) \quad (60)$$

$$u_{111}(0) = 0 = u_{111}(1) \quad (61)$$

and hence

$$u_{111}(y) = E_{26}e^{E_1 y} + E_{27}e^{E_1 y} + \frac{E_{28}}{E_1 - E_2} y e^{E_1 y} + \frac{E_{29}}{E_2 - E_1} y e^{E_2 y} + \frac{E_{30}}{E_{22}} e^{(R-\pi)y} + \frac{E_{31}}{E_{23}} e^{(R+\pi)y} + \frac{E_{32}}{E_{24}} e^{(R+E_1)y} + \frac{E_{33}}{E_{24}} e^{(R+E_2)y} + \frac{E_{34}}{E_{22}} \left(y - \frac{R-2\pi}{E_{22}} \right) e^{(R-\pi)y} + \frac{E_{35}}{E_{23}} \left(y - \frac{R+2\pi}{E_{23}} \right) e^{(R+\pi)y} + \frac{E_{36}}{E_{24}} \left(y - \frac{R+2E_1}{E_{24}} \right) e^{(R+E_1)y} + \frac{E_{37}}{E_{25}} \left(y - \frac{R+2E_2}{E_{25}} \right) e^{(R+E_2)y} \quad (62)$$

In view of Equations (48), (59), (62), and (56), the Equation (49) yields

$$u(y, z) = 1 + \frac{e^{Ry} - e^R}{e^R - 1} + K \left(E_{13} + E_{14}e^{Ry} + yE_{15}e^{Ry} \right) + \left(E_{16}e^{E_1 y} + E_{17}e^{E_2 y} + E_{18}e^{(R-\pi)y} + E_{19}e^{(R+\pi)y} + E_{20}e^{(R+E_1)y} + E_{21}e^{(R+E_2)y} + \left(E_{26}e^{E_1 y} + E_{27}e^{E_1 y} + \frac{E_{28}}{E_1 - E_2} y e^{E_1 y} + \frac{E_{29}}{E_2 - E_1} y e^{E_2 y} + \frac{E_{30}}{E_{22}} e^{(R-\pi)y} + \frac{E_{31}}{E_{23}} e^{(R+\pi)y} + \frac{E_{32}}{E_{24}} e^{(R+E_1)y} + \frac{E_{33}}{E_{24}} e^{(R+E_2)y} + \frac{E_{34}}{E_{22}} \left(y - \frac{R-2\pi}{E_{22}} \right) e^{(R-\pi)y} + \frac{E_{35}}{E_{23}} \left(y - \frac{R+2\pi}{E_{23}} \right) e^{(R+\pi)y} + \frac{E_{36}}{E_{24}} \left(y - \frac{R+2E_1}{E_{24}} \right) e^{(R+E_1)y} + \frac{E_{37}}{E_{25}} \left(y - \frac{R+2E_2}{E_{25}} \right) e^{(R+E_2)y} \right) \right) \cos \pi z \quad (63)$$

4.3. Shear Stress Components

The shear stress components F_1 and F_2 along the x - and z -axes, respectively, are

$$\left. \begin{aligned} \tau_x &= \left(\frac{du_0}{dy} \right)_{y=0} + \varepsilon \left(\frac{du_{11}}{dy} \right)_{y=0} \cos \pi z \\ \tau_z &= \frac{-\varepsilon}{\pi} \left(\frac{dv'_{11}}{dy} \right)_{y=0} \sin \pi z \\ F_1 &= \left(\frac{du_{11}}{dy} \right)_{y=0}, F_2 = \frac{-1}{\pi} \left(\frac{dv'_{11}}{dy} \right)_{y=0} \end{aligned} \right\} \quad (64)$$

Hence in view of Equations (63) and (38), Equation (64) yields

$$F_1 = E_1 E_{16} + E_2 E_{17} + E_{20} (E_1 + R) + E_{21} (E_2 + R) + E_{18} (R - \pi) + E_{19} (R + \pi) + K \left(\begin{aligned} &E_1 E_{26} + E_2 E_{27} + \frac{E_{28} e^{E_1}}{E_1 - E_2} + \frac{E_{29} e^{E_2}}{E_2 - E_1} + \frac{E_{30} (R - \pi)}{E_{22}} \\ &+ \frac{E_{31} (R + \pi)}{E_{23}} + \frac{E_{32} (R + E_1)}{E_{24}} + \frac{E_{33} (R + E_2)}{E_{25}} + \\ &\frac{E_{34}}{E_{22}} \left(1 - \frac{(R - \pi)(R - 2\pi)}{E_{22}} \right) + \frac{E_{35}}{E_{23}} \left(1 - \frac{(R + \pi)(R + 2\pi)}{E_{23}} \right) + \\ &\frac{E_{36}}{E_{24}} \left(1 - \frac{(R + E_1)(R + 2E_1)}{E_{24}} \right) + \frac{E_{37}}{E_{25}} \left(1 - \frac{(R + E_2)(R + 2E_2)}{E_{25}} \right) \end{aligned} \right) \quad (65)$$

$$F_2 = -\frac{1}{\pi} \left(\begin{aligned} &E_3 \pi^2 + E_4 \pi^2 + E_5 E_1 + E_6 E_2 + \\ &K (E_7 \pi^2 + E_8 \pi^2 + E_9 E_1 + E_{10} E_2 + 2E_{11} E_1 + 2E_{12} E_2) \end{aligned} \right) \quad (66)$$

4.4. Pressure

Inserting value from Equation (36) into the Equation (27) and simplifying the resulting equation to get

$$p_{11}(y) = E_{38} e^{-\pi y} + E_{39} e^{\pi y} + E_{40} e^{E_1 y} + E_{41} e^{E_2 y} + K \left(\begin{aligned} &\frac{E_{42} e^{-\pi y}}{E_{45} e^{E_2 y}} + \frac{E_{43} e^{\pi y}}{y} + \frac{E_{44} e^{E_1 y}}{E_{46} e^{E_1 y}} + \frac{E_{47} e^{E_2 y}}{E_{47} e^{E_2 y}} \end{aligned} \right) \quad (67)$$

The constants $E_{16}, E_{17}, \dots, E_{47}$ are given in Appendix A.

5. Results

In this article, a time-independent, fully established, incompressible, laminar Couette flow of a second-grade fluid with periodic injection/suction is formulated and scrutinized. The lower plate is fixed whereas upper moves with constant velocity U in the +ve x -direction. The crosswise periodic suction of the fluid through the above plate while its equivalent removal by periodic injection at the fixed plate is considered. Due to variable crosswise periodic injection/suction velocity on the plates, the flow turns to be three-dimensional. The influences of nondimensional parameters occurring in mathematical model of the governing equations on velocity field, pressure and skin friction components are visualized graphically.

The main flow velocity component is shown in Figures 2–4. The effects of injection/suction parameter α and Reynolds number Re are presented in Figures 2 and 4, respectively. The reduction in velocity component u is noted when injection/suction parameter or Reynolds number are increased. For maximum value of Reynolds number or injection/suction there is more decay. The maximum and minimum velocities occur at the plates, which are actually the velocities of the plates. Figure 3 depicts the effect of non-Newtonian parameter K on the main flow velocity component u . This figure reveals that the main flow velocity component increases exponentially with the increase of K . The impact of α , K , and Re on velocity component v are demonstrated in Figures 5–7 respectively. It is found from Figure 5 that this velocity component rises with an increase in α , the injection/suction parameter. In other words, injection/suction parameter gives a tool to boost the velocity component v . Moreover, for $\alpha \leq 0.4$, the velocity profile behaves as a linear function, of course, for $\alpha = 0.7$, the velocity profile

is parabolic. Figure 6 illustrates the effect of non-Newtonian parameter K on the cross flow velocity component v . The velocity component decreases with an increase in K , which was expected naturally. The velocity profiles are symmetric about the mid of the plates. Similar effect of Reynolds number Re on the velocity is observed from the Figure 7.

Figure 2. Effect of α on main flow velocity u along y .

Figure 3. Effect of K on main flow velocity u along y .

Figure 4. Effect of Re on main flow velocity u along y .

Figure 5. Effect of α on cross-flow velocity v along y .

Figure 6. Effect of K on cross-flow velocity v along y .

Figure 7. Effect of Re on cross-flow velocity v along y .

In Figures 8–10, the velocity component w is demonstrated for various values of α , K , and Re . The forward flow is observed for $y = 0$ to $y = 0.5$, and then, afterward, there is backward flow. Actually, the dragging impact of the quicker layer employed on the fluid elements near the stationary plate is adequate to overawed the adverse pressure gradient, and therefore there is forward flow. On the other hand, due to the application of periodic suction at the moving plate, the dragging influence of the quicker layer employed on the fluid elements will be abridged, and therefore this dragging result is deficient to overwhelm the adverse pressure gradient and there is backflow. It is also witnessed that the backflow is only the optical image of the forward flow. It is clear from the Figure 8, that the velocity component w rises with growth of α in forward flow, though a reverse consequence is gotten in the

backward flow. On the other hand, the velocity component w declines with an increase in either K or Re in forward flow a rises in back flow.

Figure 8. Effect of α on cross-flow velocity w along y .

Figure 9. Effect of K on cross-flow velocity w along y .

Figure 10. Effect of Re on cross-flow velocity w along y .

The skin friction components along main and secondary flow directions versus Reynolds number Re at the lower plate are presented in Figures 11–14. Figure 11 depicts the effect of injection/suction on the skin friction component F_1 . Depending upon the value of α , the magnitude of F_1 increases for small value of Reynolds number and then decreases for large values of Reynolds numbers and tends to zero. However, the magnitude of skin friction component F_2 decreases initially (small value of Reynolds numbers), and then behaves as a constant function (Figure 13). It is found that magnitude of skin friction components have accelerating behavior by increasing non-Newtonian parameter K which was expected naturally (Figure 12). It approaches to zero for $Re \geq 150$. It is evident from Figure 14 that F_2 decreases with an increase in K . However, it attains constant value for large of Re which depends on the value of α taken in this regard.

Figure 11. Effect of α on F_1 along Re at $K = 0.1$.

Figure 12. Effect of K on F_1 along Re at $\alpha = 0.1$.

Figure 13. Effect of α on F_2 along Re at $K = 0.1$.

Figure 14. Effect of K on F_2 along Re at $\alpha = 0.7$.

The influences of α , K , and Re on pressure are shown in Figures 15–17, respectively. It is prominent from Figure 15 that with an increase in α adverse pressure rises near the fixed plate, of course, near the moving plate the favourable pressure rises. It seems that the motion of the plate and suction at the plate together provides a mechanism to grow the favourable pressure. Figure 16 specifies that pressure declines with increasing K which was predictable certainly. Figure 17 displays that there is fall in adverse pressure from $y = 0$ to round $y = 0.5$, and then, straight on, there is enrichment in favourable pressure.

Figure 15. Effect of α on pressure p along y .

Figure 16. Effect of K on pressure p along y .

Figure 17. Effect of Re on pressure p along y .

6. Closing Remarks

In the light of above discussion one can conclude the following:

- The Reynolds number and injection/suction parameter cause a reduction in main flow velocity. On the contrary, elastic parameter enhances main flow velocity.
- The transverse velocity components v and w have increasing trend with increasing injection/suction parameter. On the other hand, an opposite trend of these velocity components is noted with Reynolds number and non-Newtonian parameter.
- The elastic parameter enhances the main flow velocity profile.
- Reynolds number provides a system to stable the shear stress components.
- This study gives a better outcome as changing injection/suction velocity is assumed at both plates as in natural practice injection/suction cannot be constant in all cases.
- Viscous results in the absence of Magnetohydrodynamic (MHD) effect [12] are recovered when $K = 0$.

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Appendix A

Constants involved in this paper are

$$E_1 = \frac{R - \sqrt{R^2 + 4\pi^2}}{2}, E_2 = \frac{R + \sqrt{R^2 + 4\pi^2}}{2} \quad (A1)$$

$$E_3 = (e^\pi(-e^{E_1}E_1E_2 + e^{E_2}E_1E_2 + e^{E_1+\pi}E_1E_2 - e^{E_2+\pi}E_1E_2 + e^{E_1}E_1\pi - e^{E_1+E_2}E_1\pi - e^\pi E_1\pi + e^{E_2+\pi}E_1\pi - e^{E_2}E_2\pi + e^{E_1+E_2}E_2\pi + e^\pi E_2\pi - e^{E_1+\pi}E_2\pi)\alpha / (-e^{E_1}E_1E_2 + e^{E_2}E_1E_2 + e^{E_1+2\pi}E_1E_2 - e^{E_2+2\pi}E_1E_2 + e^{E_1}E_1\pi + e^{E_2}E_1\pi - 2e^\pi E_1\pi - 2e^{E_1+E_2+\pi}E_1\pi + e^{E_1+2\pi}E_1\pi + e^{E_2+2\pi}E_1\pi - e^{E_1}E_2\pi - e^{E_2}E_2\pi + 2e^\pi E_2\pi + 2e^{E_1+E_2+\pi}E_2\pi - e^{E_1+2\pi}E_2\pi - e^{E_2+2\pi}E_2\pi + e^{E_1}\pi^2 - e^{E_2}\pi^2 - e^{E_1+2\pi}\pi^2 + e^{E_2+2\pi}\pi^2) \quad (A2)$$

$$E_4 = -\left(\left(\begin{array}{l} e^{E_1}E_1E_2 - e^{E_2}E_1E_2 - e^{E_1+\pi}E_1E_2 + e^{E_2+\pi}E_1E_2 + E_1\pi - e^{E_2}E_1\pi - \\ e^{E_1+\pi}E_1\pi + e^{E_1+E_2+\pi}E_1\pi - E_2\pi + e^{E_1}E_2\pi + e^{E_2+\pi}E_2\pi - e^{E_1+E_2+\pi}E_2\pi \end{array} \right) \alpha \right) / \left(\begin{array}{l} -e^{E_1}E_1E_2 + e^{E_2}E_1E_2 + e^{E_1+2\pi}E_1E_2 - e^{E_2+2\pi}E_1E_2 + e^{E_1}E_1\pi + e^{E_2}E_1\pi - \\ 2e^\pi E_1\pi - 2e^{E_1+E_2+\pi}E_1\pi + e^{E_1+2\pi}E_1\pi + e^{E_2+2\pi}E_1\pi - e^{E_1}E_2\pi - e^{E_2}E_2\pi + \\ 2e^\pi E_2\pi + 2e^{E_1+E_2+\pi}E_2\pi - e^{E_1+2\pi}E_2\pi - e^{E_2+2\pi}E_2\pi \\ + e^{E_1}\pi^2 - e^{E_2}\pi^2 - e^{E_1+2\pi}\pi^2 + e^{E_2+2\pi}\pi^2 \end{array} \right) \quad (A3)$$

$$E_5 = -\left(\left(\begin{array}{l} -E_2\pi - e^{E_2}E_2\pi + 2e^\pi E_2\pi - e^{2\pi}E_2\pi + \\ 2e^{E_2+\pi}E_2\pi - e^{E_2+2\pi}E_2\pi + \pi^2 - e^{E_2}\pi^2 - e^{2\pi}\pi^2 + e^{E_2+2\pi}\pi^2 \end{array} \right) \alpha \right) / \left(\begin{array}{l} e^{E_1}E_1E_2 - e^{E_2}E_1E_2 - e^{E_1+2\pi}E_1E_2 + e^{E_2+2\pi}E_1E_2 - e^{E_1}E_1\pi - e^{E_2}E_1\pi + \\ 2e^\pi E_1\pi + 2e^{E_1+E_2+\pi}E_1\pi - e^{E_1+2\pi}E_1\pi - e^{E_2+2\pi}E_1\pi + e^{E_1}E_2\pi + e^{E_2}E_2\pi - \\ 2e^\pi E_2\pi - 2e^{E_1+E_2+\pi}E_2\pi + e^{E_1+2\pi}E_2\pi + e^{E_2+2\pi}E_2\pi - e^{E_1}\pi^2 + \\ e^{E_2}\pi^2 + e^{E_1+2\pi}\pi^2 - e^{E_2+2\pi}\pi^2 \end{array} \right) \quad (A4)$$

$$E_6 = -\left(\pi \left(\frac{-E_1 - e^{E_1} E_1 + 2e^{\pi} E_1 - e^{2\pi} E_1 + 2e^{E_1+\pi} E_1 -}{e^{E_1+2\pi} E_1 + \pi - e^{E_1} \pi - e^{2\pi} \pi + e^{E_1+2\pi} \pi} \right) \alpha \right) / \left(\begin{array}{l} -e^{E_1} E_1 E_2 + e^{E_2} E_1 E_2 + e^{E_1+2\pi} E_1 E_2 - e^{E_2+2\pi} E_1 E_2 + e^{E_1} E_1 \pi + \\ e^{E_2} E_1 \pi - 2e^{\pi} E_1 \pi - 2e^{E_1+E_2+\pi} E_1 \pi + e^{E_1+2\pi} E_1 \pi + e^{E_2+2\pi} E_1 \pi - \\ e^{E_1} E_2 \pi - e^{E_2} E_2 \pi + 2e^{\pi} E_2 \pi + 2e^{E_1+E_2+\pi} E_2 \pi - e^{E_1+2\pi} E_2 \pi - \\ e^{E_2+2\pi} E_2 \pi + e^{E_1} \pi^2 - e^{E_2} \pi^2 - e^{E_1+2\pi} \pi^2 + e^{E_2+2\pi} \pi^2 \end{array} \right) \quad (A5)$$

$$E_7 = -(e^{\pi}(e^{E_1} E_{11} - e^{\pi} E_{11} - e^{E_1} E_1 E_{11} + e^{E_1} E_{12} - e^{\pi} E_{12} - e^{E_2} E_1 E_{12} + e^{E_1} E_{11} \pi + e^{E_2} E_{12} \pi)) / (-E_1 + e^{2\pi} E_1 + \pi + e^{2\pi} \pi - 2e^{E_1+\pi} \pi) - (e^{\pi}(e^{E_2} E_1 - e^{\pi} E_1 - e^{E_1} E_2 + e^{\pi} E_2 + e^{E_1} \pi - e^{E_2} \pi)((2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi)) (2(-e^{E_1}(1 + E_1)E_{11} - e^{E_2}(1 + E_2)E_{12})\pi - (-E_{11} - E_{12})(e^{-\pi}\pi + e^{\pi}\pi)) - (-(-e^{-\pi} + e^{\pi})(-E_{11} - E_{12}) + 2(-e^{E_1} E_{11} - e^{E_2} E_{12})\pi)(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) / ((E_1 - e^{2\pi} E_1 - \pi - e^{2\pi} \pi + 2e^{E_1+\pi} \pi)(-2(e^{E_2} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_2 + \pi))(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)) + (2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(2\pi(e^{E_2} E_2 + e^{-\pi}\pi) - (E_2 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) \quad (A6)$$

$$E_8 = -(E_{11} - e^{E_1+\pi} E_{11} + e^{E_1+\pi} E_1 E_{11} + E_{12} - e^{E_1+\pi} E_{12} + e^{E_2+\pi} E_1 E_{12} + e^{E_1+\pi} E_{11} \pi + e^{E_2+\pi} E_{12} \pi) / (-E_1 + e^{2\pi} E_1 + \pi + e^{2\pi} \pi - 2e^{E_1+\pi} \pi) - ((E_1 - e^{E_2+\pi} E_1 - E_2 + e^{E_1+\pi} E_2 + e^{E_1+\pi} \pi - e^{E_2+\pi} \pi)((2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi)) (2(-e^{E_1}(1 + E_1)E_{11} - e^{E_2}(1 + E_2)E_{12})\pi - (-E_{11} - E_{12})(e^{-\pi}\pi + e^{\pi}\pi)) - (-(-e^{-\pi} + e^{\pi})(-E_{11} - E_{12}) + 2(-e^{E_1} E_{11} - e^{E_2} E_{12})\pi)(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) / ((E_1 - e^{2\pi} E_1 - \pi - e^{2\pi} \pi + 2e^{E_1+\pi} \pi)(-2(e^{E_2} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_2 + \pi))(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)) + (2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(2\pi(e^{E_2} E_2 + e^{-\pi}\pi) - (E_2 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) \quad (A7)$$

$$E_9 = -(-E_{11} + e^{2\pi} E_{11} - E_{12} + e^{2\pi} E_{12} - 2e^{E_1+\pi} E_{11} \pi - 2e^{E_2+\pi} E_{12} \pi) / (-E_1 + e^{2\pi} E_1 + \pi + e^{2\pi} \pi - 2e^{E_1+\pi} \pi) - ((2(e^{E_2} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_2 + \pi))((2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi)) (2(-e^{E_1}(1 + E_1)E_{11} - e^{E_2}(1 + E_2)E_{12})\pi - (-E_{11} - E_{12})(e^{-\pi}\pi + e^{\pi}\pi)) - (-(-e^{-\pi} + e^{\pi})(-E_{11} - E_{12}) + 2(-e^{E_1} E_{11} - e^{E_2} E_{12})\pi)(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) / ((2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(-2(e^{E_2} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_2 + \pi))(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)) + (2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(2\pi(e^{E_2} E_2 + e^{-\pi}\pi) - (E_2 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) \quad (A8)$$

$$E_{10} = ((2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(2(-e^{E_1}(1 + E_1)E_{11} - e^{E_2}(1 + E_2)E_{12})\pi - (-E_{11} - E_{12})(e^{-\pi}\pi + e^{\pi}\pi)) - (-(-e^{-\pi} + e^{\pi})(-E_{11} - E_{12}) + 2(-e^{E_1} E_{11} - e^{E_2} E_{12})\pi)(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) / ((-2(e^{E_2} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_2 + \pi))(2\pi(e^{E_1} E_1 + e^{-\pi}\pi) - (E_1 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)) + (2(e^{E_1} - e^{-\pi})\pi - (-e^{-\pi} + e^{\pi})(E_1 + \pi))(2\pi(e^{E_2} E_2 + e^{-\pi}\pi) - (E_2 + \pi)(e^{-\pi}\pi + e^{\pi}\pi)))) \quad (A9)$$

$$E_{11} = \frac{-RE_1 E_5 (E_1 - \pi^2)}{(E_1 - E_2)}, E_{12} = \frac{-RE_2 E_6 (E_2 - \pi^2)}{(E_2 - E_1)}, E_{13} = \frac{E_{15} e^R}{e^R - 1}, E_{14} = \frac{-E_{15} e^R}{e^R - 1}, E_{15} = \frac{-R^3}{e^R - 1} \quad (A10)$$

$$E_{16} = \frac{1}{e^{E_1} - e^{E_2}} \left(\begin{array}{l} E_{18}(e^{E_2} - e^{R-\pi}) + E_{19}(e^{E_2} - e^{R+\pi}) + \\ E_{20}(e^{E_2} - e^{R+E_1}) + E_{21}(e^{E_2} - e^{R+E_2}) \end{array} \right) \quad (A11)$$

$$E_{17} = \frac{1}{e^{E_2} - e^{E_1}} \left(\begin{array}{l} E_{18}(e^{E_1} - e^{R-\pi}) + E_{19}(e^{E_1} - e^{R+\pi}) + \\ E_{20}(e^{E_1} - e^{R+E_1}) + E_{21}(e^{E_1} - e^{R+E_2}) \end{array} \right)$$

$$E_{18} = \frac{\text{Re} E_3 R}{(e^R - 1) E_{22}}, E_{19} = \frac{\text{Re} E_4 R}{(e^R - 1) E_{23}}, E_{20} = \frac{\text{Re} E_5 R}{(e^R - 1) E_{24}}, E_{21} = \frac{\text{Re} E_6 R}{(e^R - 1) E_{25}} \quad (A12)$$

$$\begin{aligned} E_{22} &= (R - \pi - E_1)(R - \pi - E_2), \quad E_{23} = (R + \pi - E_1)(R + \pi - E_2) \\ E_{24} &= R(R + E_1 - E_2), \quad E_{25} = (R + E_2 - E_1)R \end{aligned} \quad (A13)$$

$$E_{26} = \frac{1}{e^{E_1} - e^{E_2}} \left(\begin{aligned} &\frac{E_{28}}{E_2 - E_1} e^{E_1} + \frac{E_{29}}{E_1 - E_2} e^{E_2} + \frac{E_{30}}{E_{22}} (e^{E_2} - e^{R - \pi}) + \\ &\frac{E_{31}}{E_{23}} (e^{E_2} - e^{R + \pi}) + \frac{E_{32}}{E_{24}} (e^{E_2} - e^{R + E_1}) + \frac{E_{33}}{E_{25}} (e^{E_2} - e^{R + E_2}) \\ &- \frac{E_{34}}{E_{22}} \left(\frac{2(R - \pi) - R}{E_{22}} e^{E_2} + \left(1 - \frac{2(R - \pi) - R}{E_{22}} \right) e^{R - \pi} \right) \\ &- \frac{E_{35}}{E_{23}} \left(\frac{2(R + \pi) - R}{E_{23}} e^{E_2} + \left(1 - \frac{2(R + \pi) - R}{E_{23}} \right) e^{R + \pi} \right) \\ &- \frac{E_{36}}{E_{24}} \left(\frac{2(R + E_1) - R}{E_{24}} e^{E_2} + \left(1 - \frac{2(R + E_1) - R}{E_{24}} \right) e^{R + E_1} \right) \\ &- \frac{E_{37}}{E_{25}} \left(\frac{2(R + E_2) - R}{E_{25}} e^{E_2} + \left(1 - \frac{2(R + E_2) - R}{E_{25}} \right) e^{R + E_2} \right) \end{aligned} \right) \quad (A14)$$

$$E_{27} = \frac{1}{e^{E_2} - e^{E_1}} \left(\begin{aligned} &\frac{E_{28}}{E_2 - E_1} e^{E_1} + \frac{E_{29}}{E_1 - E_2} e^{E_2} + \frac{E_{30}}{E_{22}} (e^{E_1} - e^{R - \pi}) + \\ &\frac{E_{31}}{E_{23}} (e^{E_1} - e^{R + \pi}) + \frac{E_{32}}{E_{24}} (e^{E_1} - e^{R + E_1}) + \frac{E_{33}}{E_{25}} (e^{E_1} - e^{R + E_2}) \\ &- \frac{E_{34}}{E_{22}} \left(\frac{2(R - \pi) - R}{E_{22}} e^{E_1} + \left(1 - \frac{2(R - \pi) - R}{E_{22}} \right) e^{R - \pi} \right) \\ &- \frac{E_{35}}{E_{23}} \left(\frac{2(R + \pi) - R}{E_{23}} e^{E_1} + \left(1 - \frac{2(R + \pi) - R}{E_{23}} \right) e^{R + \pi} \right) \\ &- \frac{E_{36}}{E_{24}} \left(\frac{2(R + E_1) - R}{E_{24}} e^{E_1} + \left(1 - \frac{2(R + E_1) - R}{E_{24}} \right) e^{R + E_1} \right) \\ &- \frac{E_{37}}{E_{25}} \left(\frac{2(R + E_2) - R}{E_{25}} e^{E_1} + \left(1 - \frac{2(R + E_2) - R}{E_{25}} \right) e^{R + E_2} \right) \end{aligned} \right) \quad (A15)$$

$$\begin{aligned} E_{28} &= RE_{16}E_1(\pi^2 - E_1), \quad E_{29} = RE_{17}E_2(\pi^2 - E_2) \\ E_{30} &= \operatorname{Re}E_3E_{14}R + RE_{18}(R - \pi)(\pi^2 - (R - \pi)^2) + \operatorname{Re} \left(\frac{E_3E_{15} + \frac{R}{e^{R-1}}(E_7 - E_3R)}{e^{R-1}} \right) \\ E_{31} &= \operatorname{Re}E_4E_{14}R + RE_{19}(R + \pi)(\pi^2 - (R + \pi)^2) + \operatorname{Re} \left(\frac{E_4E_{15} + \frac{R}{e^{R-1}}(E_8 - E_4R)}{e^{R-1}} \right) \end{aligned} \quad (A16)$$

$$\begin{aligned} E_{32} &= \operatorname{Re}E_5E_{14}R + RE_{20}(R + E_1)(\pi^2 - (R + E_1)^2) + \operatorname{Re} \left(\frac{E_5E_{15} + \frac{R}{e^{R-1}}(E_9 - E_5R)}{e^{R-1}} \right) \\ E_{33} &= \operatorname{Re}E_6E_{14}R + RE_{21}(R + E_2)(\pi^2 - (R + E_2)^2) + \operatorname{Re} \left(\frac{E_6E_{15} + \frac{R}{e^{R-1}}(E_{10} - E_6R)}{e^{R-1}} \right) \\ E_{34} &= \operatorname{Re}RE_3E_{15}, \quad E_{35} = \operatorname{Re}RE_4E_{15}, \quad E_{36} = \operatorname{Re}R(E_5E_{15} + \frac{E_{11}}{e^{R-1}}) \end{aligned} \quad (A17)$$

$$\begin{aligned} E_{37} &= \operatorname{Re}R(E_6E_{15} + \frac{E_{12}}{e^{R-1}}), \quad E_{38} = -\alpha E_3, \quad E_{39} = -\alpha E_4 \\ E_{40} &= \frac{E_5E_1}{\pi^2} \left(\frac{1}{\operatorname{Re}}(E_1 - \pi^2) - \alpha E_1 \right), \quad E_{41} = \frac{E_6E_2}{\pi^2} \left(\frac{1}{\operatorname{Re}}(E_2 - \pi^2) - \alpha E_2 \right) \\ E_{42} &= -\alpha E_7, \quad E_{43} = -\alpha E_8 \\ E_{44} &= \frac{1}{\pi^2} \left(\begin{aligned} &\alpha E_5E_1(E_1 - \pi^2) + \frac{1}{\operatorname{Re}}(E_9E_1 + 3E_{11}E_1 - \pi^2(E_9E_1 + E_{11})) \\ &- \alpha E_1(E_9E_1 + 2E_{11}) \end{aligned} \right) \\ E_{45} &= \frac{1}{\pi^2} \left(\begin{aligned} &\alpha E_6E_2(E_2 - \pi^2) + \frac{1}{\operatorname{Re}}(E_{10}E_2 + 3E_{12}E_2 - \pi^2(E_{10}E_2 + E_{12})) \\ &- \alpha E_2(E_{10}E_2 + 2E_{12}) \end{aligned} \right) \\ E_{46} &= \frac{E_{11}E_1}{\pi^2} \left(\frac{1}{\operatorname{Re}}(E_1 - \pi^2) - \alpha E_1 \right), \quad E_{47} = \frac{E_{12}E_2}{\pi^2} \left(\frac{1}{\operatorname{Re}}(E_2 - \pi^2) - \alpha E_2 \right) \end{aligned} \quad (A18)$$

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