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Finite Element Analysis of Variable Viscosity Impact on MHD Flow and Heat Transfer of Nanofluid Using the Cattaneo–Christov Model

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Abstract: In this mathematical study, magnetohydrodynamic, time-independent nanofluid flow over a stretching sheet by using the Cattaneo–Christov heat flux model is inspected. The impact of the thermal, solutal boundary and gravitational body forces with the effect of double stratification on the mass flow and heat transfer phenomena is also observed. The temperature-dependent viscosity impact on heat transfer through a moving sheet with capricious heat generation in nanofluids have studied, and the viscosity of the fluid is presumed to deviate as the inverse function of temperature. With the appropriate transformations, the system of partial differential equations is transformed into a system of nonlinear ordinary differential equations. By applying the variational finite element method, the transformed system of equations is solved. The properties of the several parameters for buoyancy, velocity, temperature, stratification, and Brownian motion parameters have examined. The enhancement in the concentration and thermal boundary layer thickness of the nanofluid sheet due to the increment in the viscosity parameter, also increased the temperature and concentration of nanoparticles. Moreover, the fluid temperature declined with the increasing values of thermal relaxation parameter. This displays that the Cattaneo–Christov heat flux model provides a better assessment of temperature distribution. Moreover, confirmation of the code and precision of the numerical method has inveterate with the valuation of the presented results with previous studies.

Keywords: variable viscosity; stratification; FEM; nanofluids; MHD; heat transfer

1. Introduction

In recent years, the dynamic research work associated with the problems of heat transfer has based on constant physical characteristics of ambient fluids. However, it is known that these characteristics might be variate with temperature, particularly for fluid viscosity. For the accurate predictions of heat transfer and flow rates, it is compulsory to take into consideration this alteration in the viscosity. The flow through a stretching sheet is a momentous problem in various engineering processes with applications in different industries, such as melt-spinning, glass fiber manufacturing, rubber sheet, plastic manufacturing, wire drawing, and metallic plates cooling in baths. Sakiadis [1] and Crane [2] described the stretching problems and fluid flow with constant temperature. Grubka, and Bobba [3] have evaluated the stretching problems for the surfaces moving with a linear velocity and with variant temperature. Banks et al. [4] inspected the flow field of a stretching wall with the power-law velocity variation. El-Hakiem [5] has deliberated the impacts of transverse flow along with variable viscosity in a micropolar fluid. Elbashbeshy et al. [6] premeditated the consequences of temperature-dependent viscosity on heat transfer through an incessant moving sheet.

In view of all the abovementioned studies, the viscidness of the fluid has been supposed to be fixed. Although it is well known that this physical property might be variate expressively along with the temperature, to precisely visualize the flow demeanor, it is compulsory to know this disparity of the viscosity. In earlier times, water was used as the refrigerator of fluid, but then, it did not appear that water was an impeccable refrigeration fluid after scrutinizing the required properties of the final products. Furthermore, many researchers have investigated various physical parameters such as magnetic field, injection/suction, production of heat, thermal radiation, and different types of fluids. Christov [7] added the heat flux time-derivative and reformulated the law in a hyperbolic energy equation. Recently, many researchers in the current literature on the boundary layer flow of the stretching sheet have used the non-Fourier energy equation. Salahuddin et al. [8] explored the temperature profile for the magnetohydrodynamic flow of Williamson fluid by fluctuating thickness, which displayed a decrement by the heat flux model associated with the conventional energy model. Hayat et al. [9] deliberated the stagnation point flow of Jeffrey fluid towards a nonlinearly stretching sheet. The fluid temperature reduced with the growing values of the thermal relaxation parameter. Malik et al. [10] inspected the magnetohydrodynamic Casson fluid flow and recognized that the temperature depreciated with the non-Fourier energy model. Rehman et al. [11] scrutinized the Buongiorno model and exposed the characteristics of the heat transfer and forced convective flow in the permeable stretching plate in the existence of second-order slip conditions at the interface. The Cattaneo–Christov model was accomplished by Zampoli and Tibullo [12] for an incompressible fluid. Lie et al. [13] found that the influence of the Cattaneo–Christov model for Maxwell hydro-magnetic fluid originated from the vertically moving sheet.

Nanofluids are a modern type of heat transfer fluids that have been subject to emerging research in different and innovative extents of the exploration. Nanofluids are perceived as boosting the thermal conductivity of the base fluid. The transference of heat is an elementary feature of the enormous utilization via applications that are subjected to the thermal conductivity of operational liquids as a procedure to determine the capability of thermal utilization and erection. There are a lot of periodicals on nanofluids that are apprehensive of identifying their behaviors as this could be exploited when the transfer of heat enrichment intervenes, as in quite a lot of industrial utilizations like transportation, nuclear-power reactors, electricity, as well as in bio-medicine and nutrition. Nanofluids have been confirmed where heat transfer might be truncated or enlarged. Khan et al. [14] have examined the nonlinear radiation effects on the magnetohydrodynamics flow of nanofluid through a nonlinear stretching/shrinking wedge. Bagh et al. [15] scrutinized the effect of unsteady magnetohydrodynamic nanofluid flow through a radially nonlinear stretching sheet with a viscosity-dependent temperature, convective boundary conditions, and thermo-diffusion. Alsarraf et al. [16] have explored the effect of a nanoadditive shape on the fluid flow and heat transfer aspects of γ -AlOOH nano-fluid flowing through a sinusoidal wavy channel. Sarafranz et al. [17] computed the heat transfer coefficient (HTC) of Fe₃O₄ aqueous nano-suspension at several mass concentrations of 0.05–0.2%. Asadi et al. [18] deliberated the recent advances in the preparation methods and thermophysical properties measurements of oil-based nanofluids. Pourmehran et al. [19] have investigated the nanofluid flow and heat transfer between two contracting and rotating disks. Liaqat et al. [20] described the effect of a magnetic dipole on the heat transfer phenomenon of various nanoparticles on the micropolar fluid flow through a stretching sheet. Yulin et al. [21] investigated the natural convective heat transfer of water/Al₂O₃ nano-fluid in an inclined square. Sarafranz et al. [22] represented the results of various experiments conducted on the thermal performance and efficiency of an evacuated tube solar collector (ETSC). Tian et al. [23] examined the nanofluid rheological behavior of nanoparticles CuO/MWCNTs incorporated with base fluid water/EG (70:30) at the temperature 20–60 °C. Alsarraf et al. [24] investigated the impact of the nanoparticle shape on the fluid flow features of boehmite alumina nanofluid in a horizontal double-pipe minichannel heat exchanger. Liaqat et al. [25] explored the influence of multiple slips and the solutal boundary condition on magnetohydrodynamic unsteady bioconvective micropolar

nanofluid-restrictive gyrotactic microbes, mass, and heat transference impact through the sheet. Ibrahim and Shankar [26] revealed that the heat transfer and the magnetohydrodynamic boundary layer flow of nanofluids via infiltration was able to stretch the sheet in velocity, thermal, and solutal-slip boundary conditions. Ranjbarzadeh et al. [27] in an experimental study, used an eco-friendly process to produce silica nanoparticles. Moreover, the surface and atomic structures of the nanoparticles were explored through SEM and FTIR tests. Sarafraz et al. [28] explored the flow boiling heat transfer properties of MgO/therminol nanofluid as a potential coolant on a copper-made disc. Moradikazerouni et al. [29] investigated the optimal artificial neural network (ANN) designed to evaluate the thermal conductivity enhancement of SWCNTs/EG-water nanofluid using experimental data. Das [30] investigated, through an imperative mathematical analysis on the convective heat transference narration of nanofluids through an infiltrate-able stretching sheet, the existence of partially slip, thermal buoyancy, and internal heat development/captivation. Sarafraz et al. [31] conducted an experimental study to quantify the heat transfer coefficient, thermal resistance and the thermal performance of a thermosyphon heat pipe charged with zirconia-acetone nanofluid. Sohaib et al. [32] investigated the multislip impacts of a magneto-hydrodynamic mixed convection unsteady flow of micropolar nanofluid through a stretching sheet on radiation and the heat source. Uddin et al. [33] deliberated whether free convective boundary layer flow of nanofluid through an infiltrate would be able to ascend a horizontal plate in permeable medium under thermal convective boundary conditions. Sarafraz et al. [34] explored the thermal conductivity, viscosity, and boiling heat transfer coefficient of a biologically produced nano-coolant.

The above literature remains limited regarding to investigations that is explored the influence of temperature-dependent viscosity on magneto-hydrodynamic boundary layer flow to analyze the heat transfer with double stratification through a stretching sheet in nanofluids. However, no studies have investigated the effect of temperature-dependent viscosity on magneto-hydrodynamic boundary layer nanofluid flow with double stratification using the Cattaneo–Christov heat flux model. The significance of this model is that it provides a better assessment of temperature distribution. The viscosity of the fluid is presumed to deviate as the inverse function of temperature and according to this framework, Reynold’s exponential viscosity model has been considered. The variational finite element technique has been applied to find the solution of transformed assembly equations, also using MATLAB software for programming. The aims of the present study is to scrutinize the behavior of several types of existing parameters and thermal conductivity clearly through detailed discussion. Furthermore, a numeral valuation of the existing results is displayed and reflected in figures.

2. Problem Description

Two-dimensional magnetohydrodynamic incompressible boundary layer nanofluid flow through electrically navigated elongating surface of a stretching sheet was investigated in this paper. Here, a two-coordinate system was chosen with the intention of the dimensions of the surface being designated along the x -axis and y -axis. $u_w = cx$ is the velocity of the sheet, as shown in Figure 1. The stretching gives the impression that the sheet is proportionate to the distance from the origin. With the assumption that the Reynolds number is very small, a uniform magnetic of strength B_0 has been applying perpendicularly to the sheet so that the induced magnetic field could be omitted. The wall temperature was T_w which is given as $T_w = T_0 + A_1x$, the concentration was $C_w = C_0 + A_2x$, where $C_w > C_0$ and $T_w > T_0$. The stratification temperature is $T_\infty = T_0 + B_1x$ and the concentration is given as $C_\infty = C_0 + B_2x$ where T_0 and C_0 are the area the ambient temperature and concentration of the nanofluid, respectively.

Figure 1. Configuration of Flow.

According to the boundary layer conventions, the principal equations for flow can be inscribed as [35];

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = [\frac{\partial}{\partial y} \{ \mu(T) \frac{\partial u}{\partial y} \}] - \sigma B_0^2 u + g \{ \rho f_{\infty} \beta_T (1 - C_{\infty}) (T - T_{\infty}) \} + g \beta_C \{ (\rho_p - \rho f_{\infty}) (C - C_{\infty}) \} \quad (2)$$

$$u(\frac{\partial T}{\partial x}) + v(\frac{\partial T}{\partial y}) + \delta T_{\delta} = \tau \{ D_B \frac{\partial C}{\partial y} + \frac{D_T}{T_{\infty}} \frac{\partial T}{\partial y} \} (\frac{\partial T}{\partial y}) + a (\frac{\partial^2 T}{\partial y^2}) \quad (3)$$

$$u(\frac{\partial C}{\partial x}) + v(\frac{\partial C}{\partial y}) = \frac{D_T}{T_{\infty}} \frac{\partial^2 T}{\partial y^2} + D_B \frac{\partial^2 C}{\partial y^2} \quad (4)$$

where

$$T_{\delta} = u^2 \frac{\partial^2 T}{\partial x^2} + \{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \} \frac{\partial T}{\partial x} + \{ v \frac{\partial v}{\partial y} + u \frac{\partial v}{\partial x} \} \frac{\partial T}{\partial y} + 2uv \frac{\partial^2 T}{\partial x \partial y} + v^2 \frac{\partial^2 T}{\partial y^2}$$

and $a = \{ \kappa / (\rho C_p) \}_f$, where κ is the thermal conductivity of the fluid and C_p is specific heat and the proportion of capacitance of nanoparticles to the warmness capacitance of base fluid is $\tau = \{ (\rho C_p)_p / (\rho C_p)_f \}$. The velocity compartments laterally to the x-axis and y-axis are u , and v , nanofluid temperature is T , the nanoparticles volume fraction is C , the thermal relaxation time is δ , the electric conductivity of fluid is σ , the dynamic viscosity is μ , ρ is the density of the base fluid, ρ_p is the nanoparticles' density, D_B is the Brownian diffusion coefficient, and thermal diffusibility is D_T . The boundary conditions of the above mathematical model are given as [35];

$$\begin{aligned} u(x, 0) = u_w(x), \quad v(x, 0) = 0, \quad T(x, 0) = T_w, \quad C(x, 0) = C_w, \quad \text{as } y = 0. \\ u(x, \infty) = 0, \quad T(x, \infty) = T_{\infty}, \quad C(x, \infty) = C_{\infty}, \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (5)$$

where T_0 , B_1 , and B_2 are the allusion temperature and the dimensionless quantities, respectively. The temperature T_w at the surface, T_c is the curie temperature and T_{∞} is the temperature of the fluid away from the surface. The viscosity of fluid in Equation (2), which is temperature-dependent and thus might fluctuate exponentially, can be mathematically expressed as [36] $\mu(T) = \mu_0 e^{\{-H(T-T_{\infty})\}}$ (Reynolds viscosity model), where H denotes its strength that depends on $\mu(T)$ and T , where μ_0 denotes the viscosity of the fluid at temperature T_{∞} . Normally, the stream function ψ is stated as, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. To solve the system of Equations (1)–(4) with boundary conditions (5), we substitute the similarity transformations as shown below [37];

$$\begin{aligned} \xi = y \{ (u_w) / v_x \}^{\frac{1}{2}}, \quad \psi = \tilde{f}(\xi) \{ (u_w) v_x \}^{\frac{1}{2}} \\ \tilde{\theta}(\xi) = (T - T_{\infty}) / (T_w - T_0), \quad \tilde{\phi}(\xi) = (C - C_{\infty}) / (C_w - C_0) \end{aligned} \quad (6)$$

The components of velocity for the stream function $\psi(\eta, \xi)$ are

$$u = \frac{\partial \psi}{\partial y} = cx\tilde{f}'(\xi), \quad v = -\frac{\partial \psi}{\partial x} = -\sqrt{cv}\tilde{f}(\xi) \quad (7)$$

By applying similarities (6) and (7) and then using Maclaurin's series, we attain the expression as [36]

$$e^{-\lambda\tilde{\theta}} = 1 - \lambda\tilde{\theta} + O(\lambda^2) \quad (8)$$

According to similarity transformations (6) and (7), the following dimensionless system is now accomplished.

$$(1 - \lambda\tilde{\theta})\frac{d^3\tilde{f}}{d\xi^3} - \lambda\frac{d\tilde{\theta}}{d\xi}\frac{d^2\tilde{f}}{d\xi^2} + \tilde{f}\frac{d^2\tilde{f}}{d\xi^2} - \left(\frac{d\tilde{f}}{d\xi}\right)^2 - M\frac{d\tilde{f}}{d\xi} + \lambda_1\tilde{\theta} + \lambda_2\tilde{\phi} = 0, \quad (9)$$

$$\frac{1}{Pr}\frac{d^2\tilde{\theta}}{d\xi^2} + \tilde{f}\frac{d\tilde{\theta}}{d\xi} - S_1\frac{d\tilde{f}}{d\xi} + Nb\frac{d\tilde{\theta}}{d\xi}\frac{d\tilde{\phi}}{d\xi} + Nt\left(\frac{d\tilde{\theta}}{d\xi}\right)^2 - \alpha_T\tilde{\theta}_{\alpha T} = 0, \quad (10)$$

$$\frac{d^2\tilde{\phi}}{d\xi^2} + PrLe\left\{\tilde{f}\frac{d\tilde{\phi}}{d\xi} - \tilde{\phi}\frac{d\tilde{f}}{d\xi} - S_2\frac{d\tilde{f}}{d\xi}\right\} + \{Nt/Nb\}\frac{d^2\tilde{\theta}}{d\xi^2} = 0, \quad (11)$$

where

$$\tilde{\theta}_{\alpha T} = \tilde{f}^2\frac{d^2\tilde{\theta}}{d\xi^2} - \tilde{f}\frac{d\tilde{f}}{d\xi}\frac{d\tilde{\theta}}{d\xi} + \tilde{\theta}\left(\frac{d\tilde{f}}{d\xi}\right)^2 + S_1\left(\frac{d\tilde{f}}{d\xi}\right)^2 - \tilde{f}\tilde{\theta}\frac{d^2\tilde{f}}{d\xi^2} - \tilde{f}S_1\frac{d^2\tilde{f}}{d\xi^2}$$

Moreover, the boundary conditions are

$$\begin{aligned} \tilde{f}(0) = 0, \quad \frac{d\tilde{f}(0)}{d\xi} = 1, \quad \tilde{\theta}(0) = 1 - S_1, \quad \text{and} \quad \tilde{\phi}(0) = 1 - S_2 \\ \frac{d\tilde{f}(\infty)}{d\xi} \rightarrow 0, \quad \tilde{\theta}(\infty) \rightarrow 0, \quad \text{and} \quad \tilde{\phi}(\infty) \rightarrow 0 \end{aligned} \quad (12)$$

The dimensionless parameters involved in Equations (9)–(11) are describes as

$$\begin{aligned} P_r &= \nu/\alpha, \quad Re_x = \frac{cx^2}{\nu}, \quad M = \sigma B_0^2/c\rho \\ S_1 &= B_1/A_1 \text{ and } S_2 = B_2/A_2, \quad \lambda_1 = Gr/Re_x^2 \\ \lambda_2 &= Gc/Re_x^2, \quad \alpha_T = c\sigma, \quad Le = \alpha/D_B \\ Nt &= \{\tau D_T(T_w - T_0)\}/\nu T_\infty \\ Nb &= \{\tau D_B(C_w - C_0)\}/\nu T_\infty \end{aligned}$$

where P_r is the Prandtl number, Re_x is the Reynolds number, M is the magnetic parameter, S_2 , and S_1 are the solutal and thermal stratification parameters, λ_1 is the thermal buoyancy parameters, λ_2 is the solutal buoyancy parameter, α_T is the thermal relaxation parameter, λ is the variable viscosity parameter, Le is the Lewis number, Nt is the thermophoresis parameter, and Nb is the Brownian motion parameter. Moreover,

$$\begin{aligned} Gr &= \{g\rho_f\beta_T(T - T_0)(1 - C_\infty)x^3\}/(\rho_f\nu^2) \\ Gc &= \{g\beta_C(\rho_p - \rho_{f\infty})(C_w - C_0)x^3\}/(\rho_f\nu^2) \end{aligned}$$

It can be noticed that in the case of the un-stratification of nanofluids, the system of Equations (9)–(11) is reduced to the model and the most substantial corporeal measurements are

the skin friction coefficient, the Nusselt number, the Sherwood number and the explanation of these dimensionless corporal measurements is given as (Akbar et al. [38])

$$C_f(Re_x)^{1/2} = \frac{d\tilde{f}(0)}{d\tilde{\zeta}}$$

$$Nu/(Re_x)^{1/2} = -\frac{d\tilde{\theta}(0)}{d\tilde{\zeta}}$$

$$Sh_x/(Re_x)^{1/2} = -\frac{d\tilde{\phi}(0)}{d\tilde{\zeta}}$$

3. Interpretation of Method

An efficient scheme, that is, the finite element technique, was executed to find the mathematical resolution of the nonlinear scheme of differential Equations (9)–(11) by using the boundary conditions (12). This method is more efficient and reliable than other numerical techniques, such as the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), and the finite-difference method (FDM). Additionally, it is very proficient and has been applied to study miscellaneous problems in fluid mechanics and computational fluid dynamics, solid mechanics, mass transfer, heat transfer, and in many other fields. The inclusive features of the finite element method are described by Reddy [39] and Gupta et al. Swapna et al. [40,41] described that with the finite element technique, a boundary value problem can be solved efficiently and accurately. To resolve the scheme of Equations (9)–(11), firstly we have to consider

$$\frac{d\tilde{f}}{d\tilde{\zeta}} = \tilde{g} \quad (13)$$

Equations (9)–(11) are converted into the following form:

$$(1 - \lambda\tilde{\theta}) \frac{d^2\tilde{g}}{d\tilde{\zeta}^2} - \lambda \frac{d\tilde{\theta}}{d\tilde{\zeta}} \frac{d\tilde{g}}{d\tilde{\zeta}} + \tilde{f} \frac{d\tilde{g}}{d\tilde{\zeta}} - \tilde{g}^2 - M\tilde{g} + \lambda_1\tilde{\theta} + \lambda_2\tilde{\phi} = 0, \quad (14)$$

$$\frac{1}{Pr} \frac{d^2\tilde{\theta}}{d\tilde{\zeta}^2} + \tilde{f} \frac{d\tilde{\theta}}{d\tilde{\zeta}} - S_1\tilde{g} + Nb \frac{d\tilde{\theta}}{d\tilde{\zeta}} \frac{d\tilde{\phi}}{d\tilde{\zeta}} + Nt \left(\frac{d\tilde{\theta}}{d\tilde{\zeta}} \right)^2 - \alpha_T \tilde{\theta}_{\alpha T} = 0, \quad (15)$$

$$\frac{d^2\tilde{\phi}}{d\tilde{\zeta}^2} + PrLe \left\{ \tilde{f} \frac{d\tilde{\phi}}{d\tilde{\zeta}} - \tilde{\phi}\tilde{g} - S_2\tilde{g} \right\} + \{Nt/Nb\} \frac{d^2\tilde{\theta}}{d\tilde{\zeta}^2} = 0, \quad (16)$$

The correspondent boundary conditions are abbreviated to the subsequent form:

$$\begin{aligned} \tilde{f}(0) = 0, \quad \tilde{g}(0) = 1, \quad \tilde{\theta}(0) = 1 - S_1, \quad \text{and} \quad \tilde{\phi}(0) = 1 - S_2 \\ \tilde{g}(\infty) \rightarrow 0, \quad \tilde{\theta}(\infty) \rightarrow 0, \quad \text{and} \quad \tilde{\phi}(\infty) \rightarrow 0 \end{aligned} \quad (17)$$

3.1. Variational Formulations

The variational form of Equations (13)–(16) over a typical two-nodded domain $(\tilde{\zeta}_b, \tilde{\zeta}_{b+1})$ can be described as

$$\int_{\tilde{\zeta}_b}^{\tilde{\zeta}_{b+1}} s_1 \left\{ \frac{d\tilde{f}}{d\tilde{\zeta}} - \tilde{g} \right\} d\tilde{\zeta} = 0, \quad (18)$$

$$\int_{\tilde{\zeta}_b}^{\tilde{\zeta}_{b+1}} s_2 \left\{ (1 - \lambda\tilde{\theta}) \frac{d^2\tilde{g}}{d\tilde{\zeta}^2} - \lambda \frac{d\tilde{\theta}}{d\tilde{\zeta}} \frac{d\tilde{g}}{d\tilde{\zeta}} + \tilde{f} \frac{d\tilde{g}}{d\tilde{\zeta}} - \tilde{g}^2 - M\tilde{g} + \lambda_1\tilde{\theta} + \lambda_2\tilde{\phi} \right\} d\tilde{\zeta} = 0, \quad (19)$$

$$\int_{\tilde{\zeta}_b}^{\tilde{\zeta}_{b+1}} s_3 \left\{ \frac{1}{Pr} \frac{d^2\tilde{\theta}}{d\tilde{\zeta}^2} + \tilde{f} \frac{d\tilde{\theta}}{d\tilde{\zeta}} - S_1\tilde{g} + Nb \frac{d\tilde{\theta}}{d\tilde{\zeta}} \frac{d\tilde{\phi}}{d\tilde{\zeta}} + Nt \left(\frac{d\tilde{\theta}}{d\tilde{\zeta}} \right)^2 - \alpha_T \tilde{\theta}_{\alpha T} \right\} d\tilde{\zeta} = 0, \quad (20)$$

$$\int_{\zeta_b}^{\zeta_{b+1}} s_4 \left\{ \frac{d^2 \tilde{\phi}}{d\zeta^2} + PrLe \left\{ \tilde{f} \frac{d\tilde{\phi}}{d\zeta} - \tilde{\phi} \tilde{g} - S_2 \tilde{g} \right\} + \{Nt/Nb\} \frac{d^2 \tilde{\theta}}{d\zeta^2} \right\} d\zeta = 0, \quad (21)$$

where s_1, s_2, s_3 , and s_4 are the arbitrary weight functions that may be observed as variation in $\tilde{f}, \tilde{g}, \tilde{\theta}, \tilde{\phi}$, respectively and the domain (ζ_b, ζ_{b+1}) represents the length of the boundary layer region.

3.2. Finite Element Formulation

Let the domain be described into the three-noded quadratic element of the finite element technique which is obtained from Equations (18)–(21) by bartering the FEM estimations as follows:

$$\tilde{f} = \sum_{j=1}^3 \tilde{f}_j \psi_j, \quad \tilde{g} = \sum_{j=1}^3 \tilde{g}_j \psi_j, \quad \tilde{\theta} = \sum_{j=1}^3 \tilde{\theta}_j \psi_j, \quad \tilde{\phi} = \sum_{j=1}^3 \tilde{\phi}_j \psi_j,$$

with $s_1 = s_2 = s_3 = s_4 = \psi_i$ where $(i = 1, 2, 3)$ and the quadratic shape functions ψ_i are given as;

$$\begin{aligned} \psi_1^b &= \frac{(\zeta_{b+1} - \zeta)(\zeta_{b+1} + \zeta_b - 2\zeta)}{(\zeta_{b+1} - \zeta_b)^2}, & \psi_2^b &= \frac{4(\zeta_{b+1} - \zeta)(\zeta - \zeta_b)}{(\zeta_{b+1} - \zeta_b)^2} \\ \psi_3 &= -\frac{(\zeta - \zeta_b)(\zeta_{b+1} + \zeta_b - 2\zeta)}{(\zeta_{b+1} - \zeta_b)^2}, & \zeta_b \leq \zeta \leq \zeta_{b+1} \end{aligned} \quad (22)$$

The FEM model equations are in matrix form, therefore given by

$$\begin{bmatrix} \{M^{11}\} & \{M^{12}\} & \{M^{13}\} & \{M^{14}\} \\ \{M^{21}\} & \{M^{22}\} & \{M^{23}\} & \{M^{24}\} \\ \{M^{31}\} & \{M^{32}\} & \{M^{33}\} & \{M^{34}\} \\ \{M^{41}\} & \{M^{42}\} & \{M^{43}\} & \{M^{44}\} \end{bmatrix} \begin{bmatrix} \{\tilde{f}\} \\ \{\tilde{g}\} \\ \{\tilde{\theta}\} \\ \{\tilde{\phi}\} \end{bmatrix} = \begin{bmatrix} \{b_1\} \\ \{b_2\} \\ \{b_3\} \\ \{b_4\} \end{bmatrix} \quad (23)$$

where T_{mn} and r_m ($m, n = 1, 2, 3, 4, 5$) are defined as

$$\begin{aligned} M_{ij}^{11} &= \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \frac{d\psi_j}{d\zeta} d\zeta, & M_{ij}^{12} &= -\int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, & M_{ij}^{13} &= M_{ij}^{14} = 0, & M_{ij}^{21} &= 0. \\ M_{ij}^{22} &= -(1 - \lambda\theta) \int_{\zeta_b}^{\zeta_{b+1}} \frac{d\psi_i}{d\zeta} \frac{d\psi_j}{d\zeta} d\zeta - \lambda \int_{\zeta_b}^{\zeta_{b+1}} \psi_i^2 \left(\frac{d\psi_j}{d\zeta} \right)^2 d\zeta + \int_{\zeta_b}^{\zeta_{b+1}} \tilde{f} \psi_i \frac{d\psi_j}{d\zeta} d\zeta - \int_{\zeta_b}^{\zeta_{b+1}} \tilde{g} \psi_i \psi_j d\zeta - M \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, \\ M_{ij}^{23} &= \lambda_1 \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, & M_{ij}^{24} &= \lambda_2 \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, & M_{ij}^{31} &= 0, & M_{ij}^{32} &= -(S_1 + S_2) \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, \\ M_{ij}^{33} &= -\frac{1}{Pr} \int_{\zeta_b}^{\zeta_{b+1}} \frac{d\psi_i}{d\zeta} \frac{d\psi_j}{d\zeta} d\zeta + \int_{\zeta_b}^{\zeta_{b+1}} \tilde{f} \psi_i \frac{d\psi_j}{d\zeta} d\zeta + Nb \int_{\zeta_b}^{\zeta_{b+1}} \psi_i^2 \left(\frac{d\psi_j}{d\zeta} \right)^2 d\zeta + Nt \int_{\zeta_b}^{\zeta_{b+1}} \tilde{\theta} \psi_i \frac{d\psi_j}{d\zeta} d\zeta \\ &\quad - \alpha_T \int_{\zeta_b}^{\zeta_{b+1}} \psi_i \psi_j d\zeta, & M_{ij}^{34} &= M_{ij}^{41} = M_{ij}^{42} = 0, & M_{ij}^{43} &= \frac{Nt}{Nb} \int_{\zeta_b}^{\zeta_{b+1}} \frac{d\psi_i}{d\zeta} \frac{d\psi_j}{d\zeta} d\zeta, \\ M_{ij}^{44} &= -\int_{\zeta_b}^{\zeta_{b+1}} \frac{d\psi_i}{d\zeta} \frac{d\psi_j}{d\zeta} d\zeta + PrLe \int_{\zeta_b}^{\zeta_{b+1}} \tilde{f} \psi_i \frac{d\psi_j}{d\zeta} d\zeta - PrLe \int_{\zeta_b}^{\zeta_{b+1}} \tilde{g} \psi_i \psi_j d\zeta. \end{aligned}$$

and

$$b_i^1 = 0, \quad b_i^2 = -(1 - \lambda\tilde{\theta}) \left(\psi_i \frac{d\tilde{g}}{d\zeta} \right)_{\zeta_b}^{\zeta_{b+1}}, \quad b_i^3 = -\frac{1}{Pr} \left(\psi_i \frac{d\tilde{\theta}}{d\zeta} \right)_{\zeta_b}^{\zeta_{b+1}}, \quad b_i^4 = -\left(\psi_i \frac{d\tilde{\phi}}{d\zeta} \right)_{\zeta_b}^{\zeta_{b+1}},$$

where $\tilde{f} = \sum_{j=1}^3 \tilde{f}_j \psi_j$, $\tilde{g} = \sum_{j=1}^3 \tilde{g}_j \psi_j$, $\tilde{\theta} = \sum_{j=1}^3 \tilde{\theta}_j \psi_j$ and $\tilde{\phi} = \sum_{j=1}^3 \tilde{\phi}_j \psi_j$ are considered to be unknown. Subsequently, the accumulation of element equations, a substantial system of nonlinear equations, is achieved. Moreover, an iterative structure is needed to resolve the system of nonlinear differential equations for an efficacious solution. At an inferior level of iterations, the functions $\tilde{f}, \tilde{g}, \tilde{\theta}$, and $\tilde{\phi}$ are assumed to be known to linearize the framework and need to be accepted and proceeded for high levels until the exactness of 0.00005 is achieved. To accomplish autonomous work, a mesh affect-ability repetition was made. Table 1 shows the concurrence of the consequences; it is intended to enhance the

number of components that are prolonged as ($n = 60, 100, 180, 300, 420, 500$ and 700). From the results shown in Table 1, there is no substantial disparity in the output of functions $\tilde{f}$, $\tilde{g}$, $\tilde{\theta}$, and $\tilde{\phi}$ with the increasing number of components beyond 500; thus, the last results are stated for ($n = 500$) elements.

Table 1. FEM convergence results of $\tilde{f}(\xi)$, $\tilde{h}(\xi)$, $\tilde{\theta}(\xi)$, and $\tilde{\phi}(\xi)$ for a different number of elements when $Pr = M = 1$, $S_1 = 0.2$, $Le = 1$, $Nb = Nt = 0.1$, $S_2 = 0.2$, $\alpha_T = 0.05$, $\lambda = 0.5$, $N = 0.09$.

Number of Elements	$\tilde{f}(3)$	$\tilde{h}(3)$	$\tilde{\theta}(3)$	$\tilde{\phi}(3)$
60	0.86976	0.04045	0.02302	0.13808
100	0.87027	0.04057	0.02315	0.13809
180	0.87049	0.04062	0.02321	0.13809
300	0.87056	0.04064	0.02323	0.13810
420	0.87058	0.04065	0.02324	0.13810
500	0.87059	0.04065	0.02324	0.13810
700	0.87060	0.04065	0.02324	0.13810

4. Results and Discussion

This section was prepared to inspect the presentation of the dimensionless axial velocity profile $\tilde{f}'(\xi)$, and the temperature distribution $\tilde{\theta}(\xi)$ together with the concentration profile $\tilde{\phi}(\xi)$, under the influence of different emerging parameters such as the viscosity parameter λ , Hartman number M , Grashof number Gr , thermophoresis parameter Nt , Brownian motion parameter Nb , Lewis number Le , and Prandtl number Pr as shown in all the graphs. Furthermore, the acquired results of Skin friction and the Nusselt number were compared with the existing results of [42–45] and presented in the form of tables. Moreover, all the other using parameters are preserved to be fixed throughout the problem as $Pr = M = \lambda_1 = 1$, $S_1 = 0.2$, $Le = 1$, $Nb = Nt = 0.1$, $S_2 = 0.2$, $\alpha_T = 0.05$, $\lambda_2 = 0.09$. The aspect of the existing outputs and assessment of skin friction was done with accurate results that confirm the legitimacy of the finite element technique. Table 2 shows the results of heat transference proportion which were assimilated through the finite element technique and are compared with the results of previous studies as well as with the exact solutions of [42–44]. To ensure the accuracy of the current precise values, the conclusions attained by the finite element method for the Nusselt number for the unsteady and steady flow were compared with the calculated results from earlier studies, are shown in Table 3, which shows the current results, that is, creditable conventionality supported by current results and earlier available research that approves the intensity and exactness of the current results that were attained by the finite element method (FEM). To determine the consistency and corroborations of the results, the existing results were comprehensively assesses with [42,44–46] and were efficaciously simulated, as shown in Table 3.

Table 2. Assessment of skin friction coefficient for different values of M , when $Pr = \lambda_1 = 1$, $S_1 = 0.2$, $Le = 1$, $Nb = Nt = 0.1$, $S_2 = 0.2$, $\alpha_T = 0.05$, $\lambda = 0.5$, $\lambda_2 = 0.09$.

M	Mudassar et al. [43] (a)	Liaqat et al. [42]	Bagh et al. [44]	FEM (Our results) (b)	Error $ \left(\frac{b-a}{a}\right) \times 100$
0.0	1.000000	1.0000078	1.0000080	1.0000080	0.00002
0.2	1.095445	1.0954462	1.0954458	1.0954460	0.00001
0.5	1.224745	1.2247452	1.2247446	1.2247450	0.00001
1.0	1.414214	1.4142142	1.4142132	1.4142138	0.00002
1.2	1.483240	1.4832385	1.4832393	1.4832390	0.00003
1.5	1.581139	1.5811392	1.5811384	1.5811389	0.00001
2.0	1.732051	1.7320515	1.7320504	1.7320510	0.00002

Table 3. Comparison of $-\tilde{\theta}'(0)$ for various values of Pr when $Pr = M = \lambda_1 = 1$, $S_1 = 0.2$, $Le = 1$, $Nb = Nt = 0.1$, $S_2 = 0.2$, $\alpha_T = 0.05$, $\lambda = 0.5$, $\lambda_2 = 0.09$.

Pr	Ishak et al. [45] (a)	Liaqat et al. [42]	Bagh et al. [44]	Dulal Pal. [46]	FEM (Our results) (b)	Error $ (\frac{b-a}{a}) \times 100$
0.72	0.808631	0.808633	0.8086339299	-	0.8086339297	0.0003
1.00	1.000000	1.000008	1.0000080213	1.0000	1.0000080210	0.0008
3.00	1.923682	1.923677	1.9236777221	1.9236	1.9236777219	0.0003
10.0	3.720673	3.720668	3.7206681683	3.7207	3.7206681679	0.0002
100	12.294083	12.294051	12.294051659	12.2940	12.294051661	0.0003

Tables 4 and 5 show the comparison of $-\tilde{\theta}'(0)$, and $-\tilde{\phi}'(0)$ for the doubly stratified and unstratified cases; all the displayed results in this study are applicable to limited cases and are in favorable agreement with the previously published work. Here, it is shown that current results are completely concurrent and the grid invariance test is proficient to endure an exactness up to five decimal digits. The velocity profile in Figure 2 illustrates the impacts of the viscosity parameters λ and M on the velocity profile $\tilde{f}'(\xi)$ for both the linear and nonlinear sheets.

Table 4. Comparison of $-\tilde{\theta}'(0)$, and $-\tilde{\phi}'(0)$ for different values of α_T when $Pr = 1$, $M = 0.49$, $\lambda_1 = 0.3$, $S_1 = 0.2$, $Le = 1$, $Nb = Nt = 0.2$, $S_2 = 0.2$, $\alpha_T = 0.0$, $\lambda = 0.0$, $\lambda_2 = 0.09$.

α_T	Abbasi et al. [47]		FEM (Our Results)	
	$-\tilde{\theta}'(0)$	$-\tilde{\phi}'(0)$	$-\tilde{\theta}'(0)$	$-\tilde{\phi}'(0)$
0.0	0.82852	0.37977	0.828512	0.379840
0.05	-	-	0.855730	0.357125
0.1	-	-	0.882813	0.334570

Table 5. Comparison of $-\tilde{\theta}'(0)$, and $-\tilde{\phi}'(0)$ for different values of Nb and Nt when $Pr = 1$, $M = 0.49$, $\lambda_1 = 0.3$, $S_1 = 0.0$, $Le = 1$, $Nb = Nt = 0.2$, $S_2 = 0.0$, $\alpha_T = 0.0$, $\lambda = 0.0$, $\lambda_2 = 0.09$.

$Nb = Nt$	Khan and Pop [48]		Anwar et al. [49]		FEM (Our Results)	
	$-\tilde{\theta}'(0)$	$-\tilde{\phi}'(0)$	$-\tilde{\theta}'(0)$	$-\tilde{\phi}'(0)$	$-\tilde{\theta}'(0)$	$-\tilde{\phi}'(0)$
0.1	0.9524	2.1294	0.9524	2.1294	0.952380	2.129392
0.2	0.3654	2.5152	0.3654	2.5152	0.365561	2.515220
0.3	0.1355	2.6080	0.1355	2.6088	0.135517	2.608822

Figure 2. The influence of Magnetic penetrability (M) on Velocity profile.

With the increasing values of M , the velocity profile declines but there is an opposite effect on concentration and temperature profiles that can be seen from Figures 3 and 4. The declaration in the velocity field is due to magnetic field that generally induces Lorentz forces which act as resistive forces and consequently oppose the fluid motion. Moreover, there is an increment in the concentration of nanoparticles and temperature; the main reason for this is the enhancement in the viscous effect with the higher viscosity, which consequently causes decrements in the fluid velocity [50].

Figure 3. The influence of Magnetic penetrability (M) on concentration profile.

Figure 4. The influence of Magnetic penetrability (M) on Temperature profile.

Figure 5 clearly shows that the temperature profile declines as there is an increment in values of α_T up to a certain distance ξ while the momentum boundary layer thickness increases so that the heat transference process from the sheet to fluid decreases with increasing values of α_T . Figures 6 and 7 demonstrate that the temperature and nanoparticle concentration decline with the increasing values of thermal and solutal stratification parameters S_1 , and S_2 , while the thickness of the thermal boundary layer increases maximally [36].

Figure 5. The influence of thermal relaxation parameter α_T on Temperature profile.

Figure 6. The influence of Thermal Stratification S_1 on Temperature profile.

Figure 7. The influence of Solutal Stratification S_2 on Concentration profile.

It is clear that with the increasing values of S_1 and S_2 , the decrement in temperature is possible so it can be controlled by controlling the values of the stratification parameters. Figures 8 and 9 show the impact of the viscosity parameter λ and Nb on the concentration and temperature profiles; by increasing the values of the Brownian motion parameter, the thermal boundary layer thickness increases in both cases while the effect is opposite in the concentration and temperature profiles, as in the presence of nanoparticles, the nanofluids have a better thermal conductivity compared to water as the base fluid [37].

Figure 8. The influence of the Brownian motion parameter (Nb) on the Concentration profile.

Figure 9. The influence of the Brownian motion parameter (Nb) on the Temperature profile.

From Figure 10, it can be seen that as the values of the thermophoresis parameter Nt increase, a remarkable increment in the concentration profile is seen, and this was found throughout the boundary layer. The thermophoretic forces under the steady temperature gradient are produced by the motion of the particle and the natural phenomenon of the molecules moving from the hotter area towards the cooler region, so as a result, the temperature of the fluid increases.

Figure 10. The influence of the thermophoresis parameter (Nt) on the Concentration profile.

Figure 11 shows the impact of the thermophoresis parameter Nt and the viscosity parameter λ on the dimensionless temperature profile. For the different values of the viscosity parameter, for constant viscosity parameter ($\lambda = 0$) and for ($\lambda = 0.5$), with the increasing viscosity, the temperature profile shows an intensifying performance [5].

Figure 11. The influence of thermophoresis parameter (Nt) on Temperature profile.

From Figures 12 and 13 one can perceive an intensification in the velocity profile as the values of the buoyancy parameters λ_1 and λ_2 are cumulative. Nevertheless, the extensiveness of the momentum boundary layer deteriorates. For $\lambda_1 = 0.5$, the buoyancy forces associated with the nanoparticle types in equation (13) reduce. However, the non zero values of λ_1 show that the velocity profile is exceedingly close to the stretching wall. Thus, in turn, the momentum grows inside the boundary layer so that it elevates the fluid velocity. Moreover, from Figures 12 and 13, the elevation in the viscid parameter creates an obstacle in the fluid motion and because of this, the fluid velocity declines and the thickness of the momentum boundary layer increases. Figures 14–16 reveals the impact of the magnetic parameter M and the viscosity parameter on the Nusselt number (Nur), Sherwood number (Shr) and on the Skin friction. It can be seen that with the increasing values of the magnetic parameter M , the values of the Nusselt number (Nur), the Sherwood number (Shr), and the Skin friction coefficient

decrease. Moreover, this reduces the thermal boundary layer in the case of the Nusselt number (Nur) and the Sherwood number (Shr) but the effect is opposite on the Skin friction coefficient. The magnetic parameter M is characteristic of fluid flow, which causes instantaneous heat and mass transferes by convection. Lastly, it was found that the heat transfer rate declines as the values of the magnetic parameter M increase. The observed decrement in temperature profile is due to temperature difference reasonably decaying between the surface and away from the sheet. Thus, the change in values of the thermal stratification parameter causes an increase in the density of the fluid layers, which leads the dense particles toward the surface and creates more magnetohydrodynamic interaction, due to which a decrement occurs in heat transference.

Figure 12. The influence of the Buoyancy parameter (λ_1) on the Velocity profile.

Figure 13. The influence of the Buoyancy parameter (λ_2) on the Velocity profile.

Figure 14. The influence of Magnetic penetrability (M) on Nusselt number.

Figure 15. The influence of Magnetic penetrability (M) on the Sherwood number.

Figure 16. The influence of Magnetic penetrability (M) on Skin friction.

5. Concluding Remarks

This article incorporates the theoretical study associated with the magnetohydrodynamic flow and heat transfer of the doubly stratified nanofluid with buoyancy forces and temperature-dependent viscosity impacts and also used a Cattaneo–Christov Heat flux model. The mathematical model (partial differential equations) was transformed into the system of ordinary differential equations by using the suitable transformations. The finite element method (FEM) was selected to solve the consequential boundary value problem and the subsequent results were compared with the previous studies. Furthermore, the graphical pictures demonstrating the influence of numerous evolving parameters on the momentum, mass, and heat transference properties were discussed. Below are the foremost conclusions from this study:

- The increasing values of M and the viscosity parameter decrease the nanofluid velocity magnitude; however, there is a contrary impact on the concentration and temperature distributions.
- The increase in the concentration and thermal boundary layer thickness of the nanofluid sheet due to the increment in the viscosity parameter also increases the temperature and concentration of nanoparticles.
- Fluid temperature declines with increasing values of the thermal relaxation parameter. This shows that the Cattaneo–Christov heat flux model provides a better assessment of temperature distribution.
- The magnitude of the Nusselt number and the Sherwood number declines, and the thermal boundary layer thickness is reduced with increasing values of magnetic M and viscosity parameters.
- The increasing values of the Brownian motion parameter Nb cause a decline in the nanoparticle concentration profile whereas for thermophoresis parameter Nt , the results are contrary.
- The thermal relaxation parameter effectively increases the rate of heat transfer of the flow.

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Nomenclature

μ_0	Viscosity of fluid
S_1	Thermal Stratification
S_2	Solutal Stratification
B_1, B_2	Dimensionless parameters
λ_1, λ_2	Buoyancy parameters
D_B	Brownian diffusion coefficient
Nb	Brownian motion parameter
T_c	Curie temperature
Le	Lewis number
κ	Thermal conductivity of fluid
ρ_p	Nanoparticle's density
C_p	Specific heat

Pr	Prandtl number
Gr	Thermal Grashof number
Nt	Thermophoresis parameter
α_T	Thermal relaxation parameter
D_T	Thermal diffusibility
u, v	Velocity components
U_w	Velocity of sheet
T_w	Temperature at surface
Re_x	Local Reynold number
T	Non-dimensional temperature
M	Magnetic penetrability
σ	Conductivity of fluid
δ	Thermal relaxation parameter
λ	Variable viscosity parameter
ρ	Fluid density
T_∞	Temperature away from the surface

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