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Radiation and Magnetic Field Impacts on Time-Dependent Mixed Convection Flow and Heat Transmission of Maxwellian Fluid Past A Stretching Sheet

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Abstract: The current study was devoted to explicating the impacts of heat transmission in an unsteady mixed convection flow of an upper convected Maxwell (UCM) fluid passing over a continuously stretching surface under the influence of radiation and magnetic field. Appurtenant similarity transmutations were adopted in order to express the constitutive boundary layer Equations of flow and heat transmission in non-dimensionalized form. The reduced system of partial differential Equations was solved by implementing the implicit finite difference method (IFDM). Our center of attention was to scrutinize the behavior of influential flow parameters on some significant features of flow and heat transmission, which were briefly examined, discussed, and presented in both graphical and tabular formats. Finally, a comparison was established with existing literature in limiting cases to support the present results, and a good agreement was found, corroborating our work. It was predicted that the thermal diffusion rate could be controlled by varying the Prandtl number. Moreover, a rise in radiation and magnetic field parameters reduced the skin friction coefficient and led to enhance the heat transmission rate at the surface. The outcomes of the study might have viable implementations in order to improve the quality of industrial products.

Keywords: stretching sheet; Maxwell fluid; heat transfer; implicit finite difference method; mixed convection

1. Introduction

The fact-finding concerning the flow and heat transmission over mobile surfaces has remarkable significance regardless of its relevance in various technological processes, such as fiber manufacturing in glass and polymers, metallurgy, glass blowing, paper production, hot rolling, condensation phenomena, fiber spinning, and hot rolling [1]. In all mentioned cases, the most commonly used coolant liquid is water. The quality of end product mainly confides in on heat transmission rate and coefficient of skin friction. Therefore, the flow field and heat transmission studies can be of considerable significance. The pioneering work on the flow of a viscous fluid and heat transmission over a stretchable moving plate was reported by Sakiadis [2], who opened new doors for researchers. Initially, the analytical solution for a flow problem over the contracting sheet was presented by Crane [3]. Later, numerous studies were carried out to extend the research under various aspects for a better explanation of experiments and fluid behaviors [4–10].

In recent times, the flow analysis of non-Newtonian fluids has been discussed vastly due to its significance in industrial engineering. Most of the industrial fluids, such as paints, glues, polymers, and printing ink, are non-Newtonian in nature. Due to the complexity, familiar Navier–Stokes Equations cannot exhibit all characteristics of such fluids in an appropriate way. The majority of available models for such fluids are basic, like power laws with grades two or three [11–17]. These models are frequently utilized in fluids modeling, but they cannot foretell the impacts of elasticity. A lot of researchers have introduced the upper convected Maxwell (UCM) fluid model, which anticipates stress relaxation. This rheological model also debars the complicated impacts of shear-dependent viscosity, thus enabling one to concentrate merely on the impacts of fluid elasticity [18–23]. Thermoplastic polymers and various metals in the proximity of their melting point and most of the geo-materials behave as typical Maxwell fluids. The mathematical model of the Maxwell fluid has been employed vastly to characterize the polymeric solutions/fluids and also to express the rheology of crude oil, glycerin, flour dough, toluene, etc.

The study concerning the impacts of thermal radiation and magnetic field on mixed convective flows is a new facet in the context of stretchable surfaces. Radiation effects play a key role in nuclear reactors, space technology, missile re-entry, hypersonic flights, and high-temperature procedures. Thence, many authors have concentrated on this subject as a source of energy transmission, and several attempts are made in order to improve the idea of radiative transfer in such processes. Cess [24] was the first to probe thermal radiation impacts with free convection heat transmission along a vertical plate by employing a perturbation technique. Hossain and Takhar [25] treated a similar problem of a viscous fluid with uniform temperature by utilizing the Keller Box finite difference scheme. Cheng and Ozisik [26] deployed a normal expansion method to treat the radiation effects for an isotropically scattering fluid. Raptis [27] and Hossain et al. [28] explored radiation effects on free convection flow in permeable media. T. Hayat et al. [29] proposed an analytical solution by employing the homotopy analysis method (HAM) to examine the radiation and magnetic field effects over mixed convection stagnation point flow. The impacts of radiation on MHD-Rayleigh flow along with uniform surface heat flux were evaluated by Duwairi and Duwairi [30]. The state-of-the-art is briefly encapsulated in the studies [31–35].

The majority of the aforementioned investigations considered both the flow and temperature fields in a steady state. However, in some instances, including sudden flat sheet stretching or gradual change in its temperature may render the flow field unsteady. Impetuous stretch in the sheet's surface immediately generates an inviscid flow, but it takes a bit longer to develop in the viscous layer close to the sheet. A very few studies have explored the scenarios wherein the stretching force, together with surface temperature, alter with time. Some studies [36–41] employed a similarity method to investigate unsteady stretching surface at constant temperature and derive ordinary differential Equations from the time-dependent boundary layer Equations. Liu and Anderson [42] analyzed the thermal attributes of a liquid film operated by an irregular stretching surface with recommended temperature changes of the stretching sheet. Lately, similarity solutions were obtained for heat transfer as well as unsteady flow across a stretching sheet at varying conditions [43–47]. Bhattacharyya et al. [48] examined the impacts of slip condition on the stagnation point flow of an unsteady boundary layer across a stretching sheet.

It is perceived that a lot of work has been done on UCM fluids, and numerous methods are introduced to compute the specific type of problems. However, no efforts have been made yet to find the solution of heat transmission of an unsteady mixed convection flow of Maxwellian fluids past a continuing stretched sheet in the presence of radiation and magnetic field effects. The focal point of the ongoing study was to fill-up this gap in the existing literature. The exact solution was computed, and further implicit finite difference method (IFDM) method was implemented in order to get the solution to the designed problem. In addition, graphical results were illustrated and analyzed.

2. Problem Development

Let us take into account a two dimensional (2-D) unsteady laminar flow of Maxwellian fluid past over a surface stretching continuously with velocity $U_w(x)$ in the plane $y = 0$. The x - axis is taken along the sheet, and the y -axis is normal to the surface. We presume that $U_w(x)$ varies linearly, while surface temperature $T_w(x)$ has a quadratic variation with the distance x measured from O . The flow geometry is presented in Figure 1. Keeping in view the above-stated assumptions, the fluid flow can be described by the following Equations. The boundary layer flow model and concentration field are presented as [49].

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda^* \left[u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right] = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u + g\beta^*(T - T_\infty), \quad (2)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\alpha}{\rho c_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p} \frac{\partial q}{\partial y} \quad (3)$$

The suitable boundary constraints for the problem can be written as

$$\begin{aligned} u &= U_w(x), \quad v = 0, \quad T = T_w(x) = T_\infty + \Delta T(x) \quad \text{at } y = 0, \\ u &\rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (4)$$

where u and v are the velocity components in the x and y directions, respectively, λ is the relaxation time, T is the fluid temperature inside the boundary layer, t is time, ρ is the fluid density, c_p is the specific heat capacity at constant pressure of the fluid, where B_0 is the strength of the magnetic field, q_r is the radiative heat flux, α and ν are the thermal diffusivity and the kinematic viscosity, respectively.

Figure 1. The flow diagram and coordinate system.

We consider the following form of stretching velocity of the surface and temperature

$$U_w(x) = ax, \quad \Delta T(x) = bx^2 \quad (5)$$

Making use of the Rosseland approximation for radiation for an optically thick layer [50], we have

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y}, \quad (6)$$

where σ^* and k^* denote Stefan–Boltzmann constant and mean absorption coefficient, respectively. We can express the term T^4 as a linear function of temperature. It is recognized by expanding T^4 in a Taylor series about T_∞ and neglecting higher terms, then one can write

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4. \quad (7)$$

From Equations (3), (6), and (7), one obtains

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\alpha}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3\rho c_p k^*} \frac{\partial^2 T}{\partial y^2}. \quad (8)$$

Employing the non-dimensionalized quantities:

$$\psi = x \sqrt{av\xi} f(\eta, \xi), \quad \eta = y \sqrt{\frac{a}{v\xi}}, \quad \xi = 1 - e^\tau, \quad \tau = at, \quad \theta = \frac{T - T_\infty}{\Delta T(x)}, \quad (9)$$

Here, τ , represents dimensionless time, $a > 0$, and ψ denotes stream function as $u = \partial\psi/\partial y$, and $v = -\partial\psi/\partial x$, continuity Equation is automatically satisfied as a consequence, and the corresponding other concentration Equations are transmuted as follow:

$$f''' + \xi(ff'' - f'^2 - Mf') + \beta\xi(2ff'f'' - f^2f''') + \frac{1}{2}\eta(1-\xi)f'' + \xi\lambda = \xi(1-\xi)\frac{\partial f'}{\partial \xi}, \quad (10)$$

$$\frac{1}{\text{Pr}} \left(1 + \frac{4}{3}R_d\right)\theta'' + \xi(f\theta' - 2f'\theta) + \frac{1}{2}\eta(1-\xi)\theta' = \xi(1-\xi)\frac{\partial \theta}{\partial \xi}, \quad (11)$$

Suitable boundary constraints are:

$$f'(0, \xi) = 1, \quad f(0, \xi) = 0, \quad \theta(0, \xi) = 1, \\ \theta(\infty, \xi) = 0, \quad f'(\infty, \xi) = 0, \quad (12)$$

Here, $\beta = a\lambda^*$, $M = \frac{\sigma B_0^2}{a\rho}$, and $\lambda = \frac{Gr_x}{\text{Re}_x^2}$ are the relaxation time, magnetic field parameter, and mixed convection parameter, respectively. The local Grashof number Gr_x , the local Reynold number Re_x , the Prandtl number Pr , and the radiation parameter R_d are, respectively, defined as $Gr_x = \frac{g\beta^*(T-T_\infty)x^3}{\nu^2}$, $\text{Re}_x = \frac{U_w x}{\nu}$, $\text{Pr} = \frac{\nu}{\alpha}$, and $R_d = \frac{4\sigma^* T_\infty^3}{ak^*}$.

The physical parameters C_f and Nu are defined as:

$$C_f = \frac{\tau_w(x)}{\rho(U_w(x))^2} \text{ and } Nu_x = \frac{q_w(x)x}{\alpha\Delta T(x)}, \quad (13)$$

where the wall share stress τ_w , the heat transfer, and mass transfer are given by q_w as:

$$\tau_w(x) = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \text{ and } q_w(x) = - \left(\left(\alpha + \frac{16\sigma^* T_\infty^3}{3k^*} \right) \frac{\partial T}{\partial y} \right)_{y=0}, \quad (14)$$

By utilizing similarity transformations (5), skin friction coefficient C_f and Nusselt number Nu can be expressed as:

$$\xi^{\frac{1}{2}} \text{Re}_x^{\frac{1}{2}} C_f = f''(0, \xi), \quad (15)$$

$$\xi^{-\frac{1}{2}} Re_x^{-\frac{1}{2}} Nu_x = -\left(1 + \frac{4}{3} R_d\right) \theta'(0, \xi). \quad (16)$$

The Equations (10)–(12) authorize reductions to ODE systems of the governing profiles $f'(\eta)$ and $\theta(\eta)$ in the initial unsteady-state flow at $\xi = 0$, for a small time and final steady-state flow at large times given by $\xi = 1$.

3. Exact Solution

3.1. Initial Steady Flow

As $\xi = 0$ ($\tau = 0$) and also $R_d = 0$, the Equations (10) and (11) reduces to:

$$f''' + (1/2)\eta f'' = 0, \quad (17)$$

$$\frac{1}{Pr} \theta'' + (1/2)\eta \theta' = 0, \quad (18)$$

With boundary constraints

$$\begin{aligned} f'(0, 0) = 1, \quad f(0, 0) = 0, \quad \theta(0, 0) = 1, \\ f(\infty, 0) = 0, \quad \theta(\infty, 0) = 0. \end{aligned} \quad (19)$$

The analytical results for the Equations (13) and (14) are specified as:

$$f(\eta, 0) = \eta + \frac{2}{\sqrt{\pi}} \left(1 - e^{(-\eta^2/4)}\right) - \eta \operatorname{erf}\left(\frac{\eta}{2}\right), \quad (20)$$

$$\theta(\eta, 0) = 1 - \frac{1}{\sqrt{Pr}} \operatorname{erf}\left(\frac{Pr \eta}{2}\right), \quad (21)$$

The error function $\operatorname{Erf}(z)$ is defined as:

$$\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \quad (22)$$

3.2. Final Unsteady State Flow

For $\xi = 1$ ($\tau \rightarrow \infty$), corresponding Equations (10) and (11) are reduced to

$$f''' + \lambda \theta + \beta f(2f'f'' - ff''') + (ff'' - f'^2 - Mf') = 0, \quad (23)$$

$$\frac{1}{Pr} \left(1 + \frac{4}{3} R_d\right) \theta'' - (2f'\theta - f\theta') = 0, \quad (24)$$

Suitable boundary constraints

$$\begin{aligned} f'(0, 1) = 1, \quad f(0, 1) = 0, \quad \theta(0, 1) = 1, \\ f'(\infty, 1) = 0, \quad \theta(\infty, 1) = 0, \end{aligned} \quad (25)$$

Equation (23) affirms the exact results at $\beta = 0$ and $\lambda = 0$ as:

$$f(\eta, 1) = \frac{1}{b} \left(1 - e^{(-b\eta)}\right), \quad (26)$$

as $b = \sqrt{1 + M}$. Assume $c = \left(1 + \frac{4}{3} R_d\right)$.

Invoking Equation (25) into Equations (23) and (24) and familiarizing the change $\xi = -\frac{\text{Pr}}{c}e^{-b\eta}$, shown as:

$$b^2\xi\frac{d^2\theta}{d\xi^2} + \left(b^2 - \frac{\text{Pr}}{c} - \xi\right)\frac{d\theta}{d\xi} + 2\theta = 0 \quad (27)$$

$$\theta\left(-\frac{\text{Pr}}{c}\right) = 1, \theta'(0) = 0. \quad (28)$$

Equation (26) may be changed in the Confluent hypergeometric “Kummer’s function”, and the results are satisfying the Equations (23) and (24) as

$$\theta(\eta, 1) = \frac{\text{Kummer}\left[\left(\frac{2b^2c+\text{Pr}}{b^2c}\right), \left(\frac{b^2c+\text{Pr}}{b^2c}\right), \left(-\frac{\text{Pr}e^{-b\eta}}{b^2c}\right)\right]\left(-\frac{\text{Pr}}{c}e^{-b\eta}\right)^{\frac{\text{Pr}}{b^2c}}}{\text{Kummer}\left[\left(-\frac{2b^2c-\text{Pr}}{b^2c}\right), \left(\frac{b^2c+\text{Pr}}{b^2c}\right), \left(-\frac{\text{Pr}}{b^2c}\right)\right]\left(-\frac{\text{Pr}}{c}\right)^{\frac{\text{Pr}}{b^2c}}}.$$
 (29)

4. Implicit Finite Difference Method (IFDM)

We recast Equations (10) and (11) as a set of simultaneous Equations by introducing the variables U, V, P :

$$f' = U, f'' = V, \theta = W, \theta' = P. \quad (30)$$

$$V' + \xi(fV - U^2 - MU) + \beta\xi[2fUV - f^2V] + \frac{1}{2}\eta(1 - \xi)V + \xi\lambda W = \xi(1 - \xi)\frac{\partial U}{\partial \xi}, \quad (31)$$

$$\frac{1}{\text{Pr}}\left(1 + \frac{4}{3}R_d\right)P' + \frac{1}{2}\eta(1 - \xi)P + \xi(fP - 2U\theta) = \xi(1 - \xi)\frac{\partial \theta}{\partial \xi}, \quad (32)$$

$$\begin{aligned} f(0, \xi) &= 0, U(0, \xi) = 1, U(\infty, \xi) = 0, \\ \theta(0, \xi) &= 1, \theta(\infty, \xi) = 0. \end{aligned} \quad (33)$$

We now place a net on the (η, ξ) -plane defined by

$$\begin{aligned} \eta_0 &= 0, \eta_j = \eta_{j-1} + h_j, j = 1, 2, 3, \dots, J, \eta_J = \eta_\infty, \\ \xi^0 &= 0, \xi^n = \xi^{n-1} + k^n, n = 1, 2, 3, \dots \end{aligned} \quad (34)$$

If g_j^n denotes the value of any variable at (η_j, ξ^n) , then the variables and derivatives of Equations (27) and (28) at $(\eta_{j-1/2}, \xi^{n-1/2})$ are replaced by

$$\begin{aligned} g_{j-1/2}^{n-1/2} &= \frac{1}{4}(g_j^n + g_{j-1}^n + g_j^{n-1} + g_{j-1}^{n-1}), \\ \left(\frac{\partial g}{\partial \eta}\right)_{j-1/2}^{n-1/2} &= \frac{1}{2h_j}(g_j^n - g_{j-1}^n + g_j^{n-1} - g_{j-1}^{n-1}), \\ \left(\frac{\partial g}{\partial \xi}\right)_{j-1/2}^{n-1/2} &= \frac{1}{2k^n}(g_j^n - g_j^{n-1} + g_{j-1}^n - g_{j-1}^{n-1}), \end{aligned} \quad (35)$$

We now show the finite-difference approximation of Equations (31) and (32) for the mid-point $(\eta_{j-1/2}, \xi^n)$, as below

$$h_j^{-1}(f_j^n - f_{j-1}^n) = U_{j-1/2}^n, \quad (36)$$

$$h_j^{-1}(U_j^n - U_{j-1}^n) = V_{j-1/2}^n, \quad (37)$$

$$\begin{aligned} &h_j^{-1}\left(V_j^n - V_{j-1}^n\right) - \frac{\beta\xi^{n-1/2}}{2}\left[\left(f^2\right)_{j-1/2}^n h_j^{-1}\left(V_j^n - V_{j-1}^n\right) + \left(f^2\right)_{j-1/2}^n h_j^{-1}\left(V_j^{n-1} - V_{j-1}^{n-1}\right) + \right. \\ &\left. \left(f^2\right)_{j-1/2}^{n-1} h_j^{-1}\left(V_j^n - V_{j-1}^n\right) + 4\xi^{n-1/2}(fUV)_{j-1/2}^n\right] \\ &+ \xi^{n-1/2}\left[(fV)_{j-1/2}^n - (U^2)_{j-1/2}^n\right] - \xi^{n-1/2}MU_{j-1/2}^n + \frac{\eta}{2}(1 - \xi^{n-1/2})V_{j-1/2}^n - 2\alpha_n(1 - \alpha_n)U_{j-1/2}^n \\ &= R_{j-1/2}^{n-1/2}, \end{aligned} \quad (38)$$

$$h_j^{-1}(\theta_j^n - \theta_{j-1}^n) = P_{j-1/2}^n, \quad (39)$$

$$\begin{aligned} & \frac{1}{Pr} \left(1 + \frac{4}{3} R_d \right) h_j^{-1} \left(P_j^n - P_{j-1}^n \right) + \frac{\eta}{2} (1 - \xi^{n-1/2}) P_{j-1/2}^n + \xi^{n-1/2} \left[(fP)_{j-1/2}^n - 2(U\theta)_{j-1/2}^n \right] - 2\alpha_n (1 - \alpha_n) \theta_{j-1/2}^n \\ & = T_{j-1/2}^{n-1} \end{aligned} \quad (40)$$

where

$$\begin{aligned} \alpha_n &= \xi^{n-1/2} / k^n, \\ R_{j-1/2}^{n-1} &= -L_{j-1/2}^{n-1} + 2\alpha_n (1 - \alpha_n) U_{j-1/2}^{n-1}, \\ L_{j-1/2}^{n-1} &= h_j^{-1} \left(V_j^{n-1} - V_{j-1}^{n-1} \right) + \frac{\beta \xi^{n-1/2}}{2} \left[2(fUV)_{j-1/2}^{n-1} - (f^2)_{j-1/2}^{n-1} h_j^{-1} \left(V_j^{n-1} - V_{j-1}^{n-1} \right) \right] + \xi^{n-1/2} \left[(fV)_{j-1/2}^{n-1} - (U^2)_{j-1/2}^{n-1} \right] \\ & - M \xi^{n-1/2} U_{j-1/2}^{n-1} + \frac{\eta}{2} (1 - \xi^{n-1/2}) V_{j-1/2}^{n-1}, \\ T_{j-1/2}^{n-1} &= -M_{j-1/2}^{n-1} + 2\alpha_n (1 - \alpha_n) \theta_{j-1/2}^{n-1}, \\ M_{j-1/2}^{n-1} &= \frac{1}{Pr} \left(1 + \frac{4}{3} R_d \right) h_j^{-1} \left(P_j^{n-1} - P_{j-1}^{n-1} \right) + \xi^{n-1/2} \left[(fP)_{j-1/2}^{n-1} - 2(U\theta)_{j-1/2}^{n-1} \right] + \frac{\eta}{2} (1 - \xi^{n-1/2}) P_{j-1/2}^{n-1}. \end{aligned} \quad (41)$$

The boundary constraints are

$$f_0^n = 0, U_0^n = 1, \theta_0^n = 1, U_J^n = 0, \theta_J^n = 0. \quad (42)$$

If f_j^{n-1} , U_j^{n-1} , V_j^{n-1} , θ_j^{n-1} , P_j^{n-1} are presumed to be well-known for $0 \leq j \leq J$, Equations (36)–(40) form a system of $5J + 5$. Equations for the solution of $5J + 5$ have unknowns f_j^n , U_j^n , V_j^n , θ_j^n , P_j^n , $j = 0, 1, 2, \dots, J$. Newton's linearization method is implemented to linearize this system of Equations, and Equations are solved by employing the Keller-box elimination scheme. The procedure is initiated with $\xi = 0$, and we begin with a set-up of estimated unconditionally convergent profiles for f , U , V , θ , and P , which are exerted in the Keller-box scheme along with second-order precision to advance gradually along the boundary-layer. In order to compute these functions, the difference in procedure is taken less than 10^{-7} , i.e., $|\delta f^i| \leq 10^{-7}$, where the superscript i denotes the number of iterations. To achieve speedy convergence, we take $\eta_j = \sinh(j/a)$ in order to save calculations time and space.

5. Numerical Results and Discussions

Heat transmission of unsteady mixed convection flow of an upper convected Maxwellian fluid past a continuously stretching surface under the influence of radiation and magnetic field impacts was studied. Exact solutions for initial steady and final unsteady state flows were attained, and, finally, the implicit finite difference scheme was utilized to solve the designed problem. In order to examine the flow behavior of Maxwell fluid for velocity and temperature profiles, a rigorous numerical computation was accomplished for numerous values of parameters, which narrated the flow characteristics. Moreover, Table 1 is presented to establish a comparison with existing literature (Abel et al. (2012) [23]) and Sadeghy et al. (2006) [19]) in order to support the results of the proposed study, and a good agreement was found, corroborating our work. Table 2 shows the values of $f''(0, \xi)$ and $-\theta'(0, \xi)$ for different values of flow parameters.

Table 1. Comparison of $f''(0)$ for various values of β and M when $\lambda = 0$ and $\xi = 1$.

	Sadeghy et al. [19]	Subhas et al. [23]	Present Results	Subhas et al. [21]	Present Results
	$M = 0.0$			$M = 0.2$	
0.0	−1.0000000	−0.999962	−0.999962	−1.095445	−1.095151
0.2	−1.0549000	−1.051948	−1.052001	−1.188270	−1.188321
0.4	−1.1008400	−1.101850	−1.101843	−1.275878	−1.275888
0.6	−1.0015016	−1.150163	−1.150133	−1.358733	−1.358771
0.8	−1.1987200	−1.196692	−1.196634	−1.437369	−1.437357
1.2	—	−1.285257	−1.285269	−1.512280	−1.512280
1.6	—	−1.368641	−1.368630	−1.095445	−1.095445
2.0	—	−1.447617	−1.447654	−1.188270	−1.188119

Table 2. $-\theta'(0, \xi)$ and $f''(0, \xi)$ exact solutions found through the different procedure for the ξ as $Pr = 1, M = 0.5, R_d = 0.5, \lambda = 1.0$, and $\beta = 0.5$.

ξ	$f''(0, \xi)$	$-\theta'(0, \xi)$
0.0	−0.56784	0.86725
0.1	−0.59342	0.97241
0.2	−0.61735	1.06797
0.3	−0.64177	1.16383
0.4	−0.66621	1.25733
0.5	−0.69096	1.35039
0.6	−0.71573	1.44164
0.7	−0.74067	1.53248
0.8	−0.76554	1.62206
0.9	−0.79034	1.71184
1.0	−0.81459	1.80201

Impacts of relaxation time parameter β on skin friction and heat transfer rate are displayed in Figure 2a,b. A decline in skin friction and the heat transfer rate was observed with the rise in values of β .

Figure 2. Graph of (a) $\frac{1}{2} Re_x^{1/2} C_{fx}$, (b) $Nu_x Re_x^{-1/2}$ with ξ for changing values of β while $Pr = \lambda = 1.0$ and $R_d = M = 0.5$.

Figure 3a,b exhibit the influence of different mixed convection parameter λ on skin friction and heat transfer rate. These figures narrate that both $\frac{1}{2} Re_x^{1/2} C_f$ and $Nu_x Re_x^{-1/2}$ amplified with the rise in values of ξ and λ . It seemed quite logical that due to boost in buoyancy impact, velocity had to rise, and thus, as a consequence, the wall shear stress increased. When $\xi = 0$, Equations (10) and (11) had an exact solution in the form of Equations (25) and (28), assisting us to validate our numerical results.

Figure 3. Graph of (a) $\frac{1}{2} Re_x^{1/2} C_{fx}$, (b) $Nu_x Re_x^{-1/2}$ with ξ for changed values of λ while $R_d = M = \beta = 0.5$, and $Pr = 1.0$.

From Figure 4a,b, it could be clearly predicted that an enhancement in the radiation parameter R_d led to reducing the coefficient of local skin friction and increased the heat transmission rate at the surface. This phenomenon could be recognized from the fact that due to the rise in radiation parameter R_d , the ambient fluid temperature decreased, and Roseland's mean absorption coefficient k increased, diminishing the skin friction and augmenting the rate of heat transmission at the surface.

Figure 4. Graph of (a) $\frac{1}{2} Re_x^{1/2} C_{fx}$, (b) $Nu_x Re_x^{-1/2}$ with ξ for changed values of R_d while $M = \beta = 0.5$ and $Pr = \lambda = 1.0$.

Variations in skin friction and heat transfer against different values of ξ for different Prandtl number Pr are sketched in Figure 5a,b. Figure 5a indicates a reduction in skin friction, while Figure 5b shows an elevation in heat transmission rate with rising values of ξ . It was deduced that buoyancy forces got higher as Pr became smaller, elevating the fluid pressure. Therefore, velocity enhanced in the neighborhood of heated surface, and hence a boost in skin friction was witnessed. Moreover, we observed that, for lower Pr values, the influence of the warmed surface was intense within the fluid; thus, there was a lower Nusselt number.

Figure 5. Graph of (a) $\frac{1}{2} Re_x^{1/2} C_{fx}$, (b) $Nu_x Re_x^{-1/2}$ with ξ for changed values of Pr while $R_d = M = \beta = 0.5$, and $\lambda = 1.0$.

It is evident from Figure 6 that an increase in the magnetic field parameter M turned down the local skin friction. This was due to the fact that with the elevation of M , the Lorentz force became stronger and retarded the flow, and the coefficient of local skin friction reduced consequently.

Figure 6. Graph of $\frac{1}{2} Re_x^{1/2} C_{fx}$ with ξ for changed values of M while $Pr = \lambda = 1.0$, and $R_d = 0.5$.

Figure 7a,b unveil the impact of different Pr values on $f'(\eta, \xi)$ and $\theta(\eta, \xi)$ with variation in ξ . Figure 7a discloses a velocity rise in the nearby region of the heated surface due to high fluid conductivity for smaller Pr values. It was because of high fluid conductivity, causing significant density differences and, consequently, a high buoyancy force. Figure 7b indicates a decline in temperature distribution with a rise in Pr, as higher values of Pr slowed down the thermal diffusion.

Figure 7. (a) Velocity $f'(\eta, \xi)$, (b) Temperature $\theta(\eta, \xi)$ against η when $Pr = 1.0, 3.0$, and 7.0 , $\xi = 0.1, 0.3, 0.7, 1.0$, while $R_d = M = \beta = 0.5$ and $\lambda = 1.0$.

The impacts of mixed convection parameter λ for various values of ξ on velocity and temperature profiles are sketched in Figure 8a,b. Since we knew that $\lambda = 0$ represents the case of forced convection and rise in λ intensifies the buoyancy force, it resulted in a rise in fluid velocity and the decline in temperature in the neighborhood of surface.

Figure 8. (a) Velocity $f'(\eta, \xi)$, (b) Temperature $\theta(\eta, \xi)$ against η for $\lambda = 0.0, 1.0$, and 2.0 , $\xi = 0.1, 0.3, 0.7, 1.0$, while $R_d = M = \beta = 0.5$ and $Pr = 1.0$.

Figure 9a presents the effects of the magnetic field parameter M on velocity and temperature profiles. It was obvious that with the rise in values of M , the Lorentz force became stronger and retarded the flow of fluid. Figure 9b illustrates the variation in the magnetic parameter M on the temperature profile $\theta(\eta, \xi)$. It showed that the magnetic-field parameter had an increasing effect on the thermal boundary layer thickness. In Figure 10a,b, it is shown that by increasing the radiation parameter R_d , and by keeping the other parameters constant, the velocity and temperature distribution was diminished.

Figure 9. (a) Velocity $f'(\eta, \xi)$, (b) Temperature $\theta(\eta, \xi)$ against η for $M = 0.0, 1.0$, and 2.0 , $\xi = 0.1, 0.3, 0.7, 1.0$, while $R_d = \beta = 0.5$ and $Pr = \lambda = 1.0$.

Figure 10. (a) Velocity $f'(\eta, \xi)$, (b) Temperature $\theta(\eta, \xi)$ against η for $R_d = 0.3, 0.5$, and 0.7 , $\xi = 0.1, 0.3, 0.7, 1.0$, while $M = \beta = 0.5$ and $Pr = \lambda = 1.0$.

6. Concluding Remarks

In the current study, a laminar two dimensional mixed convectional unsteady flow of a Maxwell fluid and heat transmission due to a continuous stretching sheet was analyzed under the effect of radiation and magnetic field. A series solution was given by employing finite difference technique, which relies on temperature profiles, skin friction, heat transfer, the magnetic field parameter M , radiation parameter R_d , Prandtl number Pr , and relaxation time parameter β . It was observed that the solution was valid for the whole temporal ($0 \leq \tau < \infty$) and spatial domain ($0 \leq \eta < \infty$).

The major findings of our study are as under.

1. A decline in skin friction and the heat transfer rate was noted with a rise in values of the relaxation time parameter.
2. It was depicted that skin friction decreased with rising values of mixed convection parameter λ and ξ , while the heat transmission rate showed the totally opposite behavior.

3. The rise in radiation parameter R_d reduced the local skin friction coefficient and led to enhance the heat transfer rate at the surface.
4. It was noticed that higher values of the Prandtl number Pr reduced the temperature and thermal boundary layer thickness rapidly.
5. A rise in velocity and decline in heat transfer rate were witnessed with lower Pr values.
6. The magnetic field parameter M turned down the local skin friction.
7. The thermal diffusion rate could be controlled by varying the Prandtl number.

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Nomenclature

Pr	Prandtl number	B_0	Magnetic field
q_r	Radiative heat flux	t	Time
C_{fx}	Skin friction coefficient	R_d	Radiation parameter
Nu_x	Local Nusselt number	T	Temperature
Re_x	Local Reynolds number	Re	Reynold number
c_p	Specific heat capacity	M	Magnetic field parameter
Gr_x	Local Grashof number	ν	Kinematic viscosity

Greek Symbols

β	Relaxation time parameter	σ^*	Stefan–Boltzmann constant
ρ	Fluid density	ψ	Stream function
α	Thermal diffusivity	τ_w	Wall shear stress
k	Mean absorption coefficient	λ	Mixed convection parameter

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