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A New Generalized Definition of Fractional Derivative with Non-Singular Kernel

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Abstract: This paper proposes a new definition of fractional derivative with non-singular kernel in the sense of Caputo which generalizes various forms existing in the literature. Furthermore, the version in the sense of Riemann–Liouville is defined. Moreover, fundamental properties of the new generalized fractional derivatives in the sense of Caputo and Riemann–Liouville are rigorously studied. Finally, an application in epidemiology as well as in virology is presented.

1. Introduction

Fractional derivative is the generalization of the classical derivative of integer order. It has been recently used to study the impact of memory on the dynamics of various systems from different fields such as epidemiology [1,2], virology [3–5], ecology [6–8] and economics [9]. On the other hand, it has been shown that the membranes of cells of biological systems have a fractional-order electrical conductance [10]. Furthermore, the fractance is an electrical circuit with non-integer order impedance [11]. Additionally, fractional differential equations are currently used to model and solve a variety of biological and engineering problems [12–17].

In recent years, the definition of fractional derivative has drawn attention several researchers. In 2015, Caputo and Fabrizio [18] presented a new fractional derivative with non-singular kernel. In 2016, Atangana and Baleanu [19] remarked that the fractional derivative proposed in [18] cannot produce the original function when the order of derivative is equal to zero. To solve this problem, they proposed a new definition of fractional derivative based on Mittag–Lefler function. In 2020, Al-Refai [20] defined the weighted Atangana–Baleanu fractional derivative in a Caputo sense and he used the Laplace transform to solve an associated linear fractional differential equation.

The main purpose of this study is to propose a new definition of fractional derivative that generalizes the above mentioned fractional derivatives with non-singular kernel for both Caputo and Riemann–Liouville types. To do this, Section 2 is devoted to the definition for both types and some fundamental properties. The Laplace transform and fractional integral corresponding to new generalized derivative are given in Sections 3 and 4. Finally, an application is presented in the last section.

2. The New Fractional Derivative

In this section, we define our new fractional derivative and establish their properties.

Let $H^1(a,b)$ be the Sobolev space of order one defined as follows:

$$H^1(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}.$$

Definition 1. *Let $\alpha \in [0,1)$, $\beta, \gamma > 0$, and $f \in H^1(a,b)$.*
The new generalized fractional derivative of order α of Caputo sense of the function $f(t)$ with respect to the weight function $w(t)$ is defined as follows:

$$^C D_{a,t,w}^{\alpha,\beta,\gamma} f(t) = \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \int_a^t E_\beta[-\mu_\alpha(t-x)^\gamma] \frac{d}{dx}(wf)(x)dx, \tag{1}$$

where $w \in C^1(a,b)$, $w, w' > 0$ on $[a,b]$, $N(\alpha)$ is a normalization function obeying $N(0) = N(1) = 1$, $\mu_\alpha = \dfrac{\alpha}{1-\alpha}$ and $E_\beta(t) = \sum\limits_{k=0}^{+\infty} \dfrac{t^k}{\Gamma(\beta k + 1)}$ is the Mittag-Leffler function of parameter β.

It is very important to note that the above definition includes many special cases existing in the literature. For example,

1. When $w(t) = 1$, $\beta = \gamma = 1$, we obtain the Caputo–Fabrizio fractional derivative [18] given by

$$^C D_{a,t,1}^{\alpha,1,1} f(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t \exp[-\mu_\alpha(t-x)] f'(x)dx.$$

2. When $w(t) = 1$, $\beta = \gamma = \alpha$, we get the Atangana–Baleanu fractional derivative [19] given by

$$^C D_{a,t,1}^{\alpha,\alpha,\alpha} f(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t E_\alpha[-\mu_\alpha(t-x)^\alpha] f'(x)dx.$$

3. When $\beta = \gamma = \alpha$, we obtain the weighted Atangana–Baleanu fractional derivative that recently defined in [20], and it is given by

$$^C D_{a,t,w}^{\alpha,\alpha,\alpha} f(t) = \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \int_a^t E_\alpha[-\mu_\alpha(t-x)^\alpha] \frac{d}{dx}(wf)(x)dx.$$

On the other hand, it is not hard to show that the new generalized fractional derivative of Caputo sense has the following properties:

(i) $^C D_{a,t,w}^{\alpha,\beta,\gamma}(c_1 f(t) + c_2 g(t)) = c_1 {}^C D_{a,t,w}^{\alpha,\beta,\gamma} f(t) + c_2 {}^C D_{a,t,w}^{\alpha,\beta,\gamma} g(t)$ holds for all scalars c_1, c_2 and functions $f, g \in H^1(a,b)$. This implies that the new generalized fractional derivative is a linear operator.
(ii) $^C D_{a,t,1}^{\alpha,\beta,\gamma}(c) = 0$, for all constant function $f(t) = c$.
(iii) $^C D_{a,t,w}^{0,\beta,\gamma} f(t) = \frac{1}{w(t)} \int_a^t E_0(0) \frac{d}{dx}(wf)(x)dx = \frac{1}{w(t)}(w(t)f(t) - w(a)f(a))$.

From the last property, we observe that when the derivative order is equal to zero, we do not recover the original function, unless $f(a)$ is null. To avoid this problem, we present the following new definition.

Definition 2. *Let $\alpha \in [0,1)$, $\beta, \gamma > 0$, and $f \in H^1(a,b)$.*
The new generalized fractional derivative of order α of Riemann–Liouville sense of the function $f(t)$ with respect to the weight function $w(t)$ is given by

$$^R D_{a,t,w}^{\alpha,\beta,\gamma} f(t) = \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \frac{d}{dt} \int_a^t E_\beta[-\mu_\alpha(t-x)^\gamma] w(x) f(x)dx. \tag{2}$$

Equation (2) shows that the new generalized fractional derivative of Riemann–Liouville sense of zero order recovers the original function. Indeed,

$$^R D_{a,t,w}^{0,\beta,\gamma} f(t) = \frac{1}{w(t)} \frac{d}{dt} \int_a^t E_0(0) w(x) f(x) dx = f(t).$$

Furthermore, the new generalized fractional derivative in the sense of Riemann–Liouville is a linear operator. In fact, for all scalars c_1, c_2 and functions $f, g \in H^1(a, b)$, we have

$$
\begin{aligned}
^R D_{a,t,w}^{\alpha,\beta,\gamma} (c_1 f(t) + c_2 g(t)) &= \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \frac{d}{dt} \int_a^t E_\beta[-\mu_\alpha(t-x)^\gamma] w(x) (c_1 f(t) + c_2 g(t)) dx. \\
&= c_1 {}^R D_{a,t,w}^{\alpha,\beta,\gamma} f(t) + c_2 {}^R D_{a,t,w}^{\alpha,\beta,\gamma} g(t).
\end{aligned}
$$

Theorem 1. *Let wf be an analytic function. Then*

$$^R D_{a,t,w}^{\alpha,\beta,\gamma} f(t) = {}^C D_{a,t,w}^{\alpha,\beta,\gamma} f(t) + \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} E_\beta[-\mu_\alpha(t-a)^\gamma](wf)(a). \tag{3}$$

Proof. Since wf is an analytic function, we have
$$(wf)(x) = \sum_{n=0}^{+\infty} \frac{(wf)^{(n)}(t)}{n!}(x-t)^n \text{ and}$$

$$
\begin{aligned}
^R D_{a,t,w}^{\alpha,\beta,\gamma} f(t) &= \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \frac{d}{dt} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^n(-\mu_\alpha)^k (wf)^{(n)}(t)}{n!\Gamma(\beta k+1)} \int_a^t (t-x)^{\gamma k+n} dx \\
&= \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \frac{d}{dt} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^n(-\mu_\alpha)^k (wf)^{(n)}(t)(t-a)^{\gamma k+n+1}}{n!\Gamma(\beta k+1)(\gamma k+n+1)} \\
&= \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^n(-\mu_\alpha)^k}{n!\Gamma(\beta k+1)(\gamma k+n+1)} (wf)^{(n+1)}(t)(t-a)^{\gamma k+n+1} \\
&\quad + \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^n(-\mu_\alpha)^k}{n!\Gamma(\beta k+1)} (wf)^{(n)}(t)(t-a)^{\gamma k+n} \\
&= \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^n(-\mu_\alpha)^k}{n!\Gamma(\beta k+1)} (wf)^{(n+1)}(t) \int_a^t (t-x)^{\gamma k+n} dx \\
&\quad + \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} (wf)^{(n)}(t)(t-a)^n \sum_{k=0}^{+\infty} \frac{(-\mu_\alpha)^k}{\Gamma(\beta k+1)} (t-a)^{\gamma k} \\
&= {}^C D_{a,t,w}^{\alpha,\beta,\gamma} f(t) + \frac{N(\alpha)}{1-\alpha} \frac{1}{w(t)} E_\beta[-\mu_\alpha(t-a)^\gamma](wf)(a).
\end{aligned}
$$

This completes the proof. $\square$

3. Laplace Transform of the New Derivative

In this section, we determine the Laplace transform of the generalized fractional derivative of both types, Caputo and Riemann–Liouville.

Lemma 1. *The Laplace transform of $E_\beta(-\mu_\alpha t^\gamma)$ is given by*

$$\mathcal{L}\{E_\beta(-\mu_\alpha t^\gamma)\}(s) = \frac{1}{s} \sum_{k=0}^{+\infty} \left(\frac{-\mu_\alpha}{s^\gamma}\right)^k \frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}. \tag{4}$$

In particular, when $\gamma = \beta$, we have

$$\mathcal{L}\{E_\beta(-\mu_\alpha t^\beta)\}(s) = \frac{s^{\beta-1}}{s^\beta + \mu_\alpha}, \quad \left|\frac{\mu_\alpha}{s^\beta}\right| < 1. \tag{5}$$

Proof. We have

$$
\begin{aligned}
\mathcal{L}\{E_\beta(-\mu_\alpha t^\gamma)\}(s) &= \mathcal{L}\left\{\sum_{k=0}^{+\infty}\frac{(-\mu_\alpha t^\gamma)^k}{\Gamma(\beta k+1)}\right\}(s) \\
&= \sum_{k=0}^{+\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\beta k+1)}\mathcal{L}\left\{t^{\gamma k}\right\}(s) \\
&= \frac{1}{s}\sum_{k=0}^{+\infty}\left(\frac{-\mu_\alpha}{s^\gamma}\right)^k\frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}.
\end{aligned}
$$

□

According to Lemma 1, we can easily get the following result.

Theorem 2. *The Laplace transform of $^C D_{a,t,w}^{\alpha,\beta,\gamma}$ is given by*

$$\mathcal{L}\{w(t)\,^C D_{0,t,w}^{\alpha,\beta,\gamma}f(t)\}(s) = \frac{N(\alpha)[s\mathcal{L}\{w(t)f(t)\}(s) - w(0)f(0)]}{(1-\alpha)s}\sum_{k=0}^{+\infty}\left(\frac{-\mu_\alpha}{s^\gamma}\right)^k\frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}. \tag{6}$$

In particular, we have

$$\mathcal{L}\{w(t)\,^C D_{0,t,w}^{\alpha,\beta,\beta}f(t)\}(s) = \frac{N(\alpha)}{1-\alpha}\frac{s^\beta\mathcal{L}\{w(t)f(t)\}(s) - s^{\beta-1}w(0)f(0)}{s^\beta + \mu_\alpha}. \tag{7}$$

Further, the Laplace transform of $^R D_{a,t,w}^{\alpha,\beta,\gamma}$ is given by

$$\mathcal{L}\{w(t)\,^R D_{0,t,w}^{\alpha,\beta,\gamma}f(t)\}(s) = \frac{N(\alpha)}{1-\alpha}\mathcal{L}\{w(t)f(t)\}(s)\sum_{k=0}^{+\infty}\left(\frac{-\mu_\alpha}{s^\gamma}\right)^k\frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}. \tag{8}$$

In particular, we have

$$\mathcal{L}\{w(t)\,^R D_{0,t,w}^{\alpha,\beta,\beta}f(t)\}(s) = \frac{N(\alpha)}{1-\alpha}\frac{s^\beta\mathcal{L}\{w(t)f(t)\}(s)}{s^\beta + \mu_\alpha}. \tag{9}$$

Obviously, we have the following remark.

Remark 1. *When $w(t) = 1$ and $\beta = \gamma = \alpha$, we obtain the Laplace transform of the Atangana–Baleanu fractional derivatives in the sense of Caputo and Riemann–Liouville calculated in [19].*

4. Fractional Integral Associated to the New Derivative

This section focuses on the definition of fractional integral corresponding to the new generalized derivative.

Theorem 3. *The following fractional differential equation:*

$$^R D_{0,t,w}^{\alpha,\beta,\beta}y(t) = f(t) \tag{10}$$

has a unique solution given by

$$y(t) = \frac{1-\alpha}{N(\alpha)}f(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\frac{1}{w(t)}\int_0^t (t-x)^{\beta-1}w(x)f(x)dx. \tag{11}$$

Proof. We have

$$w(t)^R D_{0,t,w}^{\alpha,\beta,\beta}y(t) = w(t)f(t).$$

By passage to Laplace transform and applying Theorem 2, we find

$$
\begin{aligned}
\mathcal{L}\{w(t)y(t)\}(s) &= \frac{1-\alpha}{N(\alpha)}\mathcal{L}\{w(t)f(t)\}(s) + \frac{1-\alpha}{N(\alpha)}\frac{\mu_\alpha}{s^\beta}\mathcal{L}\{w(t)f(t)\}(s) \\
&= \frac{1-\alpha}{N(\alpha)}\mathcal{L}\{w(t)f(t)\}(s) + \frac{1-\alpha}{N(\alpha)}\frac{\mu_\alpha}{\Gamma(\beta)}\mathcal{L}\{t^{\beta-1}*(wf)(t)\}(s)
\end{aligned}
$$

The passage to the inverse Laplace leads to

$$
w(t)y(t) = \frac{1-\alpha}{N(\alpha)}w(t)f(t) + \frac{1-\alpha}{N(\alpha)}\frac{\mu_\alpha}{\Gamma(\beta)}\left(t^{\beta-1}*(wf)(t)\right).
$$

Thus, $y(t) = \frac{1-\alpha}{N(\alpha)}f(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\frac{1}{w(t)}\int_0^t (t-x)^{\beta-1}w(x)f(x)dx.$ □

Definition 3. *When $\gamma = \beta$, we define the generalized fractional integral corresponding to new fractional derivative as follows*

$$I_{a,t,w}^{\alpha,\beta,\beta}f(t) = \frac{1-\alpha}{N(\alpha)}f(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\frac{1}{w(t)}\int_a^t (t-x)^{\beta-1}w(x)f(x)dx. \tag{12}$$

This generalized fractional integral coincides with the Atangana–Baleanu fractional integral when $w(t) = 1$ and $\gamma = \beta = \alpha$, and with the weighted Atangana–Baleanu fractional integral defined by Al-Refai [20] when $\gamma = \beta = \alpha$. Additionally, we recover the original function when $\alpha = 0$ and also the ordinary integral when $\alpha = 1$.

On the other hand, we have

$$
\begin{aligned}
^R D_{a,t,w}^{\alpha,\beta,\gamma}f(t) &= \frac{N(\alpha)}{1-\alpha}\frac{1}{w(t)}\sum_{k=0}^{+\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\beta k+1)}\frac{d}{dt}\int_a^t (t-x)^{\gamma k}(wf)(x)dx \\
&= \frac{N(\alpha)}{1-\alpha}\frac{1}{w(t)}\sum_{k=0}^{+\infty}\frac{\gamma k(-\mu_\alpha)^k}{\Gamma(\beta k+1)}\int_a^t (t-x)^{\gamma k-1}(wf)(x)dx.
\end{aligned}
$$

Hence,

$$^R D_{a,t,w}^{\alpha,\beta,\gamma}f(t) = \frac{N(\alpha)}{1-\alpha}\sum_{k=0}^{+\infty}\frac{\Gamma(\gamma k+1)}{\Gamma(\beta k+1)}(-\mu_\alpha)^k \mathcal{I}_{a,t,w}^{\gamma k}f(t), \tag{13}$$

where

$$\mathcal{I}_{a,t,w}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{1}{w(t)}\int_a^t (t-x)^{\alpha-1}(wf)(x)dx, \tag{14}$$

which denotes the weighted Riemann–Liouville fractional integral of order α. Consequently, the generalized derivative in the sense of Riemann–Liouville can be represented by an infinite series whose general term contains the weighted Riemann–Liouville integral.

5. Application

Mathematical modeling in epidemiology has become an effective tool for understanding and describing the dynamics of infectious diseases. It currently used to predict the evolution of coronavirus disease 2019 (COVID-19) in many countries. The first epidemiological model was

introduced by Ross [21,22] to study transmission of Malaria in early 1900. Based on Ross' ideas, Kermack and Mckendrick [23] presented a susceptible-infected-recovered (SIR) compartmental model in order to explain the evolution of the plague in island of Bombay over the period 17 December 1905 to 21 July 1906. This classical SIR model was extended by many researchers to describe other infectious diseases (see for example [24,25]). In this section, we consider the following model:

$$\begin{cases} S'(t) &= A - \mu S(t) - \kappa S(t)I(t), \\ I'(t) &= \kappa S(t)I(t) - (r+\mu)I(t), \\ R'(t) &= rI(t) - \mu R(t), \end{cases} \tag{15}$$

where $S(t)$, $I(t)$ and $R(t)$ are the number of susceptible, infected, and removed individuals at time t, respectively. The parameters A, μ, κ and r represent the recruitment rate, the natural death rate, the infection rate and the removal rate, respectively.

Let $T(t)$ be the total population. Then $T(t) = S(t) + I(t) + R(t)$ and

$$T'(t) = A - \mu T(t). \tag{16}$$

Clearly, the solution of (16) is given by

$$T(t) = \frac{A}{\mu} + \left(T(0) - \frac{A}{\mu}\right)e^{-\mu t}. \tag{17}$$

For simplicity, we denote ${}^{C}D_{0,t,w}^{\alpha,\beta,\beta}$ by $\mathcal{D}_{t,w}^{\alpha,\beta}$. When a disease spreads within a community, individuals acquire knowledge about this disease. It is more reasonable to replace the classical derivative by $\mathcal{D}_{t,w}^{\alpha,\beta}$. Then (16) becomes

$$\mathcal{D}_{t,w}^{\alpha,\beta}T(t) = A - \mu T(t). \tag{18}$$

In the following, we are interested to solve this last fractional differential equation which plays a significant role in epidemiology as well as in virology. In particular for human immunodeficiency virus (HIV) infection, $T(t)$ can represent the concentration of healthy CD4^{+} T cells that are produced at rate A and die at rate μ.

Applying Laplace transform to (18), we obtain

$$\mathcal{L}\{w(t)\mathcal{D}_{t,w}^{\alpha,\beta}T(t)\} = A\mathcal{L}\{w(t)\} - \mu\mathcal{L}\{w(t)T(t)\}.$$

From Theorem 2, we have

$$\mathcal{L}\{w(t)T(t)\} = \frac{N(\alpha)w(0)T(0)s^{\beta-1}}{[N(\alpha) + \mu(1-\alpha)]s^{\beta} + \alpha\mu} + \frac{A(1-\alpha)s^{\beta} + A\alpha}{[N(\alpha) + \mu(1-\alpha)]s^{\beta} + \alpha\mu}\mathcal{L}\{w(t)\}.$$

Let $a_\alpha = N(\alpha) + \mu(1-\alpha)$. Then

$$
\begin{aligned}
\mathcal{L}\{w(t)T(t)\} &= \frac{N(\alpha)w(0)T(0)}{a_\alpha}\frac{s^{\beta-1}}{s^\beta + \frac{\alpha\mu}{a_\alpha}} + \frac{A(1-\alpha)}{a_\alpha}\frac{s^{\beta-1}}{s^\beta + \frac{\alpha\mu}{a_\alpha}}s\mathcal{L}\{w(t)\}\\
&\quad + \frac{A\alpha}{a_\alpha}\frac{1}{s^\beta + \frac{\alpha\mu}{a_\alpha}}\mathcal{L}\{w(t)\}\\
&= \frac{N(\alpha)w(0)T(0)}{a_\alpha}\mathcal{L}\{E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)\}\\
&\quad + \frac{A(1-\alpha)}{a_\alpha}\mathcal{L}\{E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)\}\big(\mathcal{L}\{w'(t)\} + w(0)\big)\\
&\quad - \frac{A}{\mu}\mathcal{L}\{\frac{d}{dt}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)\}\mathcal{L}\{w(t)\}.
\end{aligned}
$$

The passage to the inverse Laplace leads to

$$
\begin{aligned}
w(t)T(t) &= \frac{N(\alpha)w(0)T(0)}{a_\alpha}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big) + \frac{A(1-\alpha)}{a_\alpha}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)*w'(t)\\
&\quad - \frac{A}{\mu}\frac{d}{dt}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)*w(t) + \frac{A(1-\alpha)w(0)}{a_\alpha}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big).
\end{aligned}
$$

By using integration by parts, we have

$$
\frac{d}{dt}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)*w(t) = E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)w(0) - w(t) + E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)*w'(t).
$$

Therefore,

$$
T(t) = \frac{A}{\mu} + \frac{N(\alpha)w(0)}{a_\alpha w(t)}\big(T(0) - \frac{A}{\mu}\big)E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big) - \frac{AN(\alpha)}{\mu a_\alpha w(t)}E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big)*w'(t). \tag{19}
$$

Remark 2.

(i) *For $w(t) = 1$, Equation (19) becomes*

$$
T(t) = \frac{A}{\mu} + \frac{N(\alpha)}{a_\alpha}\big(T(0) - \frac{A}{\mu}\big)E_\beta\big(-\frac{\alpha\mu}{a_\alpha}t^\beta\big). \tag{20}
$$

(ii) *For $w(t) = 1$ and $\beta = \alpha = 1$, Equation (19) coincides with that in (17).*

Now, we study numerically the impact of the order of new fractional derivative on the dynamics behavior of the solution given by (20). For the case of HIV infection, we choose $A = 10$ cells μL^{-1} day^{-1}, $\mu = 0.0139$ day^{-1} and $T(0) = 600$ cells μL^{-1}. For simplicity, we take $N(\alpha) = 1$.

Figure 1 shows that when $\alpha = \beta = 1$ the graph of (20) coincides with that of the ordinary differential equation given by (17). In addition, when the order of fractional derivative increases the solution given by (20) converges rapidly to the steady state $\frac{A}{\mu}$.
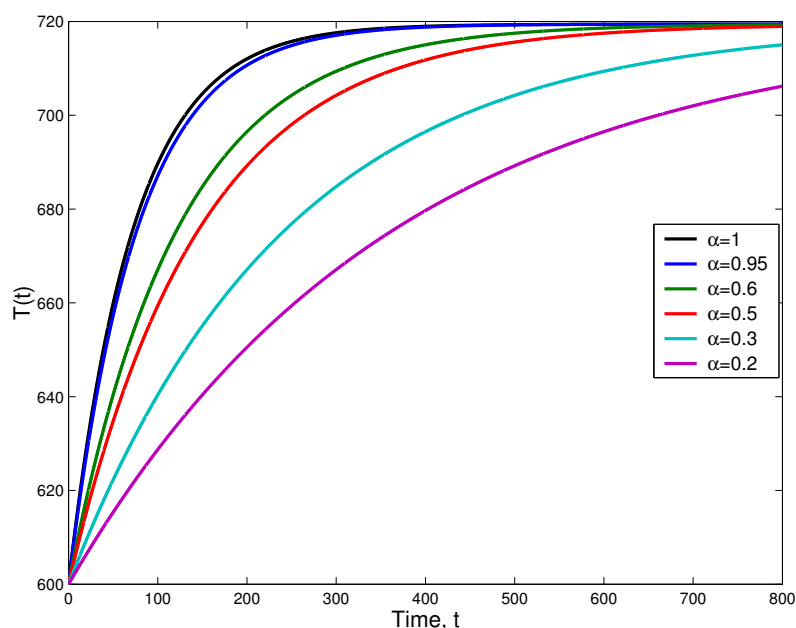
Figure 1. Graph of (20) for different values of α and $\beta = 1$.

6. Conclusions

In this work, we have proposed a new fractional derivative with non-singular kernel which includes the Caputo–Fabrizio fractional derivative, the Atangana–Baleanu fractional derivative and the recent weighted Atangana–Baleanu fractional derivative presented [20]. We have derived some fundamental properties of this new generalized derivative and applied it to a model in epidemiology as well as in virology. In addition, we have studied numerically the impact of the order on the dynamical behavior of the biological model.

Modeling other biological systems with memory or having hereditary properties using the new fractional derivative, and also the determination of other important properties of this new derivative, will be the subject of our future works.

Conflicts of Interest: The author declares no conflict of interest.

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