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Joint Optimization of Preservation Technology, Hybrid Payment Policies, and Prepayment Discounts for Non-Instantaneously Deteriorating Items with Shortages

El-Awady Attia and Md Sharif Uddin *

Industrial Engineering Department, College of Engineering in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia; e.attia@psau.edu.sa
* Correspondence: uddinm@sunybroome.edu or s.uddin@psau.edu.sa

Abstract

Retailers of non-instantaneously deteriorating items must jointly set inventory, preservation technology, and payment decisions. Preservation technology reduces deterioration, but excessive investment increases operational costs, making the determination of an optimal preservation level essential for maximizing profit. Although preservation technology, hybrid payment schemes, and prepayment discounts have been studied individually, their joint treatment alongside partially backlogged shortages remains largely unexplored. To address this gap, this study develops an inventory model that simultaneously incorporates preservation technology investment, a hybrid payment structure, advance payment combined with trade credit, optionally supplemented by a prepayment discount, and partially backlogged shortages for non-instantaneously deteriorating items. A classical optimization approach is employed, yielding quasi-closed-form solutions for the shortage and replenishment timing across four trade credit scenarios, while the profit-maximizing preservation investment level is identified through sensitivity analysis. Numerical examples and sensitivity analysis show that increasing the number of prepayment installments lowers the discount rate offered by the supplier; because this forgone discount outweighs the benefit of retaining capital longer, the retailer's profit falls. Profit responds most strongly to purchasing cost, the advance payment period, and lead time. These results give retailers a practical basis for balancing preservation investment, payment structure, and shortage policy to maximize profitability. In the sensitivity analysis, profit varies by more than 45% over the tested range of the purchasing cost and by up to 21% depending on the number of prepayment installments negotiated with the supplier. That gives retailers a concrete ranked basis for prioritizing which contract terms to negotiate first.

Keywords: inventory model; non-instantaneous deterioration; preservation technology; advance payment; delay-in-payments; discounts; shortage

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1. Introduction

Globally, more than one billion tons of food are wasted every year, representing 19% of all food available to consumers, while a further 13% is lost after harvest and before it ever reaches retail [1]. This scale of loss underscores how inefficient inventory and preservation practices for perishable and non-instantaneously deteriorating products translate directly into lost revenue, wasted resources, and weaker sustainability performance for retailers. Developing a realistic and efficient inventory management system is essential to the success of any business, especially with increasing market competition and the need to reduce costs [2,3]. The inventory models become more complicated when dealing with perishable products that cannot be stored under normal conditions for long periods. These products gradually start to deteriorate after a while. Deterioration is the decay, evaporation, damage, or decreasing utility of products [4]. Deterioration is one of the important factors that regularly affect inventory costs, alongside ordering, holding, and shortage costs. However, some products do not deteriorate immediately; this has led to a distinct line of research known as non-instantaneous deteriorated products [5]. A representative example is the potato; after harvest, tubers pass through a dormancy period during which sprouting does not occur, and only once dormancy is broken do respiration, weight loss, and physiological breakdown accelerate [6]. Comparable behavior is reported for other root crops and for many fruits held under controlled storage. This structure should not, however, be assumed for all fresh products. Highly perishable leafy vegetables such as spinach and lettuce lose quality from the moment of harvest and remain marketable for only about 10–14 days, even at 0 °C [7]; thus, the non-instantaneous assumption is a poor approximation for them. The model developed in this study is therefore intended for products with an identifiable and non-negligible fresh period, and its accuracy degrades as that period approaches zero. In today's competitive markets, it is critical to consider non-instantaneous deterioration (NID). Management of product deterioration involves placing products under specific storage conditions, such as temperature and humidity control, sterilization, modified atmosphere packaging, or any other approved processing. Therefore, instantaneous or non-instantaneous products of both types need energy-efficient technology to manage deterioration; i.e., preservation technology to prolong the life of deteriorating products and slow their decay [8,9].

The implementation of such preservation technology requires high investments. Investment in these technologies is considered one of the most important future trends in food security and food industry. It contributes to reducing food waste, improving quality, and extending food shelf-life, leading to better profits and lower costs. On the other hand, overinvestment in preservation technologies is a problem that can negatively impact operational efficiency and sustainability, as using technologies that exceed the actual needs of the product or target market can lead to unjustified increases in investment and operating costs without generating tangible economic benefits. Furthermore, the excessive use of modern technologies increases the organization's environmental impact due to the increased energy consumption. Therefore, to ensure investment efficiency and ensure the best economic and environmental trade-off, a balanced approach based on in-depth analyses is required. Such an approach can be employed using realistic mathematical inventory modeling that considers non-instantaneous deteriorated products, investment in preservation technology, shortage for considering product freshness, and different payment schemes. According to [10], such inventory models are an effective strategic tool for encouraging best practices in sustainable supply chains. It directly contributes to improving operational efficiency and reducing waste by identifying optimal ordering and storage quantities, thus minimizing losses, especially for perishable products. Such models can enhance environmental sustainability by reducing the number of shipments, energy consumption, and reducing emissions related to storage and transportation operations [11,12]. Other models try to reduce carbon emissions by incorporating carbon emissions, preservation technology, and NID into an Economic Order Quantity (EOQ) [13,14]. Furthermore, it helps reduce product storage time, reducing the need for long-term preservation technologies with high environmental impact [15]. By balancing the costs of ordering, holding, and shortage, the model can provide a basis for informed financial and environmental decisions and improve the effectiveness of investments in preservation technologies.

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Beyond preservation investment, the timing of replenishment also depends on how stock-outs are managed, since not all unmet demand during a shortage is necessarily lost. Shortage and backlogging allow unsatisfied demand during a stock-out to be fulfilled later when the next lot arrives. Customers wait at a rate that declines with the length of the waiting time, so only part of the shortage is recovered and the rest is lost. In the present model this behavior is not left unspecified; the willingness to wait is assumed to decay exponentially in the remaining time until the next lot arrives, so that the proportion of customers still prepared to wait reduces at a constant proportional rate. The exponential form is adopted for three reasons. It is the specification most widely used in the partially backlogged deteriorating-inventory literature [16–18]. In addition, it keeps the backordered quantity and its associated cost analytically tractable within the piecewise profit function. Moreover, it spans the two extremes of interest, since a vanishing decay rate recovers complete backlogging while a large one approaches a pure lost sales system. A single parameter therefore governs the entire backlogging response of the model, determining how much of the shortage is recovered and how much is lost. In deteriorating inventory models, this partial backlogging feature links shortage costs to the length of the stock-out interval, thereby influencing the chosen cycle time and preservation level.

In addition to operational decisions, the financial policies play a critical role in inventory performance. Thus, the proposed inventory model considers a payment scheme that relies on advance and delay-in-payment. The current literature describes three types of payment systems for purchasing goods by a retailer: instant payment, prepayment, and delay-in-payment. Additionally, there may be some combination of these payment strategies used in certain cases [19]. To increase sales, a supplier may offer an advanced payment scheme to a retailer such that the retailer can prepay a portion of the total paid amount in multiple installments. This is a very interesting offer for retailers. It reduces some of the retailers' risk, such as receiving products in due time and not paying a large amount at one time. More importantly, in the interim period, the retailer sells the product and can pay for the installments afterwards. Delay-in-payment also plays an important role in making business policy decisions [20]. In delay-in-payment schemes, retailers obtain time to earn sales revenue to maximize their profit. At the same time, the supplier obtains an opportunity to earn interest from the retailer. Thus, both participants benefit from the advantages of the trade credit policy. In addition to these credit policies, suppliers sometimes attempt to entice customers (retailers) to use different approaches; for instance, by offering a discount on the total purchase cost. Some customers may not be interested in accepting trade credit policy every time. Therefore, in that case, discounts are a widely used marketing policy for attracting customers and boosting product sales quickly. Numerous discount policies have been studied over the years by a significant number of researchers. Price discounts are popular among those policies [21]. The proposed model develops a synergy between advance payments and discounts on multiple prepayments alongside trade credit, a combination whose compatibility is rarely examined in the existing literature, as detailed next.

For this topic, a research gap is arising. Although extensive research has been conducted on deteriorating inventory systems, preservation technology investment, and trade-credit-based payment policies, existing studies largely examine these dimensions in isolation or under restrictive assumptions. The literature on preservation technology primarily focuses on either instantaneous or NID without fully integrating advanced financial mechanisms such as hybrid advance and delay-in-payment schemes. Similarly, studies on advance payment and trade credit policies often neglect deterioration control through preservation investment or consider simplified deterioration structures without shortages or partial backlogging. Moreover, while discounts have been widely studied as pricing incentives, their compatibility with advance payment schemes, particularly discounts on multiple prepayments, remains largely unexplored. Two recent studies [22,23] combine preservation technology with trade credit or delay-in-payment mechanisms and confirm the field remains active. However, neither incorporates a supplier discount tied to the number of prepayment installments, the specific gap this paper addresses. Existing models that incorporate trade credit and preservation technology typically assume fixed or simplified payment structures and rarely analyze the joint interaction among preservation investment, hybrid payment policies, shortages, and pricing decisions within a single-warehouse framework. Such a single-warehouse framework is particularly relevant for small and medium-sized retailers, who typically lack the capital or storage capacity to operate a rented or secondary warehouse and must instead coordinate ordering, preservation, and payment decisions within a single facility. Consequently, there is a clear need for an integrated inventory model that simultaneously considers NID, preservation technology investment, advance and delay-in-payment mechanisms, discount based prepayment incentives, and partial backordering to provide more realistic and actionable managerial insights. Operationally, treating these decisions in isolation leads a retailer to misprice the cost of prepayment. A preservation technology plan sized without reference to the payment structure can leave capital tied up in advance payments precisely when it is needed to fund the preservation investment itself. In addition, a payment plan negotiated without reference to shortages can lock in an installment schedule that is no longer optimal once backordering is allowed. An integrated model is what lets a retailer see these trade-offs together, rather than optimizing one lever while inadvertently worsening another.

It is worth noting at the beginning that advance payment is evaluated here strictly as a cost-and-profit mechanism. The model is deterministic, and uncertainty of demand is not considered. Therefore, one cannot claim that prepayment may mitigate demand risk, stabilize supply, or even reduce the probability of order cancellations. Its effect enters the model through exactly two channels; the cyclic capital cost the retailer incurs by committing funds before delivery, and the discount the supplier grants in return. The trade-off between these two channels is what the analysis quantifies. To translate this gap into concrete lines of inquiry, this study addresses the following research questions:

- RQ1: How should a retailer jointly determine the optimal preservation technology investment and shortage/replenishment policy for non-instantaneously deteriorating items under a hybrid advance-and-delay payment scheme?
- RQ2: How does a supplier's discount on a multiple prepayments policy interact with preservation investment and shortage decisions to shape the retailer's optimal payment strategy and total profit?
- RQ3: Among the operational and financial parameters of a hybrid payment scheme, which ones actually govern the retailer's profit, and is the resulting ordering stable enough to reduce the joint model to a small set of dominant decision levers?

To address the aforementioned challenges and research gaps, this study proposes an integrated inventory framework that simultaneously considers product deterioration behavior, preservation technology investment, and alternative payment and discount policies. By jointly modeling operational and financial decision variables, this study aims to capture their interdependence and assess their combined impact on the retailer's profitability and managerial performance. In particular, the proposed approach provides a comprehensive analysis of how preservation efforts, payment incentives, and shortage related decisions interact within a unified decision-making environment. The major issues addressed in this study are summarized as follows:

Theoretical contributions:

- Develop an inventory model for non-instantaneously deteriorating items incorporating preservation technology investment.
- Jointly analyze advance-payment, trade credit, and discount-based prepayment policies.
- Investigate the interaction between shortages, discount strategies, and payment incentives.
- Formulate the retailer's profit as a piecewise function across four trade credit scenarios and derive quasi-closed-form optimal solutions via Hessian-based concavity proofs.

Managerial contributions:

- Determine an optimum balance among shortages, advance payments, and trade credits or discounts.
- Provide managerial insights through numerical and sensitivity analyses.

The other sections of this study are as follows. In Section 2, a brief literature review is presented to highlight the research gap, while in Section 3, notations and assumptions for the anticipated model are presented. Moreover, the basic model is presented in two cases: Section 3.3.1 supports a model with advance payment and trade credit approaches with theoretical implications, while Section 3.3.4 presents advance payments from a discount approach. Section 4 presents a numerical analysis, and a sensitivity analysis is provided in Section 5. A conclusion with limitations and future scopes for the proposed study finalizes the paper in Section 6.

2. Literature Review

Deteriorating inventory research began with [24], who added a constant decay rate to the classical EOQ. Dye [25] keeps this constant rate assumption and proves, through fractional programming, that preservation investment is profitable only when the non-deterioration period is short. Constant decay is analytically simple, but it forces the spoilage hazard to remain flat over the whole storage life. Weibull models were introduced essentially because their shape parameter allows that hazard to be increasing, decreasing, or constant, so a single family can represent products whose spoilage accelerates with age, as well as those whose losses are concentrated early. This flexibility also allows the distribution to be fitted to the observed shelf-life characteristics of a particular product rather than assumed. Many researchers, like [16,26–28], adopt a two-parameter Weibull distribution that starts after a fresh interval; they show that the optimal cycle lengthens and the preservation budget rises as the shape parameter increases. Storage geometry has also evolved. Researchers like [13,14,16,25,27,28] treat a single warehouse, while studies such as [12,17,29] examine two warehouses: an owned space with low holding cost and an unlimited rented space with higher cost. These two-warehouse studies assume Last In First Out (LIFO) dispatch and demonstrate that the rented warehouse is only used when the order size exceeds the owned capacity, but the resulting cost saving is sensitive to the deterioration rate in each storage zone.

2.1. Preservation Technology

Product deterioration is one of the imperative issues in a competitive business. Therefore, it is a common situation where each retailer wants to minimize the rate of product deterioration as much as possible, as it is not possible to stop completely (as it is a natural phenomenon). The problem represents real-world scenarios for perishable goods like food products, pharmaceuticals, and chemicals, where deterioration begins after a threshold period and demand follows complex, non-linear patterns [26]. Preservation technology is modelled as a capital expense that lowers the effective deterioration rate. Studies such as [25–27] treat the investment as a continuous decision variable with diminishing returns; they prove that a unique positive preservation budget exists when the non-deterioration period is short and the holding or carbon penalty is high. Authors like [13,14,30] extend the idea to green settings and show that the optimal preservation level falls as the carbon tax rises because longer cycles become less attractive. Das et al. [16] compares concave and exponential preservation functions and finds the exponential form yields slightly higher profit, while [12] links preservation effort to the non-deterioration interval instead of to the decay rate. Through these studies, technology is always assumed to work instantly and its cost is deterministic; random failure or learning effects are still unexplored.

Chang et al. [31] developed a deteriorating inventory model under advance cash credit payment schemes with partial backlogging, demonstrating the value of integrating financial and operational decisions for deteriorating items. Dye and Hsieh [32] specifically examined the effects of preservation technology investment (PTI), showing that the overall profit per product time is a concave function of PTI. Dye [25] then extended this, showing a monotonic relation between NID and optimal solutions given non-instantaneous deteriorating substances with PTIs. Later, Mashud et al. [9] projected an inventory model with preservation technology for NID items under the effect of trade credit procedures. In that model, a classical optimization technique was used to obtain the closed-form solution that showed the retailer's profit. Li et al. [33] proposed a joint pricing model for NID items using preservation technology. In that paper, they proposed two cases on the relationship between the holding period and NID period. In this proposed model, a non-instantaneous deteriorating inventory problem and advance and delay-in-payment under preservation technology are discussed. Shortages are endorsed and backlogged partially with a fixed backlogging amount, which is absent in prior studies (see Table 1 for a comparative summary against related studies). Building on this, Li et al. [33] and [34] developed an inventory model that integrates pricing, replenishment, and preservation technology investment decisions for deteriorating items. Li et al. [33] concluded that investment in preservation technology is not always optimal for retailers. Others, such as [27], integrated a profit-maximizing inventory model for NID items that incorporates learning effects, preservation technology investment, and price- and time-dependent demand, jointly optimizing key decision variables and validated through numerical and sensitivity analyses. Recently, [26] developed two distinct inventory models for products exhibiting quadratic demand patterns and NID following a Weibull distribution. The first model examines conventional inventory management without preservation interventions, while the second incorporates preservation technologies to mitigate deterioration effects. The study aims to quantify cost savings achievable through preservation technology adoption, demonstrating through analytical methods and numerical simulations that preservation strategies extend shelf-life, reduce waste, and lower total inventory costs for perishable goods in demand-sensitive environments. Most recently, Liao et al. [35] linked preservation technology investment for non-instantaneous deteriorating items with expiration dates to an order-size-triggered advance payment scheme, while Alharbi [36] solved a NID and preservation technology model for green products using a metaheuristic (dragonfly) algorithm. Across this body of work, preservation technology is almost always modelled as an isolated investment decision, decoupled from the payment and financing terms under which the retailer actually acquires the inventory, an interaction that the present study addresses directly.

2.2. Advance Payment

One more vital factor in inventory management is the advance payment scheme, which many firms use; however, its effect on inventory control tactics is scarcely considered in the collected works. Lashgari [37] presented a model capturing both advance and delay-in-payments with three different scenarios: without shortage, partial backordering, and full backordering. Diabat et al. [38] modelled stock dependent demand with partial downstream delay-in-payment and partial upstream advance payment under the effect of shortages with partial backordering. In this manner, Taleizadeh [39] established an inventory model considering two situations: full prepayment and partial prepayment with partial delayed payment. The results showed that there is no effect of the length of the advance payment period on the inventory cycle. This finding may seem counterintuitive, but it should be examined carefully rather than accepted at face value. In fact, it is due to the underlying structure. In this class of models, the prepayment horizon enters the objective function only through the cost of capital committed before delivery, which scales the effective unit purchase cost by a factor of the form one plus a small multiple of that horizon, the multiple collecting the interest rate, the prepaid fraction, and the number of installments. Because that factor stays close to unity at realistic interest rates and prepaid fractions, the prepayment horizon shifts the level of the retailer's cost while barely perturbing the first-order conditions that fix the cycle. The present model reproduces the same behavior; in sensitivity analysis, a twenty percent change in the lead time alters the cyclic capital cost substantially; yet, the optimal replenishment and shortage timings move by less than one tenth of one percent. Therefore, this result reflects the separation of the financing term from the ordering decision rather than a true independence between prepayment and inventory policy. It is unlikely to hold in models where the prepayment period also affects demand, deterioration, or the order quantity. Later, [40] proposed a collaborative approach between a retailer and supplier with the effect of advance payments to optimize profits. They found that advance payment financing enhances profit when companies are not in financial distress. Chang et al. [41] provided a comprehensive idea about payment systems for an EPQ model. They proposed an advance-cash-credit payment (ACC) where some portion of purchase costs were paid in advance, some at the time of product receipt, and the balance in credit policy. Mishra [42] proposed a sustainable inventory model with advance payments under environmental emissions, while [15] exposed the influences of advanced payments in a two-warehouse sustainable inventory system using a trade credit strategy. Recently, Chang [31] developed an inventory model for perishable goods in which suppliers impose an ACC payment and retailers allow customers cash–credit terms. Chiu et al. [14] proposed the use of advance payment policy, where the supplier provides a price discount that depends on whether the retailer pays the full purchasing cost upfront in a single installment or pays a partial amount in advance through multiple installments, with the discount rate varying by the number of installments. Nath et al. [43] developed an EOQ model that jointly integrates three factors: imperfect quality items in received lots, deteriorating items with constant deterioration rate, and advance payment policy with cash discount facility. Their objective is to determine the optimal ordering quantity that maximizes total profit per unit time under both full and partial advance payment schemes. Across this body of work, advance payment is generally analyzed either in isolation or jointly with a single complementary mechanism, such as trade credit or a purchase discount, but its combined interaction with preservation technology investment and partial backlogging under deteriorating inventory is rarely examined jointly, a gap this study addresses directly. It is worth noting that a supplier discount tied to the number of prepayment installments, the mechanism this paper terms "discount on multiple prepayments", has begun to appear in a number of recent studies. Khan et al. [44] examined variable prepayment installments under a power demand pattern and carbon cap and price regulation, while a related study [45] considered installment-linked prepayment discounts within a sustainable vendor–buyer supply chain under carbon allowance policy.

Neither study, however, incorporates preservation technology investment, shortages, or a comparison against trade credit, all of which are jointly addressed in the present paper.

2.3. Delay-in-Payment (DIP)

Delay-in-payment is a process where one representative asks another party to pay the purchase price within a certain time limit. It may happen in a retail supplier or retailer–customer relationship. All agents have the power to offer DIP for certain goods except the end person/customer. The most exciting and well-known DIP used among retailers and customers is when a retailer offers a fixed period for its customers to repay purchase amounts. On the other hand, DIP (trade credit) demonstrates a commanding source of short-term external financing for retailers and results in a lower unit purchasing cost. During this whole period, customers do not have to pay any interest to the retailer. In this period, the customer has no financial pressure. However, in some other contracts with customers, under certain terms and situations, interest is required.

In the current literature, Goyal [46] first addressed the common use of trade credit policy in an EOQ model with allowable DIP; interest is charged, measured on the acquisition cost of properties transacted inside the allowable DIP period. Aggarwal [47] modified this model by considering exponential deterioration of products as an additional factor. Furthermore, Md Mashud [8] proposed an inventory model for NID that considers trade credit, preservation technology, and dependent demand, while [48] also provided an inventory model aimed at two warehouses, which included advance and delay-in-payments. However, the two-warehouse solution may not be affordable for every retailer. Chung et al. [49] provided a model for trade credit, with both quantity and cash discounts for a single warehouse. Bearing this in mind, a single-warehouse system with some realistic attributes (advance payment, trade credit strategy and discount, shortages) is considered in this proposed model. Recently, [50] proposed a deteriorating inventory model considering DIP. Moreover, the effect of DIP was highly considered in different sustainable inventory models. Jauhari et al. [51] integrated circular economy, green investing, and trade credit financing and demonstrated that strategic delay of payment not only increases joint benefit but also reduces emissions under the pressure of a carbon tax. In addition, [30] used a power function for preservation technology investment costs, differing from the exponential functions commonly employed in prior literature. The model aims to maximize retailer profit by optimizing pricing, replenishment cycles, and preservation technology investments across varying greenness levels. In [52], they considered the delay of payment with credit in a three-level supply chain. One of the limitations of their model is the ignorance of discounts for advance payment. Kaushik et al. [28] developed an inventory model for deteriorating items with ramp-type demand under a two-tier permissible delay payment system. The model considers a finite time horizon with two distinct interest rate charges: a lower rate after the first credit period, and a higher rate after the second credit period.

Most DIP literature examines how trade credit affects ordering and preservation decisions in deteriorating inventory systems. Studies such as [52–54] introduce single or progressive credit periods during which buyers earn interest on sales before paying suppliers; they show that longer credit lengthens the optimal cycle and reduces the need for preservation because capital costs fall. Others such as [28] extended this to a two-tier interest structure and prove that retailers will delay payment up to the higher-rate threshold only when the deterioration penalty is moderate. Studies such as [43] and [29] combine advance and delayed payments, revealing that partial prepayment with cash discount can raise profit if the discount rate exceeds the loan rate. Throughout these works, shortages are considered and demand is generally kept constant; none jointly incorporate preservation technology investment alongside credit-based payment and shortage decisions, an interaction this paper addresses directly. Most recently, Bansal et al. [55] extended multi-level trade credit to imperfect and gradually deteriorating items under carbon emission considerations, though without shortages or preservation technology investment.

2.4. Hybrid Payment

A hybrid payment scheme splits the procurement cost into an advance portion and a delayed portion. The buyer pays a preset fraction of the order value before or at delivery, while the supplier grants a credit period for the remaining balance. This mixed structure reduces default risk for the supplier and offers the retailer partial cash-flow relief. In deteriorating inventory models, the scheme alters the timing of cash outflows, so the optimal cycle and preservation investment adjust to balance interest charges, early-payment discounts, and spoilage costs. Some authors like [52] considered the hybrid payment policy; they considered advance payment, cash payment, and credit payment in both upstream (supplier–retailer) and downstream (retailer–customer) relationships in a three-level supply chain. Limi and Rangarajan [12] considered the hybrid payment scheme that mixes upfront and delayed payments to provide a way to manage finances, giving more flexibility in cash flow and improving operational efficiency. Khan et al. [53] investigated the influence of the hybrid payment schemes (advance installments + cash-on-delivery) on optimal replenishment decisions and total inventory costs. Jaggi et al. [54] investigated how different payment scenarios—e.g., full payment, partial payments, or progressive payment—impact total inventory costs. Progressive payment means multiple credit periods with increasing interest rates. They used three-tier progressive policy, in which the first period is interest-free, interest is charged in the second period, and for the third period, a higher interest is charged. They found that progressive trade credit is more beneficial for retailers than single-period credit, providing flexibility to delay payments when cash flow is constrained. Khan et al. [29] developed an EOQ model that integrates a hybrid payment scheme combining partial advance payment and partial delay payment, both linked to order size; quantity discounts on purchase cost; limited own-warehouse capacity necessitating rented warehouse usage; and partial backlogging during stockout periods. They highlighted that flexible hybrid payment schemes enable retailers to optimize profits by selecting order sizes that balance prepayment costs, credit benefits, and capacity constraints. Shekhar et al. [56] recently combined hybrid cash-advance payment policies with prepayment discounts in a two-warehouse sustainable setting, though their model does not consider preservation technology or shortages, both of which they flag as future extensions. While these studies demonstrate the value of hybrid payment structures for cash-flow management, none jointly incorporate preservation technology investment or a supplier discount tied to the number of prepayment installments, leaving the financial–operational interaction explored in this paper largely unaddressed.

2.5. Backlogging

Shortage is a situation that occurs when the inventory level cannot satisfy the customer demand, resulting in unmet demand for a specified period. In this scenario, the company does not have enough stock to satisfy immediate customer needs, leading to either lost sales or backordering. Shortages can be temporary and are characterized by the accumulation of unmet demand during stock-out periods. When shortages occur, customers who are willing to wait for the product are considered backlogged. Their unmet demands are temporarily stored as backorders, which are fulfilled later once the inventory is replenished. Specifically, partial backordering refers to situations where only a fraction of the unmet demand is backlogged, and the rest can be considered as lost sales. Many authors, such as [16,26,27], adopted partial backlogging rates that decrease exponentially or hyperbolically with waiting time; they show that longer shortage intervals raise the optimal preservation budget because the firm must protect a smaller, slower-moving inventory. A parallel and largely separate stream has studied the structure of partial backordering itself rather than its interaction with deterioration. San José et al. [57] analyzed an inventory system in which the backordered fraction decays exponentially in the waiting time and obtained the optimal policy in closed form; the same group later generalized the result to a wider class of customer impatience functions [58], while Pentico and Drake [59] give the corresponding treatment for the deterministic EOQ under a constant backordering fraction. These studies establish the analytical properties of the exponential impatience function adopted in the present work, but they do so in settings without deterioration, preservation investment, or payment structure. Singh and Rathore [17] coupled stock-dependent demand with partial backlogging under inflation and found that higher customer impatience (larger backlogging parameter) shortens the cycle and increases preservation spending to reduce spoilage during the subsequent recovery phase. Khan et al. [29,53] embed capacity-driven shortages in two-warehouse settings; when the owned store is full, excess demand is backlogged while rented space is filled, so the backlogging fraction interacts with the higher holding cost to determine whether extra capacity is economical. Across these studies, backlogging is typically analyzed in isolation from preservation technology investment and hybrid payment decisions; how a deterministic backlogging fraction jointly interacts with these financial and preservation choices remains comparatively underexplored.

Table 1. An assessment of the proposed work with prior studies.

Authors	Payment System	Deterioration Form	Shortage Type	Preservation Technology	Discount
[17]		CD	PB		No
[60]	Advance	CD	PB	No	No
[39]	Advance and Partial Delay	No	PB	No	Yes
[61]	Advance	Time-Varying	PB	No	No
[62]	Advance and Partial trade credit	Time-varying	PB	No	No
[18]	Hybrid partial prepayment, partial trade credit	CD	PB	No	No
[8]	Delay-in-payments	NI	PB	Yes	No
[63]	Cash, advance and credit payments	No	No	No	Yes
[42]	Advance	NI	No	Yes	Yes
[48]	Advance and delay-in-payments	CD	PB	No	No
[14]	Advance payment	NI		Yes	Yes
[52]	Hybrid payment policy	NI		Yes	No
[12]	Hybrid payment policy	NI		Yes	No
[16]		NI	PB	Yes	No
[35]	Advance (order-size linked)	NI	No	Yes	No
[56]	Hybrid cash-advance	General (perishable)	No	No	Yes
[55]	Multi-level trade credit	NI (gradual)	No	No	No
[36]	No	NI	No	Yes	
[44]	Advance (installment-linked discount)	No	No	No	Yes
[45]	Advance (installment-linked discount)	No	No	No	Yes
This paper	Advance and Delay	NI	PB	Yes	Yes

N.B. Constant Deterioration = CD; Partially Backlogged = PB; Non-instantaneous = NI.

Taken together, these studies are not merely complementary; literature contains unresolved tensions that directly motivate the present model. First, the evidence on when preservation investment pays is contradictory; Dye [25] concludes that preservation is profitable only when the non-deterioration period is short, Das et al. [16] finds that the choice of preservation function itself changes the optimal profit, and Limi and Rangarajan [12] ties preservation effort to the length of the non-deterioration interval rather than to the decay rate, so these studies do not agree even on what the investment acts upon. Second, the treatment of deterioration is split between constant rate and Weibull formulations, and none of the works cited here compare the two under an identical payment structure; it therefore remains unclear how much of the reported profit difference is attributable to the deterioration model rather than to the financial assumptions. Third, the payment literature is internally inconsistent about prepayment; Taleizadeh et al. [39] reports no effect of the prepayment horizon on the cycle, whereas Nath and Sen [43] and Khan et al. [29] find that partial prepayment with a cash discount raises profit whenever the discount rate exceeds the loan rate. As argued in Section 2.2, this divergence follows from whether the financing term is separable from the ordering decision, a distinction the literature has not made explicit. Fourth, and common to almost all of these works, the backlogging fraction is treated as a modelling convenience rather than a decision-relevant quantity, and its interaction with the payment structure is not examined at all.

The gaps this study closes, and those it does not, can therefore be stated precisely. It closes the joint treatment gap; preservation technology investment, a hybrid advance and delay payment structure, a supplier discount tied to the number of prepayment installments, and partially backlogged shortages are optimized within a single deterministic profit function, and Table 1 positions the model against the closest prior studies on each of these dimensions. Several gaps remain open. Demand is deterministic, so questions of prepayment under demand risk lie outside the scope of the model; a single product and a single warehouse are assumed, so capacity and assortment interactions are not addressed; and the deterioration rate is constant, so the constant rate versus Weibull question raised above is left unresolved.

3. Mathematical Formulations

3.1. Problem Description

Before formalizing the model, it is useful to describe the overall decision problem the retailer faces. A retailer stocks a non-instantaneously deteriorating product and must jointly decide: (i) how much to invest in preservation technology to slow deterioration, (ii) which payment arrangement to use with the supplier, either an advance payment combined with trade credit (Case 1) or an advance payment combined with a prepayment discount alongside trade credit (Case 2), and (iii) how to time the onset and duration of shortages, which are partially backlogged rather than fully lost. Within Case 1, the retailer's optimal policy further depends on how the supplier's permissible delay period M relates to the shortage timing, giving rise to four distinct payment scenarios developed in Section 3.3.2. Figure 1 summarizes how these decisions and the solution approach fit together; the detailed notation, assumptions, and derivations follow in the remainder of this section.

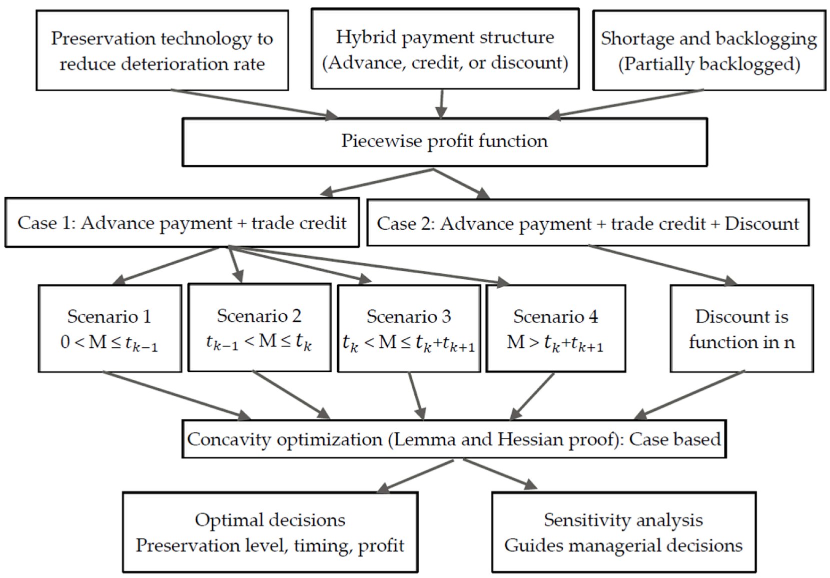


Figure 1. Overview of the model components and solution approach.

3.2. Model Assumptions

- The deterioration rate θ is constant, and ($0 < \theta < 1$). This reflects common practice in fresh-food and pharmaceutical supply chains, where a retailer's own records show deterioration proceeding at an approximately steady rate once it begins, rather than accelerating or decelerating sharply within a single order cycle.
- Once deterioration occurs on products, there is no chance of replacement or repair during the complete cycle. Only a single type of deteriorating item is considered. An insignificant lead time is considered with an infinite replenishment rate for the total planning horizon. This matches the reality that most perishable retailers cannot reprocess or restore deteriorated stock (e.g., wilted produce or expired pharmaceuticals) and instead absorb it as a loss, and that a single-item, single-cycle framework is the standard starting point for isolating the payment and preservation trade-offs this paper studies.
- The retailer proceeds with the percentage σ of the total purchase cost to the supplier through several n equally spaced installments inside the lead time L, accumulating the lot by compensating the residual purchasing cost (similar to the assumption used by Mashud et al. [15]). This imitates installment-based procurement contracts commonly used by small and medium sized retailers, who negotiate a partial advance payment spread over several installments rather than a single lump sum.
- The function for preservation technology is $m(\xi) = e^{-a_1\xi}$, where $m(\xi)$ is the multiplicative factor ($0 < m(\xi) < 1$) by which preservation technology investment reduces the deterioration rate and θ is the base deterioration rate that applies with no preservation effort, so the resultant, effective deterioration rate entering the inventory-depletion equation (Equation (2)) is $\theta m(\xi)$; a_1 is the coefficient governing how effectively investment translates into this reduction [34,64]. Because $m(\xi)$ is strictly decreasing in ξ, greater preservation investment yields a proportionally slower rate of spoilage, matching the physical role of preservation technology in delaying deterioration. The identifier $m(\xi)$ continuously satisfies the first-order and second-order derivatives concerning preservation technology investment ξ.
- For this study, the partial backordered are noted as $e^{-v(t_k+t_{k+1}-t)}$ for $v > 0$, where $(t_k + t_{k+1} - t)$ is the customers' wait time, which will be filled from the next lot. This represents customers who are willing to wait for a limited time after a stockout before giving up and taking their business elsewhere. This pattern is widely documented at retail checkouts for backordered perishable and non-perishable goods alike.
- In an alternative case (Section 3.3.4), the supplier offers a discount price based on the number of prepayment installments alongside trade credit; the corresponding assumptions for this discount-linked case are detailed separately in that section. This captures suppliers who use a prepayment discount rather than trade credit as their preferred incentive for securing upfront cash flow, a mechanism increasingly offered alongside or instead of trade credit in supplier–retailer contracts.

3.3. Mathematical Model Formulation

In the proposed framework, two types of models are proposed to provide managerial insights. In the first case, for advance payment and securing payment, a trade credit approach has been applied. In the second case, an advance payment through a price discount is offered to entice customers toward the business alongside trade credit.

3.3.1. Case I: When an Advance Payment and Trade Credit Are Jointly Offered

Let us assume a trade where the retailer prepays a part σ of the buying cost (formerly the product supply) per the wish of the supplier within the bounded lead time L by n equal, multiple installments. The retailer obtains the lot at time $t = 0$ per deal with the supplier. The retailer takes the supplier's offer to pay the residual portion $(1 - \sigma)$ of the purchasing cost within the managed offer. The managed offer is a trade credit strategy. After some time, R units are exploited to partially satisfy the backlogged demand; subsequently, the on-hand inventory level becomes S. Throughout the time interval $[0, t_{k-1}]$, the inventory level decreases in arrears to link to the customer's demand D. It drops to zero at $t = t_k$, due to the combined effect of demand and deterioration. Shortages appear afterwards, and the total demand in the period $[t_k, t_k + t_{k+1}]$ is partially backordered. Figure 2 displays the graphical representation of inventory contrasted with time in which the inventory level fluctuates between $0 \le t \le t_k + t_{k+1}$, represented by $I(t)$.

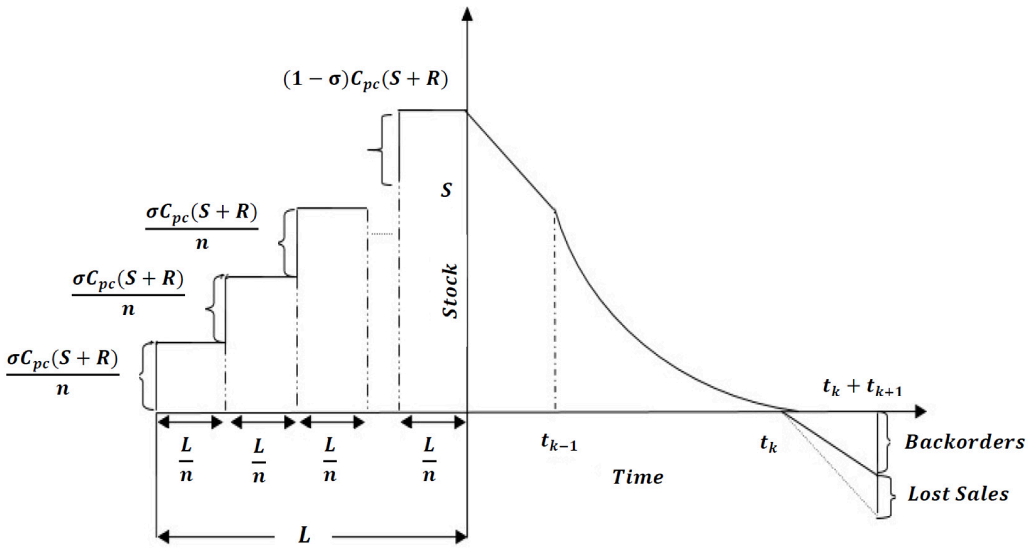


Figure 2. Diagram of inventory against time.

The whole inventory scenario can be explained by using the following differential equations.

The first stage, denoted Phase 1 (pre-deterioration depletion), covers the interval during which the stock depletes solely due to demand before deterioration sets in:

$$\frac{dI_1(t)}{dt} = -D, 0 < t \le t_{k-1} \tag{1}$$

where $I_1(t) = S$ at $t = 0$.

The second differential equation, governing Phase 2 (depletion under deterioration), is:

$$\frac{dI_2(t)}{dt} + \theta m(\xi) I_2(t) = -D \quad t_{k-1} < t \le t_k \tag{2}$$

where $I_2(t) = 0$ at $t = t_k$.

After plugging in the conditions in Equations (1) and (2), the results are:

$$I_1(t) = S - Dt, 0 < t \le t_{k-1} \tag{3}$$

and

$$I_2(t) = \frac{D}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_k - t)} - 1 \right), t_{k-1} < t \le t_k \tag{4}$$

Here, it is clear that $I(t)$ is continuous at $t = t_{k-1}$; hence, $I_1(t_{k-1}) = I_2(t_{k-1})$, resulting in:

$$S = Dt_{k-1} + \frac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) \tag{5}$$

As the on-hand stock has been exhausted at point $t = t_k$, shortages appear, marking the onset of Phase 3 (shortage/backlogging). This can be expressed through the differential equation as:

$$\frac{dI_3(t)}{dt} = -De^{-v(t_k + t_{k+1} - t)} \qquad t_k < t \le t_k + t_{k+1} \tag{6}$$

with $I_3(t) = 0$ at $t = t_k$, one has the solution as:

$$I_3(t) = -\frac{D}{v}\left[e^{-v(t_k+t_{k+1}-t)} - e^{-vt_{k+1}}\right] \qquad t_k < t \le t_k + t_{k+1} \tag{7}$$

By using the conditions $I_3(t) = -R$ at $t = t_k + t_{k+1}$, the aggregate quantity that needs to be backlogged is:

$$R = -I_3(t_k + t_{k+1}) = \frac{D}{v}\left[1 - e^{-vt_{k+1}}\right] \tag{8}$$

Therefore, the order quantity is:

$$\begin{aligned} Q &= S + R \\ &= Dt_{k-1} + \frac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) + \frac{D}{v}\left[1 - e^{-vt_{k+1}}\right] \end{aligned} \tag{9}$$

The different costs associated with the total inventory system are:

Ordering cost (OC): For any retail modelling, this is the most important basic cost. Products can be ordered through phone calls, emails, or other means. The corresponding charge is known as the ordering cost. It is always the cost per order, and it is generally considered as follows:

$$OC = A \tag{10}$$

Inventory holding cost (HC): To hold products, the basic cost is known as the per-unit holding time cost per product. The cost is designed centered on the average inventory per unit time. The calculation is:

$$\begin{aligned} HC &= c_{IHC}\int_0^{t_{k-1}} I_1(t)dt + c_{IHC}\int_{t_{k-1}}^{t_k} I_2(t)dt \\ &= c_{IHC}\left[St_{k-1} - \frac{Dt_{k-1}^2}{2} - \frac{D}{\theta^2 m(\xi)^2}\left(1 - e^{\theta m(\xi)(t_k - t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\right)\right] \end{aligned} \tag{11}$$

Backorder cost (BC): Even though shortages are present, some consumers will wait for the retailer due to the company's early commitment or brand image. These consumers wait for the retailer, and profit is gained from the consumers even though it is delayed due to shortages. Therefore, the backorder cost is as follows:

$$\begin{aligned} BC &= c_{BC}\int_{t_k}^{t_k+t_{k+1}} I_3(t)dt \\ &= c_{BC}\tfrac{D}{v}\left[\int_{t_k}^{t_k+t_{k+1}} \left[e^{-v(t_k+t_{k+1}-t)} - e^{-vt_{k+1}}\right]dt\right] \\ &= c_{BC}\tfrac{D}{v}\left[\tfrac{1}{v} - \left(\tfrac{1}{v} + t_{k+1}\right)e^{-vt_{k+1}}\right] \end{aligned} \tag{12}$$

Opportunity cost due to lost sales (LSC): Some consumers do not wait for the retailer and have their demands fulfilled from the other retailers during a shortage. This is the cost resulting from those lost profits. The cost can be calculated as:

$$\begin{aligned} LSC &= c_{LSC}\int_{t_k}^{t_k+t_{k+1}}\left\{1-e^{-v(t_k+t_{k+1}-t)}\right\}D\,dt \\ &= c_{LSC}D\left\{t_{k+1}-\tfrac{1}{v}\left(1-e^{-vt_{k+1}}\right)\right\} \end{aligned} \tag{13}$$

Total purchasing cost (PC): This is the cost needed to purchase the product. Generally, it is the price per unit quantity if there is no discounted quantity offered. The purchasing cost is:

$$\begin{aligned} PC &= c_{PC}Q \\ &= c_{PC}\left[Dt_{k-1}+\tfrac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k-t_{k-1})}-1\right)+\tfrac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] \end{aligned} \tag{14}$$

Sales Revenue: The total complete cycle's sales revenue is calculated through total sales in the whole cycle, with the products' selling price such that:

$$SR = p\left[Dt_k+\frac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] \tag{15}$$

Preservation technology cost (PTC): This is the cost of controlling the rate of deterioration. This can be found as follows:

$$PTC = \xi(t_k+t_{k+1}) \tag{16}$$

Cyclic Capital cost: The cyclic capital cost from Figure 2 for the purchaser in an advance payment scheme obtaining the order is (using similar procedures as [3,60]):

$$\begin{aligned} CCC &= \left\{\begin{array}{l} \left(\frac{I_p\sigma c_{PC}}{n}Q\times n\times\frac{L}{n}\right)+\left(\frac{I_p\sigma c_{PC}}{n}Q\times(n-1)\times\frac{L}{n}\right)+\left(\frac{I_p\sigma c_{PC}}{n}Q\times(n-(n-2))\times\frac{L}{n}\right) \\ \left(\frac{I_p\sigma c_{PC}}{n}Q\times(n-(n-1))\times\frac{L}{n}\right) \end{array}\right\} \\ &= \left(\frac{I_p\sigma c_{PC}}{n}Q\frac{L}{n}\right)[n+(n-1)+\cdots\cdots+2+1] = \frac{n+1}{2n}I_pL\sigma c_{PC}Q \\ &= \frac{n+1}{2n}I_pL\sigma c_{PC}\left[Dt_{k-1}+\frac{D}{\theta m(\xi)}(e^{\theta m(\xi)(t_k-t_{k-1})}-1)+\frac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] \end{aligned} \tag{17}$$

The two limiting cases confirm the derivation: at $n = 1$, a single prepayment is held for the whole lead time, and as $n \to \infty$, payment becomes continuous with average commitment $L/2$. At $n = 1$: $CCC = \frac{2}{2}I_pL\sigma c_{PC}Q = I_pL\sigma c_{PC}Q$ (single prepayment held for the whole lead time). At $n \to \infty$: $CC \to \frac{1}{2}I_pL\sigma c_{PC}Q$ (continuous payment, average commitment $L/2$).

Before turning to the scenario-specific derivations, it is useful to preview the overall structure of the retailer's profit. In every scenario, the cycle profit is built from the same components introduced above: sales revenue, ordering cost, purchasing cost, holding cost, backorder cost, lost-sale cost, preservation technology cost, and cyclic capital cost, averaged over the cycle length (t_k + t_{k+1}). What changes from one scenario to the next is only the interest-earned and interest-charged terms, IE_i and IC_i, which depend on where the trade credit period M falls relative to the shortage timing; the remaining cost and revenue structure stays fixed. Formally:

$$\Pi^i(t_k,t_{k+1}) = \frac{1}{t_k+t_{k+1}}[SR-OC-PC-HC-BC-LSC-PTC-CCC+IE_i-IC_i],$$

3.3.2. Inventory Conditions for Different Scenarios

According to the rate of permissible delays in the payment (M) granted by the supplier to the retailer, four altered scenarios arise, shown as:

Scenario 1: $0 < M \le t_{k-1}$
Scenario 2: $t_{k-1} < M \le t_k$
Scenario 3: $t_k < M \le t_k + t_{k+1}$

Scenario 4: $M > t_k + t_{k+1}$

Interest earned:

In this section, the retailer's total interest will be discussed according to three different trade credit periods. For the total interest earned from the sales of positive stocks in the chain within the allowed period, the retailer deposits the revenues in an interest-bearing account at any bank.

Scenario 1: $0 < M \leq t_{k-1}$

The total interest earned within the mentioned scenario is measured in Equation (18):

$$\begin{aligned} IE_1 &= pI_e(1-\sigma)\int_0^{t_k}\int_0^t D\,du\,dt + \frac{pI_e(1-\sigma)Dt_k}{v}\left[1-e^{-vt_{k+1}}\right] \\ &= \frac{pI_e(1-\sigma)Dt_k^2}{2} + \frac{pI_e(1-\sigma)Dt_k}{v}\left[1-e^{-vt_{k+1}}\right] \end{aligned} \tag{18}$$

The total payable interest is:

$$\begin{aligned} IC_1 &= c_{PC}I_p\int_M^{t_{k-1}} I_1(t)dt + c_{PC}I_p\int_{t_{k-1}}^{t_k} I_2(t)dt \\ &= c_{PC}I_p\left[S(t_{k-1}-M) - D\frac{\left(t_{k-1}{}^2 - M^2\right)}{2}\right] \\ &\qquad - \frac{c_{PC}I_pD}{(\theta m(\xi))^2}\left[\left(1-e^{\theta m(\xi)(t_k-t_{k-1})}\right) + \theta m(\xi)(t_k-t_{k-1})\right] \end{aligned} \tag{19}$$

Therefore, the total profit is:

$$\Pi^1(t_k, t_{k+1}) = \frac{X1}{t_k + t_{k+1}}$$

with $X1 = (SR - OC - PC - HC - BC - LSC - PTC - CCC + IE_1 - IC_1)$

$$\begin{aligned} X1 = {} & p\left[Dt_k + \frac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] - A - c_{PC}\left[Dt_{k-1} + \frac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\right) + \frac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] \\ & - c_{IHC}\left[St_{k-1} - \frac{Dt_{k-1}^2}{2} - \frac{D}{\theta^2 m(\xi)^2}\left(1 - e^{\theta m(\xi)(t_k-t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\right)\right] \\ & - C_{BC}\frac{D}{v}\left[\frac{1}{v} - \left(\frac{1}{v} + t_{k+1}\right)e^{-vt_{k+1}}\right] - c_{LSC}D\left\{t_{k+1} - \frac{1}{v}\left(1-e^{-vt_{k+1}}\right)\right\} \\ & - \xi(t_k + t_{k+1}) - \frac{n+1}{2n}I_pL\sigma c_{pc}\left[Dt_{k-1} + \frac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\right) + \frac{D}{v}\left[1-e^{-vt_{k+1}}\right]\right] \\ & + \frac{pI_e(1-\sigma)Dt_k^2}{2} + \frac{pI_e(1-\sigma)Dt_k}{v}\left[1-e^{-vt_{k+1}}\right] - c_{PC}I_p\left[S(t_{k-1}-M) - D\frac{\left(t_{k-1}{}^2 - M^2\right)}{2}\right] \\ & - \frac{c_{PC}I_pD}{(\theta m(\xi))^2}\left[\left(1-e^{\theta m(\xi)(t_k-t_{k-1})}\right) + \theta m(\xi)(t_k-t_{k-1})\right] \end{aligned} \tag{20}$$

The resultant problem can be expressed mathematically as:

Problem 1. $\Pi^1(t_k, t_{k+1}) = \frac{X1}{t_k+t_{k+1}}$ with the condition: $0 < M \leq t_{k-1}$.

Scenario 2: $t_{k-1} < M \leq t_k$

The interest earned for this case is formulated in Equation (21) as follows:

$$\begin{aligned} IE_2 &= pI_e(1-\sigma)\int_0^{t_k}\int_0^t D\,du\,dt + \frac{pI_e(1-\sigma)Dt_k}{v}\left[1-e^{-vt_{k+1}}\right] \\ &= \frac{pI_e(1-\sigma)Dt_k^2}{2} + \frac{pI_e(1-\sigma)Dt_k}{v}\left[1-e^{-vt_{k+1}}\right] \end{aligned} \tag{21}$$

The payable interest in this case is:

$$\begin{aligned} IC_2 &= c_{PC}I_p\int_M^{t_k} I_2(t)dt \\ &= c_{PC}I_p\left[\frac{D}{(\theta m(\xi))^2}\left(e^{\theta m(\xi)(t_k-M)} - 1\right) - \frac{D(t_k-M)}{\theta m(\xi)}\right] \end{aligned} \tag{22}$$

Hence, the total profit is:

$$\Pi^2(t_k, t_{k+1}) \;=\; \frac{X2}{t_k+t_{k+1}}$$

where $X2 = (SR - OC - PC - HC - BC - LSC - PTC - CCC + IE_2 - IC_2)$

$$\begin{aligned}
X2 = p\Big[Dt_k + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] - A - c_{PC}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] \\
-c_{IHC}\Big[St_{k-1} - \tfrac{Dt_{k-1}^2}{2} - \tfrac{D}{\theta^2 m(\xi)^2}\Big(1 - e^{\theta m(\xi)(t_k-t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\Big)\Big] \\
-C_{BC}\tfrac{D}{v}\Big[\tfrac{1}{v} - \Big(\tfrac{1}{v} + t_{k+1}\Big)e^{-vt_{k+1}}\Big] - c_{LSC}D\Big\{t_{k+1} - \tfrac{1}{v}\big(1-e^{-vt_{k+1}}\big)\Big\} \\
-\xi(t_k+t_{k+1}) - \tfrac{n+1}{2n}I_pL\sigma c_{pc}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] \\
+\tfrac{pI_e(1-\sigma)Dt_k^2}{2} + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big] - c_{PC}I_p\Big[\tfrac{D}{(\theta m(\xi))^2}\Big(e^{\theta m(\xi)(t_k-M)} - 1\Big) - \tfrac{D(t_k-M)}{\theta m(\xi)}\Big]
\end{aligned} \tag{23}$$

Thus, the equivalent mathematical form of the above equations is as follows:
Problem 2. $\Pi^2(t_k, t_{k+1}) \;=\; \frac{X2}{t_k+t_{k+1}}$ Subject to constraints $t_{k-1} < M \le t_k$

Scenario 3: $t_k < M \le t_k + t_{k+1}$

Interest earned in this case is the same as in previous cases, as shown in Equation (24):

$$\begin{aligned}
IE_3 &= pI_e(1-\sigma)\int_0^{t_k}\int_0^{t} Ddudt + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big] \\
&= \tfrac{pI_e(1-\sigma)Dt_k^2}{2} + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big]
\end{aligned} \tag{24}$$

Since the permissible delay period is smaller than the total cycle length, the retailer does not need to pay any extra interest. Thus, interest paid in this case is zero (IC_3=0).

The updated profit is presented as:

$$\Pi^3(t_k, t_{k+1}) \;=\; \frac{X3}{t_k+t_{k+1}},$$

with $X3 = (SR - OC - PC - HC - BC - LSC - PTC - CCC + IE_3)$

$$\begin{aligned}
X3 = p\Big[Dt_k + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] - A - c_{PC}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] \\
-c_{IHC}\Big[St_{k-1} - \tfrac{Dt_{k-1}^2}{2} - \tfrac{D}{\theta^2 m(\xi)^2}\Big(1 - e^{\theta m(\xi)(t_k-t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\Big)\Big] \\
-C_{BC}\tfrac{D}{v}\Big[\tfrac{1}{v} - \Big(\tfrac{1}{v} + t_{k+1}\Big)e^{-vt_{k+1}}\Big] - c_{LSC}D\Big\{t_{k+1} - \tfrac{1}{v}\big(1-e^{-vt_{k+1}}\big)\Big\} \\
-\xi(t_k+t_{k+1}) - \tfrac{n+1}{2n}I_pL\sigma c_{pc}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k-t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1-e^{-vt_{k+1}}\big]\Big] \\
+\tfrac{pI_e(1-\sigma)Dt_k^2}{2} + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big]
\end{aligned} \tag{25}$$

Thus, the corresponding optimization problem is as follows:
Problem 3. $\Pi^3(t_k, t_{k+1}) \;=\; \frac{X3}{t_k+t_{k+1}}$ subject to constraints $t_k < M \le t_k + t_{k+1}$.

Scenario 4: $M > t_k + t_{k+1}$

Within this condition, the total interest earned can be measured using the following mathematical Equation (26):

$$\begin{aligned}
IE_4 &= pI_e(1-\sigma)\int_0^{t_k}\int_0^{t} Ddudt + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big] + pI_e(1-\sigma)D(t_k+t_{k+1})(M - t_k - t_{k+1}) \\
&= \tfrac{pI_e(1-\sigma)Dt_k^2}{2} + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1-e^{-vt_{k+1}}\big] + pI_e(1-\sigma)D(t_k+t_{k+1})(M - t_k - t_{k+1})
\end{aligned} \tag{26}$$

In this case, the delay period is the same as in scenario 3. Therefore, the retailer does not need to compensate for any interest in the supplier's account under this scenario; i.e., (IC_4 = 0).

Therefore, the modified profit is:

$$\Pi^4(t_k, t_{k+1}) \ = \ \frac{X4}{t_k+t_{k+1}}$$
$$\text{with } X4 = (SR - OC - PC - HC - BC - LSC - PTC - CCC + IE_4)$$

$$\begin{aligned}
X4 = {} & p\Big[Dt_k + \tfrac{D}{v}\big[1 - e^{-vt_{k+1}}\big]\Big] - A - c_{PC}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1 - e^{-vt_{k+1}}\big]\Big] \\
& - c_{IHC}\Big[St_{k-1} - \tfrac{Dt_{k-1}^2}{2} - \tfrac{D}{\theta^2 m(\xi)^2}\Big(1 - e^{\theta m(\xi)(t_k - t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\Big)\Big] \\
& - C_{BC}\tfrac{D}{v}\Big[\tfrac{1}{v} - \Big(\tfrac{1}{v} + t_{k+1}\Big)e^{-vt_{k+1}}\Big] - c_{LSC}D\Big\{t_{k+1} - \tfrac{1}{v}\big(1 - e^{-vt_{k+1}}\big)\Big\} \\
& - \xi(t_k + t_{k+1}) - \tfrac{n+1}{2n}I_p L\sigma c_{pc}\Big[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\Big(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\Big) + \tfrac{D}{v}\big[1 - e^{-vt_{k+1}}\big]\Big] \\
& + \tfrac{pI_e(1-\sigma)Dt_k^2}{2} + \tfrac{pI_e(1-\sigma)Dt_k}{v}\big[1 - e^{-vt_{k+1}}\big] + pI_e(1-\sigma)D(t_k + t_{k+1})(M - t_k - t_{k+1})
\end{aligned} \tag{27}$$

Therefore, the mathematical form of the above equation is:
Problem 4. $\Pi^4(t_k, t_{k+1}) \ = \ \frac{X4}{t_k+t_{k+1}}$ subject to constraints $M > t_k + t_{k+1}$.

3.3.3. Theoretical Implications

The profit function comprises four disjoint profit functions based on different trade credit periods, although the profit function consists of the mentioned cost function obtained from Equations (10)–(17). Thus, one can mathematically write the profit of the retailer in a piecewise function such as:

$$\Pi^i(t_k, t_{k+1}) = \frac{1}{t_k+t_{k+1}}[SR - OC - PC - HC - BC - LSC - PTC - CCC + IE_i - IC_i],$$
$$\forall\, i = 1, 2, 3, 4$$
$$\Pi(t_k, t_{k+1}) = \begin{cases} \Pi^1(t_k, t_{k+1}),\ when\ 0 < M \le t_{k-1} \\ \Pi^2(t_k, t_{k+1}), when\ t_{k-1} < M \le t_k \\ \Pi^3(t_k, t_{k+1}), when\ t_k < M \le t_k + t_{k+1} \\ \Pi^4(t_k, t_{k+1}), when\ M > t_k + t_{k+1} \end{cases}$$

It is possible to derive some continuity relations from the above expression, which are $\Pi^1(t_k, t_{k+1}) = \Pi^2(t_k, t_{k+1})$, which is continuous at point t_{k-1}

$\Pi^2(t_k, t_{k+1}) = \Pi^3(t_k, t_{k+1})$, which is continuous at point t_k

$\Pi^3(t_k, t_{k+1}) = \Pi^4(t_k, t_{k+1})$, which is continuous at point $(t_k + t_{k+1})$

The maximum value of the profit function $\Pi(t_k, t_{k+1})$ can be found by examining the points (t_k, t_{k+1}) directly inside the conditions (i.e., $0 < M \le t_{k-1}$, $t_{k-1} < M \le t_k$, $t_k < M \le t_k + t_{k+1}$, $M > t_k + t_{k+1}$). The optimal form will be measured $(t_k{}^*, t_{k+1}{}^*)$, such that

$$\Pi^*(t_k, t_{k+1}) = max\Big\{\Pi^{*1}(t_k, t_{k+1}), \Pi^{*2}(t_k, t_{k+1}), \Pi^{*3}(t_k, t_{k+1}), \Pi^{*4}(t_k, t_{k+1})\Big\}$$

The optimal values of t_k and t_{k+1} that optimize the above profit function will be the required decision variables for a provided data set. We consider the fourth piecewise profit function for theoretical derivations although all the functions' derivation techniques are the same. It is selected because it is structurally the most financially favorable regime for the retailer. Since M exceeds the full cycle length in this case, no interest is ever charged (IC_4 = 0), and interest is earned on the full cycle's sales revenue for the longest span among the four scenarios. This scenario is therefore expected to yield the highest profit among the four, and it is used as the base case throughout the remaining numerical analysis.

$\Pi^4(t_k, t_{k+1})$ contains exponential terms in t_k and t_{k+1}, both from the deterioration dynamics and from the backlogging term $e^{vt_{k+1}}$. Setting its partial derivatives to zero yields transcendental equations with no closed-form solution. To obtain a tractable, quasi-closed-form expression that supports the concavity proof and a closed-form solution procedure, the exponential terms are approximated by a truncated Taylor series, retaining the first three terms. This truncated form is used only for the analytical derivation in this subsection. All numerical results reported later in the paper are computed from the exact, untruncated profit function, so the truncation does not affect any reported value. By expanding all the exponential terms of the profit function (27) in Taylor's series and taking the first three terms of the series, one obtains:

$$\begin{aligned}\Pi^4(t_k,t_{k+1}) = \tfrac{1}{t_k+t_{k+1}}\{ &\left[pD - D\left(1+\tfrac{n+1}{2n}I_pL\sigma\right)c_{PC} + pI_e(1-\sigma)DM - \xi\right](t_k+t_{k+1}) \\ -\tfrac{1}{2}D[c_{IHC}+pI_e(1-\sigma)]t_k^2 &- \tfrac{1}{2}D\left[vp - v\left(1+\tfrac{n+1}{2n}I_pL\sigma\right)c_{PC} + c_{BC} + vc_{LSC} + 2pI_e(1-\sigma)\right]t_{k+1}^2 \\ &+\tfrac{1}{2}c_{BC}vDt_{k+1}^3 - pI_e(1-\sigma)Dt_kt_{k+1} - \tfrac{1}{2}pI_e(1-\sigma)vDt_kt_{k+1}^2 \\ &-\tfrac{1}{2}\theta m(\xi)D(t_k-t_{k-1})^2\left[\left(1+\tfrac{n+1}{2n}I_pL\sigma\right)c_{PC} + c_{IHC}t_{k-1}\right] - A\}\end{aligned} \tag{28}$$

$$\frac{\partial\Pi^4(t_k,t_{k+1})}{\partial t_k} = -\frac{X}{(t_k+t_{k+1})^2} + \frac{1}{t_k+t_{k+1}}\frac{\partial X}{\partial t_k} \tag{29}$$

$$\frac{\partial^2\Pi^4(t_k,t_{k+1})}{\partial {t_k}^2} = \frac{2X}{(t_k+t_{k+1})^3} - \frac{2}{(t_k+t_{k+1})^2}\frac{\partial X}{\partial t_k} + \frac{1}{t_k+t_{k+1}}\frac{\partial^2 X}{\partial {t_k}^2} \tag{30}$$

$$\frac{\partial\Pi^4(t_k,t_{k+1})}{\partial t_{k+1}} = -\frac{X}{(t_k+t_{k+1})^2} + \frac{1}{t_k+t_{k+1}}\frac{\partial X}{\partial t_{k+1}} \tag{31}$$

$$\frac{\partial^2\Pi^4(t_k,t_{k+1})}{\partial {t_{k+1}}^2} = \frac{2X}{(t_k+t_{k+1})^3} - \frac{2}{(t_k+t_{k+1})^2}\frac{\partial X}{\partial t_{k+1}} + \frac{1}{t_k+t_{k+1}}\frac{\partial^2 X}{\partial {t_{k+1}}^2} \tag{32}$$

For optimization, we know the necessary conditions:

$$\frac{\partial\Pi^4(t_k,t_{k+1})}{\partial t_k} = 0 \tag{33}$$

$$\frac{\partial\Pi^4(t_k,t_{k+1})}{\partial t_{k+1}} = 0 \tag{34}$$

From Equations (33) and (34), Equations (29)–(32) are reduced to the following forms:

$$X = (t_k+t_{k+1})\frac{\partial X}{\partial t_k} \tag{35}$$

$$\frac{\partial^2\Pi^4(t_k,t_{k+1})}{\partial t_k^2} = \frac{1}{t_k+t_{k+1}}\frac{\partial^2 X}{\partial t_k^2} \tag{36}$$

$$X = (t_k+t_{k+1})\frac{\partial X}{\partial t_{k+1}} \tag{37}$$

$$\frac{\partial^2\Pi^4(t_k,t_{k+1})}{\partial t_{k+1}^2} = \frac{1}{t_k+t_{k+1}}\frac{\partial^2 X}{\partial t_{k+1}^2} \tag{38}$$

By amalgamating Equations (35) and (37), we obtain:

$$\frac{\partial X}{\partial t_k} = \frac{\partial X}{\partial t_{k+1}} \tag{39}$$

where

$$\begin{aligned}\tfrac{\partial X}{\partial t_k} = &\left[pD - D\left(1+\tfrac{n+1}{2n}I_pL\sigma\right)c_{PC} + pI_e(1-\sigma)DM - \xi\right] - D[c_{IHC}+pI_e(1-\sigma)]t_k - pI_e(1-\sigma)Dt_{k+1} - \tfrac{1}{2}pI_e(1-\sigma)vDt_{k+1}^2 \\ &-\theta m(\xi)D\left[\left(1+\tfrac{n+1}{2n}I_pL\sigma\right)c_{PC} + c_{IHC}t_{k-1}\right](t_k-t_{k-1})\end{aligned} \tag{40}$$

and

$$\frac{\partial X}{\partial t_{k+1}} = \left[pD - D\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + pI_e(1-\sigma)DM - \xi\right] - pI_e(1-\sigma)Dt_k$$
$$-D\left[vp - v\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{BC} + vc_{LSC} + 2pI_e(1-\sigma) + pI_e(1-\sigma)vt_k\right]t_{k+1} + \frac{3}{2}c_{BC}vDt_{k+1}^2 \quad (41)$$

Performing Equation (39), we have:

$$\lambda_1 t_{k+1}^2 - \lambda_2(t_k)\, t_{k+1} + \lambda_3(t_k) = 0$$

where

$$\lambda_1 = \frac{1}{2}Dv(3c_{BC} + pI_e(1-\sigma)),$$
$$\lambda_2(t_k) = D\left[vp - v\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{BC} + vc_{LSC} + pI_e(1-\sigma)\right] + DpI_e(1-\sigma)vt_k \quad (42)$$

and

$$\lambda_3(t_k) = D\left\{\theta m(\xi)\left[\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{IHC}t_{k-1}\right](t_k - t_{k-1}) + c_{IHC}t_k\right\}$$

By executing some calculations of Equation (42), we obtain:

$$t_{k+1} = \frac{\lambda_2(t_k) - \sqrt{\lambda_2(t_k)^2 - 4\lambda_1\lambda_3(t_k)}}{2\lambda_1} \text{ (the smaller root; the larger root is extraneous)} \quad (43)$$

Thus, t_{k+1} is a function of t_k.

Executing the implicit differentiation of Equation (42) relating to t_k, one has:

$$\frac{dt_{k+1}}{dt_k} = \frac{D\left[\theta m(\xi)\left((1 + \frac{n+1}{2n} I_p L\sigma)c_{PC} + c_{IHC}t_{k-1}\right) + c_{IHC} - pI_e(1-\sigma)vt_{k+1}\right]}{\sqrt{\lambda_2(t_k)^2 - 4\lambda_1\lambda_3(t_k)}} \quad (44)$$

Now, by utilizing the relation of Equation (42) and after brief calculation, we can write Equation (35) in the following form:

$$X - (t_k + t_{k+1})\frac{\partial X}{\partial t_k} = 0, \qquad t_{k+1} \text{ given by Equation (43)} \quad (45)$$

Stimulated by Equation (45), let us consider an auxiliary function, $\phi(t_k)$, $t_k \in \left[t_{k-1}, \grave{t}_k\right])$, such that:

$$\phi(t_k) = X - (t_k + t_{k+1})\frac{\partial X}{\partial t_k} \qquad t_k \in \left[t_{k-1}, \grave{t}_k\right] \quad (46)$$

Then, accumulating Equations (42) and (44), the first-order derivative of $\phi(t_k)$ regarding t_k is:

$$\frac{d\phi(t_k)}{dt_k} > 0 \qquad \text{throughout } t_k \in \left[t_{k-1}, \grave{t}_k\right] \quad (47)$$

Thus, $\varphi(t_k)$ is strictly an increasing function of $t_k \in \left[t_{k-1}, \grave{t}_k\right]$.

At $t_k = t_{k-1}$ from Equation (43),

$$t_{k+1}|_{t_k = t_{k-1}} = \frac{\lambda_2(t_{k-1}) - \sqrt{\lambda_2(t_{k-1})^2 - 4\lambda_1\, D\, c_{IHC}t_{k-1}}}{2\lambda_1} \quad (48)$$

Moreover,

$$\phi(t_{k-1}) < 0, \qquad \text{obtained from Equation (46) at } t_k = t_{k-1} \text{ using Equation (48)} \quad (49)$$

Since $\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{IHC}t_{k-1} > 0$ (call this expression Ω); as C_{PC}, C_{IHC}, t_{k-1}, I_p, L, and σ are all positive, Ω is a sum of strictly positive terms and is therefore always positive; no $\Omega \le 0$ case can arise, the optimum value of t_k is obtained from the unique root of $\varphi(t_k) = 0$ on $[t_{k-1}, \grave{t}_k]$; there is only one case to consider.

Because φ is continuous and strictly increasing on $[t_{k-1}, \grave{t}_k]$, where $\grave{t}_k$ denotes any point within the domain where Equation (43)'s discriminant $\lambda_2(t_k)^2 - 4\lambda_1\lambda_3(t_k)$ remains non-negative (so that $t_{k+1}(t_k)$, and hence $\varphi(t_k)$, is real-valued there), with $\varphi(t_{k-1}) < 0$ (Equation (49)) and $\varphi(\grave{t}_k) > 0$ for a sufficiently large such t_k; the Intermediate Value Theorem guarantees a unique $t_k' \in [t_{k-1}, t_k]$ with $\varphi(t_k') = 0$; this t_k' is the unique solution of Equations (33) and (34), and t_{k+1}' follows from Equation (43).

Considering the above arguments, the following results are obtained.

Lemma 1. *There exists a unique pair of values* $(t_k^*, t_{k+1}^*) = (t_k', t_{k+1}')$ *that satisfy Equations (40) and (41).*

Let us denote the optimal value of (t_k, t_{k+1}) as (t_k^*, t_{k+1}^*). Then, one can acquire the succeeding result.

Proposition 1. *At* $(t_k, t_{k+1}) = (t_k^*, t_{k+1}^*)$, *the required global maximum value is obtained. Hence, the profit function* $\Pi^4(t_k, t_{k+1})$ *becomes concave.*

Proof. From the Lemma, the pair of values (t_k^*, t_{k+1}^*) that maximize profit function $\Pi^4(t_k, t_{k+1})$ can be characterized as follows:

$$(t_k^*, t_{k+1}^*) \text{ is the unique root of } \phi(t_k) = 0 \text{ on } [t_{k-1}, \grave{t}_k], \text{ paired with } t_{k+1}^* \text{ from Equation (43):} \tag{50}$$

At $(t_k, t_{k+1}) = (t_k^*, t_{k+1}^*)$

$$\left[\frac{\partial^2 X}{\partial t_k^2}\right]_{(t_k^*, t_{k+1}^*)} = -D\left[c_{IHC} + pI_e(1-\sigma) + \theta m(\xi)\left(\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{IHC}t_{k-1}\right)\right] < 0 \tag{51}$$

and

$$\left[\frac{\partial^2 X}{\partial t_{k+1}^2}\right]_{(t_k^*, t_{k+1}^*)} = -D\left[vp - v\left(1 + \frac{n+1}{2n} I_p L\sigma\right)c_{PC} + c_{BC} + vc_{LSC} + 2pI_e(1-\sigma) + pI_e(1-\sigma)vt_k^*\right] + 3c_{BC}vDt_{k+1}^* < 0 \tag{52}$$

Next, by compelling the second-order partial derivative of $X(t_k, t_{k+1})$, respecting t_k and t_{k+1}, we have:

$$\left[\frac{\partial^2 X}{\partial t_k\, \partial t_{k+1}}\right]_{(t_k^*, t_{k+1}^*)} = -pI_e(1-\sigma)D\left(1 + vt_{k+1}^*\right) \neq 0$$

Which is strictly negative whenever $\sigma < 1$ and $I_e > 0$, so the cross partial of X does not vanish. At a stationary point, the Hessian of Π^4 is $1/(t_k, t_{k+1})$ times the Hessian of X (the $X/(t_k, t_{k+1})^2$ terms cancel), so concavity requires both $\partial^2 X/\partial t_k^2 < 0$ and the determinant condition:

$$\left[\frac{\partial^2 X}{\partial t_k^2}\right]\left[\frac{\partial^2 X}{\partial t_{k+1}^2}\right] - \left[\frac{\partial^2 X}{\partial t_k\, \partial t_{k+1}}\right]^2 > 0$$

Furthermore, at point $(t_k, t_{k+1}) = (t_k^*, t_{k+1}^*)$, the determinant of the Hessian matrix is positive and definite, confirming the inequality above.

Therefore, at $(t_k, t_{k+1}) = (t_k^*, t_{k+1}^*)$, one obtains the global maximum concerning t_k and t_{k+1} for the optimization problem defined in Equation (28), which completes the proof. □

Remark 1. *This concavity result holds only within the region where Scenario 4 applies ($t_k + t_{k+1} < M$); the full piecewise envelope Π is continuous but has a convex kink at $t_k + t_{k+1} = M$ and is not globally concave. This is why the four scenarios are optimized separately, each on its own smooth region, rather than as a single global problem. Intuitively, this concavity reflects a diminishing returns structure common to all four scenarios. Extending the positive-stock period t_k initially captures more sales revenue, but each additional unit of time also adds holding cost on inventory that is, on average, already partly deteriorated. Therefore, the marginal revenue from a longer stock period falls faster than its marginal cost rises once t_k passes its optimum. The same logic applies to the shortage length t_{k+1}. A longer shortage period recovers more backordered demand at first, but the exponential impatience function $e^{-v(t_k+t_{k+1}-t)}$ means each additional unit of waiting time converts a shrinking share of that demand into a completed sale and a growing share into a lost sale. Therefore, the marginal benefit of extending the shortage period also erodes. These two diminishing-returns effects, which are common to all four scenarios through the same holding cost and backordering, result in a negative definite Hessian in the smooth region of each scenario. However, the scenario-specific interest term IC_i changes the location of the optimum.*

3.3.4. Case 2: When a Discount Is Applied to Prepayment Alongside Trade Credit

There are different approaches to offer discounts to the retailer for the purchase of goods. However, in an environment where the retailer purchases goods from the supplier by prepayment of the purchase cost, to obtain the maximum benefit, it is very effective to offer a discount to motivate prepayment of the entire amount (and as quickly as possible). The discount facility from the supplier regarding prepayment of the retailer’s purchase cost introduced in this paper remains rare in the literature and, to the best of our knowledge, has not previously been examined jointly with preservation technology investment, shortages, and a comparison against trade credit, although the “discount on installments” mechanism itself has recently appeared in isolation in [44] and [45]. This hinges on the constant amount of installment (n) inside the extent of the lead time. If the number of installments increases, then the discount rate will decrease. The retailer obtains a maximum (β%, $\alpha = \beta/\mathrm{n}$, when $0 \le \beta \le 100$) discount when they repay the purchase cost at a time within the lead time’s commencement.

Thus, the prepayment discount is:

$$\begin{aligned} \varpi_{DIS} &= c_{pc} Q \alpha \\ &= \tfrac{c_{pc}\beta}{n}\left[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) + \tfrac{D}{v}\left[1 - e^{-vt_{k+1}}\right]\right] \end{aligned} \tag{53}$$

Exploiting Equation (53) in Equation (27), the resultant profit function is given by Equation (54):

$$\begin{aligned} &\Pi^5(t_k, t_{k+1}) = \tfrac{X5}{t_k + t_{k+1}}, \\ &\text{where } X5 = (SR - OC - PC - HC - BC - LSC - PTC - CCC + \varpi_{DIS} + IE_4) \\ &= p\left[Dt_k + \tfrac{D}{v}\left[1 - e^{-vt_{k+1}}\right]\right] - A - c_{PC}\left[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) + \tfrac{D}{v}\left[1 - e^{-vt_{k+1}}\right]\right] \\ &-c_{IHC}\left[St_{k-1} - \tfrac{Dt_{k-1}^2}{2} - \tfrac{D}{\theta^2 m(\xi)^2}\left(1 - e^{\theta m(\xi)(t_k - t_{k-1})} + \theta m(\xi)(t_k - t_{k-1})\right)\right] \\ &-C_{BC}\tfrac{D}{v}\left[\tfrac{1}{v} - \left(\tfrac{1}{v} + t_{k+1}\right)e^{-vt_{k+1}}\right] - c_{LSC} D\left\{t_{k+1} - \tfrac{1}{v}\left(1 - e^{-vt_{k+1}}\right)\right\} \\ &-\xi(t_k + t_{k+1}) - \tfrac{n+1}{2n} I_p L\sigma c_{pc}\left[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) + \tfrac{D}{v}\left[1 - e^{-vt_{k+1}}\right]\right] \\ &+\tfrac{c_{pc}\beta}{n}\left[Dt_{k-1} + \tfrac{D}{\theta m(\xi)}\left(e^{\theta m(\xi)(t_k - t_{k-1})} - 1\right) + \tfrac{D}{v}\left[1 - e^{-vt_{k+1}}\right]\right] + \tfrac{pI_e(1-\sigma)Dt_k^2}{2} \\ &+\tfrac{pI_e(1-\sigma)Dt_k}{v}\left[1 - e^{-vt_{k+1}}\right] + pI_e(1-\sigma)D(t_k + t_{k+1})(M - t_k - t_{k+1}) \end{aligned} \tag{54}$$

Discount implementation and theoretical derivations:

For Case 2, the implementation of the discount procedure is analogous to Case 1 for avoiding redundancy in the theoretical derivations. The process of testing the concavity of the above problem is similar to Case 1. Therefore, the proof is ignored; preferably, some numerical examples are set to validate the model. Equation (54) is derived under the Scenario 4 trade credit condition (M > $t_k + t_{k+1}$), the same condition used for Case 1's theoretical derivation in Section 3.3.1. This condition is also the practically relevant one: Numerical analysis show that, among Case 1's four scenarios, Scenario 4 yields the highest profit. Case 2 is therefore illustrated under the same condition for direct comparability with Case 1's best-performing scenario; Case 2's own numerical results are reported separately.

4. Numerical Examples

The data used here are similar data to those used by [9], with some additional and revised data to validate the anticipated model.

The optimal solutions for the proposed model with different scenarios, together with all profit function plots and other graphical illustrations, were obtained using a developed "Python 3" implementation (NumPy/Matplotlib) under "Google colab." working environment. For each numerical example, the corresponding trade credit scenario is solved independently over its own feasible domain, as an adoption of case-based optimization. A "nested grid search" is applied directly to the exact, untruncated profit function (Equation (27)) and refined over successive rounds until convergence for all four scenarios similarly. The Taylor-truncated closed form of Section 3.3.3 (Equations (40)–(52)) is used to prove, analytically, that Scenario 4's profit function possesses a unique interior maximum and is concave there; it is not used to compute any reported value. At the grid search optimum, the theoretical concavity conditions (Equations (51) and (52)) retain the correct (negative) sign, consistent with the qualitative picture established in Section 3.3.3.

The nested grid search proceeds as follows. Starting from a bounding box covering the whole feasible domain of (t_k, t_{k+1}), a 260 × 260 grid is evaluated on the exact profit function. The highest scoring point is identified, and the search box is shrunk to a small neighborhood around it; this is repeated for eight rounds. Each round narrows the box by a factor of about 6/260. Therefore, after eight rounds, the box width is reduced by roughly 8×10^{-14}, well below floating-point resolution, at a cost of 540,800 function evaluations. Because the search recenters on the best-scoring point in the whole box every round rather than following a local gradient, it is not sensitive to the starting region and cannot become trapped in a spurious stationary point, unlike a local method such as Newton–Raphson.

Before working through the Case 1 Examples, it is worth stating what each scenario means for the retailer in practice. The four scenarios differ only in where the supplier's permissible delay period M falls relative to the retailer's own replenishment timeline. In Scenario 1 ($0 < M \leq t_{k-1}$), the credit period is so short that it expires before the product even begins to deteriorate. The retailer gains only a brief, low-value financing benefit and still finances most of the cycle out of pocket. In Scenario 2 ($t_{k-1} < M \leq t_k$), the credit period extends into a deterioration phase, giving the retailer more time to sell stock before settling with the supplier. In Scenario 3 ($t_k < M \leq t_k + t_{k+1}$), the credit period reaches into the shortage phase itself, so a portion of the backordered demand is also financed before payment is due. In Scenario 4 ($M > t_k + t_{k+1}$), the credit period outlasts the entire cycle. Therefore, the retailer collects and holds all sales revenue, including the trade credit share, for the full cycle and beyond, before any payment is owed. Operationally, a retailer facing a short credit period (Scenario 1 or 2) gains comparatively little financing value and should consider trade credit less heavily against other terms such as price or discount. Whereas a retailer who is able to negotiate a longer credit period (Scenario 3 or, best of all, Scenario 4) captures substantially more value from trade credit, as the results below confirm.

Example 1.

$A = \$1500$, $D = 100$, $C_{IHC} = \$20$, $C_{BC} = \$15$, $C_{LSC} = \$10$, $C_{PC} = \$55$, p = \$100, $a_1 = \$0.8$, $\theta = 0.25$, $v = 2.5$, $t_{k-1} = 0.2$, $I_e = \$0.06$, $I_p = \$0.09$, $M = 0.08$ *year*, $\sigma = 0.6$, $n = 7$, $\xi = 5$, $L = 0.25$ *year*.

With M = 0.08 year (Scenario 1), the model yields S = 106.342, R = 16.851, t_k = 1.062, t_{k+1}= 0.219, and an optimal profit of Π = 2110.926, as reported in Table 2. Figure 3 shows the optimal profit (the red point) and joint concavity of the total profit function in t_k and t_{k+1} for this scenario. Panels (a) and (b) are the 2D profit curves against the stock period and the shortage period, respectively, each with the other variable held at its optimum, while panel (c) is the corresponding 3D joint profit surface over both variables.

Table 2. Details of outcomes in each trade credit period.

Trade Credit Period Condition	S (Units)	R (Units)	t_k (Years)	t_{k+1} (Years)	$\Pi^i(t_k,t_{k+1})$ (Currency/Time)
$0 < M \le t_{k-1}$	106.342	16.851	1.062	0.219	2110.926
$t_{k-1} < M \le t_k$	107.798	16.365	1.076	0.210	2185.634
$t_k < M \le t_k + t_{k+1}$	121.932	15.635	1.217	0.198	2316.168
$M > t_k + t_{k+1}$	106.314	13.668	1.061	0.167	2360.535

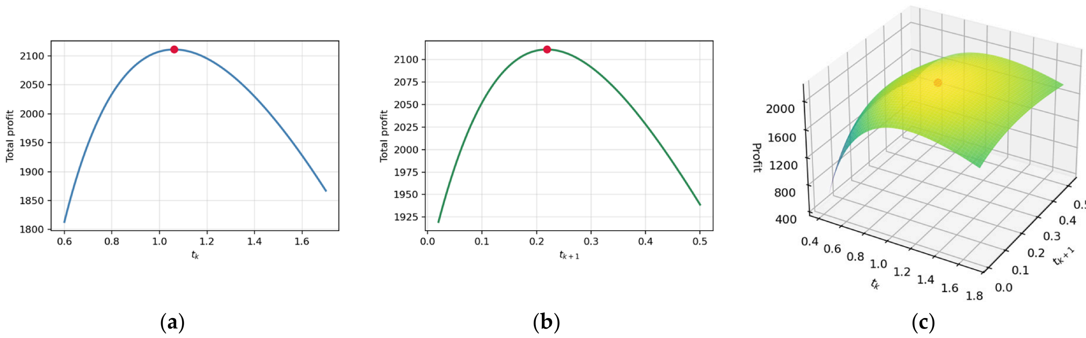


Figure 3. Concavity of the total profit function in (**a**) t_k, (**b**) t_{k+1}, and (**c**) both for Scenario 1.

Example 2.

All the inputs used in this example are the same as in example 1, except for the permissible delay-in-payments $M = 0.3$ *year*.

With M = 0.30 year (Scenario 2), the optimal solution is S = 107.798, R = 16.365, t_k= 1.076, t_{k+1}= 0.210, and Π = 2185.634 (Table 2). Figure 4 shows the optimal profit (the red point) and the corresponding joint concavity, in which panels (a) and (b) are the 2D profit curves against the stock period and the shortage period, respectively, and panel (c) is the joint 3D surface.

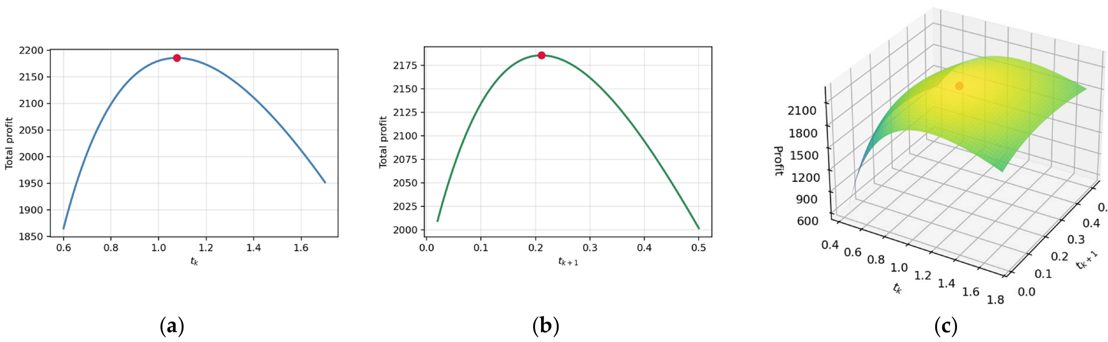
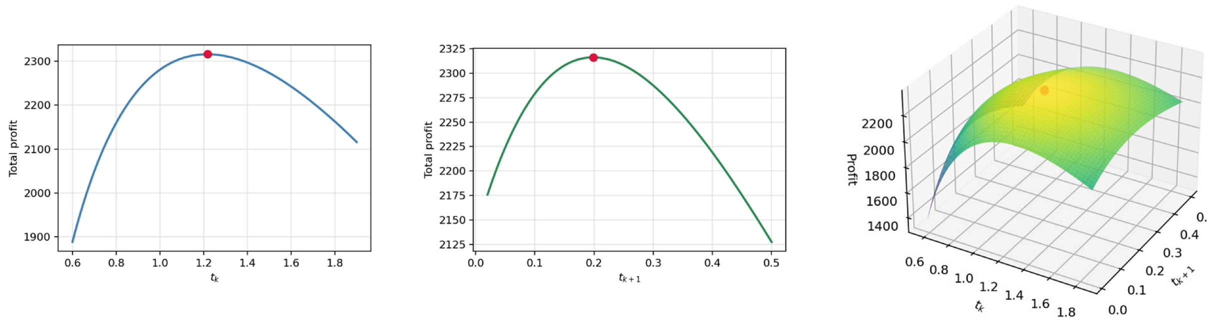


Figure 4. Concavity of the total profit function in (**a**) t_k, (**b**) t_{k+1}, and (**c**) both for Scenario 2.

Example 3.

All the inputs used in this example are identical to those in example 1, except for the permissible delay-in-payments $M = 1.35$ *year*.

With M = 1.35 years (Scenario 3), the optimal solution is S = 121.932, R = 15.635, t_k = 1.217, t_{k+1} = 0.198, and Π = 2316.168 (Table 2). Figure 5 shows the optimal profit (the red point) and the corresponding joint concavity. As in Figure 3, panels (a) and (b) are the 2D profit curves against the stock period and the shortage period, respectively, and panel (c) is the joint 3D surface.



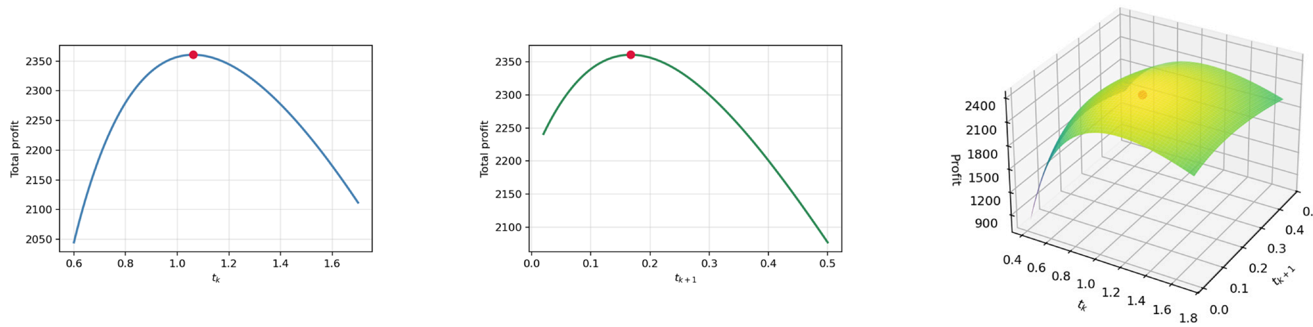
(**a**) (**b**) (**c**)

Figure 5. Concavity of the total profit function in (**a**) t_k, (**b**) t_{k+1}, and (**c**) both for Scenario 3.

Example 4.

All the inputs used in this example are the same as in example 1, except for the permissible delay-in-payments $M = 1.5$ *year*.

With M = 1.50 years (Scenario 4), the optimal solution is S = 106.314, R = 13.668, t_k= 1.061, t_{k+1} = 0.167, and Π = 2360.535 (Table 2). Figure 6 shows the optimal profit (the red point) and the corresponding joint concavity, in which panels (a) and (b) are the 2D profit curves against the stock period and the shortage period, respectively, and panel (c) is the joint 3D surface. All four examples' results are summarized in Table 2 above.

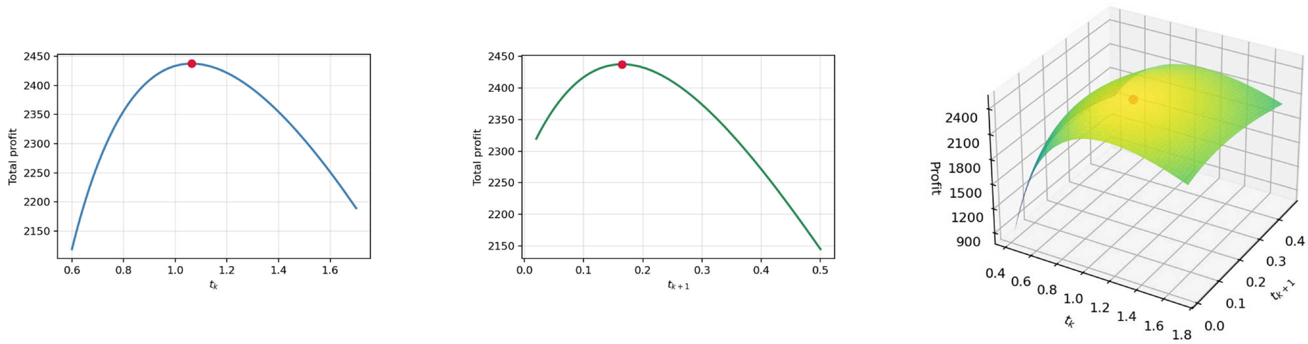


(**a**) (**b**) (**c**)

Figure 6. Concavity of the total profit function in (**a**) t_k, (**b**) t_{k+1}, and (**c**) both for Scenario 4.

Example 5. (For Case 2)

If a β% discount is available to the retailer based on the number of installments, the total profit will vary. Suppose the available percentage of discount is β% = 10%, and the data of example 4 with n = 7. The optimal solution is S = 106.445, R = 13.510, t_k = 1.0627, t_{k+1} = 0.1648, and Π = 2437.286. Compared to the results of example 4, the variations of S, R, t_k, and t_{k+1} are small. However, it is significant in the total profit, reaching a 3.25% increase due to the discount. Figure 7 shows the the optimal profit (the red point) and the corresponding joint concavity, in which panels (a) and (b) are the 2D profit curves against the stock period and the shortage period, respectively, and panel (c) is the joint 3D surface, here for Case 2.



(**a**) (**b**) (**c**)

Figure 7. Concavity of the total profit function in (**a**) t_k, (**b**) t_{k+1}, and (**c**) both for Case 2.

Using the same data and varying n to represent different prepayments, Table 3 is presented below to show the effects of the discount on prepayments. The permissible delay period is M = 1.5 year, satisfying the Scenario 4 condition ($M > t_k+t_{k+1}$) under which Equation (54) is derived, consistent with Example 4.

Table 3. Effect of the number of installments on the discount rate and profit.

No of Installments (n)	CCC	t_k	t_{k+1}	Discount Rate	$\Pi^i(t_k,t_{k+1})$
1	88.946	1.069	0.153	10% (Full discount)	2867.579
5	53.435	1.063	0.164	2%	2465.928
10	48.990	1.062	0.166	1%	2415.810
⋮	⋮	⋮	⋮	⋮	⋮
100	44.988	1.062	0.167	0.1%	2370.722
⋮	⋮	⋮	⋮	⋮	⋮
1000	44.588	1.062	0.167	0.01%	2366.215

From the above Table 3, it is clear that the rise in the number of installments decreases the total discount rate while the cyclic cost also declines. If the retailer pays all the purchase costs at the same time, they can enjoy a full discount; the discount becomes less significant with a higher number of installments. The decline in profit as n increases is observed numerically in Table 3 rather than analytically proven for all parameter combinations. A general analytical proof is left for future work.

5. Sensitivity Analysis

To provide managerial insights into the parameter variations, a sensitivity analysis was performed by altering an explicit constraint from −20% to +20% while keeping others constant. The consequences are shown in the subsequent tables:

Table 4 shows that profit increases monotonically with the credit period M, rising from 2287.862 (at M = 1.20) to 2432.535 (at M = 1.80). This increase is driven entirely by the credit bonus term in Scenario 4's interest-earned expression, which grows linearly in M once t_k and t_{k+1} are fixed; it is not associated with any increase in cyclic capital cost. Actually, CCC remains constant at 50.907 for M between 1.35 and 1.80, since it depends only on Q, C_{PC}, n, σ, and L, none of which vary with M in this range. Only at M = 1.20, where the credit period becomes binding (M = t_k + t_{k+1}), does the retailer's optimal order quantity shift to satisfy this tighter constraint, which in turn reduces CCC to 49.752. Retailers therefore benefit unambiguously from a longer credit period within Scenario 4's feasible range without incurring any additional capital cost as a consequence.

Table 4. The effect of M on profit (baseline M = 1.5 years with ±20% variation around it).

M	CCC	t_k	t_{k+1}	$\Pi(t_k,t_{k+1})$
1.8	50.908	1.061	0.167	2432.535
1.65	50.908	1.061	0.167	2396.535
1.5	50.908	1.061	0.167	2360.535
1.35	50.908	1.061	0.167	2324.535
1.2	49.752	1.037	0.163	2287.862

From Table 5, the impact of purchasing cost C_{PC} on profit is the largest of any parameter examined; a +20% change cuts profit 45.6%; a −20% change raises it 46.0%. As C_{PC} rises t_k decreases (from 1.077 at C_{PC} = 44 to 1.039 at C_{PC} = 66) while t_{k+1} increases (from 0.139 to 0.211), a higher purchasing cost leads the retailer to shorten the positive-stock period and accept a longer backordering period instead, which reduces the peak on-hand stock S (from 107.867 to 104.068 units). The cyclic capital cost CCC rises with C_{PC} (from 40.595 to 61.323), since CCC is directly proportional to the purchase price, compounding the negative effect of a higher C_{PC} on profit.

Table 5. The effect of C_{PC} on profit (baseline C_{PC} = \$55 with ±20% variation around it).

C_{PC}	CCC	t_k	t_{k+1}	$\Pi(t_k,t_{k+1})$
66	61.323	1.039	0.211	1284.413
60.5	56.103	1.051	0.186	1820.727
55	50.908	1.061	0.167	2360.535
49.5	45.739	1.070	0.152	2902.939
44	40.595	1.077	0.139	3447.330

Table 6 shows that when the lead time increases, the total profit of the retailer decreases. In a real sense, this happens because a longer lead time allows the retailer to reimburse purchasing costs for a longer time, which is comparable to the observations found by [65]. A parallel increase in lead time and the cyclic cost is noticed from the table.

Table 6. The effect of L on profit (baseline L = 0.25 years with ±20% variation around it).

L	CCC	t_k	t_{k+1}	$\Pi(t_k,t_{k+1})$
0.3	61.091	1.061	0.167	2352.249
0.275	55.999	1.061	0.167	2356.392
0.25	50.908	1.061	0.167	2360.535
0.225	45.816	1.062	0.167	2364.678
0.2	40.725	1.062	0.167	2368.821

Table 7 shows that the base deterioration rate θ has a very small effect on profit within the range tested. The profit varies by less than 0.2% across $\theta \in [0.20, 0.30]$ (from 2362.344 to 2358.730). As θ increases, t_k decreases marginally (from 1.063 to 1.060), while t_{k+1} remains essentially unchanged at 0.167. This indicates that, at the preservation-investment level used in this example (ξ = 5), the model is robust to moderate variation in the underlying deterioration rate.

Table 7. The effect of θ on profit (baseline θ = 0.25 with ±20% variation around it).

θ	CCC	t_k	t_{k+1}	$\Pi(t_k, t_{k+1})$
0.3	50.860	1.060	0.167	2358.730
0.275	50.884	1.061	0.167	2359.632
0.25	50.908	1.061	0.167	2360.535
0.225	50.931	1.062	0.167	2361.439
0.2	50.955	1.063	0.167	2362.344

Table 8 states that when the retailer raises its investment ξ to preserve its products, then t_k increases. This is because the best utilization of ξ reduces the deterioration rate and thus maximizes the fresh product period. Additionally, as the products stay fresh for a longer period, the retailer can profit. Therefore, the profit $\Pi(t_k, t_{k+1})$ increases. This positive relationship is observed within the tested range, $\xi \in [4, 6]$, because $m(\xi) = e^{-a1\xi}$ approaches its lower asymptote as ξ grows. The marginal deterioration rate reduction from each additional unit of preservation investment shrinks, so diminishing returns to preservation investment may emerge at substantially higher investment levels than those examined here, potentially yielding a profit-maximizing interior ξ^*. Figure 8 confirms this numerically; sweeping ξ well beyond the Table-8-tested range shows profit continuing to rise, flattening, and then turning down, with an interior optimum at $\xi^* \approx 7.5$. This is consistent with the diminishing returns mechanism already noted above. The practical implication for a retailer is that ξ is a genuine decision variable with its own optimum, not a lever that is safe to keep increasing.

Table 8. The effect of ξ on profit (baseline ξ = 5 with ±20% variation around it).

ξ	CCC	t_k	t_{k+1}	$\Pi(t_k, t_{k+1})$
6	51.040	1.066	0.167	2364.525
5.5	50.986	1.064	0.167	2363.019
5	50.908	1.061	0.167	2360.535
4.5	50.791	1.058	0.168	2356.604
4	50.618	1.052	0.169	2350.538

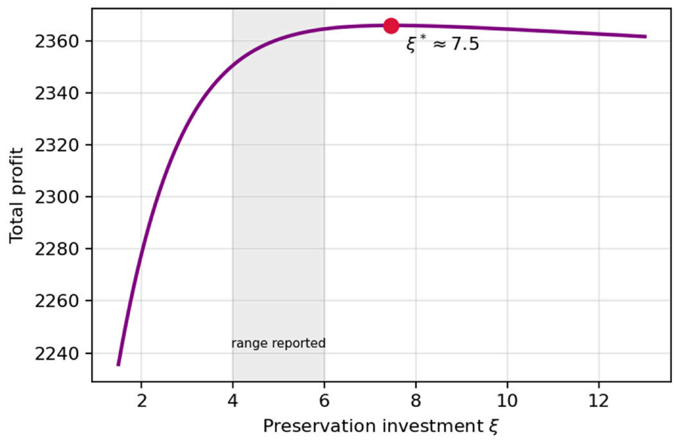


Figure 8. Profit as a function of preservation investment ξ with indication for ξ^* (at the red dot).

Graphical comparison:

Here, graphical illustrations show the variances amongst the proposed models and nonappearance of some model factors, such as those without advance payment, reducing the number of installments, and credit strategy.

Some observations from the above Figures:

Figure 9 shows that profit without advance payment (Π = 2719.669) is substantially higher, by approximately 15%, than profit with advance payment (Π = 2360.535), across the full range of t_k considered. This gap reflects the cyclic capital cost (CCC) the retailer incurs by prepaying a portion of the purchase cost, together with the reduced interest earned factor $(1 - \sigma)$ that applies whenever an advance payment is made. On a raw profit basis, therefore, advance payment is costly to the retailer, since it requires committing capital to the supplier before the goods are received rather than at delivery. Section 6 discusses the retailer's practical options where advance payment cannot be avoided.

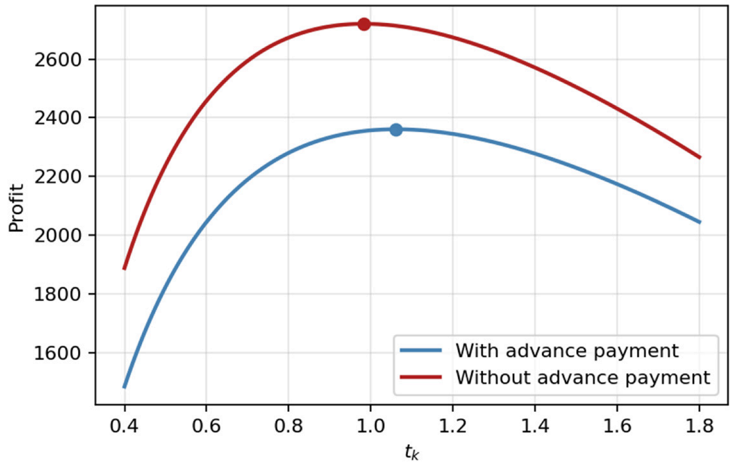


Figure 9. Comparison of the total profit with and without advance payment as a function of t_k, showing the optimum reached in each case.

As shown in Figure 10, it is worth noting that as the number of installments rises, the total profit declines. Splitting the prepayment into more installments does reduce the cyclic capital cost, since the retailer retains capital longer, but it simultaneously reduces the discount granted by the supplier. As Table 3 shows, the discount effect dominates; the reduction in CCC from $n = 1$ to $n = 5$ is far smaller than the revenue lost through the reduced discount rate.

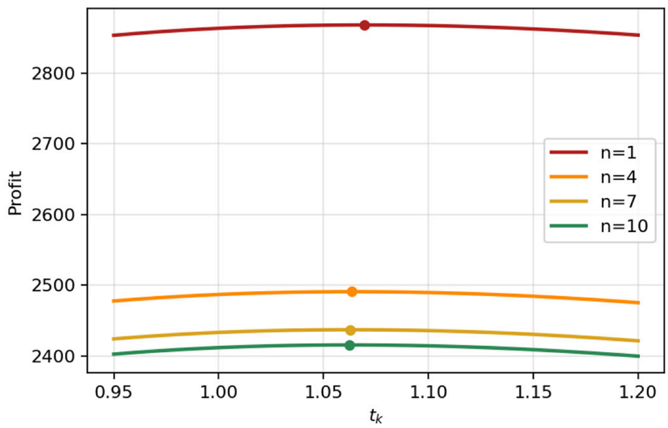


Figure 10. Effect of the number of prepayment installments on total profit as a function of t_k.

Figure 11 shows that trade credit provides a uniformly higher profit than the absence of a credit policy across the entire range of t_k considered, with the gap widening from approximately 201 profit units at $t_k = 0.4$ to a maximum of about 281 units near the optimum before narrowing slightly toward $t_k = 1.8$. Unlike the advance payment comparison in Figure 9, there is no point within this range where the two policies produce comparable profit. Extending the credit period is unambiguously beneficial to the retailer, provided the supplier is willing to grant it.

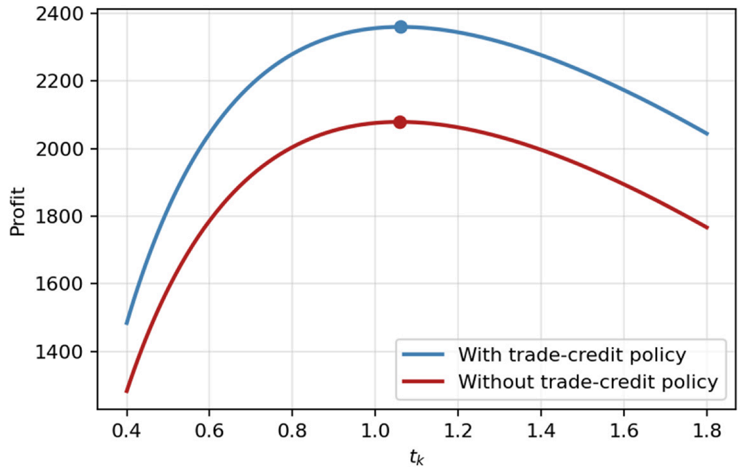


Figure 11. Comparison of total profit with and without trade credit as a function of t_k.

6. Managerial Implications

These findings are particularly relevant for retailers of perishable goods, such as fresh food, pharmaceuticals, and specialty chemicals, where deterioration begins only after a threshold period and where purchasing costs, preservation investment, and payment terms jointly determine profitability. The numerical and sensitivity results carry several implications for retailers managing non-instantaneously deteriorating inventory. Among the parameters examined, profit is considerably more sensitive to the purchasing cost C_{PC} than to any other parameter. A 20% increase in C_{PC} reduces profit by 45.6%, and a 20% decrease increases it by 46.0%. It represents the largest sensitivity of any parameter examined. The permissible delay period M is the next most influential parameter, with a 20% variation altering profit by about 3%, while equivalent variations in L, ξ, or θ each alter profit by well under 1%. This indicates that supplier price negotiation is the retailer's most consequential lever for profit improvement, followed by the terms of the trade credit agreement, and both should be prioritized over adjustments to preservation investment or lead time.

The results further suggest that preservation technology investment, although its effect on profit is comparatively small within the considered range, should not be taken as an overhead cost. As shown in Table 8, increasing ξ consistently raises profit by extending the period during which the product remains fresh. This analysis indicates that preservation investment functions as a real profit lever rather than an expense to be minimized.

With respect to payment policy, the results indicate that advance payment should not be evaluated on the basis of raw profit alone. Although the retailer's profit is higher in the absence of advance payment, as illustrated in Figure 9, advance payment reduces profit relative to trade credit or no financing arrangement at all. Figure 9 shows a substantial gap (approximately 15%) between profit without advance payment (Π = 2719.669) and profit with it (Π = 2360.535), driven by the cyclic capital cost of prepaying part of the purchase cost and by the reduced interest-earned factor that applies whenever $\sigma > 0$. This cost follows directly from committing capital to the supplier before the goods are received, so advance payment should not be recommended as a way to ease a retailer's working-capital position. Paying in advance requires more capital committed upfront, not less. Where a retailer is required to make an advance payment—for example, because the supplier makes it a condition of the order—the number of installments used to spread that payment should be chosen carefully. As reported in Table 3, increasing the installment count lowers the discount rate offered by the supplier, and this reduction outweighs any benefit from retaining capital for longer, so profit falls monotonically as the installment count rises (from Π = 2867.579 at $n = 1$ to Π = 2415.810 at $n = 10$). Retailers should therefore negotiate for the fewest installments their cash-flow position allows, accepting a higher installment count only where the resulting smoother payment schedule is worth more to the business than the discount forgone. Since the discount rate $\alpha = \beta/n$ decreases rapidly and then levels off, most of the discount benefit is achieved by $n = 5$ (2% compared with 10% at $n = 1$; Table 3). Thus, $n = 1 - 5$ provides the most meaningful and practical trade-off between discount and cash-flow smoothing, while additional installments offer limited benefit.

Finally, the comparison between trade credit and its absence, shown in Figure 11, indicates that trade credit yields uniformly higher profit than a no-credit arrangement across the full range of cycle lengths examined, with the profit gap remaining substantial throughout. Likewise, although profit increases with the permissible delay period M, as shown in Table 4, this increase reflects only the credit-bonus interest-earned term and is not accompanied by any corresponding increase in cyclic capital cost within Scenario 4's feasible range. Retailers should therefore treat a longer or more accessible credit period as an unambiguous benefit, subject only to what the supplier is willing to offer.

7. Conclusions

This study develops an integrated inventory model in which a retailer facing non-instantaneous deterioration and partial backordering jointly decides preservation technology investment, shortage timing, and how to structure payment to the supplier through advance payment, trade credit, or a prepayment discount. Advance payment reduces the retailer's achievable profit relative to trade credit or no financing arrangement since it requires committing capital to the supplier before delivery rather than after; where an advance payment is required, splitting it into fewer installments preserves more of the supplier's discount and yields higher profit than a larger installment count. These findings directly answer the research questions. In response to RQ1, the piecewise, scenario-based formulation in Section 3 yields a well-defined optimal shortage/replenishment policy (t_k, t_{k+1}) for any given level of preservation investment under the hybrid advance and delay payment scheme, and Sections 4 and 5 show that this policy and the resulting profit respond predictably as preservation investment and the other model parameters vary. In response to RQ2, the results show that increasing the number of prepayment installments lowers both the discount rate offered by the supplier and the retailer's profit. In response to RQ3, the sensitivity analysis shows that the retailer's optimal profit responds most strongly to the purchasing cost, followed at a considerable distance by the trade credit period, while lead time, preservation investment, and the deterioration rate each alter profit by less than 1% over the range tested, giving managers a ranked set of levers for adjusting the policy.

Additionally, this paper provided these significant contributions:

- This paper shows how preservation technology investment controls the deterioration rate and quantifies its effect on profit over the range examined so that a retailer can select an investment level appropriate to its return targets.
- It simultaneously achieves the optimal balance among shortages, trade credit, and advance payment decisions through preservation technology investment.
- Where offered, the prepayment discount always yields higher profit than prepayment without a discount (a 3.25% increase in Example 5 relative to Example 4), though profit still falls monotonically as the number of installments increases, and trade credit without any advance payment requirement remains the most profitable option overall whenever the retailer is able to avoid prepayment entirely.

On the other hand, several limitations bound the broad view of the proposed model. The model is deterministic, single product, and single echelon. It does not incorporate demand uncertainty, storage or capacity constraints, competition among multiple products for shared warehouse space, or coordination with multiple suppliers or retailers, all of which would require a materially different solution approach. In addition, deterioration is modelled at a constant rate once the fresh period ends; the model does not compare this against a Weibull model. Extending the framework to multiple suppliers and multiple retailers is left for future work, potentially informed by multi-objective network-design approaches that jointly optimize profitability and emissions reduction across a closed-loop supply chain [66] or, more broadly, by circular-economy principles that have been shown to strengthen socio-ecological resilience in other resource-constrained settings [67]. Besides, connecting the fresh period's length to the retailer's willingness to accept advance payment and identifying the transaction-cost threshold at which a prepayment discount would cease to dominate trade credit are left for future research.

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Notations and Abbreviations

Symbol	**Description**
C_{PC}	Per unit purchasing cost per order,
C_{IHC}	Per unit inventory holding cost per order,
C_{BC}	Per unit shortage cost per order,
C_{LSC}	Per unit lost sale cost per order,
D	Demand rate
a_1	Represents the coefficient of investment for preservation technology
A	Per order replenishment cost,
β	Maximum discount percentage in case of advance payment
v	Parameter used for backlogging,
θ	Rate of deterioration,
M	Time period where supplier's offered permissible delay-in-payments to the retailer,
$m(\xi)$	Represents the function for preservation technology
L	Lead time duration for when the retailer will pay the prepayments,
n	Number of equal prepayment installments within the lead time,
σ	Required portion of the purchasing cost needed to pay, with numerous prepayments ($0 < \sigma < 1$),
I_e	Earned interest rate,
I_p	Payable interest rate,
R	Per cycle highest shortage amount,
S	The amount of stock at the beginning of the cycle,
Q	Total order amount per cycle,

p	Per unit item selling price,
ξ	Per unit preservation technology cost,
t_{k-1}	Deterioration starting point,
α	Discount rate for prepayment,
$\Pi(t_k, t_{k+1})$	Total profit per unit time,
Decision variables:	
t_{k+1}	Time period of shortage,
t_k	Shortage starting point,

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