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Bifurcation Analysis and Symmetry Properties in a 3-D Chaotic Financial Firm System Driven by Cyclic Reinvestment Perturbations

Bob Foster [1], Susan Purnama [1], Muhamad Deni Johansyah [2], Sundarapandian Vaidyanathan [3], Rameshbabu Ramar [4], Volodymyr Rusyn [5], Bogdan Markovych [6] and Aceng Sambas [7,8,9,*]

1 Faculty of Business and Economics, Universitas Informatika dan Bisnis Indonesia, Bandung 40285, Indonesia; bobriset@unibi.ac.id (B.F.); susanpurnama675@gmail.com (S.P.)
2 Department of Mathematics, Universitas Padjadjaran, Jatinangor Sumedang 45363, Indonesia; muhamad.deni@unpad.ac.id
3 Centre for Control Systems, Vel Tech University, Avadi Chennai 600062, Tamil Nadu, India; sundar@veltech.edu.in
4 Department of Electronics and Communication Engineering, V.S.B. Engineering College, Karur 639111, Tamil Nadu, India; rameshbaburamar@vsbec.org
5 Department of Radio Engineering and Information Security, Yuriy Fedkovych Chernivtsi National University, 58012 Chernivtsi, Ukraine; rusyn_v@ukr.net
6 Department of Applied Mathematics, Lviv Polytechnic National University, 79013 Lviv, Ukraine
7 Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Kuala Terengganu 21300, Malaysia
8 Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 600077, Tamilnadu, India
9 Department of Mechanical Engineering, Universitas Muhammadiyah Tasikmalaya, Tasikmalaya 46196, Indonesia
* Correspondence: acengs@umtas.ac.id

Abstract

Financial systems are highly nonlinear and often influenced by investment cycles, market fluctuations, and external financing activities, which can generate complex dynamic behaviors. This paper proposes a new three-dimensional chaotic financial firm model by extending the classical Bouali financial system through the inclusion of a cyclic reinvestment perturbation represented by a sinusoidal nonlinear term. The proposed modification aims to capture state-dependent nonlinear reinvestment feedback in reinvestment decisions caused by changing economic conditions and business cycles. The dynamical characteristics of the model are investigated using equilibrium analysis, local stability theory, Lyapunov exponents, the Kaplan–Yorke dimension, bifurcation analysis, and multistability analysis. Numerical results show that the system possesses three equilibrium points, all of which are unstable under the selected parameter setting. The system exhibits chaotic dynamics confirmed by a positive maximum Lyapunov exponent and a fractal attractor characterized by a Kaplan–Yorke dimension greater than two. Comparative results indicate that the new model demonstrates richer nonlinear dynamical behavior than existing financial firm systems reported in the literature. Furthermore, bifurcation and Lyapunov spectrum analyses disclose multiple transitions between periodic and chaotic states as system parameters vary, while multistability analysis reveals the coexistence of symmetric periodic and chaotic attractors under identical parameter values but different initial conditions. Finally, total amplitude control and offset boosting techniques are implemented to regulate attractor size and position while preserving the underlying chaotic behavior. These findings demonstrate that cyclic reinvestment perturbations significantly enrich the nonlinear dynamics and symmetry properties of financial firm systems, providing a useful framework for studying complex financial behaviors and chaos control.

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1. Introduction

Financial firms operate in highly dynamic environments where profitability, investment decisions, and borrowing activities are continuously influenced by internal and external economic factors [1]. As a result, financial systems often exhibit instability characterized by irregular fluctuations in capital accumulation, cash flow, and debt obligations [2]. Such instability becomes particularly pronounced during business cycles, where periods of economic expansion are followed by contractions, leading to significant variations in investment behavior and corporate performance [3]. Firms typically respond to favorable economic conditions by increasing reinvestment and expanding their operations, whereas economic downturns often trigger reductions in investment and heightened financial risk [4]. These cyclical changes can create complex interactions among financial variables, making the overall behavior of firms difficult to predict using conventional linear models [5].

Another important mechanism underlying financial dynamics is the leverage effect, whereby firms utilize debt-financing to amplify investment capacity and enhance shareholder returns [6,7]. While borrowing can stimulate growth and profitability, excessive reliance on debt may also increase vulnerability to financial distress and market uncertainty [8]. The interaction between reinvestment, profitability, and debt-financing forms a nonlinear feedback structure in which changes in one variable continuously influence the others [9]. Such feedback mechanisms can generate self-sustained oscillations, abrupt transitions between different operating regimes, and even chaotic behavior [10]. Consequently, nonlinear dynamical models have become valuable tools for investigating the complex evolution of financial firms, as they provide deeper insights into instability, cyclical fluctuations, and the emergence of unpredictable financial dynamics that cannot be adequately captured by traditional equilibrium-based approaches [11–14]. Chaotic systems are of significant interest due to their irregular behavior, high sensitivity on initial conditions, and unpredictability, which make them suitable to study financial and economic models. In a financial system, chaotic dynamics represent unpredictability and highly sensitive fluctuations in the firm's capital, investment, and financial state. The small changes in initial financial conditions lead to different long-term outcomes, thereby reducing long-term predictability and increasing financial risk.

Over the past several decades, nonlinear dynamical systems have been increasingly employed to investigate the complex behavior of financial and economic systems. Unlike classical equilibrium-based models, nonlinear financial models are capable of capturing irregular fluctuations, cyclical movements, instability, and sudden transitions that frequently occur in financial environments. The interaction among key financial variables such as profit, investment, capital accumulation, and debt often generates intricate feedback mechanisms that cannot be adequately explained using linear approaches. Yadav et al. [15] developed a Bernstein wavelet-based numerical algorithm for solving fractional-order financial chaotic systems and transform fractional differential equations into algebraic equations using Bernstein wavelets and fractional integral operators. Bella [16] analyzed the emergence of chaos in a climate-augmented monetary economics model and to develop a control mechanism for stabilizing the economy and avoiding low-growth poverty traps under climate-related shocks. Pariona [17] studied chaotic behavior in financial markets using chaos theory and optimization techniques, and to assess its implications for market predic-

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tion, risk management, and financial stability. Saeidi et al. [18] proposed a fractional-order financial risk model with risk control mechanisms, investigate its chaotic characteristics, and develop optimal control strategies based on Genetic Algorithm and Particle Swarm Optimization to suppress chaos and enhance system stability. Yang and Li [19] investigated chaotic dynamics of a novel fractional-order financial system with non-constant demand elasticity and to develop an RBF neural network-based predefined-time control strategy for system stabilization, synchronization, and secure encryption applications. Almheidat [20] developed a PINN-based numerical method with L-BFGS optimization for solving variable-order fractional financial models and to analyze their chaotic dynamics under different fractional-order cases. Chen et al. [21] developed a Bayesian-regularized radial basis deep neural network for solving fractional-order chaotic financial systems and to evaluate its accuracy and reliability across multiple cases. However, existing financial firm models generally neglect periodic fluctuations in reinvestment activities caused by business cycles and market sentiment. Consequently, the influence of cyclic reinvestment perturbations on chaotic financial dynamics remains largely unexplored.

The main contributions of this work are summarized as follows:

1. A novel three-dimensional financial firm system with cyclic reinvestment perturbations is introduced to capture state-dependent nonlinear reinvestment feedback fluctuations in financial markets.
2. The new system is shown to exhibit rich nonlinear dynamics, including chaos, bifurcations, multistability, and coexisting attractors.
3. Amplitude control and offset boosting techniques are developed to regulate the attractor characteristics while preserving the underlying chaotic behavior.

The remainder of this paper is organized as follows. Section 2 introduces a new three-dimensional financial firm model with cyclic reinvestment perturbations and presents its fundamental dynamical properties, including equilibrium points and stability analysis. Section 3 investigates the nonlinear dynamics of the system through bifurcation diagrams and Lyapunov exponent spectra. Section 4 examines the existence of multistability and coexisting attractors under different initial conditions. Section 5 presents the implementation of amplitude control and offset boosting techniques for regulating the system dynamics. Finally, Section 6 concludes the paper and outlines potential directions for future research.

2. A New 3-D Chaotic Financial Firm System Driven by Cyclic Reinvestment Perturbations

In his seminal 2002 study, Bouali [22] established a theoretical bridge between corporate finance principles and nonlinear dynamical systems to characterize complex firm behavior. By subjecting the Hunt Hypothesis [23] which posits that a firm can strategically "trade on equity" to amplify shareholder returns through financial leverage to nonlinear mathematical scrutiny, Bouali [22] demonstrated that reliance on debt extends far beyond static balance sheet adjustments. Instead, the interaction between borrowing and capital accumulation introduces a sophisticated feedback mechanism and a persistent source of perturbation. This dynamic interplay fundamentally dictates the firm's long-term structural stability and can drive the transition from steady growth to complex, non-periodic financial trajectories.

Bouali's model [22] captures a firm's financial dynamics through profits, reinvestments, and debt flows. It embodies Hunt's trade-on-equity hypothesis, where borrowing boosts shareholder returns, while enforcing a cautious dividend policy: profits below a threshold fuel self-financing; above it, dividends shrink capital to pre-empt downturns. As a result, Bouali's model [22] exhibits periodic motion, chaos, and strange attractors. The mathematical framework of Bouali's model [22] captures the non-periodic, turbulent

oscillations of three endogenous state variables: profits (x), reinvestments (y), and the financial flow of borrowings (z).

The first differential equation in Bouali's model [22] describes the profit evolution equation. Bouali [22] noted that the rate of change in the firm's mass of profit (x) is defined by the following relation:

$$\frac{dx}{dt} = c(y + z) \tag{1}$$

In this formulation, c (equivalent to the reciprocal of Bouali's adjustment coefficient v) represents the adjustment velocity or the responsiveness coefficient of the firm's profitability. This parameter dictates how quickly current profits react to the total capital available to the firm, which consists of internal reinvestments (y) and external financing (z). A higher value of c suggests a more volatile firm where profits shift rapidly in response to capital injections or withdrawals.

The second differential equation in Bouali's model [22] describes the reinvestment evolution equation, which is analogous to the famous Van der Pol oscillator. The dynamics of capital reinvestment (y) is governed by a nonlinear relationship that serves as the "engine" of the system's oscillations:

$$\frac{dy}{dt} = mx + n(1 - x^2)y \tag{2}$$

As noted by Bouali [22], the differential Equation (2) bears a significant structural resemblance to the damping term of the classical van der Pol equation. In physical systems, the van der Pol oscillator is known for producing self-sustained limit cycles through a nonlinear damping mechanism.

In this financial context, the term $n(1 - x^2)y$ operates as a state-dependent regulator of capital flow:

- The Nonlinear Damping Term: The term $(1 - x^2)$ acts as a strategic switch. When profitability is below the unity threshold $(|x| < 1)$, the term $(1 - x^2)$ is positive, functioning as "negative damping" that pumps capital into the system to drive self-financing.
- Strategic Decapitalization: When profits exceed the threshold $(|x| > 1)$, the term $(1 - x^2)$ becomes negative. This represents the "braking" mechanism described by Bouali [1], where the firm pays out dividends or stockholders withdraw equity to mitigate risk against market declines, thereby reducing the reinvestment mass (y).

The external borrowing flow (z) is modeled according to the Hunt Hypothesis [23], which explores the dynamics of "trading on equity" to amplify shareholder returns:

$$\frac{dz}{dt} = -ax + by \tag{3}$$

In the differential Equation (3), the parameters a and b govern the evolution of financial obligations:

- The Interest Coefficient (a): This represents the interest rate or the debt-servicing burden relative to current profits (x). It acts as a corrective term; as profits increase, the firm possesses a higher capacity to settle obligations, thereby reducing the net flow of new debt.
- The Debt Intensity (b): This represents the debt rate proportional to the firm's self-financing or reinvestment efforts (y). It quantifies the firm's reliance on external loans to amplify its internal investment capacity.

In system form, the Bouli chaotic financial firm model can be expressed by combining the differential Equations (1)–(3) as follows:

$$\begin{cases} \dot{x} = c(y+z) \\ \dot{y} = mx + n(1-x^2)y \\ \dot{z} = -ax + by \end{cases} \tag{4}$$

It is useful for us to simplify the notations by defining $X = (x, y, z)^T$, the 3-D state of the Bouali financial firm model (4).

Bouali [1] demonstrated that the 3-D financial firm model in (4) exhibits a chaotic attractor for the parameter values $a = 0.1$, $b = 0.6$, $c = 0.25$, $m = 0.04$, and $n = 0.02$. For these parameter values and the initial state $X(0) = (0.03, 0.03, 0.03)^T$, the Lyapunov exponents (LEs) of Bouali's financial firm model (4) were computed over 1×10^4 seconds as $LE_1 = 0.02006$, $LE_2 = 0$ and $LE_3 = -0.31099$. The maximum Lyapunov exponent of Bouali's chaotic financial firm model (4) is therefore found as $LE_{\max} = 0.02006$. Consequently, the Kaplan–Yorke dimension of Bouali's chaotic financial firm model (4) is calculated as follows:

$$D_K = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 2 + \frac{0.02006 + 0}{0.31009} = 2.06450 \tag{5}$$

This shows that the chaotic attractor of Bouali's financial firm model (4) is fractal.

Johansyah et al. [24] introduced a new financial firm model by modifying the second differential equation of Bouali's financial firm model (4) by replacing the nonlinear damping term in it and modifying it as follows:

$$\frac{dy}{dt} = mx + n(1-x^4)y \tag{6}$$

This modification is motivated by Johansyah et al., [24] who showed that Bouali's model [22] captures nonlinear profit–reinvestment interactions, and that replacing the damping term with an x^4 nonlinearity strengthens the feedback mechanism. This enables the financial firm model to represent more complex and realistic dynamical behavior.

The resulting Johansyah financial firm model [24] can be stated as follows:

$$\begin{cases} \dot{x} = c(y+z) \\ \dot{y} = mx + n(1-x^4)y \\ \dot{z} = -ax + by \end{cases} \tag{7}$$

We define $X = (x, y, z)^T$ to designate the three-dimensional state of the Johansyah financial firm model (7).

Johansyah et al. [3] showed that the modified financial firm model in (7) exhibits a chaotic attractor for the parameter values $a = 0.2$, $b = 0.6$, $c = 0.2$, $m = 0.05$, and $n = 0.03$. For these parameter values and the initial state $X(0) = (0.03, 0.03, 0.03)^T$, the Lyapunov exponents (LEs) of the Johansyah's modified financial firm model (7) were computed over 1×10^4 seconds as $LE_1 = 0.02268$, $LE_2 = 0$ and $LE_3 = -0.25570$. The maximum Lyapunov exponent of the Johansyah's modified chaotic financial firm model (7) is therefore found as $LE_{\max} = 0.02268$. Consequently, the Kaplan–Yorke dimension of Johansyah's modified chaotic financial firm model (7) is calculated as follows:

$$D_K = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 2 + \frac{0.02268 + 0}{0.25570} = 2.08870 \tag{8}$$

This shows that the chaotic attractor of Johanyah's modified financial firm model (7) is fractal.

In this work, we propose a new financial firm model by modifying the second differential equation of Bouali's financial firm model (4) by adding a nonlinear term in it representing the cyclic reinvestment perturbations $(q\sin(y))$ and modifying it as follows:

$$\frac{dy}{dt} = mx + n(1 - x^2)y + q\sin(y) \tag{9}$$

In actual financial environments, reinvestment decisions are frequently influenced by business cycles, market sentiment, and periodic economic shifts, which induce oscillatory behavior in capital reinvestment. To reflect these dynamics, we incorporate a sinusoidal nonlinear term $q\sin(y)$ into the second differential equation of Bouali's financial firm model. This term represents cyclic reinvestment perturbations, where the parameter q denotes the intensity of investment fluctuations affecting the firm's capital dynamics.

From a dynamical systems perspective, this addition strengthens the nonlinear feedback mechanism and enriches the system's phase space. Consequently, the modified model captures more complex phenomena such as bifurcations, coexisting attractors, and chaotic motion, thereby enhancing the realism and analytical flexibility of the original framework.

The resulting new financial firm model can be stated as follows:

$$\begin{cases} \dot{x} = c(y + z) \\ \dot{y} = mx + n(1 - x^2)y + q\sin(y) \\ \dot{z} = -ax + by \end{cases} \tag{10}$$

We define $X = (x, y, z)^T$ to designate the three-dimensional state of the new financial firm model (10), where the reinvestment variable y and argument of the sine function are dimensionless.

In practical financial systems, reinvestment decisions are often adjusted according to the firm's current reinvestment level, reflecting managerial responses, capital allocation policies, liquidity conditions, and investment confidence. To capture this endogenous decision-making process, we incorporate the bounded nonlinear function q sin(y) into the reinvestment equation. Since the additional term depends only on the reinvestment variable y, it represents a state-dependent nonlinear feedback mechanism rather than an externally imposed periodic force. The parameter q measures the strength of this nonlinear feedback. Such bounded nonlinear feedback enriches the system dynamics while preserving the autonomous character of the financial model. The positive values of *q sin(y)* correspond to reinforcement of reinvestment activity, whereas negative values represent a reduction in reinvestment.

The financial system (10) incorporates a bounded nonlinear feedback term q sin(y) into the reinvestment dynamics. This term models endogenous cyclic adjustments in the firm's reinvestment policy arising from the current reinvestment state rather than from an externally imposed time-periodic signal. The parameter q controls the strength of this feedback. Because the feedback is nonlinear and bounded, it enriches the phase-space structure of the financial system and facilitates the emergence of multistability, coexisting attractors, bifurcations, and chaotic behavior while maintaining the autonomous nature of the model. Moreover, the parameter q controls the strength (amplitude) of the nonlinear feedback, allowing a systematic investigation of how different feedback intensities influence the financial system dynamics.

In this research work, we exhibit that the modified financial firm model in (10) displays a chaotic attractor for the parameter values $a = 0.3$, $b = 0.6$, $c = 0.4$, $m = 0.05$, $n = 0.02$, and $q = 0.2$. For these parameter values and the initial state $X(0) = (0.03, 0.03, 0.03)^T$, the

Lyapunov exponents (LEs) of the new modified financial firm model (10) were computed over 1×10^4 seconds as $LE_1 = 0.04684$, $LE_2 = 0$ and $LE_3 = -0.21983$. The maximum Lyapunov exponent of the new modified chaotic financial firm model (10) is therefore found as $LE_{\max} = 0.04684$. Consequently, the Kaplan–Yorke dimension of the new modified chaotic financial firm model (10) is calculated as follows:

$$D_K = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 2 + \frac{0.04684 + 0}{0.21983} = 2.21307 \tag{11}$$

This shows that the chaotic attractor of the new modified financial firm model (10) is fractal.

Table 1 presents a comparative analysis of the Lyapunov exponents (LE), Maximum Lyapunov Exponent (MLE), and Kaplan dimension (D_K) for the three chaotic financial firm systems: the Bouali system [22], the Johansyah system [24], and the system presented in Equation (10). It is evident from the numerical results that the financial firm system (10), driven by cyclic reinvestment perturbations, exhibits a significantly rich nonlinear dynamical behavior and chaos compared to the existing models [22,24].

Table 1. Comparison of maximum Lyapunov exponents (MLE) and Kaplan dimensions (D_K) between the new financial firm system and established models.

Metric	Bouali's Financial Firm System [22]	Johansyah's Financial Firm System [24]	Our Financial Firm System
MLE	$LE_{\max} = 0.02006$	$LE_{\max} = 0.02268$	$LE_{\max} = 0.04684$
Kaplan dimension	$D_K = 2.06450$	$D_K = 2.08870$	$D_K = 2.21307$

The equilibrium points or the rest points of the system (10) are obtained by solving these equations:

$$c(y + z) = 0 \tag{12a}$$

$$mx + n(1 - x^2)y + q\sin(y) = 0 \tag{12b}$$

$$-ax + by = 0 \tag{12c}$$

From Equation (12c), we get

$$y = \frac{ax}{b} \tag{13}$$

From Equation (12a) and (13), we get

$$z = -y = -\frac{ax}{b} \tag{14}$$

Thus, the equilibrium points of the system (10) have the general form

$$E = \left(x, \frac{ax}{b}, -\frac{ax}{b}\right)^T, \tag{15}$$

where x satisfies the equation

$$f(x) = mx + \frac{an}{b}(1 - x^2)x + q\sin\left(\frac{ax}{b}\right) = 0 \tag{16}$$

We note that f is an odd function of x. Thus, if $x \neq 0$ is a root of (16), then $-x$ is also a root.

This calculation shows that the roots of the system (10) are symmetric about the plane $x = 0$.

For the chaotic case, the parameter values are $a = 0.3$, $b = 0.6$, $c = 0.4$, $m = 0.05$, $n = 0.02$, and $q = 0.2$.

Thus, the equilibrium points of the system (10) are given by

$$E = (x, 0.5x, -0.5x)^T \tag{17}$$

where x satisfies the equation

$$f(x) = 0.06x - 0.01x^3 + 0.2\sin(0.5x) = 0 \tag{18}$$

A simple calculation via MATLAB R2020a shows that Equation (18) has three roots, *viz.* $x = 0$, and $x = \pm 3.4306$. Thus, the system (10) has three equilibrium points in total given by $P_0 = (0, 0, 0)^T$, $P_1 = (3.4306, 1.7153, -1.7153)^T$, and $P_2 = (-3.4306, -1.7153, 1.7153)^T$.

Next, we calculate the stability of the equilibrium points P_0, P_1, and P_2.

The Jacobian matrix of the system (10) is obtained as

$$J(x, y, z) = \begin{bmatrix} 0 & c & c \\ m - 2nxy & n(1 - x^2) + q\cos(y) & 0 \\ -a & b & 0 \end{bmatrix} \tag{19}$$

For the chaotic case, the parameter values are $a = 0.3$, $b = 0.6$, $c = 0.4$, $m = 0.05$, $n = 0.02$, and $q = 0.2$.

Thus, the Jacobian matrix takes the following simplified form:

$$J(x, y, z) = \begin{bmatrix} 0 & 0.4 & 0.4 \\ 0.05 - 0.04xy & 0.02(1 - x^2) + 0.2\cos(y) & 0 \\ -0.3 & 0.6 & 0 \end{bmatrix} \tag{20}$$

At the equilibrium point $P_0(0, 0, 0)$

$$J(P_0) = \begin{bmatrix} 0 & 0.4 & 0.4 \\ 0.05 & 0.22 & 0 \\ -0.3 & 0.6 & 0 \end{bmatrix} \tag{21}$$

The eigenvalues of the matrix (21) are given by

$$\lambda_1 = 0.3049,\ \lambda_{2,3} = -0.04249 \pm j0.35229 \tag{22}$$

Since the eigenvalues of the matrix (21) has one positive real and one complex conjugate pair with negative real part, the equilibrium point P_0 has unstable behavior.

AT the equilibrium point $P_{1,2}(3.4306, 1.7153, -1.7153)$

$$J(P_{1,2}) = \begin{bmatrix} 0 & 0.4 & 0.4 \\ -0.1853 & -0.2442 & 0 \\ -0.3 & 0.6 & 0 \end{bmatrix} \tag{23}$$

The eigenvalues of the matrix (23) are given by

$$\lambda_1 = -0.3310,\ \lambda_{2,3} = 0.0434 \pm j0.4700 \tag{24}$$

Since the eigenvalues of the matrix (23) has one negative real and one complex conjugate pair with positive real part, the equilibrium points $P_{1,2}$ are unstable.

The phase portraits of the new financial firm system (10) are shown in Figure 1.

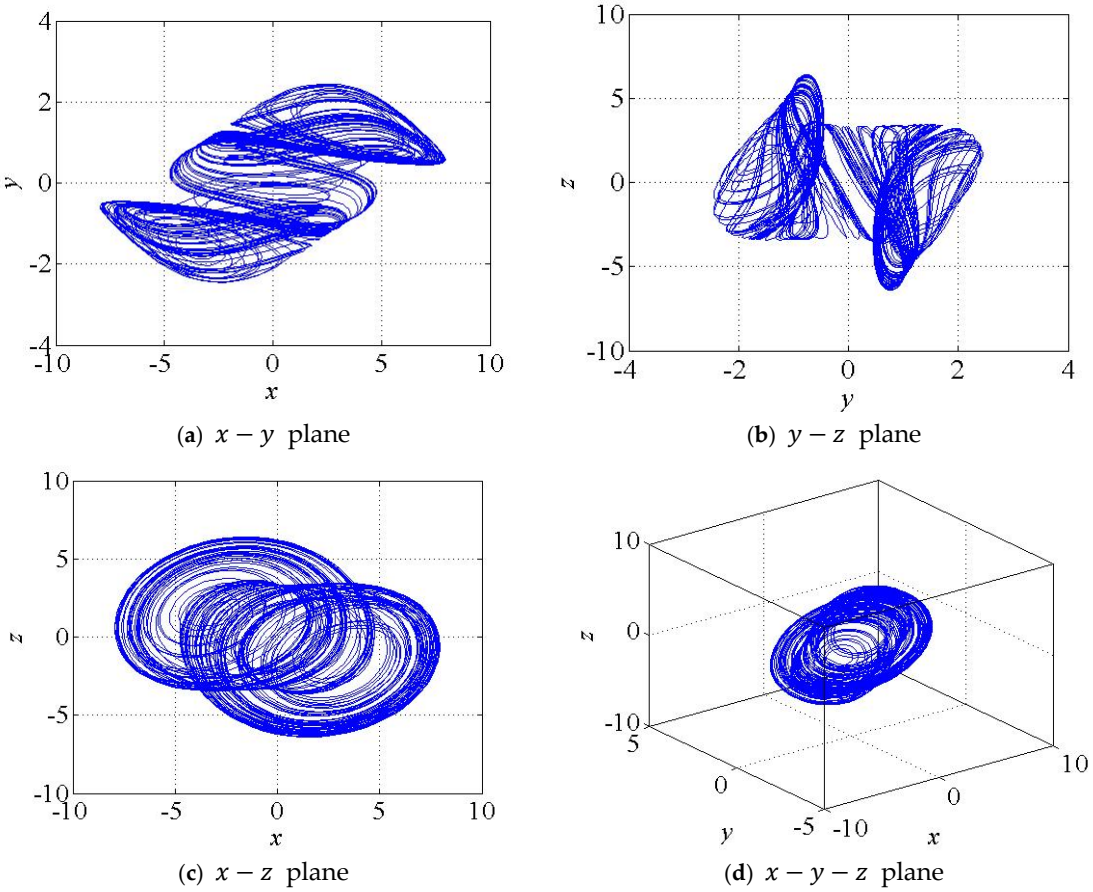


Figure 1. The phase portraits of the new multi-echelon chaotic system (10) in various planes for the initial conditions (0.03, 0.03, 0.03). We confirm that the overlapping content in Figure does not affect the scientific understanding or interpretation of results.

3. Dynamic Analysis of Financial Firm System

The bifurcation diagram and Lyapunov Exponent (LE) spectrum are the core tools used to study the chaotic behavior present in the system (10). The bifurcation diagram gives the details about the qualitative changes in the system's behavior as the parameter varies. The bifurcation analysis is mainly used to identify the transition between stable state to periodic oscillations and chaotic attractors. The LE is the tool that provides the quantitative measure to detect the chaos in the system. A positive LE value indicates the sensitivity on initial conditions and confirms the presence of chaos in the system. This section focuses on the combined study of bifurcation and the LE spectrum has been employed to investigate the complex dynamics of the new finance system. All numerical results, including phase portraits, bifurcation diagrams, Lyapunov exponents, and multistability analyses, were obtained using the classical fourth-order Runge–Kutta (RK4) integration scheme with an appropriate fixed time step.

Figure 2 illustrates the bifurcation diagram and LE spectrum of the system (10) as a function of parameter a over the region $a \in [0 - 0.6]$. The bifurcation analysis reveals that the system undergoes alternating transitions between periodic and chaotic states as the parameter a varies. The periodic behaviors are observed over the regions $a \in [0.2 - 0.25]$, and $a \in [0.43 - 0.6]$, where the bifurcation exhibits distinct single branch structures. In contrast, the chaotic regions are observed in the regions $a \in [0.05 - 0.2]$, and $a \in [0.25 - 0.35]$, which are characterized by dense bifurcation distributions. Figure 2b demonstrates that

the chaotic intervals in Figure 2a are further confirmed by the presence of positive largest LE values, indicating the emergence of chaos in the system. The phase projections of the system for various values of a are given in Figure 3.

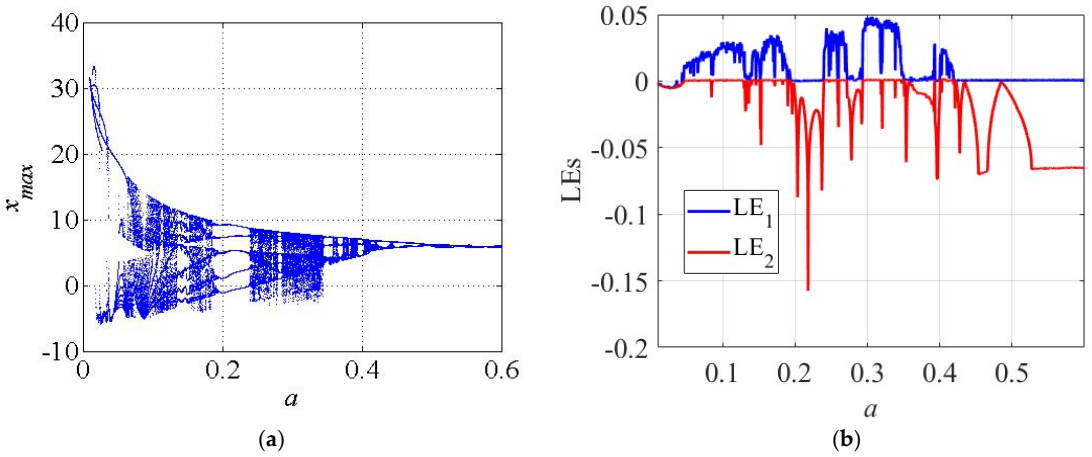


Figure 2. The dynamic analysis of the system (10) for the variation in the parameter a. (**a**) Bifurcation diagram; (**b**) LE spectrum.

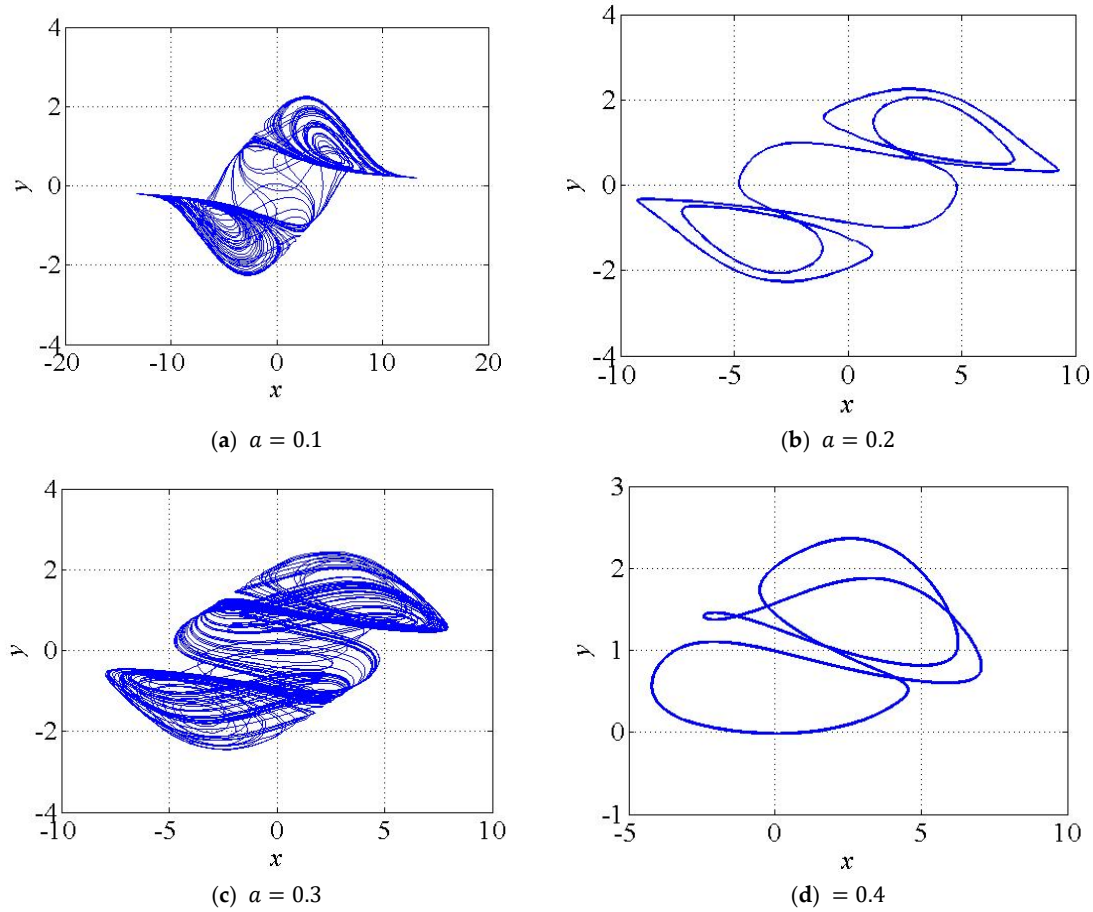


Figure 3. The phase portraits of the new financial firm system (10) for various values of the parameter a. (**a**) Chaotic attractor at $a = 0.1$, (**b**) periodic attractor at $a = 0.2$, (**c**) chaotic attractor at $a = 0.3$, and (**d**) periodic attractor at $a = 0.5$.

The bifurcation diagram and LE spectrum of the system (10) with respect to the parameter $b \in [0 - 2]$ are displayed in Figure 4. Figure 4a displays that the system exhibits periodic oscillatory behavior over the region $b \in [0 - 0.15]$, as evidenced by the single

bifurcation branch. As the parameter increases beyond $b = 0.15$, the system undergoes the transition from periodic to chaotic state and displays chaotic attractor over the region $b \in [0.151 - 1.7]$, as evidenced by the densely distributed structure. As the parameter increases further, the system returns to periodic oscillatory. This bifurcation analysis is confirmed by the LE spectrum, presented in Figure 4b. The presence of positive largest LE values over the interval $b \in [0.151 - 1.7]$ confirms the existence of chaotic attractor in the system. The largest LE value remains at zero over the remaining interval, indicating the periodic oscillation in the system. The phase projections of the system (10) for various regions of the parameter b are displayed in Figure 5.

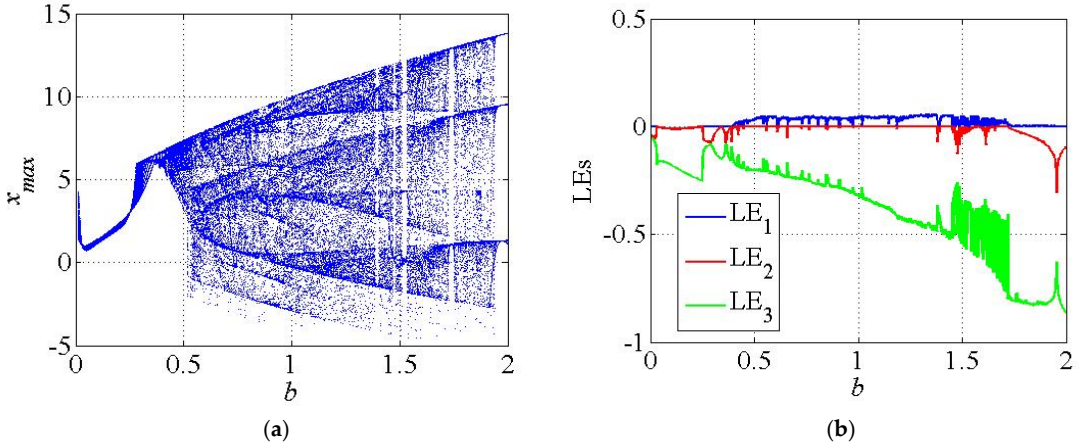


Figure 4. The dynamic analysis of the system (10) for the variation in the parameter b. (**a**) Bifurcation diagram; (**b**) LE spectrum.

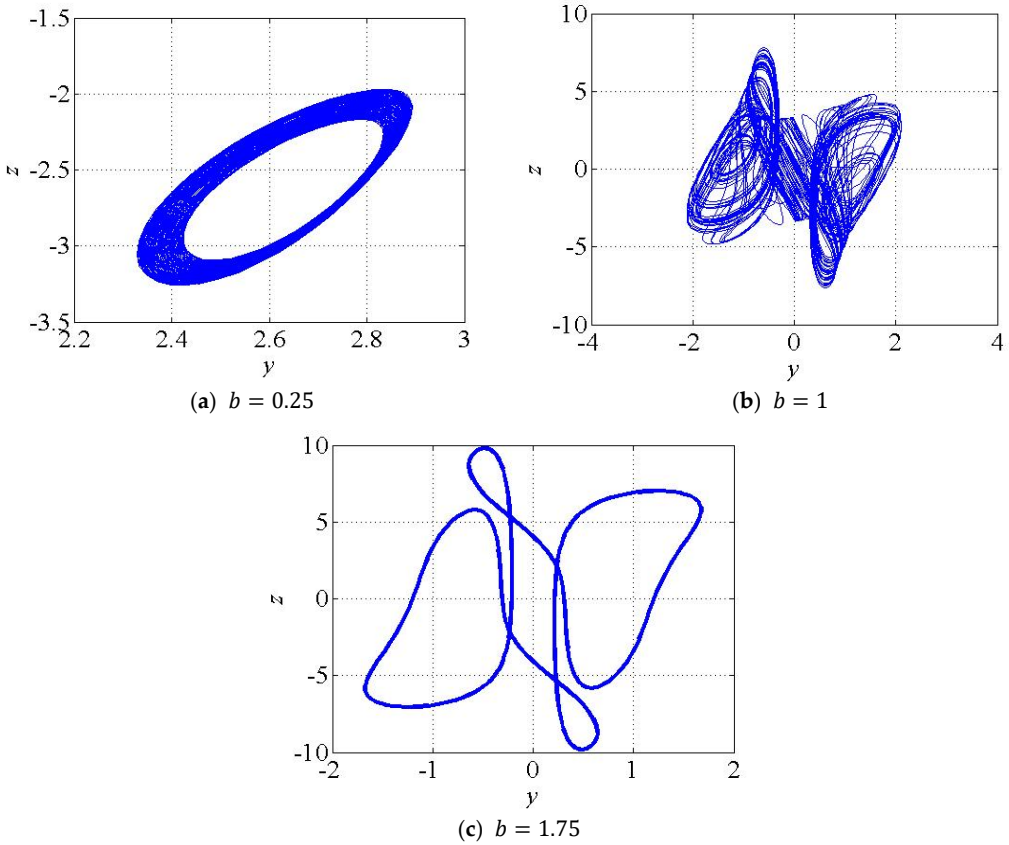


Figure 5. The phase portraits of the new financial firm system (10) for various values of the parameter b. (**a**) Periodic attractor at $b = 0.25$, (**b**) chaotic attractor at $b = 1$, and (**c**) periodic attractor at $b = 1.75$.

The bifurcation diagram and the corresponding LE spectrum as a function of parameter c is presented in Figure 6. The bifurcation diagram, shown in Figure 6a, illustrates that the system exhibits periodic oscillatory behavior over the interval $c \in [0 - 0.3]$, which is characterized by distinct single branches in the bifurcation structure. As the parameter c varies beyond c = 0.3, the bifurcation structure changes from single branch pattern to densely populated distribution over the interval $c \in [0.31 - 0.7]$, indicating the presence of chaotic attractor in the system. With further increase in the parameter c, periodic windows are detected over the region $c \in [0.71 - 1.4]$. Beyond c = 1.4, the bifurcation structure again displays a single-line pattern, signifying that the system returns to periodic oscillations for higher values of c. The bifurcation result is further verified by the LE spectrum, shown in Figure 4b. In Figure 6b, the largest LE values at almost zero over the parameter interval $c \in [0 - 0.3]$ and $c \in [0.71 - 2]$, indicating the periodic dynamics of the system. The largest LE values remain positive over the interval $c \in [0.31 - 0.7]$, confirming the sensitivity on initial conditions which is the key indicator of the chaotic behavior in the system. Figure 7 displays the phase projections of the system (10) under the various regions of parameter c.

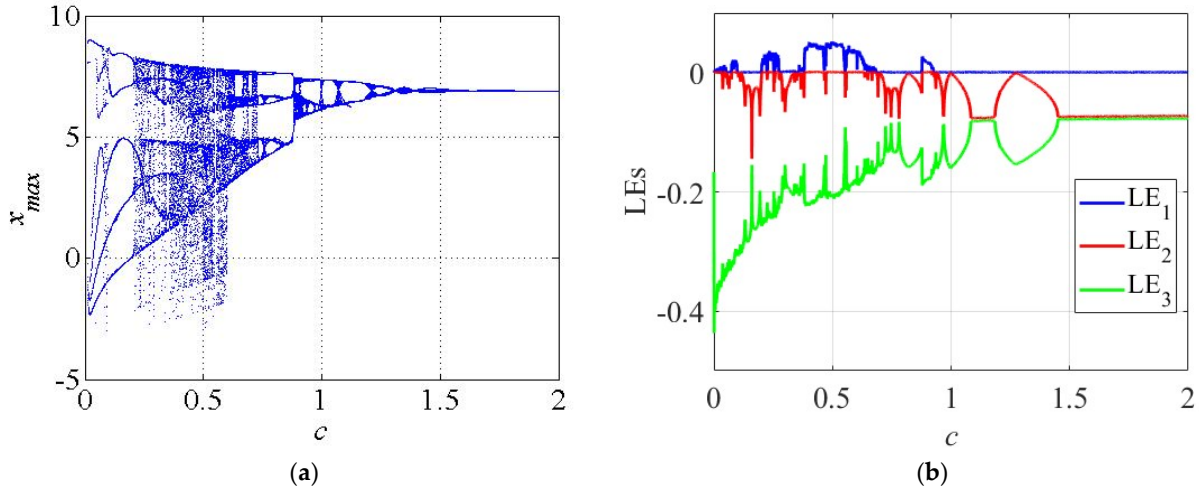
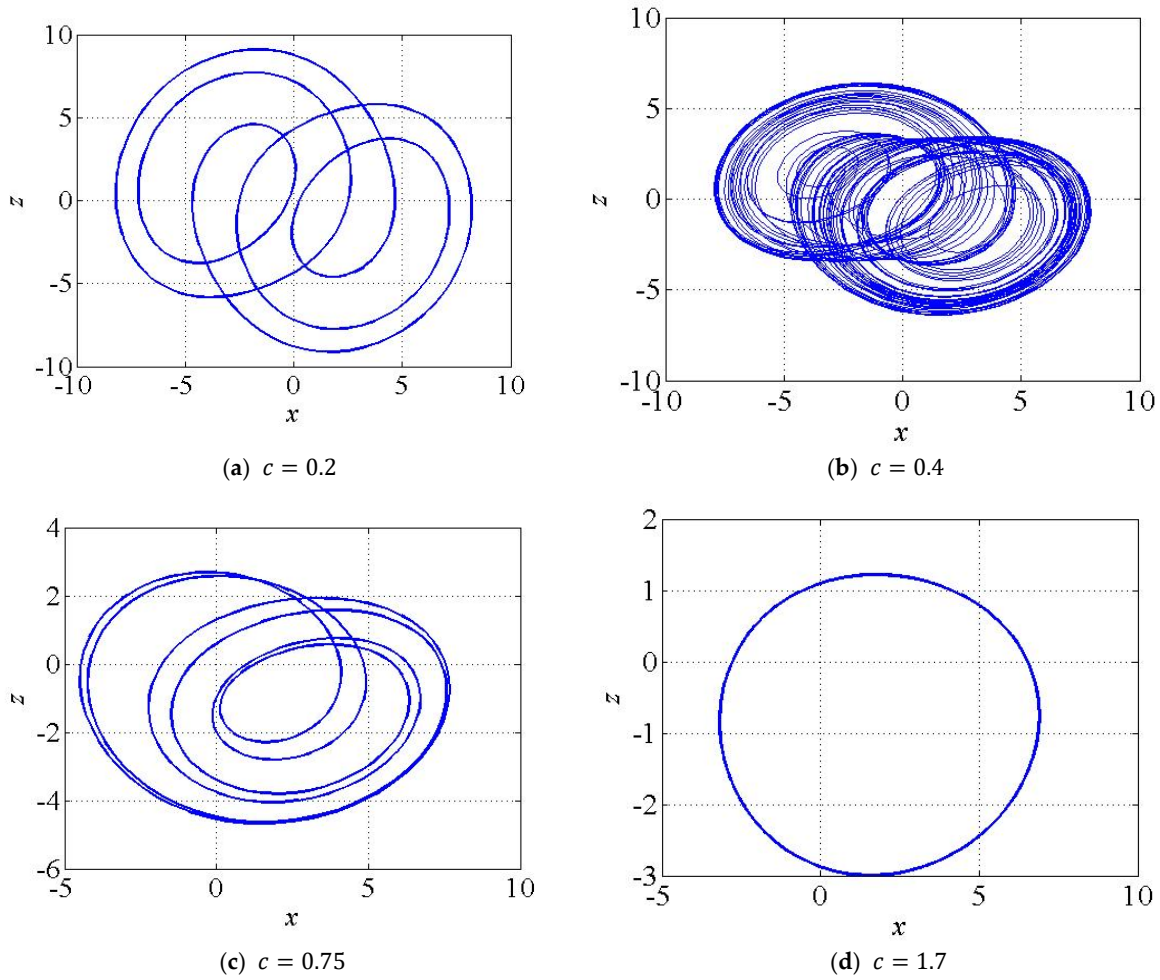


Figure 6. The dynamic analysis of the system (10) for the variation in the parameter c. (**a**) Bifurcation diagram; (**b**) LE spectrum.

Figure 8 represents the bifurcation structure and LE spectrum of the system (10) as the function of parameter m over the interval $m \in [0 - 0.15]$. Figure 5a illustrates that initially the system exhibits periodic oscillation over the narrow interval $m \in [0 - 0.01]$. With the further increase in the parameter m, the system undergoes changes from periodic to chaotic state, which is characterized by the densely populated bifurcation pattern. Furthermore, for $m > 0.09$, a periodic window is observed, indicating the appearance of periodic oscillator in the system. The existence of chaotic behavior in the region $m \in [0.011 - 0.09]$ is further supported by the positive largest LE values in the corresponding LE spectrum shown in Figure 8b. The largest LE values remain zero over the regions $m \in [0 - 0.01]$ and $m \in [0.091 - 0.15]$, indicating the presence of periodic oscillation in the system. Furthermore, Figure 8c,d shows the phase projections of the system under chaotic and periodic regions respectively.

Figure 9 illustrates the bifurcation diagram and corresponding LE spectrum with respect to the parameter n. The bifurcation structure, shown in Figure 9a, reveals that the system exhibits periodic attractors over the regions $n \in [0 - 0.0095]$ and $n > 0.045$, which is characterized by distinct single branches. The densely populated bifurcation pattern over the region $n \in [0.01 - 0.045]$ confirms the presence of chaotic attractors in the system. In Figure 9b, the largest LE values remain positive over the parameter region

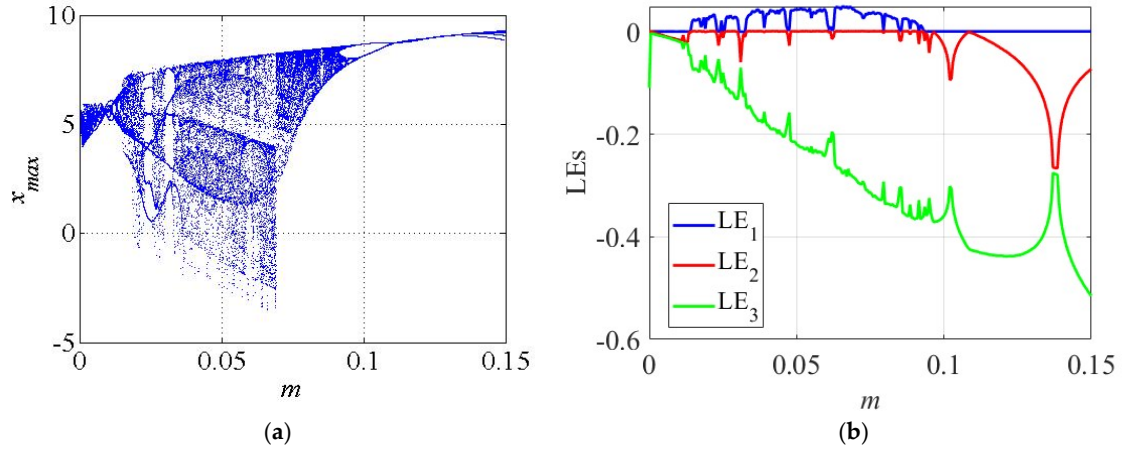
$n \in [0.01 - 0.045]$, signifying the emergence of chaotic dynamics in the system. Figure 10a,b show the phase projections of the system under chaotic and periodic regions respectively.



(**a**) $c = 0.2$ (**b**) $c = 0.4$

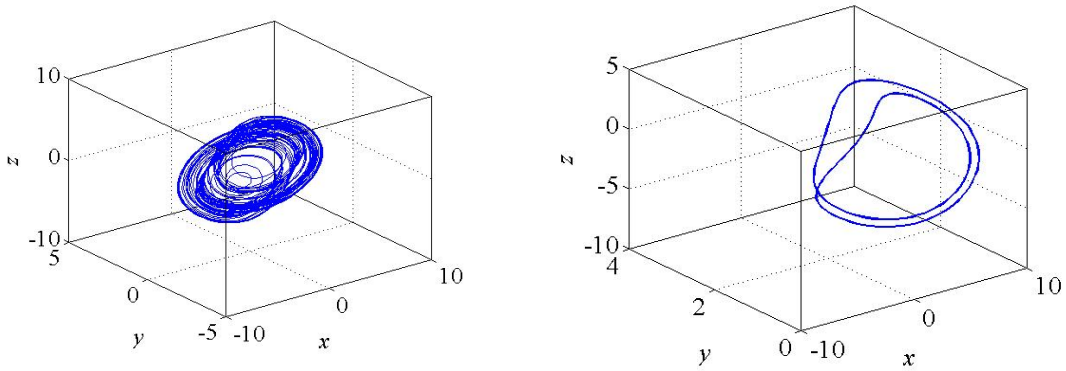
(**c**) $c = 0.75$ (**d**) $c = 1.7$

Figure 7. The phase portraits of the new financial firm system (10) for various values of the parameter c. (**a**) Periodic attractor at $c = 0.2$, (**b**) chaotic attractor at $c = 0.4$, (**c**) periodic attractor at $c = 0.75$ and (**d**) periodic attractor at $c = 1.7$.



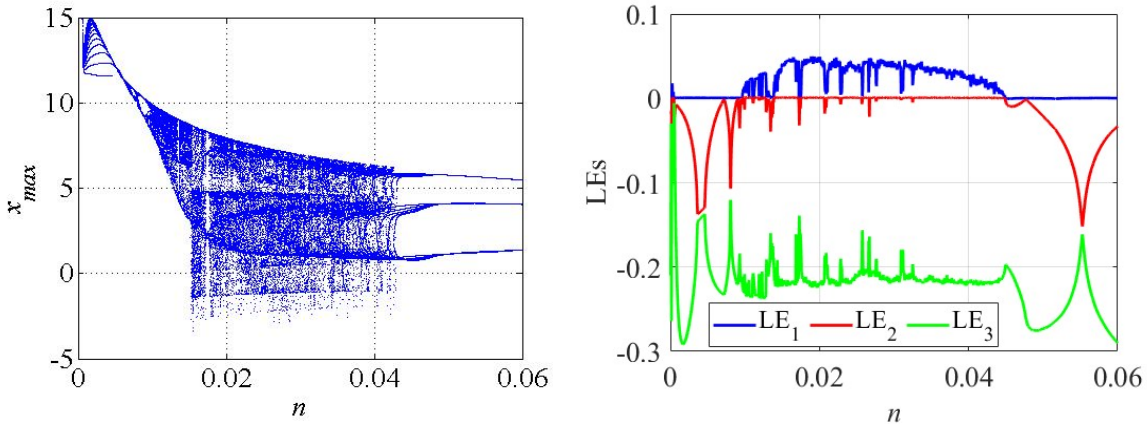
(**a**) (**b**)

Figure 8. *Cont.*



(**c**) $m = 0.05$ (**d**) $m = 0.1$

Figure 8. The dynamic analysis of the system (10) for the variation in the parameter m. (**a**) Bifurcation diagram; (**b**) LE spectrum; (**c**) chaotic attractor at $m = 0.05$; and (**d**) periodic attractor in $m = 0.1$.

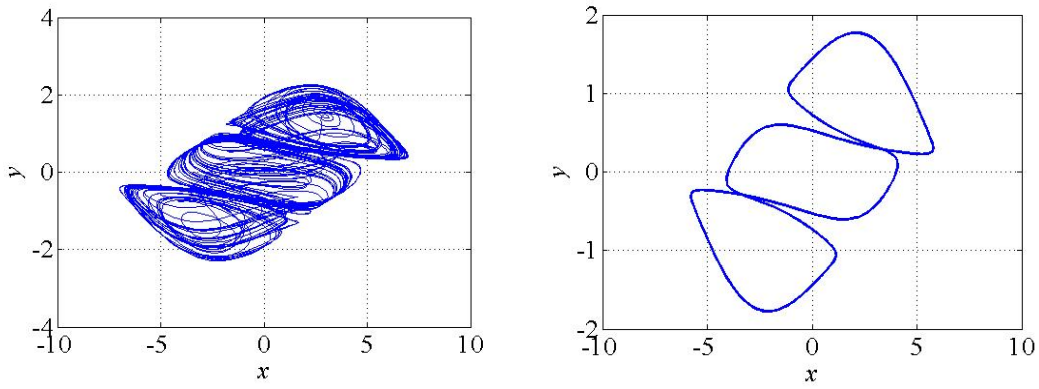


(**a**) (**b**)

Figure 9. The dynamic analysis of the system (10) for the variation in the parameter n. (**a**) Bifurcation diagram; (**b**) LE spectrum.

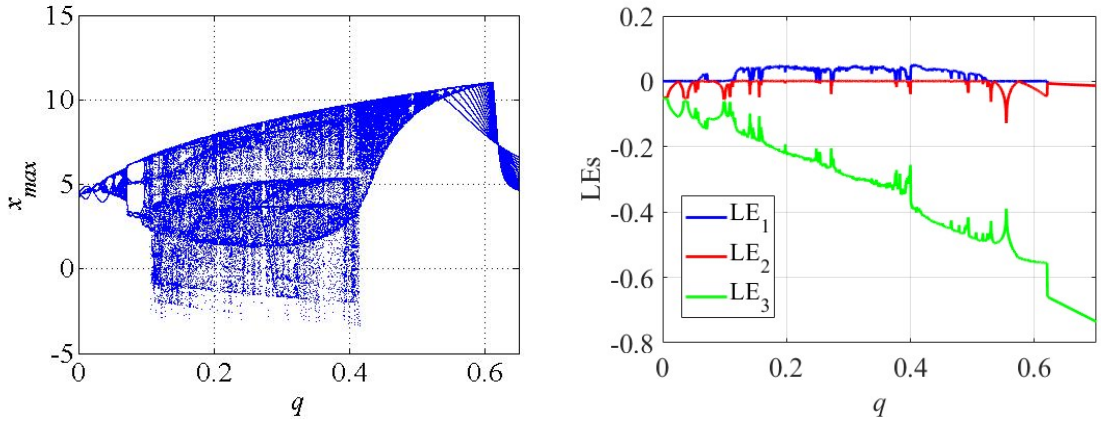
Figure 11 displays the bifurcation diagram and LE spectrum of the system (10) with respect to the parameter q. The bifurcation structure reveals that the system exhibits periodic region over the parameter region $q \in [0 - 0.16]$, followed by the chaotic region over the region $q \in [0.165 - 0.5]$. For $q > 0.5$, the system becomes stable, as indicated by the appearance of distinct single branches. The corresponding LE spectrum further confirms these dynamics, where the largest LE values remain positive over the region $q \in [0.165 - 0.5]$, indicating the presence of chaos in the system.

The phase portraits in Figure 12 illustrate the influence of the nonlinear feedback parameter q on the dynamics of the proposed financial system. For smaller values of q, the nonlinear feedback introduced through the term $qsin(y)$ is relatively weak, and the system exhibits regular periodic oscillations, corresponding to predictable variations in profitability, reinvestment, and borrowing. As q increases, the nonlinear feedback becomes stronger, enriching the interactions among the financial variables and eventually driving the system into chaotic behavior. Economically, this transition indicates that excessively strong endogenous reinvestment feedback may lead to irregular and highly sensitive financial fluctuations, making long-term planning and financial management more difficult. These results suggest that maintaining the intensity of the nonlinear reinvestment feedback within an appropriate range can help promote stable and sustainable financial performance.



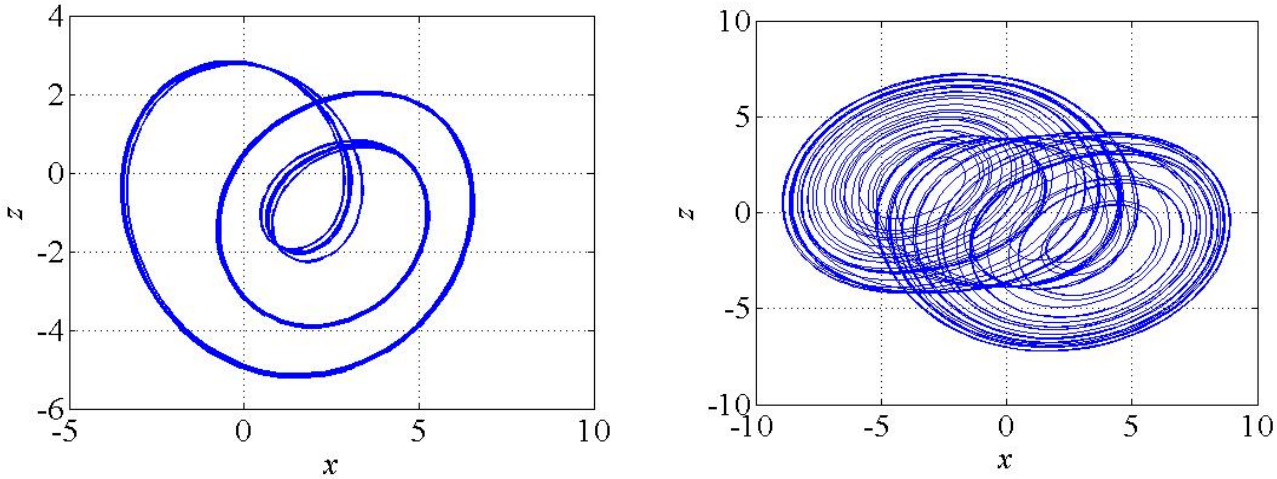
(**a**) $n = 0.03$ (**b**) $n = 0.05$

Figure 10. The phase portraits of the proposed 3-D chaotic financial firm system (10) for various values of the parameter n. (**a**) Chaotic attractor at $n = 0.03$, and (**b**) periodic attractor at $n = 0.05$.



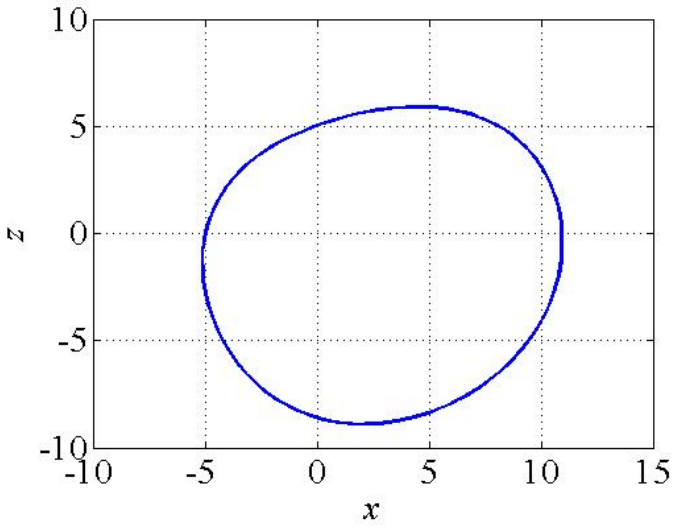
(**a**) (**b**)

Figure 11. The dynamic analysis of the system (10) for the variation in the parameter q. (**a**) Bifurcation diagram; (**b**) LE spectrum.



(**a**) $q = 0.1$ (**b**) $q = 0.3$

Figure 12. *Cont.*



(**c**) $q = 0.6$

Figure 12. The phase portraits of the proposed 3-D chaotic financial firm system (10) for various values of the parameter q. (**a**) Periodic attractor at $q = 0.1$, (**b**) chaotic attractor at $q = 0.3$, and (**c**) periodic attractor at $q = 0.6$.

4. Multistability

The multistability behavior of the financial system represents that the system may produce multiple states depending on its initial conditions. From an economic perspective, multistability means that similar financial environments may produce different outcomes due to different initial investment levels and firm capital. The presence of coexisting attractors indicates that the small changes in the initial state can cause the system to switch from one stable state to another state. Financial systems are highly sensitive on market conditions such as perturbations and management decisions, making their long term unpredictable. Therefore, examining multistability may help us to predict financial system behavior and to maintain desirable financial performance.

Multistability is an important analysis in chaotic systems in which the system exhibits more than one attractor [25]. As a result, the chaotic system evolves towards different pattern of oscillations depending on initial conditions, even when the system parameters remain fixed. When multistability is present, sensitivity to initial conditions becomes more significant because small variations in the initial conditions can cause the system to converge to different attractors.

These multiple attractors may be periodic oscillation or chaotic attractors. Understanding multistability in a chaotic system is important since the transition between the coexisting attractors influence the complexity and predictability of the real-world systems.

The bifurcation analysis is one of the powerful tools used to detect the multistability in a chaotic system. In the bifurcation diagram, the multiple attractors appear as non-overlapping branches corresponding to the same system parameter values. To demonstrate the existence of multistability, the bifurcation diagrams of the system (10) are plotted for the different initial conditions as shown in Figure 13. The non-overlapping nature of the blue and red branches in the bifurcation diagrams confirms the presence of multiple coexisting attractors in the system. Figure 13a shows the existence of multistability and coexisting attractors in the periodic regions $q \in [0 - 0.1]$ and $q \in [0.5 - 0.6]$.

Similarly, the multistability and coexisting attractors observed in the chaotic region $q \in [0.42 - 0.5]$. Figure 13b illustrates that the system has multiple coexisting attractors over the parameter range $n \in [0 - 0.018]$. The multistability behavior of the proposed

system is further verified with the attractor diagrams. Some of the coexisting attractors corresponding to the same parameter values are displayed in Figure 14. These analyses demonstrate that the system exhibits multiple coexisting attractors in both periodic and chaotic regimes.

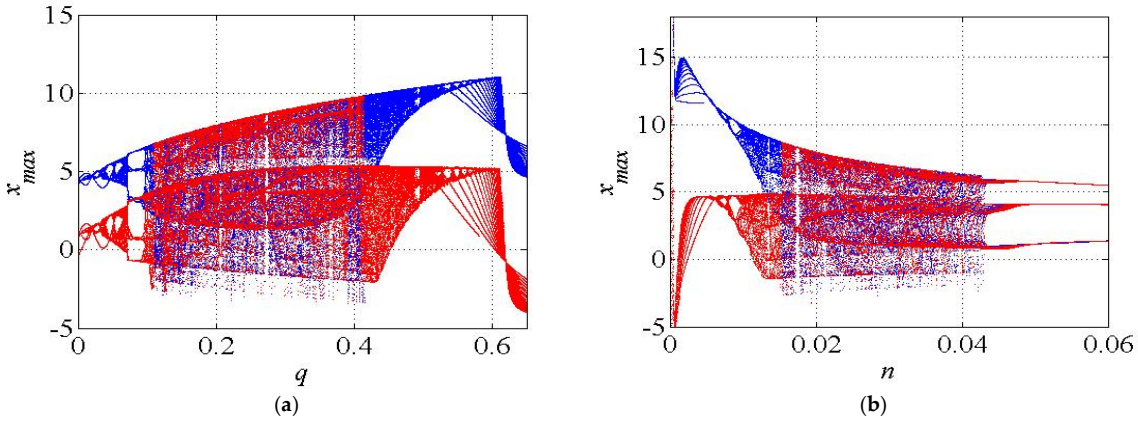


Figure 13. Multistability analysis in the proposed 3-D chaotic financial firm system (**a**,**b**) The bifurcation diagram with the distinct initial conditions (0.03, 0.03, 0.03), shown in blue branch and (−0.03, −0.03, −0.03), shown in red branch.

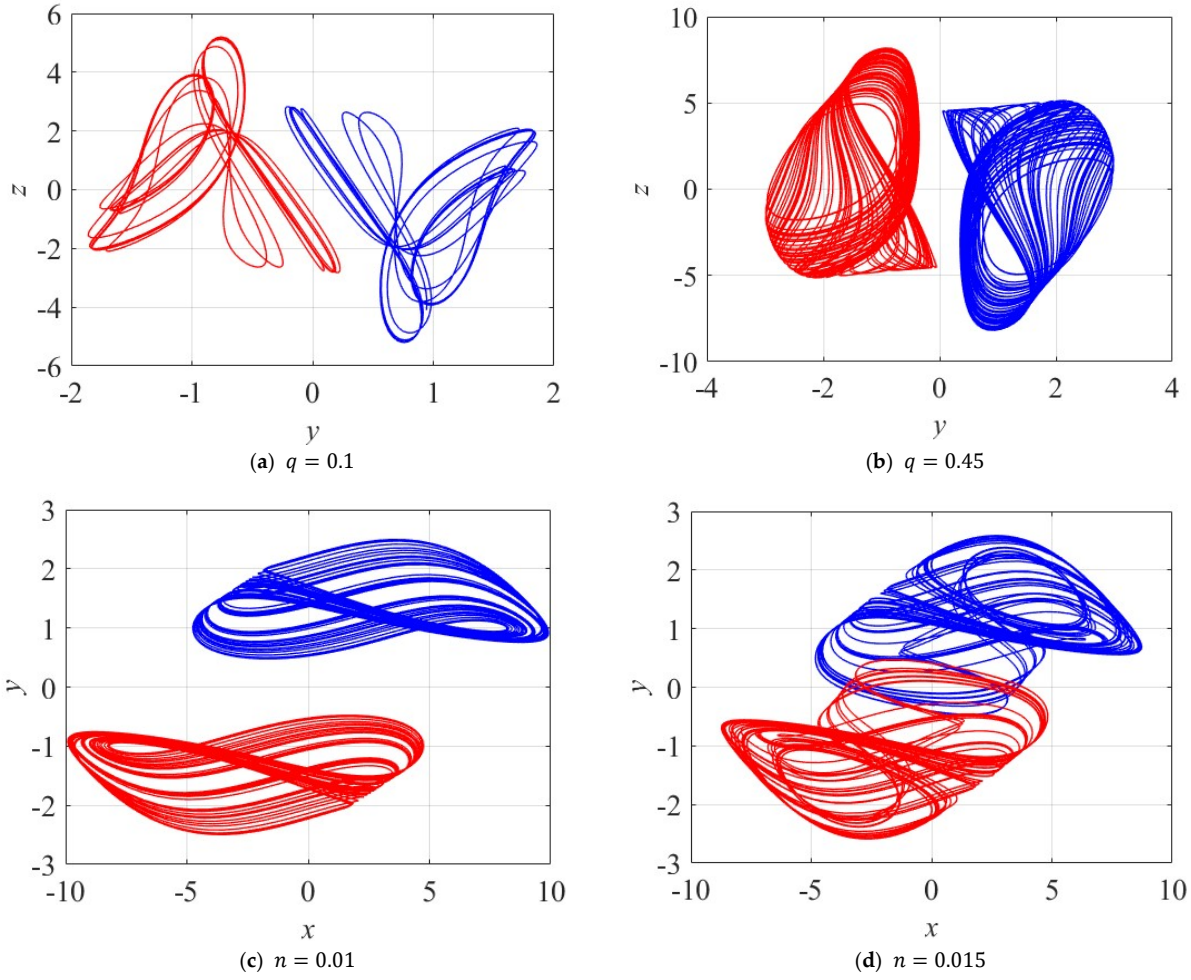


Figure 14. The coexisting attractors of the system with different initial conditions (0.03, 0.03, 0.03) (blue) and (−0.03, −0.03, −0.03) (red). (**a**) Periodic coexisting attractors at $q = 0.1$, (**b**) chaotic coexisting attractors at $q = 0.45$, (**c**) one scroll coexisting attractors at $n = 0.01$, and (**d**) two scroll coexisting attractors at $n = 0.015$.

5. Chaotic Control Analysis

In a chaotic system, the control techniques are often employed to modify the dynamic behavior of the system without disturbing its chaotic behavior. Among these techniques, amplitude control and offset boosting control are important methods used to control the size and location of the attractors in phase space. In a financial chaotic system, the amplitude control regulates the amplitude of variables such as interest rate, asset price and investment demand. The excessive amplitude of oscillations may represent extreme price volatility, significant fluctuations in investment demand and increased market instability. Therefore, controlling the amplitude of oscillations may help us to reduce or adjust the intensity of financial fluctuations. It should be emphasized that the present work is a theoretical study of the nonlinear dynamics of a three-dimensional financial firm system. The primary objective is to investigate how the proposed state-dependent nonlinear feedback mechanism influences the system's bifurcation structure, chaotic dynamics, multistability, symmetry properties, amplitude control, and offset boosting. The amplitude control and offset boosting techniques presented in this paper are introduced as theoretical mathematical tools for modifying the scale and equilibrium (baseline) of the model variables while preserving their qualitative dynamical behavior. They are intended to provide insight into the intrinsic dynamics of the proposed financial system and should not be interpreted as implementable financial control strategies or direct economic policy mechanisms for financial risk management or volatility regulation.

Offset boosting control shifts the operating point of asset price, interest rates and investment demand. For instance, asset price may operate around different operating points under changing economic conditions. The offset boosting control is a useful mechanism for modeling regime shifts, changes in market equilibrium levels, and variations in economic operating conditions.

Chaotic amplitude control [26,27] is a method used to regulate the amplitude of the state variables of the system by introducing a scaling parameter into the system equation. By varying the scaling parameter, the amplitude of the variable can be adjusted uniformly in phase space. Let us apply $x \to \beta x$, $y \to \beta y$ and $z \to \beta z$ into the original system (10), where β is the scaling parameter. The resultant transformed system is represented by Equation (25) through which the amplitude of all the state variables can be adjusted simultaneously. Thus, the system (25) is known as a total amplitude control (TAC) system. Through the variation of β, the amplitude of state variable is uniformly adjusted by the factor $\frac{1}{\beta}$. The amplitude of this increases when $\beta < 1$, whereas the amplitude decreases when $\beta > 1$.

$$\begin{cases} \dot{x} = c(y+z) \\ \dot{y} = mx + n(1-\beta^2 x^2)y + \frac{q}{\beta}\sin(\beta y) \\ \dot{z} = -ax + by \end{cases} \tag{25}$$

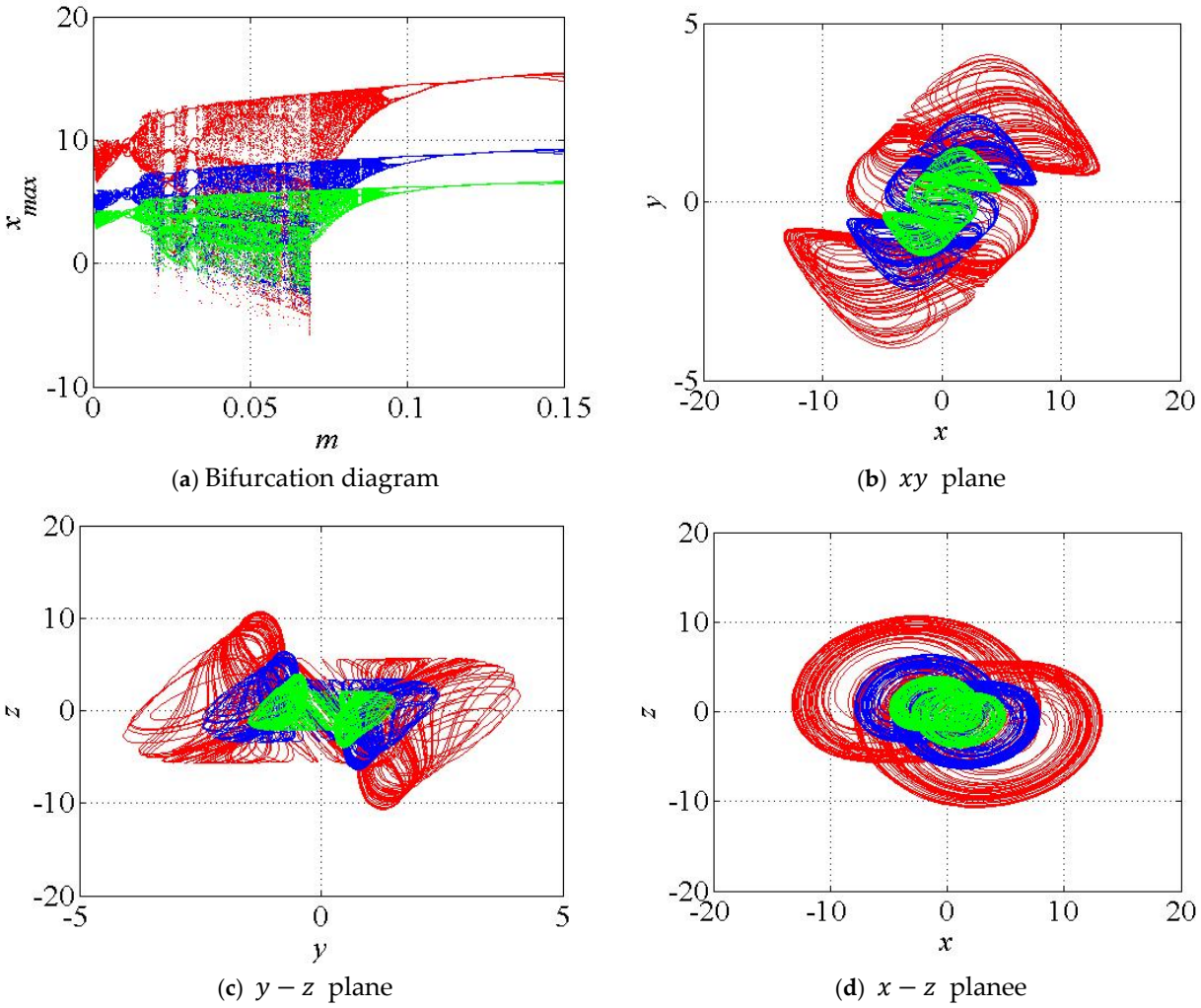
Figure 15a displays the bifurcation diagram of the TAC system with different scaling parameters $\beta = 1$ (original system), shown in blue, $\beta = 0.6$, shown in red, and $\beta = 1.4$ shown in green. It shows that the control methodology effectively regulates the amplitude of the state variables without modifying the chaotic and periodic regions of the system. The phase projections of the system in $x-y$, $y-z$ and $x-z$ planes respectively are given in Figure 15b–d. These plots illustrate that the control scheme regulates the amplitude (or) size of all the state variables effectively.

Offset boosting [28,29] is another control methodology used to shift the average value or DC value of the system in the phase space without destroying its inherent chaotic dynamics. This can be effectively achieved by adding a constant booster parameter into the

system equation. Let us apply $z \to z + \delta$ into the original system (10), the location of the state variable z can be adjusted in a controlled manner.

$$\begin{cases} \dot{x} = c(y + z + \gamma) \\ \dot{y} = mx + n(1 - x^2)y + q\sin(y) \\ \dot{z} = -ax + by \end{cases} \tag{26}$$

Equation (26) represents the offset boosted version of the original system (10). Since the state variable z appears only once in the original system, the offset boosting along with the state variable z can be easily achieved.



(**a**) Bifurcation diagram

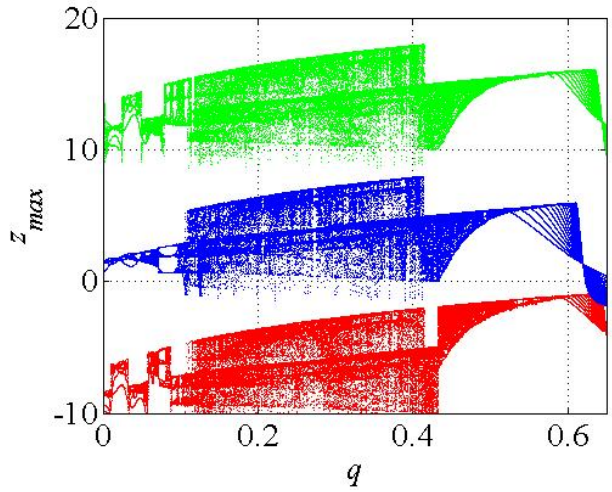
(**b**) xy plane

(**c**) $y - z$ plane

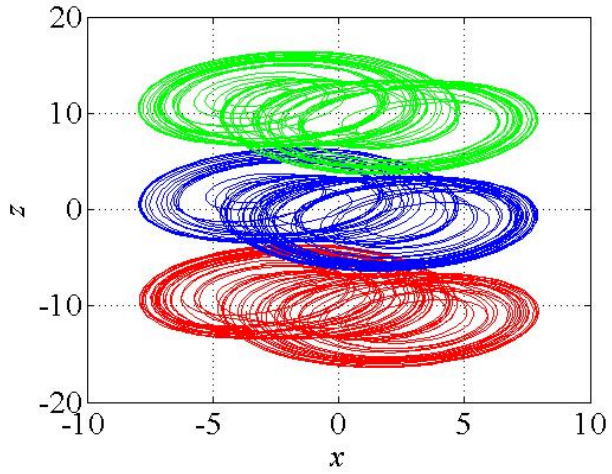
(**d**) $x - z$ planee

Figure 15. Total amplitude control analysis. (**a**) Bifurcation diagram of the system (25); (**b**–**d**) amplitude regulated attractors in xy, yz and xz plane respectively. The plots are obtained with the scaling parameter $\beta = 1$ (original system), shown in blue, $\beta = 0.6$, shown in red, and $\beta = 1.4$ shown in green.

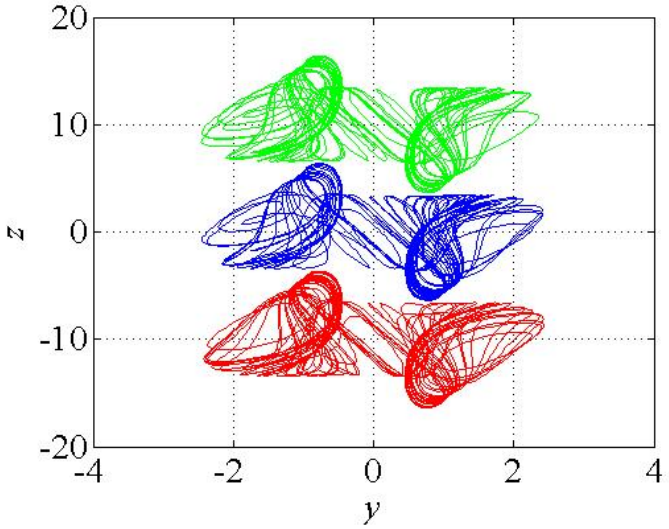
The bifurcation diagram of the system (26) is displayed in Figure 16a, in which the blue branch corresponds to the original system, the red branch corresponds to $\gamma = +10$ and the green branch corresponds to $\gamma = -10$. Figure 16a illustrates that the location of the state variable z is shifted without changing the chaotic and periodic regimes of the system. Figure 16b,c show the phase projections of the system with various booster parameters.



(**a**) Bifurcation diagram



(**b**) xz plane



(**c**) yz plane

Figure 16. Offset boosting control analysis. (**a**) Bifurcation diagram of the system (26), (**b**,**c**) offset boosted attractors in xz and yz plane respectively. The plots are obtained with the boosting parameter $\gamma = 0$ (original system), shown in blue, $\gamma = +10$, shown in red, and $\gamma = -10$ shown in green.

6. Conclusions

This paper introduced a novel three-dimensional chaotic financial firm system by incorporating a cyclic reinvestment perturbation into the classical financial dynamics framework. The introduced sinusoidal perturbation was designed to represent fluctuations in reinvestment activities arising from business cycles and changing economic conditions. The proposed modification enriches the nonlinear feedback mechanism of the financial system and generates rich dynamical behaviors compared with existing financial firm models.

The dynamical properties of the proposed system were investigated through equilibrium analysis, local stability analysis, Lyapunov exponents, Kaplan–Yorke dimension, and bifurcation analysis. Numerical results confirmed the existence of chaotic dynamics characterized by a positive maximum Lyapunov exponent and a fractal attractor. Furthermore, bifurcation and Lyapunov spectrum analyses revealed multiple transitions between periodic and chaotic regimes as system parameters varied. The model was also shown to exhibit multistability and coexisting attractors, indicating a strong dependence on initial conditions and a rich nonlinear structure.

In addition, total amplitude control and offset boosting techniques were successfully applied to regulate the size and location of attractors without destroying the intrinsic dynamics of the system. These findings demonstrate that cyclic reinvestment perturbations play a significant role in shaping complex financial dynamics and provide a useful framework for studying instability, multistability, and chaos control in financial systems. Future work may focus on synchronization, fractional-order extensions, data-driven parameter estimation, and machine learning-assisted prediction of financial dynamics. Estimation of the parameter q from observed economic data constitutes an important direction for future research.

Limitations and Future Work: The present study is primarily a theoretical investigation of the nonlinear dynamics of a three-dimensional financial firm system. Consequently, several limitations should be acknowledged. First, the proposed model has not been empirically calibrated or validated using real-world financial data. Second, the bifurcation analysis is restricted to one-parameter diagrams, which provide sectional views of the system dynamics; a comprehensive two-parameter bifurcation analysis, together with numerical continuation and rigorous bifurcation classification, would provide a more complete understanding of the global organization of periodic, chaotic, and multistable regions in the parameter space. Third, the economic interpretations presented in this work are qualitative and intended to provide conceptual insight into the underlying nonlinear dynamics rather than quantitative predictions. Finally, the proposed state-dependent nonlinear feedback term $\mathrm{qsin}(\mathrm{y})$ contains no independent frequency parameter, with the parameter q governing only the strength of the nonlinear feedback. Addressing these limitations through empirical validation and more comprehensive bifurcation analyses constitutes an important direction for future research.

Author Contributions: Conceptualization, B.F., M.D.J. and A.S.; Software, S.P., S.V. and R.R.; Validation, S.P. and B.M.; Formal analysis, S.P., V.R. and B.M.; Investigation, S.V. and R.R.; Resources, V.R. and B.M.; Data curation, M.D.J.; Writing—original draft, B.F., S.V. and R.R.; Writing—review and editing, B.F., V.R. and A.S.; Supervision, M.D.J. and A.S. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare that they have no conflicts of interest with other people or organizations that could influence the content of the paper.

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