

Article

Advanced Thick FGM Plate–Cylindrical Shells in Supersonic Air Flow by Navier–Stokes Equation Analytical–Numerical Flow Model

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Abstract

The thermal vibrations of a thick-walled functionally graded material (FGM) plate–cylindrical shells in unsteady supersonic flow with a Navier–Stokes equation analytical–numerical flow model and third-order shear deformation theory (TSDT) displacement models are investigated. The aerodynamic pressure load can be provided by using the Navier–Stokes equation analytical–numerical flow model. The data regarding the effect of the aerodynamic pressure load and TSDT model of the motion equation on the thermal stress and displacement of the FGM plate–cylindrical shells in unsteady supersonic flow are calculated with the generalized differential quadrature (GDQ) method. The Navier–Stokes equation analytical–numerical flow model, TSDT model, and advanced shear correction coefficient provide an additional effect on the values of displacement and stress.

Keywords: unsteady supersonic flow; CFD; thick-walled FGM; plate–cylindrical shells; TSDT

1. Introduction

Some aerodynamic–thermal investigations were carried out for coupled elastic structures in supersonic flow. In 2024, Wu et al. [1] applied the higher-order shear deformation theory (HSDT) of displacement and pressure loading in aerodynamic–supersonic flow to study the aero–elastic flutter of a functionally graded material (FGM) nano-pipe. In 2023, Tamura et al. [2] used nonlinear optics and condensed matter to investigate supersonic flow across an acoustic horizon around a sink. Ben-Youssef et al. [3] in 2022 applied Novozhilov’s strain displacement theory, first-order piston theory, and the finite element method (FEM) to investigate the vibrations of thin-walled cylindrical shells in supersonic air flow. In 2020, Xue et al. [4] applied the first-order shear deformation theory (FSDT) of displacement and piston theory for aerodynamic pressure to study the flutter and active control of FGM piezoelectric plates in supersonic flow. In 2020, Rahmanian and Javadi [5] studied the free vibration and aero–elastic stability of porous FGM cylindrical shells in supersonic flow. In 2020, Lin et al. [6] applied the nonlinear third-order piston theory for aerodynamic pressure to investigate the thermal flutter of a shape memory alloy panel in supersonic flow. In 2018, Guruswamy [7] applied Reynolds-averaged Navier–Stokes equations (RANSs) based on aero–elasticity to study the short period oscillation (SPO) for transport in supersonic flow. In 2018, Hammargren [8] applied a computational fluid dynamics (CFD) aerodynamic models to study the rocket in sounding flow. In 2017, Liu and Luo [9] used unsteady RANS equations to study a low-Reynolds-number propeller for unmanned aerial vehicles (UAVs). In 2016, Shahverdi et al. [10] used the classical plate theory (CPT) of displacement,



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first-order piston theory of aerodynamic pressure, and generalized differential quadrature (GDQ) method to study the aero–thermo–elasticity of FGM plates in supersonic flow. In 2015, Quan et al. [11] applied classical shell theory of displacement and Ilyushin nonlinear supersonic theory to study and control the flutter of FGM elastic–foundation cylindrical panels in supersonic flow. In 2015, Liu et al. [12] applied third-order shear deformation theory (TSDT) of displacement and first-order piston theory for aerodynamic pressure to study the nonlinear vibration of FGM circular–cylindrical shells in supersonic flow. In 2011, Navazi and Haddadpour [13] applied the CPT of displacement, linear piston theory of pressure distribution, and viscous aerodynamic heating to study the nonlinear aero–thermo–elasticity of FGM plates in supersonic flow.

There is seldom an instance in recent research concerning the dynamic characteristics of plate–shell structures, which account for the aerodynamic loads and thermal effects incorporated with the same values of displacements, shear rotations, and temperatures in two jointed sides of plates–cylindrical shells. Some GDQ results have been presented in the FGM shells and plates by using the third-order piston theory [6] and first-order piston theory [14] for the calculation of the aerodynamic pressure load. In 2022, Hong [15] presented the advanced thermal vibration of the FGM plate with the TSDT and advanced shear correction coefficient. In 2025, Hong [16] presented the thermal vibration note of a thick-walled FGM plate–cylindrical shells in unsteady supersonic flow with the aerodynamic pressure provided traditionally by using the nonlinear third-order piston theory model. It is alternative and untraditional to studying the thermal stresses and displacements of the nonlinear TSDT vibration of the thick-walled FGM plate–cylindrical shells in unsteady supersonic flow with the aerodynamic pressure provided directly by using the Navier–Stokes equation analytical–numerical flow model as displayed in Figure 1, which were made of a plate with length a and width b , and a jointed shell with length a for the plate–cylindrical shell structure. The effects of the nonlinear coefficient term of the TSDT models and Navier–Stokes equation analytical–numerical flow model for the aerodynamic pressure load on the thermal stress and the displacement of the thick-walled FGM plate–cylindrical shells in unsteady supersonic flow are investigated. The highlights of this article are to study the aerodynamic pressure load by directly using the Navier–Stokes equation analytical–numerical flow model and comparing it with the piston theory model, to apply the GDQ method on the computations for the Navier–Stokes equation analytical–numerical flow model, to investigate the parameter effects of the environment temperature and Mach number on the time response values of displacement and stress.

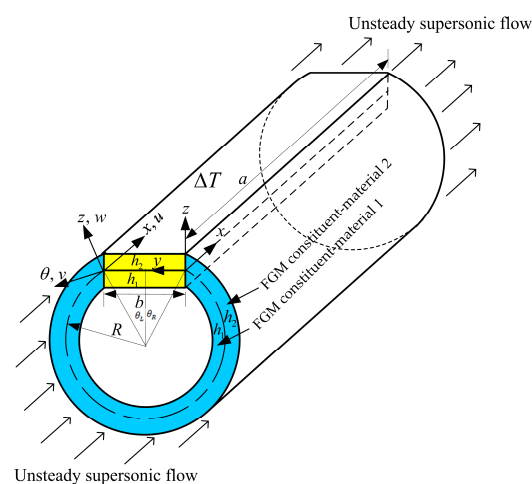


Figure 1. Supersonic-flow in Navier–Stokes equations analytical–numerical flow model over the out-er-surface of FGMs. Yellow is used for the plate, and blue is used for the shell.

2. Materials and Methods

The thermal stress of inter-laminate structure is usually induced by the thermal load due to increasing temperature [17]. The material properties of power law function of FGM plate–cylindrical shells are considered in power law index R_n and in functions of environment temperature T . The time-dependence of displacements u , v , and w of thick-walled FGM plates and circular–cylindrical shells are assumed to be in the coefficient $c_1 = 4/(3h^{*2})$ term of TSDT models, with h^* as the total thickness. The TSDT model of motion equations in plate–cylindrical shells are applied with Hamilton total energy principle for the differential volume $dV = dx dy dz$ of plate in x, y, z coordinate system and $dV = dx R d\theta dz$ of cylindrical shell in x, θ, z coordinate system, where R is middle-surface radius of shells [18]. The external loads are represented in terms of thermal loads with ΔT and mechanical loads (pp_1, p_2, q); e.g., $\Delta T = \frac{z}{h^*} \bar{T}_1 \sin(\pi x/a) \sin(\pi \theta) \sin(\gamma t)$ is temperature-difference expression for shells, γ is frequency of applied heat flux, t is time, and $\bar{T}_1$ is temperature amplitude. p_1 and p_2 are in-plane distributed forces. q is aerodynamic pressure load. One traditional way for calculating q value by using the nonlinear third-order piston theory model has been studied in published work [16] by using GDQ method [19,20]. Another alternative way for calculating q value can be presented directly by using the Navier–Stokes equation analytical–numerical flow model as follows in Sections 2.1 and 2.2 paragraphs. The complete presentation of the TSDT kinematics, stress–strain constitutive relations, strain–displacement relations, and material stiffness of FGMs are provided in Section Appendix A to continue the theoretical model in self-contained and reproducible manner.

2.1. Aerodynamic Pressure Load

The aerodynamic pressure load (q) can be provided by using the Navier–Stokes equation analytical–numerical flow model. For supersonic regimes where the viscous effects and shock waves are usually non-negligible, the boundary layers also strongly influence pressure loading and the turbulence modeling is typically required. In the preliminarily presented study, it is a simply ideal investigation for aerodynamic pressure load without considering the viscous effects, shock waves, boundary layers, and turbulence modeling. The RANS equations [8,21,22] can be simply ideally used with negligible field forces ($\bar{g} = 0$), inviscid flow (viscous stress tensor $\tau_{ij} = 0$), heat flux vector ($-K \frac{\partial \Delta T}{\partial x_j}$), and no yaw effect ($\frac{\partial F}{\partial y} = 0$) for air flow, as follows:

$$u_{\text{air}} \frac{\partial \rho_{\text{air}}}{\partial x} + \rho_{\text{air}} \frac{\partial u_{\text{air}}}{\partial x} + w_{\text{air}} \frac{\partial \rho_{\text{air}}}{\partial z} + \rho_{\text{air}} \frac{\partial w_{\text{air}}}{\partial z} + \frac{\partial \rho_{\text{air}}}{\partial t} = 0 \quad (1)$$

$$\rho_{\text{air}} u_{\text{air}} \frac{\partial u_{\text{air}}}{\partial x} + \frac{\partial p_{\text{air}}}{\partial x} + \rho_{\text{air}} u_{\text{air}} \frac{\partial w_{\text{air}}}{\partial z} + \rho_{\text{air}} \frac{\partial u_{\text{air}}}{\partial t} = 0 \quad (2)$$

$$\rho_{\text{air}} u_{\text{air}} \frac{\partial v_{\text{air}}}{\partial x} + \rho_{\text{air}} w_{\text{air}} \frac{\partial v_{\text{air}}}{\partial z} + \rho_{\text{air}} \frac{\partial v_{\text{air}}}{\partial t} = 0 \quad (3)$$

$$\rho_{\text{air}} u_{\text{air}} \frac{\partial w_{\text{air}}}{\partial x} + \rho_{\text{air}} w_{\text{air}} \frac{\partial w_{\text{air}}}{\partial z} + \frac{\partial p_{\text{air}}}{\partial z} + \rho_{\text{air}} \frac{\partial w_{\text{air}}}{\partial t} = 0 \quad (4)$$

$$\left(p_{\text{air}} - \frac{\rho_{\text{air}} u_{\text{air}}^2}{2} - \frac{\rho_{\text{air}} v_{\text{air}}^2}{2} - \rho_{\text{air}} u_{\text{air}} v_{\text{air}} \right) \frac{\partial u_{\text{air}}}{\partial x} + \rho_{\text{air}} u_{\text{air}} v_{\text{air}} \frac{\partial v_{\text{air}}}{\partial x} + \left(p_{\text{air}} - \frac{\rho_{\text{air}} w_{\text{air}}^2}{2} - \frac{\rho_{\text{air}} v_{\text{air}}^2}{2} \right) \frac{\partial w_{\text{air}}}{\partial z} + \rho_{\text{air}} u_{\text{air}} w_{\text{air}} \frac{\partial u_{\text{air}}}{\partial z} = K_{\text{air}} \left(\frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{\partial z^2} \right) \quad (5)$$

in which ρ_{air} is the density of air. u_{air} , v_{air} , and w_{air} are the velocity components of air in the x , y , and z directions, respectively. p_{air} is the pressure of air. t is time. $K_{\text{air}} = k/(\rho_{\infty} C_v)$ is a constant conductivity value of heat flux for air, in which k is thermal conductivity, ρ_{∞} is the density of air with subscript, ∞ is the location in the free stream at $t = 0$, and C_v is volumetric heat capacity. ΔT is the temperature difference of applied heat flux.

For the typical unsteady flow, we use the numerical computation, e.g., $\frac{\partial u_{\text{air}}}{\partial x} + \frac{\partial w_{\text{air}}}{\partial z} = -\frac{1}{u_{\text{air}}} \frac{\partial u_{\text{air}}}{\partial t} - \frac{1}{\rho_{\text{air}} u_{\text{air}}} \frac{\partial p_{\text{air}}}{\partial x}$, from Equation (2) and substitute it into Equation (5) for p_{air} term, which becomes $\left(\frac{\partial u_{\text{air}}}{\partial x} + \frac{\partial w_{\text{air}}}{\partial z}\right) p_{\text{air}} = \left(-\frac{1}{u_{\text{air}}} \frac{\partial u_{\text{air}}}{\partial t} - \frac{1}{\rho_{\text{air}} u_{\text{air}}} \frac{\partial p_{\text{air}}}{\partial x}\right) p_{\text{air}}$; thus, Equations (1)–(5) can be written in matrix form, as follows:

$$\begin{bmatrix} u_{\text{air}} & w_{\text{air}} & \rho_{\text{air}} & 0 & 0 & 0 & 0 & \rho_{\text{air}} & 0 & 0 \\ 0 & 0 & \rho_{\text{air}} u_{\text{air}} & 0 & 0 & 0 & 0 & \rho_{\text{air}} u_{\text{air}} & 1 & 0 \\ 0 & 0 & 0 & 0 & \rho_{\text{air}} u_{\text{air}} & \rho_{\text{air}} u_{\text{air}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{\text{air}} u_{\text{air}} & \rho_{\text{air}} u_{\text{air}} & 0 & 1 \\ 0 & 0 & -\frac{\rho_{\text{air}}}{2} (u_{\text{air}}^2 + v_{\text{air}}^2) - \rho_{\text{air}} u_{\text{air}} v_{\text{air}} & \rho_{\text{air}} u_{\text{air}} w_{\text{air}} & \rho_{\text{air}} u_{\text{air}} w_{\text{air}} & 0 & 0 & -\frac{\rho_{\text{air}}}{2} (w_{\text{air}}^2 + v_{\text{air}}^2) & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_{\text{air}} \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{\text{air}} \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{\text{air}} \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{u_{\text{air}}} \frac{\partial u_{\text{air}}}{\partial t} - \frac{1}{\rho_{\text{air}} u_{\text{air}}} \frac{\partial p_{\text{air}}}{\partial x} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_{\text{air}} \\ u_{\text{air}} \\ v_{\text{air}} \\ w_{\text{air}} \\ p_{\text{air}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ K_{\text{air}} \left(\frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{\partial z^2} \right) \end{bmatrix} \quad (6)$$

When the compressible flow is used for the supersonic flow, the variation change in pressure (p_{air}) and temperature difference (ΔT) cannot be negligible, so we often use and denote the Mach number (M_{∞}) in the free stream as follows:

$$M_{\infty} = \sqrt{U_{\infty}^2 + V_{\infty}^2 + W_{\infty}^2} / a_{\infty} \quad (7)$$

in which U_{∞} , V_{∞} , and W_{∞} are the velocity amplitudes of air in the free stream in the x , y , and z directions, respectively.

Natural frequency (ω_{mn}^{air}) of free vibration ($\bar{T}_1 = 0$) for air flow over upper surface of FGMs would be obtained by assuming that five unknown variables (ρ_{air} , u_{air} , v_{air} , w_{air} , and p_{air}) in the free stream known values (ρ_{∞} , U_{∞} , V_{∞} , W_{∞} , and p_{∞}) and triangular function expressions, as follows:

$$\begin{bmatrix} \rho_{\text{air}} \\ u_{\text{air}} \\ v_{\text{air}} \\ w_{\text{air}} \\ p_{\text{air}} \end{bmatrix} = \begin{bmatrix} \rho_{\infty} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2n\pi z}{h^*}\right) \\ U_{\infty} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2n\pi z}{h^*}\right) \\ V_{\infty} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2n\pi z}{h^*}\right) \\ W_{\infty} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2n\pi z}{h^*}\right) \\ p_{\infty} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2n\pi z}{h^*}\right) \end{bmatrix}_{\text{air}} \cos(\omega_{mn}^{\text{air}} t) \quad (8)$$

in which a is the axial length of thick-walled FGM, and m and n are the integer number of air mode shapes in the x and z directions, respectively. By substituting Equation (8) into Equation (6), the following FA 5×5 matrix in homogeneous equation can be obtained:

$$\begin{bmatrix} \text{FA}_{11} & \text{FA}_{12} & \text{FA}_{13} & \text{FA}_{14} & \text{FA}_{15} \\ \text{FA}_{21} & \text{FA}_{22} & \text{FA}_{23} & \text{FA}_{24} & \text{FA}_{25} \\ \text{FA}_{31} & \text{FA}_{32} & \text{FA}_{33} & \text{FA}_{34} & \text{FA}_{35} \\ \text{FA}_{41} & \text{FA}_{42} & \text{FA}_{43} & \text{FA}_{44} & \text{FA}_{45} \\ \text{FA}_{51} & \text{FA}_{52} & \text{FA}_{53} & \text{FA}_{54} & \text{FA}_{55} \end{bmatrix} \begin{bmatrix} \rho_{\infty} \\ U_{\infty} \\ V_{\infty} \\ W_{\infty} \\ p_{\infty} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

For the free vibration condition taking place at $x = \frac{a}{2}$ and $z = \frac{h^*}{2}$, Equation (9) can be reduced into the following FA 2×2 matrix in homogeneous equation:

$$\begin{bmatrix} \text{FA}_{11} & 0 \\ 0 & \text{FA}_{55} \end{bmatrix} \begin{Bmatrix} \rho_{\infty} \\ p_{\infty} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

in which

$$\text{FA}_{11} = -U_{\infty} \frac{m\pi}{a} \left[\sin\left(\frac{m\pi}{2}\right) \right]^2 \left[\cos(n\pi) \right]^2 \left[\cos\left(\omega_{mn}^{\text{air}} t\right) \right]^2 - \omega_{mn}^{\text{air}} \cos\left(\frac{m\pi}{2}\right) \cos(n\pi) \sin\left(\omega_{mn}^{\text{air}} t\right),$$

$$\text{FA}_{55} = \omega_{mn}^{\text{air}} \cos\left(\frac{m\pi}{2}\right) \cos(n\pi) \sin\left(\omega_{mn}^{\text{air}} t\right) + \frac{p_{\infty}}{\rho_{\infty} U_{\infty}} \frac{m\pi}{a}.$$

Thus, $\det \begin{bmatrix} \text{FA}_{11} & 0 \\ 0 & \text{FA}_{55} \end{bmatrix} = 0$ for any nonzero value of ρ_{∞} and p_{∞} ; that is, $\text{FA}_{11}\text{FA}_{55} = 0$; i.e.,

$$-U_{\infty} \frac{m\pi}{a} \left[\sin\left(\frac{m\pi}{2}\right) \right]^2 \left[\cos(n\pi) \right]^2 \left[\cos\left(\omega_{mn}^{\text{air}} t\right) \right]^2 - \omega_{mn}^{\text{air}} \cos\left(\frac{m\pi}{2}\right) \cos(n\pi) \sin\left(\omega_{mn}^{\text{air}} t\right) = 0, \quad (11)$$

$$\omega_{mn}^{\text{air}} \cos\left(\frac{m\pi}{2}\right) \cos(n\pi) \sin\left(\omega_{mn}^{\text{air}} t\right) + \frac{p_{\infty}}{\rho_{\infty} U_{\infty}} \frac{m\pi}{a} = 0 \quad (12)$$

After the addition of Equations (11) and (12), ω_{mn}^{air} equation can thus be found as follows:

$$\omega_{mn}^{\text{air}} = \frac{\frac{-p_{\infty}}{\rho_{\infty} U_{\infty}} \frac{m\pi}{a}}{\sqrt{1 - \frac{p_{\infty}}{\rho_{\infty} U_{\infty}^2} \cos\left(\frac{m\pi}{2}\right) \cos(n\pi)}} \quad (13)$$

with $m = 2, 4, 6, \dots; n = 1, 2, 3, \dots$; and $p_{\infty} \neq \rho_{\infty} U_{\infty}^2$.

Then, for the thermal vibration with $\Delta T = \frac{z}{h} \bar{T}_{\text{air}} \sin(\pi x/a) \sin(\pi y/b) \sin(\gamma_{\text{air}} t)$ for plate in Equation (6) of supersonic flow, p_{air} can be computed to get aerodynamic pressure load ($q = p_{\text{air}}$) by using the numerical GDQ method.

2.2. Using GDQ Method to Solve Navier–Stokes Equation Analytical–Numerical Flow Model

The GDQ method is used to approximate the derivative of functions [16,19,20]. How the GDQ weighting coefficients are constructed for grid point distribution, weighting formulation, and derivative approximation order are provided in Appendix A. And how the resulting linear algebraic systems are solved by direct solver method for air flow is provided in paragraphs. Solving Navier–Stokes equation analytical–numerical flow model (6) by using GDQ method for first-order partial derivatives of $f(x, z)$ at the typical grid point (x_i, z_j) with subscripts $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$ can be discretized as follows, by weighting coefficients $A_{i,l}^{(m)}$ and $B_{j,m}^{(1)}$ for the superscript (m) th-order derivative of $f(x, z)$; e.g., for $(m) = (1)$:

$$\left. \frac{\partial f}{\partial x} \right|_{i,j} \approx \sum_{l=1}^N A_{i,l}^{(1)} f_{l,j}, \quad (14)$$

$$\left. \frac{\partial f}{\partial z} \right|_{i,j} \approx \sum_{m=1}^M B_{j,m}^{(1)} f_{i,m} \quad (15)$$

Take the function $f(x, z)$ for $(\rho_{\text{air}}, u_{\text{air}}, v_{\text{air}}, w_{\text{air}}, \text{ and } p_{\text{air}})$ of the supersonic flow in the places for the first-order partial derivative and under vibration of time co-sinusoidal functions as follows:

$$\begin{Bmatrix} \rho_{\text{air}} \\ u_{\text{air}} \\ v_{\text{air}} \\ w_{\text{air}} \\ p_{\text{air}} \end{Bmatrix}_{i,j} = \begin{Bmatrix} \rho_{\infty} \cos(m\pi x_i) \cos(n\pi z_j) \\ U_{\infty} \sin(m\pi x_i) \cos(n\pi z_j) \\ V_{\infty} \sin(m\pi x_i) \cos(n\pi z_j) \\ W_{\infty} \sin(m\pi x_i) \cos(n\pi z_j) \\ p_{\infty} \cos(m\pi x_i) \cos(n\pi z_j) \end{Bmatrix} \cos(\omega_{mn}^{\text{air}} t) \quad (16)$$

Thus, thermal dynamic GDQ discrete equations for the supersonic air flow can be derived in the following matrix notation:

$$[KE]_{\text{air}} \{SWSI\}_{\text{air}} + [FQ]_{\text{air}} \{UVWSI\}_{\text{air}} = \{F\}_{\text{air}} \quad (17)$$

in which

$$\begin{aligned} \{SWSI\}_{\text{air}} = & \left\{ \sum_{l=1}^N A_{i,l}^{(1)} \rho_{\text{air},l,j} \sum_{m=1}^M B_{j,m}^{(1)} \rho_{\text{air},i,m} \sum_{l=1}^N A_{i,l}^{(1)} u_{\text{air},l,j} \sum_{m=1}^M B_{j,m}^{(1)} u_{\text{air},i,m} \sum_{l=1}^N A_{i,l}^{(1)} v_{\text{air},l,j} \right. \\ & \left. \sum_{m=1}^M B_{j,m}^{(1)} v_{\text{air},i,m} \sum_{l=1}^N A_{i,l}^{(1)} w_{\text{air},l,j} \sum_{m=1}^M B_{j,m}^{(1)} w_{\text{air},i,m} \sum_{l=1}^N A_{i,l}^{(1)} p_{\text{air},l,j} \sum_{m=1}^M B_{j,m}^{(1)} p_{\text{air},i,m} \right\}^t \\ \{UVWSI\}_{\text{air}} = & \left\{ \rho_{\text{air},i,j} u_{\text{air},i,j} v_{\text{air},i,j} w_{\text{air},i,j} p_{\text{air},i,j} \right\}^t \\ \{F\}_{\text{air}} = & \{F_{1\text{air}} F_{2\text{air}} F_{3\text{air}} F_{4\text{air}} F_{5\text{air}}\}^t \end{aligned}$$

where $F_{1\text{air}} = 0$, $F_{2\text{air}} = 0$, $F_{3\text{air}} = 0$, $F_{4\text{air}} = 0$, and

$$F_{5\text{air}} = -K_{\text{air}} \bar{T}_{1\text{air}} \left(\frac{\pi}{a}\right)^2 \frac{h^*}{2} \sin(\pi x_i) \sin(\pi y_j) \sin(\gamma_{\text{air}} t),$$

$[KE]_{\text{air}}$ contains the 5×10 elements, listed as follows:

$$\begin{aligned} KE_{11\text{air}} &= U_{\infty} \sin(m\pi x_1) \cos(n\pi z_1) \cos(\omega_{mn}^{\text{air}} t), \\ KE_{12\text{air}} &= W_{\infty} \sin(m\pi x_1) \cos(n\pi z_2) \cos(\omega_{mn}^{\text{air}} t), \\ KE_{13\text{air}} &= \rho_{\infty} \cos(m\pi x_1) \cos(n\pi z_3) \cos(\omega_{mn}^{\text{air}} t), \\ KE_{18\text{air}} &= \rho_{\infty} \cos(m\pi x_1) \cos(n\pi z_8) \cos(\omega_{mn}^{\text{air}} t), \\ KE_{14\text{air}} &= KE_{15\text{air}} = KE_{16\text{air}} = KE_{17\text{air}} = KE_{19\text{air}} = KE_{110\text{air}} = 0, \\ KE_{21\text{air}} &= KE_{22\text{air}} = KE_{24\text{air}} = KE_{25\text{air}} = KE_{26\text{air}} = KE_{27\text{air}} = KE_{210\text{air}} = 0, \\ KE_{23\text{air}} &= \rho_{\infty} U_{\infty} \cos(m\pi x_2) \sin(m\pi x_2) \left[\cos(n\pi z_3) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \\ KE_{28\text{air}} &= \rho_{\infty} U_{\infty} \cos(m\pi x_2) \sin(m\pi x_2) \left[\cos(n\pi z_8) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \\ KE_{29\text{air}} &= 1, \\ KE_{31\text{air}} &= KE_{32\text{air}} = KE_{33\text{air}} = KE_{34\text{air}} = KE_{37\text{air}} = KE_{38\text{air}} = KE_{39\text{air}} = KE_{310\text{air}} = 0, \\ KE_{35\text{air}} &= \rho_{\infty} U_{\infty} \cos(m\pi x_3) \sin(m\pi x_3) \left[\cos(n\pi z_5) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \\ KE_{36\text{air}} &= \rho_{\infty} W_{\infty} \cos(m\pi x_3) \sin(m\pi x_3) \left[\cos(n\pi z_6) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \\ KE_{41\text{air}} &= KE_{42\text{air}} = KE_{43\text{air}} = KE_{44\text{air}} = KE_{45\text{air}} = KE_{46\text{air}} = KE_{49\text{air}} = 0, \\ KE_{47\text{air}} &= \rho_{\infty} U_{\infty} \cos(m\pi x_4) \sin(m\pi x_4) \left[\cos(n\pi z_7) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \end{aligned}$$

$$\begin{aligned}
KE_{48\text{air}} &= \rho_{\infty} W_{\infty} \cos(m\pi x_4) \sin(m\pi x_4) \left[\cos(n\pi z_8) \cos(\omega_{mn}^{\text{air}} t) \right]^2, \\
KE_{410\text{air}} &= 1, \\
KE_{51\text{air}} &= KE_{52\text{air}} = KE_{56\text{air}} = KE_{47\text{air}} = KE_{49\text{air}} = KE_{410\text{air}} = 0, \\
KE_{53\text{air}} &= - \left[\frac{\rho_{\infty}}{2} (U_{\infty}^2 + V_{\infty}^2) + \rho_{\infty} U_{\infty} V_{\infty} \right] \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_3) \cos(\omega_{mn}^{\text{air}} t) \right]^3, \\
KE_{54\text{air}} &= \rho_{\infty} U_{\infty} W_{\infty} \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_4) \cos(\omega_{mn}^{\text{air}} t) \right]^3, \\
KE_{410\text{air}} &= 1, \\
KE_{51\text{air}} &= KE_{52\text{air}} = KE_{56\text{air}} = KE_{47\text{air}} = KE_{49\text{air}} = KE_{410\text{air}} = 0, \\
KE_{53\text{air}} &= - \left[\frac{\rho_{\infty}}{2} (U_{\infty}^2 + V_{\infty}^2) + \rho_{\infty} U_{\infty} V_{\infty} \right] \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_3) \cos(\omega_{mn}^{\text{air}} t) \right]^3, \\
KE_{54\text{air}} &= \rho_{\infty} U_{\infty} W_{\infty} \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_4) \cos(\omega_{mn}^{\text{air}} t) \right]^3, \\
KE_{55\text{air}} &= \rho_{\infty} U_{\infty} V_{\infty} \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_5) \cos(\omega_{mn}^{\text{air}} t) \right]^3, \\
KE_{58\text{air}} &= - \left[\frac{\rho_{\infty}}{2} (W_{\infty}^2 + V_{\infty}^2) \right] \cos(m\pi x_5) [\sin(m\pi x_5)]^2 \left[\cos(n\pi z_8) \cos(\omega_{mn}^{\text{air}} t) \right]^3. \\
[FQ]_{\text{air}} &\text{ contains the } 5 \times 5 \text{ elements are listed as follows,}
\end{aligned}$$

$$\begin{aligned}
FQ_{11\text{air}} &= -\omega_{mn}^{\text{air}} \tan(\omega_{mn}^{\text{air}} t), \\
FQ_{12\text{air}} &= FQ_{13\text{air}} = FQ_{14\text{air}} = FQ_{15\text{air}} = 0, \\
FQ_{22\text{air}} &= -\rho_{\infty} \omega_{mn}^{\text{air}} \cos(m\pi x_2) \cos(n\pi z_2) \sin(\omega_{mn}^{\text{air}} t), \\
FQ_{21\text{air}} &= FQ_{23\text{air}} = FQ_{24\text{air}} = FQ_{25\text{air}} = 0, \\
FQ_{33\text{air}} &= -\rho_{\infty} \omega_{mn}^{\text{air}} \cos(m\pi x_3) \cos(n\pi z_3) \sin(\omega_{mn}^{\text{air}} t), \\
FQ_{31\text{air}} &= FQ_{32\text{air}} = FQ_{34\text{air}} = FQ_{35\text{air}} = 0, \\
FQ_{44\text{air}} &= -\rho_{\infty} \omega_{mn}^{\text{air}} \cos(m\pi x_4) \cos(n\pi z_4) \sin(\omega_{mn}^{\text{air}} t), \\
FQ_{41\text{air}} &= FQ_{42\text{air}} = FQ_{43\text{air}} = FQ_{45\text{air}} = 0, \\
FQ_{55\text{air}} &= \frac{\omega_{mn}^{\text{air}} \sin(\omega_{mn}^{\text{air}} t)}{p_{\infty} \cos(m\pi x_5) \cos(n\pi z_5) [\cos(\omega_{mn}^{\text{air}} t)]^2} + \frac{m\pi}{\rho_{\infty} U_{\infty} [\cos(m\pi x_5)]^2 [\cos(n\pi z_5)]^2 [\cos(\omega_{mn}^{\text{air}} t)]^2}, \\
FQ_{51\text{air}} &= FQ_{52\text{air}} = FQ_{53\text{air}} = FQ_{54\text{air}} = 0,
\end{aligned}$$

with

$$\begin{aligned}
x_i &= 0.5 \left[1 - \cos\left(\frac{i-1}{N-1}\pi\right) \right] a, \\
z_j &= 0.5 \left[1 - \cos\left(\frac{j-1}{M-1}\pi\right) \right] h^*.
\end{aligned}$$

The boundary conditions used are as follows for the air flow:

At $x = 0$ and $x = a$ for $i = 1$ and $i = N$: $\frac{\partial \rho_{\text{air}}}{\partial x} = u_{\text{air}} = v_{\text{air}} = w_{\text{air}} = \frac{\partial p_{\text{air}}}{\partial x} = 0$; that is

$$\sum_{l=1}^N A_{1,l}^{(1)} \rho_{\text{air},l,j} = u_{\text{air},1,j} = v_{\text{air},1,j} = w_{\text{air},1,j} = \sum_{l=1}^N A_{1,l}^{(1)} p_{\text{air},l,j} = 0, \quad (18)$$

$$\sum_{l=1}^N A_{N,l}^{(1)} \rho_{\text{air},l,j} = u_{\text{air},N,j} = v_{\text{air},N,j} = w_{\text{air},N,j} = \sum_{l=1}^N A_{N,l}^{(1)} p_{\text{air},l,j} = 0 \quad (19)$$

At $z = \frac{-h^*}{2}$ for $j = 1$: $\rho_{\text{air}} = u_{\text{air}} = v_{\text{air}} = w_{\text{air}} = p_{\text{air}} = 0$, that is

$$\rho_{\text{air},i,1} = u_{\text{air},i,1} = v_{\text{air},i,1} = w_{\text{air},i,1} = p_{\text{air},i,1} = 0 \quad (20)$$

At $z = \frac{h^*}{2}$ for $j = M$: $\frac{\partial \rho_{\text{air}}}{\partial z} = u_{\text{air}} = v_{\text{air}} = w_{\text{air}} = \frac{\partial p_{\text{air}}}{\partial z} = 0$, that is

$$\sum_{m=1}^M B_{M,m}^{(1)} \rho_{\text{air},i,m} = u_{\text{air},i,M} = v_{\text{air},i,M} = w_{\text{air},i,M} = \sum_{m=1}^M B_{M,m}^{(1)} p_{\text{air},i,m} = 0 \quad (21)$$

The derivative summation terms at boundary $(1, j)$, (N, j) , $j = 2, 3, \dots, M - 1$, and at the boundary $(1, 1)$ and (N, M) , can be expressed in interior terms. Moreover, the derivative summation terms at boundary $(i, 1)$, (i, M) , $i = 2, 3, \dots, N - 1$, and at the boundary $(1, 1)$ and (N, M) , can be expressed in interior terms. For a typical interior grid point (i, j) , in which $i = 2, 3, \dots, N - 1$ and $j = 2, 3, \dots, M - 1$, the dynamic GDQ discrete Equation (17) can be rewritten into the matrix form for air flow as follows:

$$[A]_{\text{air}} \{W^*\}_{\text{air}} = \{B\}_{\text{air}} \quad (22)$$

where:

$[A]_{\text{air}}$ is a coefficient matrix with $N^{**} \times N^{**}$, in which $N^{**} = 5(N - 2)(M - 2)$.

$\{W^*\}_{\text{air}} = \{\rho_{\text{air},2,2} \rho_{\text{air},2,3} \dots \rho_{\text{air},2,M-1} \rho_{\text{air},3,2} \rho_{\text{air},3,3} \dots \rho_{\text{air},3,M-1} \dots \rho_{\text{air},N-1,2} \rho_{\text{air},N-1,3} \dots \rho_{\text{air},N-1,M-1} u_{\text{air},2,2} u_{\text{air},2,3} \dots u_{\text{air},2,M-1} u_{\text{air},3,2} u_{\text{air},3,3} \dots u_{\text{air},3,M-1} \dots u_{\text{air},N-1,2} u_{\text{air},N-1,3} \dots u_{\text{air},N-1,M-1} v_{\text{air},2,2} v_{\text{air},2,3} \dots v_{\text{air},2,M-1} v_{\text{air},3,2} v_{\text{air},3,3} \dots v_{\text{air},3,M-1} \dots v_{\text{air},N-1,2} v_{\text{air},N-1,3} \dots v_{\text{air},N-1,M-1} w_{\text{air},2,2} w_{\text{air},2,3} \dots w_{\text{air},2,M-1} w_{\text{air},3,2} w_{\text{air},3,3} \dots w_{\text{air},3,M-1} \dots w_{\text{air},N-1,2} w_{\text{air},N-1,3} \dots w_{\text{air},N-1,M-1} p_{\text{air},2,2} p_{\text{air},2,3} \dots p_{\text{air},2,M-1} p_{\text{air},3,2} p_{\text{air},3,3} \dots p_{\text{air},3,M-1} \dots p_{\text{air},N-1,2} p_{\text{air},N-1,3} \dots p_{\text{air},N-1,M-1}\}^T$ is unknown $N^{**} \times 1$ column vector.

$\{B\}_{\text{air}} = \{F_{1,\text{air}} \dots F_{1,\text{air}} F_{2,\text{air}} \dots F_{2,\text{air}} F_{3,\text{air}} \dots F_{3,\text{air}} F_{4,\text{air}} \dots F_{4,\text{air}} F_{5,\text{air}} \dots F_{5,\text{air}}\}^T$ is known $N^{**} \times 1$ column vector. Matrix $[A]_{\text{air}}$ contains 5×10 matrix $[KE]_{\text{air}}$ and 5×5 matrix $[FQ]_{\text{air}}$ that can be prepared by the elements of $[A]_{\text{air}}$ in Equation (22). This approach is different from CFD application in 2017 by Nielsen and Diskin [23]; for the GDQ approach used for the FSDT model, please refer to the study in 2017 by Hong [24], and, for its use in the TDST model, please refer to the study in 2004 by Lee et al. [25]. The Gauss elimination method [26] is used for solving the computational $\{W^*\}_{\text{air}}$ value under sinusoidal heat loads on air $\{B\}_{\text{air}}$ for linear systems (22).

2.3. Solving Dynamic Equilibrium Differential Equations

For a grid point (i, j) , the dynamic GDQ discrete equations of TSDT model of motion equation for FGM plate and cylindrical shell in unsteady supersonic flow can be expressed in following matrix form:

$$[A]\{W^*\} = \{B\} \quad (23)$$

where $[A]$ is an N^{**} by N^{**} coefficient matrix with TSDT c_1 term, vibration ω_{mnt} term, inertia term, stiffness coefficient term, and GDQ weighting parameter term. $\{W^*\}$ is an unknown column vector and $\{B\}$ is a sinusoidal external-loads vector containing supersonic aerodynamic pressure load ($q = p_{\text{air}}$) calculated from the Navier–Stokes equation analytical–numerical flow model for the FGM plate and cylindrical shell in unsteady supersonic flow.

These numerical GDQ calculations are implemented in the Lahey–Fujitsu Fortran 7.8 program. How the GDQ weighting coefficients are constructed for grid point distribution, weighting formulation, and derivative approximation order and how the resulting linear algebraic systems are solved by direct solver method vs. iterative strategy, convergence criteria, tolerance, stopping rules, etc. for FGM plates–shells which are provided in para-

graphs are shown. Newton's method [26] with tolerance accuracy 1×10^{-6} of convergence is used for solving the computational natural frequency ω_{mn} value of eigen-problem in zero-determinant fully homogeneous equation $\det [A] = 0$ for (23) under the sinusoidal vibration of displacement and shear rotations for plate; e.g.,

$$u^0 = a_{mn} \cos(m\pi x/a) \sin(n\pi y/b) \sin(\omega_{mn}t) \quad (24)$$

$$v^0 = b_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \sin(\omega_{mn}t) \quad (25)$$

$$w = c_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \sin(\omega_{mn}t) \quad (26)$$

$$\psi_x = d_{mn} \cos(m\pi x/a) \sin(n\pi y/b) \sin(\omega_{mn}t) \quad (27)$$

$$\psi_y = e_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \sin(\omega_{mn}t) \quad (28)$$

in which a_{mn} , b_{mn} , c_{mn} , d_{mn} , and e_{mn} are amplitudes with mode shape numbers m and n . And the Gauss elimination method [26] is used for solving the computational $\{W^*\}$ value under sinusoidal external loads $\{B\}$ for linear systems (23). The numerical workflow for symbols inputs $\rightarrow$ discretization $\rightarrow$ solution $\rightarrow$ outputs used in the computation of numerical method are displayed in Figure 2.

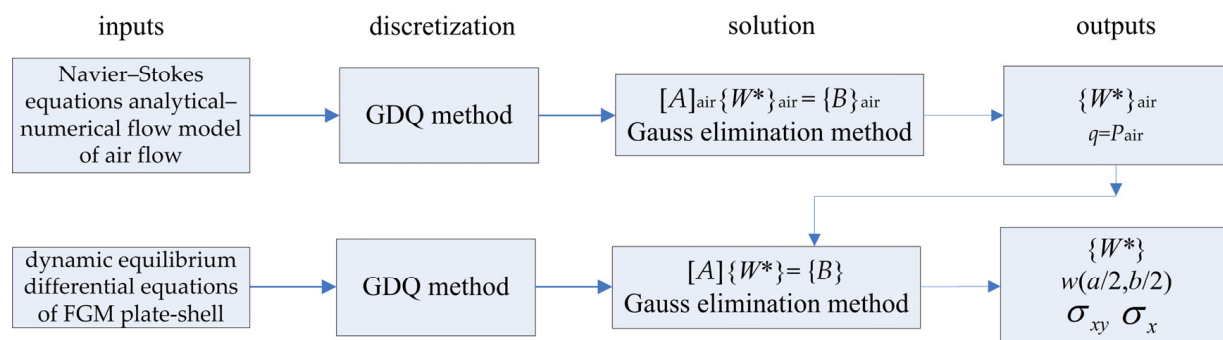


Figure 2. Numerical workflow for the computation of numerical method.

3. Numerical Results and Discussions

The FGM constituent material 1 is SUS304 with thickness h_1 , and the FGM constituent material 2 is Si_3N_4 with thickness h_2 ; these are used under supersonic air flow over the outer surface of the thick-walled FGM plates–cylindrical shells with $k = 0.002551 \text{ kgm}/(\text{smK})$, and $C_v = 132.3469 \text{ kgm}/(\text{m}^3\text{K})$ at 273.15 K , $\rho_\infty = 0.187838 \text{ kg}/\text{m}^3$, $p_\infty = 1132.96 \text{ kg}/\text{m}^2$, $T_\infty = 216.65 \text{ K}$, and $\bar{T}_{\text{air}} = 1 \text{ K}$ at altitude $50,000 \text{ ft}$ for $M_\infty = 2$, i.e., $U_\infty = 591.9216 \text{ m/s}$, and $a_\infty = 295.9608 \text{ m/s}$ in the numerical GDQ computations. Thus, we found $\omega_{21}^{\text{air}} = 10,763.79/\text{s}$ and $\gamma_{\text{air}} = 0.2618355/\text{s}$ for $a/b = 1$, $\gamma_{\text{air}} = 0.2618263/\text{s}$ for $a/b = 0.7070978$, and $\gamma_{\text{air}} = 0.2618232/\text{s}$ for $a/b = 0.5773502$ at $t = 6 \text{ s}$, in which b is the width of the FGM plates, $h^* = 1.2 \text{ mm}$, $h_1 = h_2 = 0.6 \text{ mm}$, and $\sin \theta_L = \sin \theta_R = b/2R$. The aerodynamic pressure load q (Pa) versus x/a of air on $z = 0.5h^*$, $M_\infty = 2$, $T = 100 \text{ K}$, and $\bar{T}_1 = 1 \text{ K}$ is displayed in Figure 3. They can be obtained via the traditional way using the nonlinear third-order piston theory introduced by Lin et al. [6] in 2020 and by the present Navier–Stokes equation analytical–numerical flow model, respectively. The aerodynamic pressure load q is approaching zero in uniform form vs. x/a found in the model of the third-order piston theory; it is also in small-order 10^{-14} and approaching zero but in the co-sinusoidal function form vs. x/a found in the presented Navier–Stokes equation analytical–numerical flow model. The physical meaning of the oscillatory behavior is due to the given typical vibration expression (16) of the time co-sinusoidal functions and ΔT of the time sinusoidal function. The low air pressure of order 10^{-14} Pa became a sucking pressure on the plate–shell. It would be interesting to use the presented Navier–Stokes equation analytical–numerical flow model

to study the stress in unit GPa = 10^9 Pa under thermal vibration and supersonic air flow. The $N \times M = 13 \times 13$ grid points are used further in the presented GDQ computations on the Navier–Stokes equation analytical–numerical flow model for the FGM structure under thermal vibration and supersonic air flow to find center displacement $w(a/2, b/2)$ in unit mm, stress in unit GPa, temperature in unit K, q in unit Pa, and time in unit s.

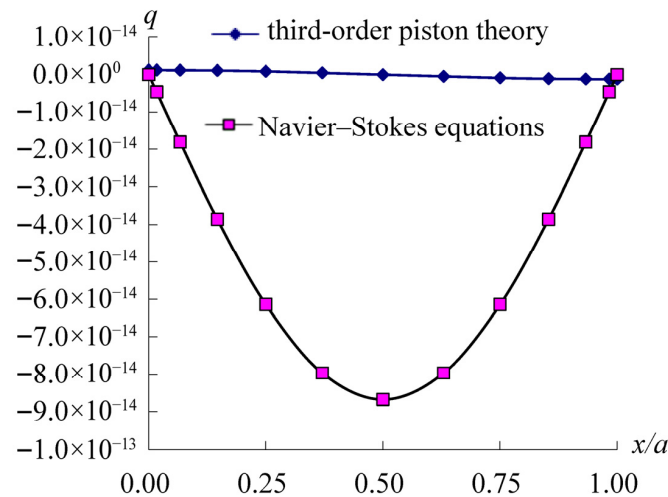


Figure 3. Comparison results of q (Pa) vs. x/a of air on $z = 0.5h^*$ at Mach no = 2.

Figure 4 displays the amplitude of $w(a/2, b/2)$ (mm) and stress at the center position of $z = 0.5h^*$ that are compared by using two presentation models of the Navier–Stokes equation analytical–numerical flow model with $\omega_{21}^{\text{air}} = 10,763.79/\text{s}$ and third-order piston theory with $\omega_{11} = 0.002500/\text{s}$. The results for the thermal vibration of the TSDT of the FGM plates–cylindrical shells during $t = 0.1 \text{ s}–3.0 \text{ s}$ are obtained in unsteady supersonic flow $M_\infty = 2$ under $\theta_L = 45^\circ$, i.e., $aa/b = 0.7070978$ with basic data in advanced shear correction coefficient k_α , $a/h^* = 5$, $a/R = 1$, $R_n = 1$, $T = 600 \text{ K}$, and $\bar{T}_1 = 100 \text{ K}$. Figure 4a displays the compared response values of $w(a/2, b/2)$ (mm) vs. t (s) for $c_1 = 0.925925/\text{mm}^2$. The $w(a/2, b/2)$ values are converging with time. In the case of the Navier–Stokes equation analytical–numerical flow model, the values converge to -0.63007 mm , and, in the case of the third-order piston theory, they converge to 0.032069 mm . The physical meaning of oscillatory behavior is due to the given typical vibration expression (24)–(28) of the time co-sinusoidal functions. Figure 4b displays the compared time responses of shear stress σ_{xy} (GPa) vs. t (s) for $\theta_L = 45^\circ$, $c_1 = 0.925925/\text{mm}^2$. The σ_{xy} values have a different tendency with time. In the case of the Navier–Stokes equation analytical–numerical flow model, there is bigger oscillating value of 0.514887 GPa , and, in the case of the third-order piston theory, there is a smaller constant value of $2.09 \times 10^{-5} \text{ GPa}$. The physical meaning of the oscillatory behavior is due to the given typical vibration expression (24)–(28) of the time co-sinusoidal functions and ΔT of the time sinusoidal function.

Comparisons of $w(a/2, b/2)$ (mm), σ_{xy} (GPa), and σ_x (GPa) vs. T (K) for $c_1 = 0.925925/\text{mm}^2$, $\theta_L = 30^\circ$, i.e., $a/b = 1$, are displayed in Figures 5 and 6 at $t = 0.1 \text{ s}$, $M_\infty = 2$, $a/h^* = 5$, $a/R = 1$, and $\bar{T}_1 = 100 \text{ K}$ by using two presentation models of the Navier–Stokes equation analytical–numerical flow model and third-order piston theory. Figure 5 displays the curves of $w(a/2, b/2)$ vs. T and $R_n = 2$; the maximum $w(a/2, b/2)$ is 10.751577 mm for the third-order piston theory on $T = 100 \text{ K}$ and $R_n = 2$. The $w(a/2, b/2)$ in $R_n = 2$ and $\theta_L = 30^\circ$ can withstand 1000 K for the two models. Figure 6 displays the stress σ_{xy} vs. T on the center position of $z = 0.5h^*$ for $M_\infty = 2$, $a/h^* = 5$. The maximum σ_{xy} is $2.7321 \times 10^{-2} \text{ GPa}$ on $T = 1000 \text{ K}$ for the Navier–Stokes equation analytical–numerical flow model; the σ_{xy} absolute values first decreasing, and then increase. The absolute smaller values of σ_{xy} versus T are constant for the third-order piston theory.

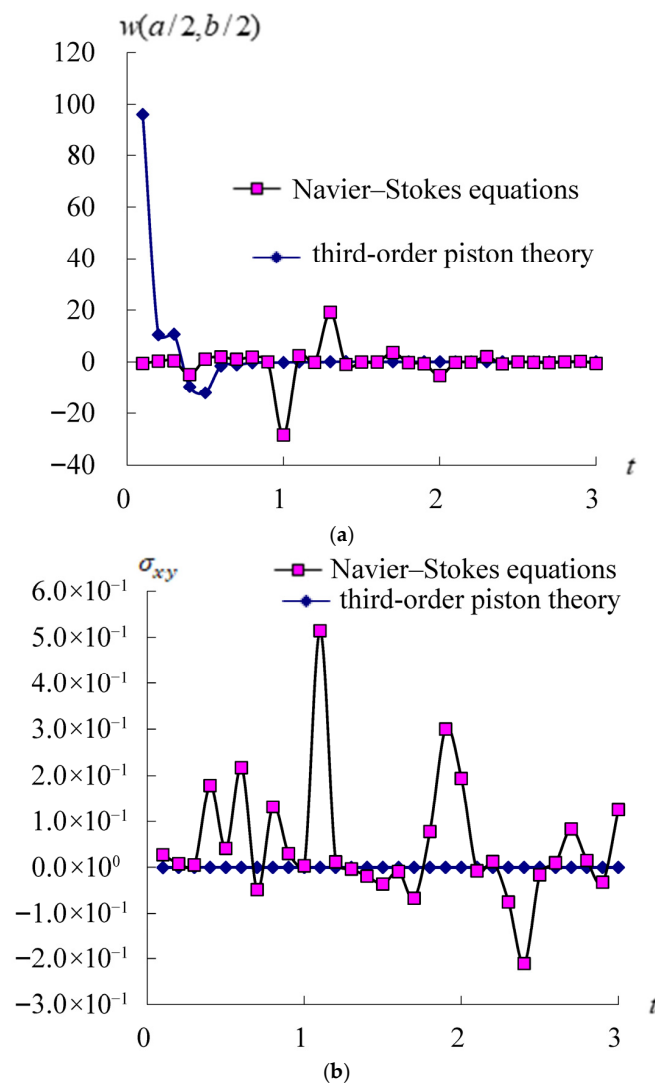


Figure 4. Comparison results of Navier–Stokes equation analytical–numerical flow model and third-order piston theory on $M_\infty = 2$ for the following: (a) $w(a/2, b/2)$ (mm) vs. t (s); and (b) σ_{xy} (GPa) vs. t (s).

Comparisons in the time response values of $w(a/2, b/2)$ (mm) and stress σ_x (GPa) vs. M_∞ during $t = 0.1$ s–3.0 s, $c_1 = 0.925925/\text{mm}^2$ under the Navier–Stokes equation analytical–numerical flow model are displayed, respectively, in Figure 7 for $a/h^* = 5$, $R_n = 1$, $T = 600$ K, $\bar{T}_1 = 100$ K, and $\theta_L = 30^\circ$. Figure 7a displays the compared response values of $w(a/2, b/2)$ vs. t in $M_\infty = 2$ and no fluid flow $M_\infty = 0$. The $w(a/2, b/2)$ values in the $M_\infty = 2$ case are oscillating and greater than that in the $M_\infty = 0$ case. Figure 7b displays the compared response values of stress σ_x vs. t in $M_\infty = 2$ and no fluid flow $M_\infty = 0$. The stress σ_x absolute values in the $M_\infty = 2$ case are oscillating and greater than that in the $M_\infty = 0$ case. That is to say, the unsteady supersonic flow over the outer surface of the FGM plate–cylindrical shells provide the additional aerodynamic effect on the values of displacement and stress under the Navier–Stokes equation analytical–numerical flow model approach. For the more computation experiences in the field of aerodynamic applications, for example, advanced computing performance, CFD design, and CFD application, please refer to the study in 2017 by Nielsen and Diskin [23]. Moreover, for the non-advanced shear correction effect included in the calculations of stress and displacement for FGM materials, please refer to the study in 2017 by Hong [24].

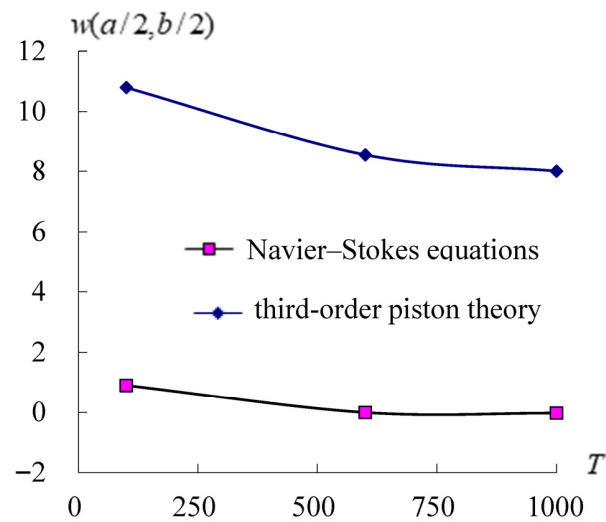


Figure 5. $w(a/2, b/2)$ (mm) vs. T (K) at $t = 0.1$ s.

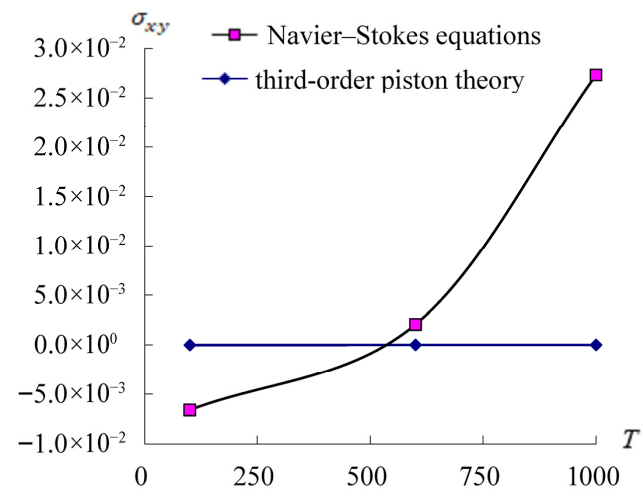


Figure 6. σ_{xy} (GPa) vs. T (K) at $t = 0.1$ s.

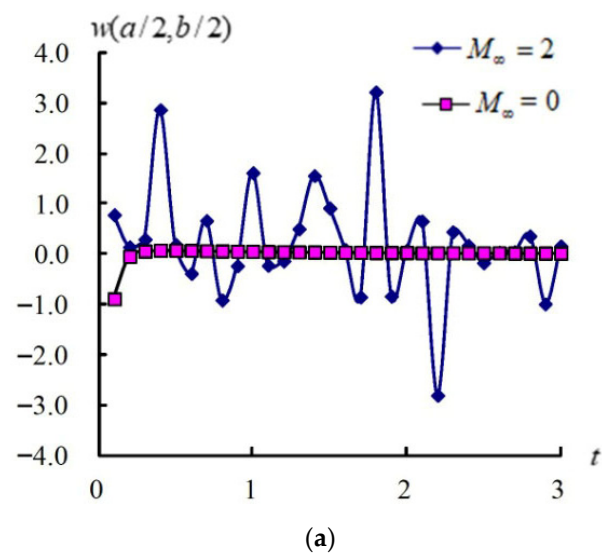


Figure 7. Cont.

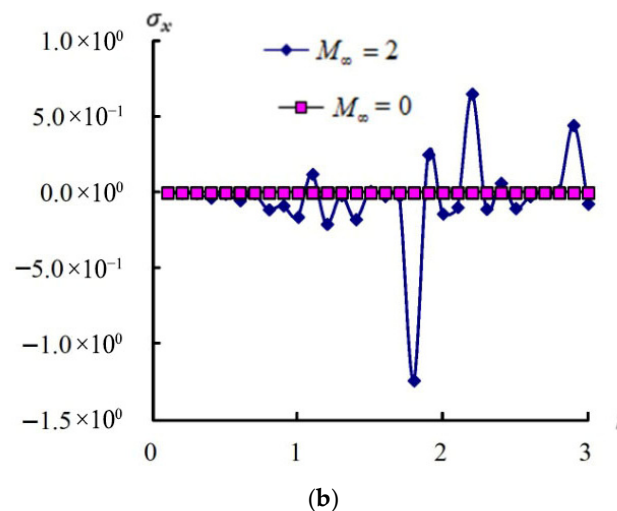


Figure 7. Comparisons results of Mach no. effects in Navier–Stokes equation analytical–numerical flow model under $\theta_L = 30^\circ$ for the following: (a) $w(a/2, b/2)$ (mm) vs. t (s); and (b) σ_x (GPa) vs. t (s).

4. Discussion

The deeper physical mechanisms are clarified and discussed as follows. For the underlying structural mechanics, e.g., the role of the shear deformation and thickness effects under the TSDT, we must study and obtain more detailed results with the $c_1 = 0.925925/\text{mm}^2$ value for the thick-walled structure than that without the c_1 value for the FSDT in the thin-walled structure. For aerodynamics, e.g., how the key flow parameters of five unknown variables ρ_{air} , u_{air} , v_{air} , w_{air} , and p_{air} influence the aerodynamic loading characteristics, these are mainly due to the fact that the effects of deflection in displacement w_{air} and the environmental temperature would cause the aerodynamic pressure $q = p_{\text{air}}$ on the FGM plate–shell. For thermo–aeroelastic coupling, e.g., how thermal loading modifies stiffness/thermal stress states and interacts with unsteady aerodynamic excitation, this mainly considers the notion that the variation change in pressure p_{air} and temperature difference ΔT cannot be negligible for air flow material properties K_{air} . The computation data with no numerical noise or stability would be valid and compared with the published CFD or aero–elastic benchmarks and experimental data, if possible, in the future. The calculated aerodynamic pressure values by Navier–Stokes equations are of order 10^{-14} Pa, which are less than value $p_\infty = 115.608$ Pa of order 10^2 Pa in the free stream. The low air pressure became a sucking pressure on the plate–shell and results in stress responses reaching around 0.5 GPa. The failure of the structure could not happen, since the stress value 0.5 GPa is lower than the FGM material strength Young’s modulus value 348.43 GPa.

5. Conclusions

The GDQ method is used to investigate displacement and stress in the thermal vibration of the thick-walled FGM plate–cylindrical shells in unsteady supersonic flow with the approach of the Navier–Stokes equation analytical–numerical flow model. Center displacements have a converging tendency versus time. The two study models, the Navier–Stokes equation analytical–numerical flow model and the third-order piston theory, demonstrate many great effects on the values of center displacement and stress. It is interesting to use the present Navier–Stokes equation analytical–numerical flow model to study the co-sinusoidal forms of displacement and stress in unit GPa under thermal vibration and supersonic air flow. The values of center displacement and stress in $M_\infty = 2$ due to the aerodynamics force are greater than those in $M_\infty = 0$ due to no air flow. The underlying structural mechanics under the TSDT study can obtain detailed results for the thick-walled FGM structure. The aerodynamics of air flow due to the key flow parameters of the five unknown variables ρ_{air} ,

u_{air} , v_{air} , w_{air} , and p_{air} are considered, and the aerodynamic pressure $q = p_{\text{air}}$ on the FGM plate–shell is obtained. Thermo–aeroelastic coupling on the stiffness and thermal stress are obtained for the FGM plate–shell by considering the material properties K_{air} of air flow and ΔT .

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Data Availability Statement: The data are all available upon request. The author declares that all the data are generated by the author and that the data are openly available.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

The TSDT kinematics used for plate [15,25] are as follows:

$$u = u^0(x, y, t) + z\psi_x(x, y, t) - c_1 z^3(\psi_x + \frac{\partial w}{\partial x}), \quad (\text{A1})$$

$$v = v^0(x, y, t) + z\psi_y(x, y, t) - c_1 z^3(\psi_y + \frac{\partial w}{\partial y}), \quad (\text{A2})$$

$$w = w(x, y, t), \quad (\text{A3})$$

in which u^0 and v^0 are displacements in x and y direction, w is transverse displacement in z direction, and ψ_x and ψ_y are the shear rotations.

The TSDT kinematics used for circular cylindrical shell with $\partial y = R\partial\theta$ are as follows:

$$u = u_0(x, \theta, t) + z\varphi_x(x, \theta, t) - c_1 z^3(\varphi_x + \frac{\partial w}{\partial x}), \quad (\text{A4})$$

$$v = v_0(x, \theta, t) + z\varphi_\theta(x, \theta, t) - c_1 z^3(\varphi_\theta + \frac{\partial w}{R\partial\theta}), \quad (\text{A5})$$

$$w = w(x, \theta, t), \quad (\text{A6})$$

in which u_0 and v_0 are displacements in x and θ direction, w is transverse displacement in z direction, and φ_x and φ_θ are the shear rotations.

The stress–strain constitutive relations used for plate in (k) th layer [17] are as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}_{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{(k)} \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_y - \alpha_y \Delta T \\ \varepsilon_{xy} - \alpha_{xy} \Delta T \end{Bmatrix}_{(k)}, \quad (\text{A7})$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}_{(k)} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_{(k)} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}_{(k)}, \quad (\text{A8})$$

in which σ_x and σ_y are normal stresses. σ_{xy} , σ_{yz} , and σ_{xz} are shear stresses. α_x and α_y are coefficients of the thermal expansion. α_{xy} is coefficient of the thermal shear. ε_x , ε_y , and ε_{xy} are in-plane strains. ε_{yz} and ε_{xz} are not-negligible shear strains. $\bar{Q}_{ij}$ is material stiffness of the FGM plates. ΔT is temperature difference expressed in $\Delta T = \frac{z}{h^*} \bar{T}_1 \sin(\pi x/a) \sin(\pi y/b) \sin(\gamma t)$ used for sinusoidal thermal loads and which meets the heat conduction $K \left(\frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{\partial y^2} + \frac{\partial^2 \Delta T}{\partial z^2} \right) = \frac{\partial \Delta T}{\partial t}$ as follows:

$$\frac{a^2}{K\pi^2[1 + (a/b)^2]} \gamma \cos(\gamma t) + \sin(\gamma t) = 0, \quad (\text{A9})$$

where $K = \kappa_{fgm} / (\rho_{fgm} C_{vfgm})$, $\kappa_{fgm} = (\kappa_1 + \kappa_2) / 2$ is thermal conductivity, $\rho_{fgm} = (\rho_1 + \rho_2) / 2$ is density, and $C_{vfgm} = (C_{v1} + C_{v2}) / 2$ is specific heat of FGM plates [15]. The computational γ value of (A9) can be solved by bisection iterative method [26] with error-accuracy 1×10^{-6} of iterative strategy.

The stress-strain constitutive relations used for circular cylindrical shell in (k) th layer [24] are as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{Bmatrix}_{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{(k)} \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_{x\theta} - \alpha_{x\theta} \Delta T \end{Bmatrix}_{(k)}, \quad (A10)$$

$$\begin{Bmatrix} \sigma_{\theta z} \\ \sigma_{xz} \end{Bmatrix}_{(k)} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_{(k)} \begin{Bmatrix} \varepsilon_{\theta z} \\ \varepsilon_{xz} \end{Bmatrix}_{(k)}, \quad (A11)$$

in which σ_x and σ_θ are normal stresses. $\sigma_{x\theta}$, $\sigma_{\theta z}$ and σ_{xz} are shear stresses. α_x and α_θ are coefficients of the thermal expansion. $\alpha_{x\theta}$ is coefficient of the thermal shear. ε_x , ε_θ , and $\varepsilon_{x\theta}$ are in-plane strains. $\varepsilon_{\theta z}$ and ε_{xz} are not-negligible shear strains. $\bar{Q}_{ij}$ is material stiffness of the FGM circular cylindrical shell. ΔT is temperature difference expressed in $\Delta T = \frac{z}{h^*} \bar{T}_1 \sin(\pi x/a) \sin(\pi \theta) \sin(\gamma t)$ used for sinusoidal thermal loads and which meets the heat conduction $K \left(\frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{R^2 \partial \theta^2} + \frac{\partial^2 \Delta T}{\partial z^2} \right) = \frac{\partial \Delta T}{\partial t}$ as follows:

$$\frac{a^2}{K\pi^2[1 + (a/R)^2]} \gamma \cos(\gamma t) + \sin(\gamma t) = 0, \quad (A12)$$

where $KK = \kappa_{fgm} / (\rho_{fgm} C_{vfgm})$, $\kappa_{fgm} = (\kappa_1 + \kappa_2) / 2$ is thermal conductivity, $\rho_{fgm} = (\rho_1 + \rho_2) / 2$ is density, and $C_{vfgm} = (C_{v1} + C_{v2}) / 2$ is specific heat of FGM shells. The computational γ value of (A12) can be solved by bisection iterative method [26] with error-accuracy 1×10^{-6} of iterative strategy.

The strain-displacement relations used for plate with assumption $\frac{\partial u^0}{\partial z} = \frac{\partial v^0}{\partial z} = \frac{\partial w}{\partial z} = \frac{\partial \psi_x}{\partial z} = \frac{\partial \psi_y}{\partial z} = 0$ are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 = \frac{\partial u^0}{\partial x} + z \frac{\partial \psi_x}{\partial x} - c_1 z^3 \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad (A13)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 = \frac{\partial v^0}{\partial y} + z \frac{\partial \psi_y}{\partial y} - c_1 z^3 \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad (A14)$$

$$\varepsilon_{zz} = 0, \quad (A15)$$

$$\begin{aligned} \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) \\ &= \frac{\partial u^0}{\partial y} + z \frac{\partial \psi_x}{\partial y} - c_1 z^3 \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial v^0}{\partial x} + z \frac{\partial \psi_y}{\partial x} - c_1 z^3 \left(\frac{\partial \psi_y}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} \right) + \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right), \end{aligned} \quad (A16)$$

$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \psi_y - 3c_1 z^2 \left(\psi_y + \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial y}, \quad (A17)$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \psi_x - 3c_1 z^2 \left(\psi_x + \frac{\partial w}{\partial x} \right) + \frac{\partial w}{\partial x}. \quad (A18)$$

The strain-displacement relations used for circular cylindrical shell with assumption $\frac{\partial v_0}{\partial z} = \frac{-v_0}{R}$, $\frac{\partial u_0}{\partial z} = \frac{-u_0}{R}$ and $\frac{\partial w}{\partial z} = \frac{\partial \varphi_x}{\partial z} = \frac{\partial \varphi_\theta}{\partial z} = 0$ are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 = \frac{\partial u_0}{\partial x} + z \frac{\partial \varphi_x}{\partial x} - c_1 z^3 \left(\frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad (A19)$$

$$\begin{aligned}\varepsilon_\theta &= \frac{1}{R} \frac{\partial v}{\partial \theta} + \frac{1}{2} \left(\frac{1}{R} \frac{\partial w}{\partial \theta} \right)^2 \\ &= \frac{1}{R} \frac{\partial v_0}{\partial \theta} + z \frac{1}{R} \frac{\partial \varphi_\theta}{\partial \theta} - c_1 z^3 \left(\frac{1}{R} \frac{\partial \varphi_\theta}{\partial \theta} + \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} \right) + \frac{1}{2} \frac{1}{R^2} \left(\frac{\partial w}{\partial \theta} \right)^2,\end{aligned}\quad (\text{A20})$$

$$\varepsilon_{zz} = 0, \quad (\text{A21})$$

$$\begin{aligned}\varepsilon_{x\theta} &= \frac{1}{R} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x} \right) \left(\frac{1}{R} \frac{\partial w}{\partial \theta} \right) = \frac{1}{R} \left[\frac{\partial u_0}{\partial \theta} + z \frac{\partial \varphi_x}{\partial \theta} - c_1 z^3 \left(\frac{\partial \varphi_x}{\partial \theta} + \frac{\partial^2 w}{\partial x \partial \theta} \right) \right] \\ &+ \frac{\partial v_0}{\partial x} + z \frac{\partial \varphi_\theta}{\partial x} - c_1 z^3 \left(\frac{\partial \varphi_\theta}{\partial x} + \frac{1}{R} \frac{\partial^2 w}{\partial x \partial \theta} \right) + \left(\frac{\partial w}{\partial x} \right) \left(\frac{1}{R} \frac{\partial w}{\partial \theta} \right),\end{aligned}\quad (\text{A22})$$

$$\varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{R} \frac{\partial w}{\partial \theta} = \frac{-v_0}{R} + \varphi_\theta - 3c_1 z^2 \left(\varphi_\theta + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) + \frac{1}{R} \frac{\partial w}{\partial \theta}, \quad (\text{A23})$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{-u_0}{R} + \varphi_x - 3c_1 z^2 \left(\varphi_x + \frac{\partial w}{\partial x} \right) + \frac{\partial w}{\partial x}. \quad (\text{A24})$$

The material stiffness $\bar{Q}_{ij}$ used for FGM plates [15] is as follows:

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{E_{fgm}}{1 - \nu_{fgm}^2}, \quad (\text{A25})$$

$$\bar{Q}_{12} = \frac{\nu_{fgm} E_{fgm}}{1 - \nu_{fgm}^2}, \quad (\text{A26})$$

$$\bar{Q}_{44} = \bar{Q}_{55} = \bar{Q}_{66} = \frac{E_{fgm}}{2(1 + \nu_{fgm})}, \quad (\text{A27})$$

$$\bar{Q}_{16} = \bar{Q}_{26} = \bar{Q}_{45} = 0, \quad (\text{A28})$$

in which $E_{fgm} = (E_2 - E_1) \left(\frac{z+h^*/2}{h^*} \right)^{R_n} + E_1$, $\nu_{fgm} = \frac{\nu_1 + \nu_2}{2}$, where E_1 and E_2 are the Young's modulus, and ν_1 and ν_2 are the Poisson's ratios of FGM plate constituent material 1 and 2.

The material stiffness $\bar{Q}_{ij}$ used for FGM circular cylindrical shell [24] is as follows:

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{E_{fgm}}{1 - \nu_{fgm}^2}, \quad (\text{A29})$$

$$\bar{Q}_{12} = \bar{Q}_{21} = \frac{\nu_{fgm} E_{fgm}}{(1 + z/R)(1 - \nu_{fgm}^2)}, \quad (\text{A30})$$

$$\bar{Q}_{44} = \frac{E_{fgm}}{2(1 + \nu_{fgm})}, \quad (\text{A31})$$

$$\bar{Q}_{55} = \bar{Q}_{66} = \frac{E_{fgm}}{2(1 + z/R)(1 + \nu_{fgm})}, \quad (\text{A32})$$

$$\bar{Q}_{16} = \bar{Q}_{26} = \bar{Q}_{45} = 0, \quad (\text{A33})$$

in which $E_{fgm} = (E_2 - E_1) \left(\frac{z+h^*/2}{h^*} \right)^{R_n} + E_1$, $\nu_{fgm} = \frac{\nu_1 + \nu_2}{2}$, where E_1 and E_2 are the Young's modulus, and ν_1 and ν_2 are the Poisson's ratios of FGM circular cylindrical shell constituent material 1 and 2.

The FGM individual material property, e.g., E_1 , E_2 , ν_1 , ν_2 , κ_1 , κ_2 , ρ_1 , ρ_2 , C_{v1} , C_{v2} , etc., usually in functions of T can be expressed by parameter P_i in expression as follows,

$$P_i = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3), \quad (\text{A34})$$

used to calculate the constituent value of material properties, in which P_0 , P_{-1} , P_1 , P_2 , and P_3 are coefficients of temperature [24].

The GDQ grid point (x_i, y_j) distribution used for plate [15] is as follows:

$$x_i = 0.5[1 - \cos(\frac{i-1}{N-1}\pi)]a, i = 1, 2, \dots, N, \quad (\text{A35})$$

$$y_j = 0.5[1 - \cos(\frac{j-1}{M-1}\pi)]b, j = 1, 2, \dots, M. \quad (\text{A36})$$

The GDQ grid point (x_i, θ_j) distribution used for shell is as follows:

$$x_i = 0.5 \left[1 - \cos\left(\frac{i-1}{N-1}\pi\right) \right] a, i = 1, 2, \dots, N, \quad (\text{A37})$$

$$\theta_j = 0.5 \left[1 - \cos\left(\frac{j-1}{M-1}\pi\right) \right] R, j = 1, 2, \dots, M. \quad (\text{A38})$$

The GDQ method used for plates–shells and air flow [16,19,20] is as follows:

$$\frac{\partial^m f(x)}{\partial x^m} \Big|_{x=x_i} = \sum_{j=1}^N C_{i,j}^{(m)} f(x_j), i = 1, 2, \dots, N, \quad (\text{A39})$$

in which superscript (m) th-order derivative of one-dimensional function $f(x)$ at grid point x_i in the x direction can be approximated by weighting parameter $C_{i,j}^{(m)}$ of summation form, and the $C_{i,j}^{(m)}$ can be expressed generally as follows:

$$C_{i,j}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \text{ for } i \neq j \text{ and } i, j = 1, 2, \dots, N, \quad (\text{A40})$$

$$C_{i,j}^{(m)} = m(C_{i,j}^{(1)} C_{i,i}^{(m-1)} - \frac{C_{i,j}^{(m-1)}}{x_i - x_j}) \text{ for } m = 2, 3, \dots, N-1, i \neq j \text{ and } i, j = 1, 2, \dots, N, \quad (\text{A41})$$

$$C_{i,i}^{(m)} = - \sum_{j=1(j \neq i)}^N C_{i,j}^{(m)} \text{ for } i = 1, 2, \dots, N, m = 1, 2, 3, \dots, N-1, \quad (\text{A42})$$

$$M^{(1)}(x_k) = \prod_{j=1(j \neq k)}^N (x_k - x_j) = (x_k - x_1)(x_k - x_2) \dots (x_k - x_N) \Big|_{x_k \neq x_j}. \quad (\text{A43})$$

The GDQ method in one-dimensional formulation can be applied to two-dimensional formulation for the derivative of function $f(x, y)$ with corresponding weighting parameter at coordinate of grid point, e.g., (x_i, y_j) , $i = 1, \dots, N$, $j = 1, \dots, M$, as follows, for first-order derivatives of $f(x, y)$ used,

$$\frac{\partial f}{\partial x} \Big|_{i,j} \approx \sum_{l=1}^N A_{i,l}^{(1)} f_{l,j}, \quad \frac{\partial f}{\partial y} \Big|_{i,j} \approx \sum_{m=1}^M B_{j,m}^{(1)} f_{i,m}, \quad (\text{A44})$$

for second-order derivatives of $f(x, y)$ used,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(2)} f_{l,j}, \quad \frac{\partial^2 f}{\partial y^2} \Big|_{i,j} \approx \sum_{m=1}^M B_{j,m}^{(2)} f_{i,m}, \\ \frac{\partial^2 f}{\partial x \partial y} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(1)} \sum_{m=1}^M B_{j,m}^{(1)} f_{l,m} \end{aligned} \quad (\text{A45})$$

for third-order derivatives of $f(x, y)$ used,

$$\begin{aligned}\frac{\partial^3 f}{\partial x^3} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(3)} f_{l,j}, \quad \frac{\partial^3 f}{\partial y^3} \Big|_{i,j} \approx \sum_{m=1}^M B_{j,m}^{(3)} f_{i,m}, \\ \frac{\partial^3 f}{\partial x \partial y^2} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(1)} \sum_{m=1}^M B_{j,m}^{(2)} f_{l,m}, \quad \frac{\partial^3 f}{\partial x^2 \partial y} \Big|_{i,j} \approx \sum_{l=1}^N A_{i,l}^{(2)} \sum_{m=1}^M B_{j,m}^{(1)} f_{l,m},\end{aligned}\quad (\text{A46})$$

And for fourth-order derivatives of $f(x, y)$ used,

$$\begin{aligned}\frac{\partial^4 f}{\partial x^4} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(4)} f_{l,j}, \quad \frac{\partial^4 f}{\partial y^4} \Big|_{i,j} \approx \sum_{m=1}^M B_{j,m}^{(4)} f_{i,m}, \\ \frac{\partial^4 f}{\partial x \partial y^3} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(1)} \sum_{m=1}^M B_{j,m}^{(3)} f_{l,m}, \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} \Big|_{i,j} \approx \sum_{l=1}^N A_{i,l}^{(2)} \sum_{m=1}^M B_{j,m}^{(2)} f_{l,m}, \\ \frac{\partial^4 f}{\partial x^3 \partial y} \Big|_{i,j} &\approx \sum_{l=1}^N A_{i,l}^{(3)} \sum_{m=1}^M B_{j,m}^{(1)} f_{l,m},\end{aligned}\quad (\text{A47})$$

in which $A_{i,l}^{(m)}, B_{j,m}^{(m)}$ are the weighting parameters for the superscript (m) th-order derivative of $f(x, y)$ in x and y directions.

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