

Pareto Efficiency in Euclidean Spaces and Its Applications in Economics

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Abstract: The aim of the first part of this paper is to show whether a set of Proper Efficient Points and a set of Pareto Efficient Points coincide in Euclidean spaces. In the second part of the paper, we show that supporting prices, which are actually strictly positive, do exist for a large class of exchange economies. A consequence of this result is a generalized form of the Second Welfare theorem. The properties of the cones' bases are significant for this purpose.

Keywords: Pareto Efficient Points; proper efficient points; bases of cones; price supportability; second welfare theorem

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1. Introduction

The aim of this section is to recall some properties of the cones in Euclidean spaces. Describing these properties is useful for enabling a better understanding of the following sections. The essential property, which is used in the following sections, is that any base of a closed cone in some Euclidean space is closed and bounded. The second section is devoted to the use of this property on the classical result of Arrow–Barankin–Blackwell. The associated result indicates whether the sets of Pareto Efficient and Proper Efficient Points of a set coincide. The main result in the last section is an application of Pareto efficiency in the field of exchange economies. The significance of the version of the Second Welfare theorem proven in the last section is that even in the well-known statement of the Second Welfare theorem, a detailed description of the positive supporting price is obtained. The reader may refer to Chapter 1 in [1] for details. The version of the Second Welfare theorem proven here is valid under certain assumptions on the exchange economies related to financial markets. For details about this frame, the reader may refer to Chapters 1 and 2 of [2]. However, it is an application of the Arrow–Barankin–Blackwell version in the field of Welfare Theorems of Exchange Economies. Specifically, it indicates that a Pareto Optimal Allocation is a Proper Efficient Point in the set of allocations. The definitions of Pareto and Weakly Pareto Optimal Allocations are given in the last section, though they are well known from related literature in mathematical economics. These definitions are given in Chapter 1 of [1]. A seminal paper on this topic was published by [3]. The statement of the essential Vector Maximization Problem in [3]—denoted as (VMP)—is the maximization of the vector function

$$f(x) = (f_1(x), \dots, f_p(x)),$$

under the assumption that $x \in X \subseteq \mathbb{R}^n$. The set X is called a set of *feasible points*. The maximization in the sense of the VMP is defined in the following way: $y \in X$ is a solution to the problem (VMP) if there is not any other $z \in X$, such that $f_i(z) \geq f_i(y)$ for any $i = 1, \dots, p$, and $f_j(z) > f_j(y)$ for some $j = 1, \dots, p$. This is actually an element of the Pareto Efficient Points of the set $f(X)$ with respect to the cone $\mathbb{R}_+^p$. If U is a function which maps any allocation $x \in \mathcal{A}$ of an exchange economy in the sense of Definition 10 to the set of the vectors $(u^1(x^1), \dots, u^I(x^I))$, this is actually the so-called *utility space* of an exchange economy. This terminology comes from [1]. Hence, any solution to the above VMP is a so-called Pareto Optimal Allocation. The set X in the above VMP is a set of allocations. We notice that even in the early steps of the evolution of the topic of vector optimization, there is a connection between it and the Welfare Theory in microeconomics. The same author seminally studied the connection between vector and scalar optimization in [4,5]. The *support points* of some convex set are studied in [6]. The set of support points is the set of Proper Efficient Points, as defined in the present paper. However, [6] is seminal in the study of cones with bounded bases in infinite-dimensional spaces, as indicated in [7]. The topic of this paper is related to the notion of a *tangent cone*, as defined in [8]. In the present paper, we simplify the method of characterizing a VMP as above using the solution to scalar minimization problems. This is achieved by identifying the properties of the base of a cone in both finite- and infinite-dimensional linear spaces. This direction of study is taken by most papers on vector optimization, which is actually related to the content of the present paper. The separation between the set of “feasible” points and the base of the cone, inducing the partial ordering of the linear space of the reference, provides results for the Pareto Efficient and Proper Efficient Points of the ‘feasible’ set. A very important paper on this topic is [9]. However, the tangent cone may be considered to be a translation of a specific cone with respect to some point in order to achieve the application of the Hyperplane Separation Theorems. This is exactly the approach we employ in the present paper. A recent paper devoted to the relationship between tangent cones and Pareto Efficient Points in infinite-dimensional spaces is [10].

A very important survey book on vector optimization is [11]. This book contains a framework part that includes essential notions about partially ordered linear spaces and, in particular, details about cones with closed and bounded bases. The second part of the book is devoted to the applications of vector-optimization operation research problems and the exact use of multi-objective optimization on them. Also, we must mention that a significant recent paper on the geometry of cones is [12]. In this paper, the authors examine the geometry of cones in a purely algebraic manner. The main notion which is studied is the core interior of a cone in general linear space. The definition of the notion in [12] is the same as the algebraic interior in [7]. As we mentioned in the second section, a novel result in [13] is the coincidence of Pareto Efficient and Proper Efficient Points of a convex set. In [12], the authors obtain results pertaining to the properties of a Proper Efficient Point set. As is well known, the cones’ interior properties are significant for the study of Pareto Efficient Points, as mentioned in the second section of the paper. A previously published paper which is devoted to the study of vector optimization through properties of the algebraic interior points of cones associated with it is [14]. The study of vector optimization is similar to the statement of problems of the VMP type, like in [3]. A similar paper to [14] is [15]. The duality between the bases and interior points of a cone is a matter of research even in recent papers, like [16–18]. Without providing too many details, the question posed in these papers is whether a cone in some normed linear space has a bounded base equivalent to the existence of interior points in the cone’s polar wedge. The definition of the polar wedge of a cone is given below. The application of Hyperplane Separation theorems includes two

cases in such cases: one is where the separated sets are bases of the cones and the set of the ‘feasible’ points, and one where the separated sets are the interior points of the cones and the set of ‘feasible’ points. Both of these cases imply some properties about the set of Pareto Efficient and the set Proper Efficient Points of the ‘feasible’ set. We also provide some details about this duality in the section where we discuss the difference between the present paper and the paper [13]. A rational question that may arise about the content of the paper is the motivation for studying Welfare Theorems in a finite-dimensional setting today since uncertainty is modified by infinite-dimensional spaces. The answer to this question may be given in [19]. This paper is devoted to the computational complexity of the calculation of equilibrium in finite-dimensional spaces, which are considered to be the consumption sets. The authors prove that these problems’ complexity is polynomial. Finite-dimensional spaces are related to the use of real data in the formulation of the associated utility functions and budget sets. The notions of equilibrium and budget sets are the ones considered in most of the relevant research. For a detailed definition of them, the reader may refer to [1]. Another interesting part of this paper is where the authors refer to the associated problems of utility maximization under the assumption of linear utility, which may be considered to be an approximation of a non-linear utility function. The same technique may be employed to study the maximization of the utility of an exchange economy. The authors prove that such a maximization problem has a solution, and they obtain a version for both of the Welfare Theorems in exchange economies. The proof that is presented in this argument may need more discussion. We also suggest the reader refer to the Chapter 1 of [1] for their statement, which we also repeat below. We also must mention that a good survey book on ordered linear spaces, which is entirely used in vector optimization, is [20]. The word Pareto in the related bibliography is after the Italian economist Vilfredo Pareto. His survey [21] marks the beginning of vector optimization and the establishment of mathematical economics.

Preliminaries

K is supposed to be a nonempty, closed subset of $\mathbb{R}^m$ and $K \neq \{0\}$. The real number $f(x)$, whenever mentioned below, is the inner product between $x \in \mathbb{R}^m$ and $f \in \mathbb{R}^m$. The set $(x - K)$, whenever mentioned below, is actually the following set: $\{y \in \mathbb{R}^m | y = x - k, k \in K\}$, where K is a wedge or a cone of $\mathbb{R}^m$.

Definition 1. K is a *wedge* of $\mathbb{R}^m$, if $K + K \subseteq K$, $\lambda \cdot K = \{\lambda \cdot k, k \in K\} \subseteq K$, for any $\lambda \in \mathbb{R}_+$.

Definition 2. K is a *cone* of $\mathbb{R}^m$, if it is a wedge and, moreover, $K \cap (-K) = \{0\}$, where $0 \in \mathbb{R}^m$ is the zero element of $\mathbb{R}^m$, and

$$(-K) = \{x \in \mathbb{R}^m | -x \in K\}.$$

Remark 1. In many parts of the literature, a cone is mentioned as a *pointed cone*, and a wedge is mentioned as a *cone*.

An example of a wedge in $\mathbb{R}^m$ is the *half-space* $H(f, a) = \{x \in \mathbb{R}^m | f \cdot x = f(x) \geq a\}$, where $\cdot$ denotes the usual inner product, and $a \in \mathbb{R}$. Then, $-H(f, a) = \{x \in \mathbb{R}^m | f \cdot (-x) \geq a\}$ is a wedge, as well. The intersection of $H(f, a)$ and $-H(f, a)$ is the hyperplane $H = \{x \in \mathbb{R}^m | f \cdot x = f(x) = a\}$. If $H(f, a)$ is nonempty, then $H(f, a) \cap -H(f, a) = H$, which is not equal to $\{0\}$, if $f \neq 0$. An example of a cone, which is not a wedge, is

$$K(f, a) = \{x \in \mathbb{R}^m | f \cdot x = f(x) \geq a \|x\|_1\},$$

where $f \neq 0, f \neq \mathbf{1}, a \in (0, 1)$ - a is a real number and $\|\cdot\|_1$ denotes the ℓ^1 norm of $\mathbb{R}^m$. $\mathbf{1}$ denotes the vector of $\mathbb{R}^m$, whose components are equal to 1.

Definition 3. A *cap* C of some closed cone K is any nonempty, closed subset of K , such that for any $x \in K \setminus \{0\}$ there exists some positive, non-zero real number $\lambda(x)$ such that $\lambda(x) \cdot x \in C$.

Example 1. A cap C may not be a convex set. If

$$K = \mathbb{R}_+^m = \{x \in \mathbb{R}^m | x_i \geq 0, i = 1, 2, \dots, m\},$$

then $S(m, 2) = \{x \in \mathbb{R}_+^m | \sum_{i=1}^m x_i^2 = \|x\|_2^2 = 1\}$ is a cap of $\mathbb{R}_+^m$. $\|\cdot\|_2$ denotes the ℓ^2 norm of $\mathbb{R}^m$. $S(m, 2)$ is a cap, because for any $x \in \mathbb{R}_+^m \setminus \{0\}$, $\frac{1}{\|x\|_2} x \in S(m, 2)$.

Definition 4. Any convex cap B of a closed cone K is a **base** of K .

Example 2. If $K = \mathbb{R}_+^m$, then $S(m, 1) = \{x \in \mathbb{R}_+^m | \sum_{i=1}^m x_i = 1\}$ is a convex cap of $\mathbb{R}_+^m$. This is true, since for any $x \in \mathbb{R}_+^m \setminus \{0\}$, $\frac{1}{\|x\|_1} x \in S(m, 1)$. $\|\cdot\|_1$ denotes the ℓ^1 norm of $\mathbb{R}^m$.

Definition 5. The set $K^0 = \{p \in K, p \cdot x \geq 0 \text{ for any } x \in K\}$ is the **polar wedge** of the cone K .

Definition 6. If $p \in K^0$ and $p \cdot x > 0$, for any $x \in K \setminus \{0\}$ is called **strictly positive** with respect to the cone K . The set of **strictly positive** linear functional with respect to K is denoted by K' .

Definition 7. $B_f = \{x \in K | f \in K^0, f \cdot x = f(x) = 1\}$ is the **base** of the cone K , with respect to f .

Example 3. $S(m, 1)$ is the base of $\mathbb{R}_+^m$, with respect to $f = \mathbf{1}$. $\mathbf{1}$ denotes the vector of $\mathbb{R}^m$, whose every component is equal to 1. Hence, $S(m, 1) = B_{\mathbf{1}}$ for the cone $\mathbb{R}_+^m$.

Any base of some closed cone K in $\mathbb{R}^m$ is convex, closed and bounded. The reader should refer to Prop.3.14 in [22] for a proof of this result. The proof of the results in the following sections relies on the Separation theorem of convex sets. The Separation theorem of convex sets may be deduced independently from the Hahn–Banach theorem in the case of finite-dimensional vector spaces. The reader should refer to [2], pp. 122–124.

2. Results

2.1. Pareto Efficient and Proper Efficient Points

Definition 8. A **Pareto Efficient Point** of some convex set A , with respect to the cone K , is any $x_0 \in A$ such that $(x_0 - K) \cap A = \{x_0\}$. The set of these points in A is denoted by $EP(A, K)$.

Definition 9. A **Proper Efficient Point** of some A , with respect to the cone K , is any $x_0 \in A$, such that $f(x) \geq f(x_0)$, for any $x \in A$ such that f is a strictly positive linear functional, with respect to the cone K . The set of these points in A is denoted by $Pos(A, K)$.

The Arrow–Barankin–Blackwell theorem’s statement in $\mathbb{R}^m$ is the following:

If K is closed cone of $\mathbb{R}^m$ and A is a nonempty, convex and compact set of $\mathbb{R}^m$, then $EP(A, K) \subseteq \overline{Pos(A, K)}$.

In the above statement, $\overline{}$ denotes the close hull of the set C .

The above theorem is the beginning point of vector optimization both in Euclidean spaces and normal spaces. The seminal references about the above result are actually [8,23]. There exist plenty of papers devoted to this seminal result, its extension in infinite-dimensional spaces, and the properties of the bases in cones of such spaces. Even though

the related literature is very rich, we mention these two works, which are the beginning of the related research. The meaning of the Arrow–Barankin–Blackwell theorem is that the solutions of a vector-optimization problem may be approximated by solutions to scalar optimization problems. These problems are characterized by the linear functionals of K' . A similar result to the Arrow–Barankin–Blackwell theorem is that any closed base B of some closed cone K in $\mathbb{R}^m$ is bounded and $0 \notin B$, as it is mentioned above. The reader should refer to Prop.3.14 in [22] for a proof of this result. Obviously, any $x_0 \in \text{Pos}(A, K)$ belongs to the set $EP(A, K)$, if $\text{Pos}(A, K) \neq \emptyset$. The proof is given below:

Lemma 1. *Any Proper Efficient Point of some nonempty set A of $\mathbb{R}^m$ is a Pareto Efficient Point of A .*

Proof. Let us suppose that there exists some $x_0 \in \text{Pos}(A, K) \setminus EP(A, K)$. Since $x_0 \in \text{Pos}(A, K)$, there exists some $g \in K'$, such that $g(a) \geq g(x_0)$ for any $a \in A$. Since $x_0 \notin EP(A, K)$, $A \cap (x_0 - K)$ contains some $x_1 \neq x_0$. This x_1 is a non-zero element of the set $(x_0 - K)$, such that $g(x_1) = g(x_0 - k_1) \geq g(x_0)$, where $k_1 \in K \setminus \{0\}$. $g(-k_1) = -g(k_1) \geq 0$, hence $0 \geq g(k_1)$. On the other hand, $g(k_1) > 0$, since $g \in K'$. This is a contradiction. Hence, such a x_1 does not exist. $\square$

An important question is whenever $\text{Pos}(A, K) \neq \emptyset$. An answer to this question is the following:

Theorem 1. *We suppose that A is a convex set of $\mathbb{R}^m$ and K is a closed cone of $\mathbb{R}^m$. B is a closed base of K . Then, any $x_0 \in A$, such that $A \cap (x_0 - B) = \emptyset$ is an element of the set $\text{Pos}(A, K)$.*

Proof. By applying the Convex Set Separation theorem,

$$f(a) \geq f(x_0) - f(b),$$

for any element of the base B and some non-zero $f \in \mathbb{R}^m$. We deduce that $f \in K'$. To prove it, we set $g = -f$. If we suppose that there exists some $k_0 \in K \setminus \{0\}$, such that $g(k_0) > 0$, then $k_0 = t_0 \cdot b_0$ for some $b_0 \in B$ and t_0 is a positive, non-zero real number. Then, $n \cdot k_0 \in K$ for any $n \in \mathbb{N}$, but this provides a contradiction since the separation inequality is violated, as far as $n \rightarrow +\infty$. Hence, such a $k_0 \in K \setminus \{0\}$ does not exist, where 0 denotes the zero element of $\mathbb{R}^m$. Therefore, f is a strictly positive functional, with respect to K . The above separation inequality implies that $f(a) \geq f(x_0)$ for any $a \in A$, namely $x_0 \in \text{Pos}(A, K)$. $\square$

The following result is the main one of this section.

Theorem 2. *We suppose that A is a convex set of $\mathbb{R}^m$, K is a closed cone of $\mathbb{R}^m$, and B is a closed base of K . If $A_0 = \{x \in A \mid A \cap (x - B) = \emptyset\}$, then $EP(A_0, K) = \text{Pos}(A_0, K)$.*

Proof. Since $x_0 \in A_0$, then $A \cap (x_0 - B) = \emptyset$. Hence, x_0 is some element of the set $\text{Pos}(A_0, K)$. The previous theorem and the previous lemma imply the validity of the conclusion. $\square$

In [13], the authors obtain the analog of this result using the notion of the *algebraic interior* points for cones. Let us suppose that X is a linear space. A point $y \in K$ is an algebraic interior point of a cone $K \subseteq X$. If for any $x \in X$, there exists some $\delta > 0$ positive real number, such that $y + t \cdot x \in K$, for any $t \in \mathbb{R}$, such that $|t| \leq \delta$. This definition was obtained by [7]. Also, it was Th.3.8.14 in [7] whose statement is the following: *If A is a complete, convex subset of a normal linear space X , then the support points of A are dense in the boundary of A , which is actually a version of the Arrow–Barankin–Blackwell theorem,*

since support points are actually Positive Efficient Points of A , according to the definition published in [7]. However, the above theorem is a result that indicates whether the set of Pareto Efficient Points and the Pareto Efficient Points coincide. The meaning of this result is actually the one examined in [13]. In [13], the authors apply the Eidelheit Separation theorem for this purpose. In this paper, we apply the Convex Set Separation theorem in finite-dimensional linear spaces. The assumption of the authors in [13], in order to obtain their ‘coincidence’ result, is that the cone K , which implies the partial ordering in the linear space, has internal points. Such an assumption in the present paper does not exist. However, any closed cone in some finite-dimensional linear space has a closed base B , which is bounded, and $0 \in B$, where 0 is the zero element of this linear space. This assertion, stated as a lemma, may be proved, bearing in mind that the dimensions of the space are finite. In the present paper, we obtain the above ‘coincidence’ theorem, using the essential property of the bases in cones for finite-dimensional linear spaces, which is that for any cone K in such a space, there exists a base B which is closed and bounded. Moreover, $0 \notin B$. As proven in detail in (Prop.3.14 in [22]), any base of a closed cone in a finite-dimensional linear space is closed, bounded, and $0 \notin B$, where 0 is the zero element of this space. In infinite-dimensional linear spaces, the bases of cones are usually unbounded.

2.2. Proper Efficiency in Mathematical Economics

The aim of this section is to answer the question of whether an allocation in Finite-Dimensional Economies is supported by a **strictly positive** price vector, with respect to a cone K . We assume that because the *consumption set* is some closed cone K , such that $\mathbb{R}_+^m \subseteq K$. The reason for this assumption is that consumption sets may include *short sales* on some financial positions. Hence, we consider a finite set of states of the world, where the long position on one unit is denoted by $e(s) = (0, \dots, 1, \dots)$ and the corresponding short position is denoted by $-e(s)$. The unit of $e(s)$ is located on the state $s \in \{1, \dots, m\}$. The main result of this section refers to the existence of a supporting price for any allocation, such that $p(s) > 0$, where $s = 1, 2, \dots$. The definition of an allocation is related to the definition of the exchange economy, given below. We also assume that the consumption set K is the same for all the individuals $\{1, \dots, I\}$. All the individuals $i = 1, \dots, I$ are endowed with some utility function $u^i : K \rightarrow \mathbb{R}_+$. A typical assumption for such a function is continuity. This assumption is related to closure properties like the one mentioned in Definition 11. The goal of any individual is to maximize their utility function on some *budget set*. Such a set in the case where the consumption set is K is defined in the following way: $B(p, \omega) = \{x \in K | p \cdot x = p \cdot \omega\}$, where $p \cdot \omega > 0$ and $\omega \in K$. Hence, any $B(p, \omega)$ is a base of a cone K . Since $p \in K^0$, because they actually are positive functionals with respect to K , we may assume that $p \in K'$. Specifically, they are strictly positive functionals with respect to K . The budget sets are closed and compact since they are actually bases of the closed cone K . A reference for all of this is Chapter 1 of [1], and especially Section 1.3.

Definition 10. The set of **allocations** in a K exchange economy is the following:

$$\mathcal{A} := \{x = (x^1, \dots, x^I) \in (K^I | \sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i)\},$$

where $\omega^i \in K$ for any $i = 1, \dots, I$.

Definition 11. A utility function $u : \mathbb{R}^m \rightarrow \mathbb{R}$ is **K uniformly convex** if it satisfies the following properties:

- (i) $u : \mathbb{R}^m \rightarrow \mathbb{R}$.
- (ii) The upper-level sets $C_u(x_0) := \{x \in \mathbb{R}^m | u(x) \geq u(x_0)\}$ are convex and closed.

(iii) $C_u(x_0) \cap (x_0 - K) = \{x_0\}$ for any $x_0 \in \mathbb{R}^m$.

Definition 12. We consider an exchange economy consisting of $m \in \mathbb{N}, m \neq 0, 1$ consumption goods. If $u^i : \mathbb{R}^m \rightarrow \mathbb{R}$ are K uniformly convex for any consumer $i = 1, \dots, I$, then we call it a K exchange economy.

Lemma 2. If $\mathbb{R}_+^m \subseteq K$, then $\mathbb{R}_+^m \subseteq K^0$. Also, for any $f \in K'$, $f(k) > 0$, for any $k \in \mathbb{R}_+^m \setminus \{0\}$.

Proof. Obvious. $\square$

Definition 13. The set of **Weakly Pareto Optimal Allocations** in some K exchange economy is the following subset of allocations: $WPOA := \{x \in \mathcal{A} \mid \text{there is no other } y \in \mathcal{A}, \text{ such that } u^i(y^i) \geq u^i(x^i) \text{ for any } i = 1, 2, \dots, I\}$.

Definition 14. The set of **Pareto Optimal Allocations** in some K exchange economy is the following subset of allocations: $POA := \{x \in \mathcal{A} \mid \text{there is not any other } y \in \mathcal{A}, \text{ such that } u^i(y^i) \geq u^i(x^i) \text{ for any } i = 1, 2, \dots, I \text{ and } u^k(y^k) > u^k(x^k) \text{ at least for one } k = 1, 2, \dots, I\}$.

The set of Weakly Pareto Optimal and Pareto Optimal Allocations are nonempty. This is assured by the assumption that the utility functions are continuous. The set of allocations is compact since it is closed and bounded. The existence of both Pareto and Weakly Pareto Optimal Allocations arises as an application of Zorn's lemma on the set of allocations. Assumption (iii) in Definition 11 is related to the monotonicity of the utility functions, which is necessary for the application of Zorn's lemma. The reader should refer to Chapter 1 of [1] and especially Sections 1.5 and 1.6 about details. The difference is that the consumption set in K exchange economies is the entire space $\mathbb{R}^m$.

Definition 15. A **supporting price** for an allocation $x = (x^1, \dots, x^I)$ is any vector $p \in \mathbb{R}_+^m$, whose every component is positive. If $u^i(y^i) \geq u^i(x^i)$, then $p \cdot y^i \geq p \cdot x^i$, for any $i = 1, \dots, I$.

The meaning of a supporting price is that any choice of consumption that improves the values of the utility functions for any $i = 1, \dots, I$ under an allocation x has an upper cost with respect to some price vector $p \in K'$.

Theorem 3. For any allocation $(x^1, \dots, x^I)$ of a K exchange economy, a supporting price exists.

Proof. By the definition of the utility functions in some K exchange economy,

$$(x^i - B_f) \cap C_{u^i}(x^i) = \emptyset,$$

for any $i = 1, \dots, I$ and for some B_f being a base of the cone K . Then, we assume that $u^i(y^i) \geq u^i(x^i)$, for any $i = 1, \dots, I$, where $(x^1, \dots, x^I)$ is an allocation of some K exchange economy. Then, $y^i \in C_{u^i}(x^i)$ and by Theorem 1, we obtain some $p_i \in K'$, such that $p_i \cdot y^i \geq p_i \cdot x^i$. Based on Lemma 2, this p_i has positive components. We consider the vector $\vee_{i=1}^I p_i = p$, namely $p(s) = \max\{p_i(s), i = 1, \dots, I\}$, for any $s = 1, \dots, m$. Then, $p \cdot y^i \geq p \cdot x^i$. $\square$

The above theorem may be considered to be a version of the Second Welfare theorem, compared to its usual form (see Th.1.6.10 in [1]).

Specifically, the last theorem implies the next one:

Theorem 4. For any Pareto Optimal (and any Weakly Pareto Optimal) allocation $(x^1, \dots, x^I)$ of a K exchange economy, a supporting price does exist.

2.3. Discussion

In a more detailed discussion about the significance of the above results, we should mention that the statement of the version of the Second Welfare theorem is the following: *In an exchange economy, where preferences (of $i = 1, \dots, I$) are strictly convex and non-satiated, any (Weakly) Pareto Optimal Allocation is supported by a non-zero price.* The meaning of supporting price is the same as that mentioned above in Definition 15. We must define the notions of strict convexity and non-satiation terms of utility functions in order to compare Theorem 4 and the above Theorem (Th.1.6.10 in [1]). We also must note that the (Th.1.6.10 in [1]) implies that the associated supporting price is positive, although not necessarily strictly positive. A utility function is *strictly convex* if for any $x, y \in \mathbb{R}^m$, such that $x \neq y$ and $u(x) \geq u(z)$, $u(y) \geq u(z)$ for some $z \in \mathbb{R}^m$, then $u(a \cdot x + (1 - a) \cdot y) > u(z)$ for any $a \in (0, 1)$, where $(0, 1)$ is the corresponding interval of real numbers. Also, a utility function $u : \mathbb{R}_+^m \rightarrow \mathbb{R}$ is *non-satiated* if for any $x \in \mathbb{R}_+^m$, there exists some $z \in \mathbb{R}_+^m$ such that $u(z) > u(x)$. As we mentioned above, the consumption set of an exchange economy in the present paper is the entire space $\mathbb{R}^m$. Hence, these definitions may be stated in the same way. As an example of strictly convex utility functions, we may mention the Cobb–Douglas utility function. In a case where $m = 2$, a Cobb–Douglas utility function is the following: $u(x, y) = x^b y^{1-b}$, where $x > 0, y > 0$, and $b \in (0, 1)$. By $(0, 1)$, we denote the corresponding open interval of the real numbers. An example of a non-satiated utility function in the case of $m = 2$ is the utility function $v(x, y) = \min\{x, y\}$, where $x, y \geq 0$. These examples are true for any $m \in \mathbb{N}$. In the case of the present study, we may replace x, y by $|x|$ and $|y|$, respectively. As mentioned at the beginning of this section, we assume that the consumption set is some cone K , such that $\mathbb{R}_+^m \subseteq K$. This assumption is related to the case of ‘short positions’ on some portfolio or some loss of income. However, the assumption of convexity (ii) about utility functions of Definition 11 is weaker than the assumption of strict convexity.

3. Further Discussion and Conclusions

The main result of this paper is that the set of Pareto Efficient Points and the set of Proper Efficient Points coincide in Euclidean spaces under conditions that are related to the bases of cones in such spaces. As a consequence, we deduce a form of the Second Welfare theorem. As a matter of fact, these results may be extended in infinite-dimensional spaces under the assumption of a bounded base for the equivalent cones.

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