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Invariance of Stationary Distributions of Exponential Networks with Prohibitions and Determination of Maximum Prohibitions

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Abstract: The paper considers queuing networks with prohibitions on transitions between network nodes that determine the protocol of their operation. In the graph of transient network intensities, a set of base vertices is allocated (proportional to the number of edges), and we raise the question of whether some subset of it can be deleted such that the stationary distribution of the Markov process describing the functioning of the network is preserved. In order for this condition to be fulfilled, it is sufficient that the set of vertices of the graph of transient intensities, after the removal of a subset of the base vertices, coincide with the set of states of the Markov process and that this graph be connected. It is proved that the ratio of the number of remaining base vertices to their total number n converges to one-half for $n \to \infty$. In this paper, we are looking for graphs of transient intensities with a minimum (in some sense) set of edges for open and closed service networks.

Keywords: product theorem; uniform distribution; Diophantine equation; transient intensity; controlled structure

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1. Introduction

The graph of transient intensities of a queuing network defines the protocol of its operation, and the removal of elements from the network leads to a decrease in possible transitions between network states. Such a change in the graph of transient intensities means a change in the protocol of the queuing network and is associated, for example, with solving problems associated with constructing a transport or computer network (see [1–6]). One of the ways to change the network protocol is to introduce blocking probabilities for transitions between network states [7–9]. However, this method does not imply the introduction of prohibitions on transitions, which is an urgent applied task and requires connectivity of the graph of transient intensity.

The motivation for the research came from numerous conversations with experts in the field of maritime transport and bus networks, which revealed the need to build mathematical models of network protocols. It was observed that the proposed mathematical models of protocols could reduce the number of rules in them for users of the rules by about half. The use of such rules is decided by the network owner, but the proposed mathematical model provides the opportunity.

When the network protocol is changed, the stationary distribution of the process of its operation remains. It is known that in stationary mode, a network may be considered as a collection of non-interacting nodes. By contrast, in non-stationary mode, there is constant interaction between the nodes of the network. Therefore, the goal in the present context

is to preserve this property, which allows us to consider their autonomous functioning instead of studying the interaction between nodes.

The authors have reported their own advancements in this area [10,11], and this paper is their continuation. In these papers, exponential networks with various protocols of operation are studied, including those with randomly changing ones, which are modeled by constructing a graph of transient intensities. The stationary distributions of the networks under consideration are calculated under the conditions of connectivity of the graph of transient intensities.

In this paper, we determine the conditions under which deleting some edges of the graph of transient intensities that determine the functioning of a queuing network does not change the stationary distribution in it. Specifically, we are considering open Jackson networks [12], including those with a total limited number of customers, and closed Gordon–Newell networks [13,14]. The search for the required conditions is based on the representation of the graph of transient intensities as a union of basic graphs. Each base graph consists of a base vertex and is complete. The intersection of the set of edges of any two base graphs is empty. This representation (decomposition) allows us to define a set of base vertices and their corresponding graphs, the removal of which preserves the total set of vertices of the graph and the connectivity of the graph of transient intensities. Graph decomposition leads to decomposition in the product theorems.

Since we were interested in graphs of transient intensities for exponential queuing networks, we limited ourselves to graphs represented by collections of basic subgraphs. This made it possible to significantly simplify the solution of graph-theoretic problems. The main result reported in this paper is the constructive removal of a certain subset of the base vertices, which preserves the stationary distribution of the Markov process describing the functioning of the network. Removing any vertex from the set of remaining base vertices violates the condition for preserving this distribution, which leads to an optimal solution in a certain sense. The set of vertices of the graph of transient intensities after removing a subset of the base vertices coincides with the set of states of the Markov process, and this graph is connected. It is proved that the ratio of the number of remaining base vertices to the total number of vertices n converges to one-half for $n \to \infty$. Additionally, it is proved that the removal of any vertex from the reduced set of base vertices leads to the non-preservation of the stationary distribution.

2. Materials and Methods

All constructions and results in this paper are based on the following theorem. Consider an open Jackson network G with a Poisson input flow with an intensity of λ, consisting of m single-server queuing systems with exponential intensity service times of $\mu_i,\ i = 1, \ldots, m$. The dynamics of customers' movement in the network is set by the route matrix $\Theta = ||\theta_{ij}||_{i,j=0}^{m}$, where θ_{ij} is the probability of switching from the ith node to the jth node after service, $\theta_{00} = 0$ and node number 0 is an external source.

It is assumed that the route matrix is indecomposable:

$$\forall i, j \in \{0, 1, \ldots, m\}\ \exists\ i_1, i_2, \ldots, i_r \in \{1, \ldots, m\} :\ \theta_{ii_1} > 0, \theta_{i_1 i_2} > 0, \ldots, \theta_{i_r j} > 0.$$

Then, the vector $\Lambda = (\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m)$ is the only solution to the balance ratio system

$$\Lambda = \Lambda\Theta. \tag{1}$$

The functioning of the network (the number of customers at the nodes) is described by a discrete Markov process $n(t)$ with multiple states $\mathbf{N} = \{\boldsymbol{n} = (n_1, \ldots, n_m) :\ n_1, \ldots, n_m \geq 0\}$ and transient intensities:

$$L(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k) = \lambda\theta_{0k},\ L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}) = \mu_k\theta_{k0},$$
$$L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i) = \mu_k\theta_{ki},\ 1 \le k \ne i \le m,\ \boldsymbol{n} \in \mathbf{N}. \tag{2}$$

Here, $\boldsymbol{e}_k \in \mathbf{N}$ has the kth coordinate 1, and the others are 0.

Consider the graph $\Gamma(\boldsymbol{n})$ with a set of vertices $V(\boldsymbol{n}) = \{\boldsymbol{n},\ \boldsymbol{n}+\boldsymbol{e}_k,\ k = \overline{1,m}\}$ and the set of edges $\{(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k),\ (\boldsymbol{n}+\boldsymbol{e}_i, \boldsymbol{n}+\boldsymbol{e}_j),\ i \ne j,\ 1 \le k, i, j \le m\},\ \boldsymbol{n} \in \mathbf{N}$, connecting them (the graphs $\Gamma(\boldsymbol{n}_1)$ and $\Gamma(\boldsymbol{n}_2)$, where $\boldsymbol{n}_1 \ne \boldsymbol{n}_2$, have no common edges).

A vertex $\boldsymbol{n}$ defines the basic graph $\Gamma(\boldsymbol{n})$, and the set of all such vertices $\mathbf{N}_0$ is the base set. Note that $\mathbf{N}_0 = \mathbf{N}$ in the case of an open network G.

For some $\mathbf{N}_* \subseteq \mathbf{N}_0$, let

$$\Gamma(\mathbf{N}_*) = \bigcup_{\boldsymbol{n} \in \mathbf{N}_*} \Gamma(\boldsymbol{n}),\ V(\mathbf{N}_*) = \bigcup_{\boldsymbol{n} \in \mathbf{N}_*} V(\boldsymbol{n}).$$

Let us describe the network G with prohibitions by mapping the graph $\Gamma(\mathbf{N}_*)$ to it (let us denote it as G_*). A missing edge $(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k)$ in the graph $\Gamma(\mathbf{N}_*)$ means that node k does not receive customers from the outside and does not leave the graph after service. A missing edge $(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i)$ in the graph $\Gamma(\mathbf{N}_*)$ means that after receiving service at node k, customers do not arrive at node i, where $k \ne i$. Such permissions and prohibitions on transitions between network nodes determine the protocol of the network's operation.

Theorem 1. *If* $\rho_i = \dfrac{\lambda_i}{\mu_i} < 1,\ i = 1, \ldots, m$, $V(\mathbf{N}_*) = \mathbf{N}$ *and a graph* $\Gamma(\mathbf{N}_*)$ *is connected, then the Markov process* $y(t)$*, describing the functioning of the network* G_**, is ergodic, and its stationary distribution* $\pi(\boldsymbol{n})$*, where* $\boldsymbol{n} \in \mathbf{N}$*, can be calculated using the formula*

$$\pi(\boldsymbol{n}) = C^{-1} \prod_{i=1}^{m} \rho_i^{n_i},\ C = \sum_{\boldsymbol{n} \in N} \prod_{i=1}^{m} \rho_i^{n_i},\ \boldsymbol{n} \in \mathbf{N}. \tag{3}$$

Proof. The proof of Formula (3) is based on the ergodicity theorem formulated in [15,16] for a discrete Markov process. Let us check its sufficient conditions.
(1) To make sure that the allocation (3) satisfies the Kolmogorov–Chapman equations

$$\sum_{\boldsymbol{n}' \in V(\mathbf{N}_0)} \pi(\boldsymbol{n}) L(\boldsymbol{n}, \boldsymbol{n}') = \sum_{\boldsymbol{n}' \in V(\mathbf{N}_0)} \pi(\boldsymbol{n}') L(\boldsymbol{n}', \boldsymbol{n}),\ \boldsymbol{n} \in \mathbf{N}, \tag{4}$$

it is sufficient to make sure that it satisfies, for $\boldsymbol{n}_0 \in \mathbf{N}_0$, the equality

$$\sum_{\mathbf{n}' \in V(\boldsymbol{n}_0)} \pi(\boldsymbol{n}) L(\boldsymbol{n}, \boldsymbol{n}') = \sum_{\boldsymbol{n}' \in V(\boldsymbol{n}_0)} \pi(\boldsymbol{n}') L(\boldsymbol{n}', \boldsymbol{n}),\ \boldsymbol{n} \in \mathbf{V}(\boldsymbol{n}_0). \tag{5}$$

Substituting the distribution (3) into Formula (5) leads to balance ratios (1), with $C < \infty$.
(2) The states of the process $y(t)$ are communicating, which follows from the assumption of connectivity of the graph $\Gamma(\mathbf{N}_*)$.
(3) The regularity condition is also fulfilled, since, for $\forall \boldsymbol{n} \in \mathbf{N}$, the inequality $\sum_{\boldsymbol{n}' \in \mathbf{N}} L(\boldsymbol{n}, \boldsymbol{n}') < \infty$ is true. Theorem 1 has been proved. □

Remark 1. *The stationary distribution (3) of the Markov process* $y(t)$*, describing the functioning of the network* G*, remained unchanged for the network* G_**.*

Remark 2. *A similar theorem can be formulated and proved for the case of multi-server nodes.*

Remark 3. *Theorem 1 may be extended to the case of a finite set of states* $\mathbf{N}$ *of a Markov process* $y(t)$ *without restrictions* $\rho_i < 1,\ i = 1, \ldots, m$, *where* $\mathbf{N}_* \subseteq \mathbf{N}_0 \subseteq \mathbf{N}$, *with* $\mathbf{N}_0$ *being a set of base vertices that also satisfies the conditions that* $V(\mathbf{N}_0) = \mathbf{N}$ *and the graph* $\Gamma(\mathbf{N}_0)$ *is connected.*

Our task is to construct a subset $\mathbf{N}_*$ of the set of base vertices $\mathbf{N}_0$ so that, on the one hand, the stationary distribution of the Markov process is preserved, and on the other hand, the ratio of the number of vertices of the set $\mathbf{N}_*$. The number of vertices of the set $\mathbf{N}_0$ tended toward $1/2$ with an increase in the number of customers on the network. To maintain a stationary distribution in an open network, it is sufficient to require the equality of $V(\mathbf{N}_*) = \mathbf{N}$ and the connectivity of the graph $\Gamma(\mathbf{N}_*)$. The solution of this graph-theoretic problem is first achieved at $m = 2$ by a heuristic method using graph drawing and extends to the general case of $m > 2$. Then, the minimality of the constructed set $\mathbf{N}_*$ is proved analytically.

3. Results

The main results in this paper are devoted to the construction of a reduced set of base vertices for graphs of transient intensities in the cases of open and closed networks. However, we found it convenient to study the case of a closed network first by studying an open network with a limited number of customers; only then did we proceed to the case of a closed network.

3.1. Open Networks

Consider an open network, the functioning of which is described by a discrete Markov process $y(t)$ with a set of states $\mathbf{N} = \{\boldsymbol{n} = (n_1, \ldots, n_m) :\ n_1, \ldots, n_m \geq 0\}$ and transient intensities (2). Let us build a network with bans. Let us introduce a subset $\mathbf{N}_*$ of the set of base vertices $\mathbf{N}_0 = \mathbf{N}$:

$$\mathbf{N}_* = \bigcup_{k=0}^{\infty} \left(\mathbf{N}_*(2k) \bigcup (2k+1)\boldsymbol{e}_m \right) \subset \mathbf{N}_0, \tag{6}$$

$$\mathbf{N}_*(k) = \{\boldsymbol{n} \in \mathbf{N} :\ n_1, \ldots, n_{m-1} \geq 0,\ n_m = k\}.$$

As an example (for the case of $m = 2$), the graphs $\Gamma(\mathbf{N})$ are constructed in Figure 1 and $\Gamma(\mathbf{N}_*)$; bold dots highlight the set of base vertices $\mathbf{N}_0$, as well as its subset $\mathbf{N}_*$.

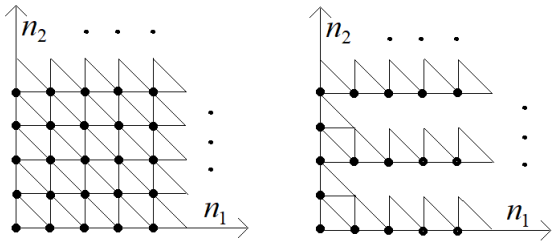


Figure 1. Graph $\Gamma(\mathbf{N}_0)$ (**left**) and graph $\Gamma(\mathbf{N}_*)$ (**right**). Sets $\mathbf{N}_0$ (**left**) and $\mathbf{N}_*$ (**right**) are highlighted with bold dots.

Theorem 2. *Consider an open network, described by transient intensities (2). The equality* $V(\mathbf{N}_*) = \mathbf{N}$ *holds, and the graph* $\Gamma(\mathbf{N}_*)$ *is connected. Removing a vertex from the set* $\mathbf{N}_*$ *violates either the equality* $V(\mathbf{N}_*) = \mathbf{N}$ *or the connectivity of the graph* $\Gamma(\mathbf{N}_*)$.

Proof. It is obvious that

$$V(\mathbf{N}_*) = \bigcup_{k=0}^{\infty} (\mathbf{N}_*(2k) \cup \mathbf{N}_*(2k+1)) = \mathbf{N}.$$

Since the graph $\Gamma(\mathbf{N}_*(2k))$ is connected and the edge $((2k+1)\boldsymbol{e}_m, (2k+2)\boldsymbol{e}_m)$ connects graphs $\Gamma(\mathbf{N}_*(2k))$ and $\Gamma(\mathbf{N}_*(2k+2))$ for $k = 0.1, \ldots$, it follows that graph $\Gamma(\mathbf{N}_*)$ is also connected.

Let us check the fulfillment of the second statement of the theorem. First, let us consider the case of $m = 2$. The following options are possible:

1. If $\boldsymbol{n} = (0, 2k)$ and $k > 0$, then the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$ does not contain a path between vertices $(0, 2k)$ and $(0, 2k+1)$ and, therefore, is disconnected. If $k = 0$, then $\boldsymbol{n} \notin V(\mathbf{N}_* \backslash \boldsymbol{n})$, and therefore, $V(\mathbf{N}_* \backslash \boldsymbol{n}) \neq V(\mathbf{N}_*)$.
2. If $\boldsymbol{n} = (0, 2k+1)$ and $k \geq 0$, then in the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$, there is no path between the vertices $(0, 2k+1)$ and $(0, 2k+2)$; therefore, the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$ is disconnected.
3. If $\boldsymbol{n} = (n_1, 2k)$, $k \geq 0$ and $n_1 > 1$, then $(n_1, 2k+1) \notin V(\mathbf{N}_* \backslash \boldsymbol{n})$, and therefore, $V(\mathbf{N}_* \backslash \boldsymbol{n}) \neq V(\mathbf{N}_*)$. If $n_1 = 1$, then the vertex $\boldsymbol{n} = (1, 2k)$ is not connected to the vertex $(2, 2k)$ in the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$, and therefore, the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$ is disconnected.

Now, let us move on to the case of $m > 2$. Checking conditions **1** and **2** when replacing $(0, 2k)$ with $(0, \ldots, 0, 2k)$ and $(0, 2k+1)$ with $(0, \ldots, 0, 2k+1)$ is similar. When checking condition **3**, we assume $\boldsymbol{n} = (n_1, \ldots, n_{m-1}, 2k)$. If $n_1 + \ldots + n_{m-1} > 1$, then it is not difficult to establish that $(n_1, \ldots, n_{m-1}, 2k+1) \notin V(\mathbf{N}_* \backslash \boldsymbol{n})$. If $n_1 + \ldots + n_{m-1} = 1$, then there is no connection between the vertices $\boldsymbol{e}_i + 2k\boldsymbol{e}_m$ and $2\boldsymbol{e}_i + 2k\mathbf{e}_m$ in the graph $\Gamma(\mathbf{N}_* \backslash \boldsymbol{n})$. Theorem 2 has been proved. □

Define the set $\Pi(2k) = \{\boldsymbol{n} \in \mathbf{N} : \ 0 \leq n_1, \ldots, n_m \leq 2k\}$, and denote as $|D|$ the number of elements of the set D.

Theorem 3. *The ratio is fair:*

$$\frac{|\mathbf{N}_* \bigcap \Pi(2k)|}{|\mathbf{N}_0 \bigcap \Pi(2k)|} \to \frac{1}{2}, \ k \to \infty. \tag{7}$$

Proof. The number of vertices $|\mathbf{N}_0 \bigcap \Pi(2k)| = (2k+1)^m$. In turn, the number of vertices $|\mathbf{N}_* \bigcap \Pi(2k)| = (2k+1)^{m-1}(k+1) + k$. Hence follows Formula (7). Theorem 3 has been proved. □

3.2. Open Networks with a Limited Total Number of Customers

Consider an open network, the functioning of which is described by a discrete Markov process $y(t)$ with a finite set of states

$$\mathbf{N} = \{\boldsymbol{n} = (n_1, \ldots, n_m) : \ n_1, \ldots, n_m \geq 0, \ \sum_{k=1}^{m} n_k \leq 2K+1\}$$

and transient intensities (2). Let us build a network with bans by defining a set of base vertices $\mathbf{N}_0 = \{\boldsymbol{n} \in \mathbf{N} : \ \sum_{k=1}^{m} n_k \leq 2K\}$. The set $\mathbf{N}_0$ is maximal in the sense that adding the vertex $\boldsymbol{n} \notin \mathbf{N}_0$ results in the ratio $V(\mathbf{N}_0 \bigcup \boldsymbol{n}) \neq \mathbf{N}$. We introduce a subset of the set of base vertices:

$$\mathbf{N}_* = \bigcup_{k=0}^{K} \mathbf{N}_*(2k) \bigcup \{(2k+1)\boldsymbol{e}_m, \ k = 0, \ldots, K-1\} \in \mathbf{N}_0,$$

$$\mathbf{N}_*(k) = \{\boldsymbol{n} \in \mathbf{N} : \ \sum_{k=1}^{m} n_k \leq 2K - k, \ n_m = k\}.$$

As an example (for the case of $m = 2$), the graphs $\Gamma(\mathbf{N})$ are constructed in Figure 2 and $\Gamma(\mathbf{N}_*)$; bold dots highlight the set of base vertices $\mathbf{N}_0$, as well as its subset $\mathbf{N}_*$.

Theorem 4. *The equality $V(\mathbf{N}_*) = \mathbf{N}$ holds, and the graph $\Gamma(\mathbf{N}_*)$ is connected. Removing a vertex from the set $\mathbf{N}_*$ violates either the equality $V(\mathbf{N}_*) = \mathbf{N}$ or the connectivity of the graph $\Gamma(\mathbf{N}_*)$.*

Proof. The proof of Theorem 4 repeats almost verbatim the proof of Theorem 2. □

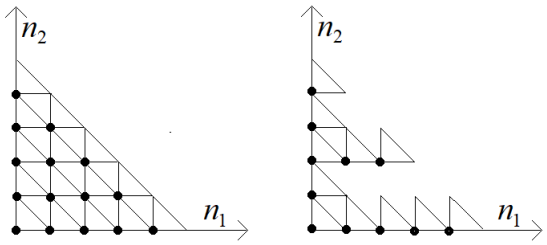


Figure 2. Graph $\Gamma(\mathbf{N})$ (**left**) and graph $\Gamma(\mathbf{N}_*)$ (**right**), Sets $\mathbf{N}_0$ (**left**) and $\mathbf{N}_*$ (**right**) are highlighted with bold dots.

Theorem 5. *The ratio is fair:*

$$\frac{|\mathbf{N}_*|}{|\mathbf{N_0}|} \to \frac{1}{2}, \; K \to \infty. \tag{8}$$

Proof. It follows from [17] that the number of solutions of the Diophantine equation $n_1 + \ldots + n_m = n$ that are non-negative integers is C^n_{n+m-1}. Then,

$$|\mathbf{N}_*(k)| = \sum_{i=0}^{2K-k} C^{m-2}_{i+m-2} = C^{m-1}_{2K-k+m-1}, \; |\mathbf{N}_*| = \sum_{k=0}^{K} |\mathbf{N}_*(2k)| + K. \tag{9}$$

From Formula (9), we obtain the ratio $1 = \mathbf{N}_*(2K) < \mathbf{N}_*(2K-1) < \ldots < \mathbf{N}_*(0)$. Therefore, there are inequalities

$$\sum_{k=0}^{K} |\mathbf{N}_*(2k)| > \sum_{k=1}^{K} |\mathbf{N}_*(2k-1)| > \sum_{k=0}^{K} |\mathbf{N}_*(2k)| - |\mathbf{N}_*(0)|. \tag{10}$$

From Formula (10), we have

$$2\sum_{k=0}^{K} |\mathbf{N}_*(2k)| - |\mathbf{N}_*(0)| < |\mathbf{N}_0| = \sum_{k=0}^{K} |\mathbf{N}_*(2k)| + \sum_{k=1}^{K} |\mathbf{N}_*(2k-1)| < 2\sum_{k=0}^{K} |\mathbf{N}_*(2k)|,$$

$$\frac{|\mathbf{N}_0|}{2} < \sum_{k=0}^{K} |\mathbf{N}_*(2k)| < \frac{|\mathbf{N}_0| + |\mathbf{N}_*(0)|}{2},$$

$$\frac{|\mathbf{N}_0|}{2} + K < |\mathbf{N}_*| < \frac{|\mathbf{N}_0| + |\mathbf{N}_*(0)|}{2} + K. \tag{11}$$

In turn, for $K \to \infty$, the asymptotic estimates are valid:

$$|\mathbf{N}_0| = \sum_{i=0}^{2K} C^{m-1}_{i+m-1} = C^m_{2K+m} \sim \frac{(2K)^m}{m!}, \; |\mathbf{N}_*(0)| = C^{m-1}_{2K+m-1} \sim \frac{(2K)^{m-1}}{(m-1)!},$$

$$\frac{K}{|\mathbf{N}_0|} \to 0, \; \frac{|\mathbf{N}_*(0)|}{|\mathbf{N}_0|} \to 0.$$

Thus, the relation (8) is fulfilled from the inequality (11). Theorem 5 is proved. □

3.3. Closed Network

Consider a closed network (Gordon–Newell network) G, in which a constant number $(2K+1,)$ of customers circulate among $m > 2$ servers. The movement of customers in the network is described by the route matrix $\Theta = ||\theta_{ij}||_{i,j=1}^{m}$, which is assumed to be indecomposable. Then, for any $B > 0$, the solution $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of the system's balance ratio $\Lambda = \Lambda\Theta$ with condition $B = \sum_{i=1}^{m} \lambda_i > 0$ exists and is unique.

The functioning of the network (the numbers of customers in nodes) is described by a discrete Markov process $y(t)$ with a finite set of states

$$\mathbf{N} = \{\boldsymbol{n} = (n_1, \ldots, n_m) : \ n_1, \ldots, n_m \geq 0, \ \sum_{k=1}^{m} n_k = 2K+1\}$$

and transient intensities:

$$L(\boldsymbol{n} + \boldsymbol{e}_i, \boldsymbol{n} + \boldsymbol{e}_k) = \mu_i \theta_{ik}, \ 1 \leq i \neq k \leq m, \ \boldsymbol{n} + \boldsymbol{e}_i, \boldsymbol{n} + \boldsymbol{e}_k \in \mathbf{N}. \tag{12}$$

Let us define the set of all base vertices $\mathbf{N}_0$ with the equality

$$\mathbf{N}_0 = \{\boldsymbol{n} \in \mathbf{N} : \ n_m > 0\}.$$

The equalities (12) are rewritten for $\boldsymbol{n} \in \mathbf{N}_0$:

$$L(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_m + \boldsymbol{e}_i) = \mu_m \theta_{mi}, \ L(\boldsymbol{n} - \boldsymbol{e}_m + \boldsymbol{e}_i, \boldsymbol{n} - \boldsymbol{e}_m + \boldsymbol{e}_k) = \mu_i \theta_{ik}, \ 1 \leq i \neq k < m. \tag{13}$$

Using the equalities (13), we define for $\boldsymbol{n} \in \mathbf{N}_0$ the set

$$V(\boldsymbol{n}) = \{\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_m + \boldsymbol{e}_i, \ 1 \leq i < m\}.$$

Consider the graph $\Gamma(\boldsymbol{n})$, where $\boldsymbol{n} \in \mathbf{N}_0$, with a set of vertices $V(\boldsymbol{n})$ and a set of edges connecting them (the graphs $\Gamma(\boldsymbol{n}_1)$ and $\Gamma(\boldsymbol{n}_2)$, where $\boldsymbol{n}_1 \neq \boldsymbol{n}_2$, do not have common edges). It is obvious that $V(\mathbf{N}_0) = \mathbf{N}$.

Let $\mathbf{N}_* \subset \mathbf{N}_0$. As in the case of an open network, we can describe a closed network G with prohibitions by mapping the graph $\Gamma(\mathbf{N}_*)$, which we denote as G_*.

Theorem 6. *If $V(\mathbf{N}_*) = \mathbf{N}$ and the graph $\Gamma(\mathbf{N}_*)$ is connected, then the Markov process $y(t)$, describing the functioning of the network G_*, is ergodic, and its stationary distribution $\pi(\boldsymbol{n})$, where $\boldsymbol{n} \in \mathbf{N}$, can be calculated using the formula*

$$\pi(\boldsymbol{n}) = C^{-1} \prod_{i=1}^{m} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}, \ C = \sum_{\boldsymbol{n} \in \mathbf{N}} \prod_{i=1}^{m} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}, \ 1 \leq i \leq m.$$

Proof. The proof of this theorem is similar to the proof of Theorem 1. □

Let us construct $\mathbf{N}_* \subset \mathbf{N}_0$ so that the stationary distribution of the Markov process is preserved and the ratio $\frac{|\mathbf{N}_*|}{|\mathbf{N}_0|} \to \frac{1}{2}$ for $K \to \infty$. Due to Theorem 6, in order to preserve a stationary distribution in a closed network, it is sufficient to require the equality $V(\mathbf{N}_*) = \mathbf{N}$ and the connectivity of the graph $\Gamma(\mathbf{N}_*)$.

To construct the set of base vertices $\mathbf{N}_*$, we compare each point of the set $\boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbf{N}$ to the point $\boldsymbol{n}' = (n_1, \ldots, n_{m-1}, 0)$, and the set of such points is de-

noted by $\mathbf{N}' = \{\boldsymbol{n}' : \sum_{k=1}^{m-1} n_k \leq 2K+1\}$. Then, to the set $\mathbf{N}_0$, we match the set $\mathbf{N}'_0 = \{\boldsymbol{n}' \in \mathbf{N}' : \sum_{k=1}^{m-1} n_k \leq 2K\}$. Let us now define

$$\mathbf{N}'_* = \bigcup_{k=0}^{K} \mathbf{N}'_*(2k) \bigcup \{(2k+1)\boldsymbol{e}_{m-1},\ k = 0, \ldots, K-1\},$$

$$\mathbf{N}'_*(k) = \{\boldsymbol{n}' \in \mathbf{N}' : \sum_{k=1}^{m-1} n_k \leq 2K-k,\ n_{m-1} = k\}.$$

Then, we construct the set $\mathbf{N}_* = \{\boldsymbol{n} \in \mathbf{N}_0 :\ \boldsymbol{n}' \in \mathbf{N}'_*\}$.

Theorem 7. *The equality $V(\mathbf{N}_*) = \mathbf{N}$ holds, and the graph $\Gamma(\mathbf{N}_*)$ is connected. Removing a vertex from the set $\mathbf{N}_*$ violates either the equality $V(\mathbf{N}_*) = \mathbf{N}$ or the connectivity of the graph $\Gamma(\mathbf{N}_*)$.*

Proof. By construction, there is a one-to-one correspondence between the sets $\mathbf{N}'_*$ and $\mathbf{N}_*$, as well as between the set $\mathbf{N}'$ and the set $\mathbf{N}$. Therefore, the graphs $\Gamma(\mathbf{N}'_*)$ and the graph $\Gamma(\mathbf{N}_*)$ are isomorphic. In turn, Theorem 4 implies that $V(\mathbf{N}'_*) = \mathbf{N}'$ and that the graph $\Gamma(\mathbf{N}'_*)$ is connected, and removing a vertex from the set $\mathbf{N}'_*$ violates at least one of these statements. The statement of the theorem is a consequence of the listed facts. Theorem 7 has been proved. □

Theorem 8. *The ratio is fair:*

$$\frac{|\mathbf{N}_*|}{|\mathbf{N}_0|} \to \frac{1}{2},\ K \to \infty. \tag{14}$$

Proof. Since

$$|\mathbf{N}'_*| = |\mathbf{N}_*|,\ |\mathbf{N}'_0| = |\mathbf{N}_0|,$$

the statement of Theorem 5 implies the statement of Theorem 8. □

4. Discussion

The most difficult question in this problem is the question of which graphs of transient intensities may be extended to the constructions given in the paper and the corresponding theorems. Apparently, a relatively simple answer to this question may be obtained by combining already-constructed graphs, especially for the case of a limited number of applications on the network. There may also be cases when the set $\mathbf{N}_0$ consists of integer rectangles, although this requires additional conditions.

5. Conclusions

In all the queuing models considered, the set of base vertices $\mathbf{N}_0$ is based on the number of vertices and, therefore, on the number of edges in the graph of transient intensities, there is a subset of base vertices containing about half the total, $\mathbf{N}_* \subset \mathbf{N}_0$. It was possible to build exponential queuing systems with prohibitions, even while maintaining a stationary distribution, by switching to a graph of transient intensities of a special type. This work began with a special case, namely, the case of $m = 2$ for an open network or $m = 3$ for a closed one.The proposed variants of the set $\mathbf{N}_*$ are not the only ones. We plan to continue our work in this direction, summarizing the results obtained and extending them to queuing systems, the functioning of which is described by Markov processes.

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