

Article

On the Generalized Inverse Gaussian Volatility in the Continuous Ho–Lee Model

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Abstract: This paper presents a new model of the term structure of interest rates that is based on the continuous Ho–Lee one. In this model, we suggest that the drift and volatility coefficients depend additionally on a generalized inverse Gaussian (GIG) distribution. Analytical expressions for the bond price and its moments are found in the new GIG continuous Ho–Lee model. Also, we compute in this model the prices of European call and put options written on bond. The obtained formulas are determined by the values of the Humbert confluent hypergeometric function of two variables. A numerical experiment shows that the third and fourth moments of the bond prices differentiate substantially in the continuous Ho–Lee and GIG continuous Ho–Lee models.

Keywords: continuous Ho–Lee model; GIG distribution; bond price; European option; Humbert hypergeometric function

1. Introduction

We consider an extension of the continuous Ho–Lee model of the interest rate dynamic, assuming that the drift and volatility depend on a GIG distribution that is independent of the Wiener process. As the continuous-time version of the discrete model (Ho and Lee [1]), the continuous Ho–Lee model was discussed for constant volatility by Hull and White [2] and Heath et al. [3], and the deterministic model was discussed by Subrahmanyam [4]. This model is Gaussian, and analytical expressions for bond, European call and put option, and option on bond prices can be found, for example, in Subsections 11.1–11.3 of Musiela and Rutkowski [5]. For the computation of exotic derivatives in the continuous Ho–Lee model, see Sections 12 and 13 of Musiela and Rutkowski [5].

The class of GIG distributions was primarily introduced by Halphen [6]. Later, it was discussed for the modeling of characteristics of dunes by Barndorff-Nielsen [7], where the GIG distributions were used as the mixing laws in the normal mean–variance mixtures. These mixtures were called the generalized hyperbolic (GH) distributions. The members of the family of GIG laws are distinguished by the shape parameter λ . The most popular GIG distributions in financial applications are the hyperbolic-inverse Gaussian (HIG) with $\lambda = 1$ and inverse Gaussian (IG) with $\lambda = -0.5$. As the mixing density, they set up the hyperbolic and normal-inverse Gaussian (NIG) distributions, respectively.

First, the hyperbolic distribution was considered as a model for the movements of financial assets by Eberlein and Keller [8], who estimated its parameters for the stocks of the main German companies and banks. The Deutsche Bank stock prices were studied with the IG distribution in Barndorff-Nielsen [9]. Küchler et al. [10] analyzed, with the HIG law, the empirical data of the 30 German stocks included in the DAX index. Together with the DAX, Bauer [11] showed that the hyperbolic distribution fits well the dynamics of the



Academic Editor: Zudi Lu

Received: 25 February 2025

Revised: 14 April 2025

Accepted: 17 April 2025

Published: 19 April 2025

Citation: Ivanov, R.V. On the Generalized Inverse Gaussian Volatility in the Continuous Ho–Lee Model. *Computation* **2025**, *13*, 100. <https://doi.org/10.3390/computation13040100>

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Dow Jones and Nikkei indices. The S&P500 index was modeled by the NIG distribution in Eriksson et al. [12]. Luciano et al. [13] showed that the hyperbolic law provides a better performance than the NIG one on long–short portfolios of the US stocks.

There are a lot of research papers in which the financial markets were tested using the whole family of GIG distributions amid the modern literature. Fajardo and Farias [14] revealed that the DAX and S&P500 indices have the harmonic ($\lambda = 0$) volatilities, but the Ibovespa, CAC, FTSE, and Nikkei ones are better described by the GH distributions with $\lambda \notin \{1, -0.5, 0\}$. Daskalaki and Katris [15] calibrated, with the GH distributions, the eight stock indices of the Eurozone and found, particularly, that the HIG and harmonic volatilities conform to the distributions of the CSE and SAX indices, respectively. Rathgeber et al. [16,17] considered different estimations of the parameters methods for the GH distributions. Comparing the original parameters from the time series with the simulated ones, Rathgeber et al. [17] obtained that the multivariate GH model is the best model for the MSCI US equity data set. Nakakita and Nakatsuma [18] modeled, with the GIG law, the volatility of the Japanese TOPIX index. The volatility of the US stock returns was simulated with the GIG distribution by Nakajima [19]. Chan et al. [20] and Sheraz and Dedu [21] examined, with the GH distribution, the returns of cryptocurrencies.

Next, it should be mentioned that the volatility of financial return is often modeled with the gamma or inverse gamma distribution. Abraham et al. [22] suggested a GARCH volatility model driven by the gamma law. Frühwirth-Schnatter and Sögner [23] simulated the stochastic volatility with the Ornstein–Uhlenbeck process generated by the gamma distribution. James et al. [24] calibrated, with the gamma distribution, the volatility of S&P500 index data. Nzokem [25] generalized the variance gamma model as a stochastic volatility model with the volatility dependent on the gamma process. Fung and Seneta [26] studied the properties of bivariate variance gamma and skew t distributions (which are the normal mean–variance mixtures with the gamma and inverse gamma mixing densities, respectively) and calibrated with them the NASDAQ index and Microsoft share price. Liu [27] fitted the DIS, IBM, and SPY data with a volatility measure determined by the gamma and inverse gamma laws. Nakajima and Omori [28] examined the volatility of S&P500 and TOPIX indices with the inverse gamma distribution. The volatility of the Australian stock index was studied with the inverse gamma law by Men et al. [29]. Various volatility prediction models based on the inverse gamma distribution were discussed in Takahashi et al. [30] for the US and Japanese financial indices. So far, as the gamma and inverse gamma distributions are different limiting cases for the class of GIG distributions (see, for example, Eberlein and von Hammerstein [31] or Ivanov [32]), the modeling of financial volatilities by the GIG law can enhance the forecast error when the gamma and inverse gamma models afford a close accuracy of prediction.

Being a Gaussian model, the continuous-time Ho–Lee model is a special case of the Heath–Jarrow–Morton (HJM) model (see Heath et al. [3] and Subsection 11.2.1 in Musiela and Rutkowski [5]). The problem of the calibration of the Gaussian HJM model to the real financial data has continued to be discussed in a number of recent studies. Using the Gaussian HJM framework, Maltsev and Pokojovy [33] forecasted the US Treasury rate movements. An extension of the HJM model with Gaussian random fields was considered in Tóth-Lakits and Arató [34]. The Euribor rate dynamic was tested under the HJM approach by Fontana et al. [35]. The power and natural gas markets were analyzed with the HJM model in Gardini and Santilli [36]. Furthermore, let us notice that the continuous Ho–Lee model is a particular case of the Hull–White model (see Hull and White [37]), which is also called the extended or generalized Vasicek one. Serafin et al. [38] calibrated, in the Hull–White framework, the USD currency exchange rates. Orlando and Bufalo [39] compared the Hull–White and CIR models for the data of overnight rates from various

countries. Van der Zwaard et al. [40] modeled the US and European market interest rates with the multivariate Hull–White model using the data of swaptions of different kinds.

Taking into account the popularity and importance of the GIG distribution in financial modeling, we aim in this paper to combine the continuous-time Ho–Lee model with the GIG-dependent mean and volatility. The suggested framework is similar to the model of Ivanov [41] for stock log-returns, where the mean and volatility were determined by the gamma and inverse gamma laws, which are the limiting cases of the GIG distribution (see, for example, Ivanov [32]). The method of the examination of the new model is often used for the analysis of normal mean–variance mixtures. Namely, it is the method of the application of special mathematical functions that was primarily proposed by Madan et al. [42] and then developed in Ano and Ivanov [43] for the variance-gamma distribution and Ivanov [44] for the hyperbolic one.

In this paper, we find formulas for the bond price, its moments, and European call and put bond option prices in the new GIG continuous Ho–Lee framework. The analytical expressions for the bond price moments can be used in the method of moments, which allows one to obtain the historical values of the parameters of the model. Such estimates could be applied by financial regulators to the objective of the supervision of banks and other financial institutions. Unlike this, the formulas for the option prices should be employed to find the implied parameters of the model. These assessments, based on the comparison of market and model values, permit us to make a prediction of the evolution of financial markets and, therefore, to accomplish the economic forecast. The information, which can be derived from the market option prices, may be leveraged to reduce the future financial risks and to administer correct prices for new financial instruments. Mathematically, the estimates of implied parameters can be made by the inversion of the option pricing model (see Section 5 in Madan et al. [42]) or the minimization of the relative errors between model and market prices (see Section 4.3 in Eberlein et al. [45]). More methods can be found in Orlando and Taglialatela [46].

The remainder of the paper is organized as follows. In Section 2, we recall the continuous Ho–Lee model, GIG distribution, and their basic properties. The main results of the work are introduced in Section 3. Section 4 numerically collates the new and Gaussian Ho–Lee interest rate models. Detailed proofs are placed in Appendices A and B.

2. Materials and Methods

2.1. Continuous-Time Ho–Lee Model

The continuous-time variant of the Ho–Lee model (see Subrahmanyam [4]) assumes that the interest rate dynamic follows the equation

$$dr_t = \alpha_t dt + \sigma_t dW_t, \quad r_0 \in \mathbb{R}, \quad (1)$$

where $(W_t)_{t \geq 0}$ is the Wiener process, adapted to the filtration on the probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, P)$, where $(\alpha_t)_{t \in [0, T]}$ and $(\sigma_t)_{t \in [0, T]}$ are deterministic processes and $\sigma_t > 0$. We imply that the usual condition

$$\int_0^T (|\alpha_t| + \sigma_t^2) dt < \infty,$$

which provides the existence of the integrals for (1), is satisfied.

If the coefficients are constants, that is, $\alpha_t \equiv \alpha$ and $\sigma_t \equiv \sigma$, then this model becomes the Merton model of the term structure of interest rates,

$$dr_t = \alpha dt + \sigma dW_t, \quad r_0 \in \mathbb{R}, \quad (2)$$

introduced in Merton [47]. See Table 1 for the connection between the continuous Ho–Lee and related models.

Table 1. Various term structure models in the framework of Equation (1). The coefficients μ_t and θ_t are time-dependent.

	Merton	Cont. Ho–Lee	Vasicek	Hull–White	Heath–Jarrow–Morton
α_t	α	determ.	$\mu - \theta r_t$	$\mu_t - \theta_t r_t$	stoch.
σ_t	σ	determ.	σ	determ.	stoch.

Equation (1) determines the evolution of the bank account $(R_t)_{t \geq 0}$, which is defined as

$$R_t = R_0 e^{\int_0^t r_s ds}, \quad R_0 > 0.$$

In accordance with the arbitrage-free property of the model (for details, see Chapter 9.5 in Musiela and Rutkowski [5] or Chapter VIII.4 in Shiryaev [48]), the bond dynamic satisfies the equality

$$B(t, T) = R_t E(R_T^{-1} | \mathcal{F}_t), \quad t \in [0, T].$$

That is, it is assumed that the discounted bond price is a martingale with respect to the initial probability measure P . Let us notice that the bond price has a simple form

$$B(t, T) = e^{\frac{\sigma^2(T-t)^3}{6} - \frac{\alpha(T-t)^2}{2} - r_t(T-t)} \quad (3)$$

in the Merton model—see Proposition 10.1.1 in Musiela and Rutkowski [5].

2.2. Generalized Inverse Gaussian Distributions

The GIG distribution $GIG = GIG(\lambda, \delta, \eta)$ has the probability density function (see Barndorff-Nielsen and Halgreen [49])

$$f_{GIG}(x) = \frac{\eta^\lambda \delta^{-\lambda}}{2K_\lambda(\delta\eta)} x^{\lambda-1} e^{-\frac{\delta^2}{2x} - \frac{\eta^2 x}{2}}, \quad x > 0, \quad (4)$$

where $K_{\zeta_1}(\zeta_2)$ is the modified Bessel function of the second kind. The restriction on the parameters in (4) is $\delta > 0$ and $\eta > 0$.

Set

$$C_{GIG} = \frac{\eta^\lambda \delta^{-\lambda}}{2K_\lambda(\delta\eta)}.$$

The characteristic function $\varphi_{GIG}(u)$ of the GIG distribution is calculated as

$$\varphi_{GIG}(u) = \int_0^\infty e^{iux} f_{GIG}(x) dx = C_{GIG} \int_0^\infty x^{\lambda-1} e^{-\frac{\delta^2}{2x} - \frac{(\eta^2 - 2iu)x}{2}} dx.$$

Formula 3.471.9 in Gradshteyn and Ryzhik [50] includes the identity

$$\int_0^\infty x^{\zeta_1-1} e^{-\frac{\zeta_2}{x} - \zeta_3 x} dx = 2 \left(\frac{\zeta_2}{\zeta_3} \right)^{\frac{\zeta_1}{2}} K_{\zeta_1}(2\sqrt{\zeta_2 \zeta_3}) \quad (5)$$

for $\zeta_2 = \text{Re } \zeta_2 + i\text{Im } \zeta_2$, $\zeta_3 = \text{Re } \zeta_3 + i\text{Im } \zeta_3$, such that $\text{Re } \zeta_2 > 0$ and $\text{Re } \zeta_3 > 0$. Leveraging (5), we see that

$$\varphi_{GIG}(u) = \left(\frac{\eta}{\sqrt{\eta^2 - 2iu}} \right)^\lambda \frac{K_\lambda(\delta\sqrt{\eta^2 - 2iu})}{K_\lambda(\delta\eta)}. \quad (6)$$

It immediately results from (6) that, if $u < \frac{\eta^2}{2}$, then the moment-generating function

$$\text{mgf}_{\text{GIG}}(u) = \left(\frac{\eta}{\sqrt{\eta^2 - 2u}} \right)^\lambda \frac{K_\lambda(\delta\sqrt{\eta^2 - 2u})}{K_\lambda(\delta\eta)}. \quad (7)$$

When $u \geq \frac{\eta^2}{2}$, then the moment-generating function of the GIG distribution does not exist. Keeping in mind (5), it is easy to find that the ς th moment of the GIG distribution for $\varsigma \geq 0$ is

$$M_{\varsigma, \text{GIG}} = E(\text{GIG})^\varsigma = \frac{\delta^\varsigma K_{\lambda+\varsigma}(\delta\eta)}{\eta^\varsigma K_\lambda(\delta\eta)}. \quad (8)$$

The basic properties and methods of the parameter estimation of the GIG distributions are given in Jørgensen [51]. Table 2 affords the connection between the three most prominent GIG distributions and the related generalized hyperbolic laws.

Table 2. The GIG and GH distributions with respect to the shape parameter λ .

	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 1$
GIG	inverse Gaussian	harmonic	hyperbolic-inverse Gaussian
GH	normal-inverse Gaussian	semi-hyperbolic	hyperbolic

The less known harmonic and semi-hyperbolic distributions were discussed, in particular, in Ivanov [32], Barndorff-Nielsen [52] and Blaesild [53].

3. Results

3.1. GIG Continuous-Time Ho–Lee Model

We introduce in this section an alternative to (1), i.e., the GIG continuous time Ho–Lee model of the term structure of interest rates. It is assumed in the new model that the drift and volatility coefficients of the Gaussian model (1) are stochastic, depending additionally on a GIG distribution. That is, the new model becomes conditionally Gaussian and the GIG continuous Ho–Lee interest rate dynamic satisfies the equation

$$dr_{\text{GIG},t} = (\alpha_t + \text{GIG}\theta_t)dt + \text{GIG}^{\frac{1}{2}}\sigma_t dW_t, \quad r_{\text{GIG},0} \in \mathbb{R}, \quad (9)$$

where $(W_t)_{t \geq 0}$ is the Wiener process, adapted to the filtration on the probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, and independent of the random variable GIG, which has a generalized inverse Gaussian distribution. It is supposed that GIG is $\mathcal{F}_0$ -measurable and the coefficients $(\alpha_t)_{t \in [0,T]}$, $(\theta_t)_{t \in [0,T]}$, $(\sigma_t)_{t \in [0,T]}$, $\sigma_t > 0$, are deterministic processes. We assume that the condition

$$\int_0^T (|\alpha_t| + |\theta_t| + \sigma_t^2)dt < \infty$$

holds for the coefficients. Because $E(\text{GIG}) < \infty$ for any set of the parameters with respect to (8), we have the integral formula

$$r_{\text{GIG},t} = r_{\text{GIG},0} + \int_0^t \alpha_s ds + \text{GIG} \int_0^t \theta_s ds + \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s dW_s \quad (10)$$

for the GIG interest rate in accordance with the theory in Chapter 4.3 in Shreve [54].

Analogously to (1) and (2), if $\alpha_t \equiv \alpha$, $\theta_t \equiv \theta$, and $\sigma_t \equiv \sigma$ are constant in (9), we call the model

$$dr_{\text{GIG},t} = (\alpha + \text{GIG}\theta)dt + \text{GIG}^{\frac{1}{2}}\sigma dW_t, \quad r_{\text{GIG},0} \in \mathbb{R}, \quad (11)$$

the GIG Merton model of the term structure of interest rates. Since the conditional variance

$$\mathbb{E}\left([r_{\text{GIG},t} - \mathbb{E}(r_{\text{GIG},t}|\text{GIG})]^2|\text{GIG}\right) = \sigma^2\text{GIG}$$

in the GIG Merton model, the new model is similar in this sense to the generalized hyperbolic one of the stock log-returns (see Eberlein [55]).

The evolution of the GIG bank account is suggested to follow the process

$$R_{\text{GIG},t} = R_{\text{GIG},0}e^{\int_0^t r_{\text{GIG},s}ds}, \quad R_{\text{GIG},0} > 0. \quad (12)$$

Similarly to (3), the bond dynamic fulfills the equation

$$B_{\text{GIG}}(t, T) = R_{\text{GIG},t}\mathbb{E}\left((R_{\text{GIG},T})^{-1}|\mathcal{F}_t\right), \quad t \in [0, T], \quad (13)$$

under the condition

$$(R_{\text{GIG},0})^{-1}B_{\text{GIG}}(0, T) = \mathbb{E}(R_{\text{GIG},T})^{-1} < \infty. \quad (14)$$

To take into account the both ordinary and GIG interest rates, one may define the compound interest rate $r_{\mathcal{C},t}$ as

$$r_{\mathcal{C},t} = r_t I_{\{\omega=0\}} + r_{\text{GIG},t} I_{\{\omega=1\}},$$

where ω is an independent $\mathcal{F}_0$ -measurable Bernoulli-distributed random variable. Therefore, the compound bond $B_{\mathcal{C}}(t, T)$ can be set by the equality

$$B_{\mathcal{C}}(t, T) = B(t, T)I_{\{\omega=0\}} + B_{\text{GIG}}(t, T)I_{\{\omega=1\}}.$$

So far, as we have for any function $\Xi: \mathbb{R}_+ \rightarrow \mathbb{R}$ that

$$\begin{aligned} \Xi(B_{\mathcal{C}}(t, T)) &= \Xi(B(t, T))(I_{\{\omega=0\}} + I_{\{\omega=1\}}) = \\ &= \Xi(B(t, T))I_{\{\omega=0\}} + \Xi(B_{\text{GIG}}(t, T))I_{\{\omega=1\}}, \end{aligned}$$

we consider only $B_{\text{GIG}}(t, T)$ instead of $B_{\mathcal{C}}(t, T)$ further in the work.

3.2. Bond Price and Its Moments

The aim of this subsection is to compute the GIG bond price that conforms to (13). Also, we find, in closed forms its moments.

Theorem 1. *The bond price $B_{\text{GIG}}(t, T)$ is calculated by the identity*

$$B_{\text{GIG}}(t, T) = e^{\text{GIG}\left(\frac{1}{2}\int_t^T \sigma_s^2(T-s)^2ds - \int_t^T (T-s)\theta_s ds\right) - r_{\text{GIG},t}(T-t) - \int_t^T (T-s)\alpha_s ds}. \quad (15)$$

For $v \in \mathbb{R}$, the moments

$$\begin{aligned} \mathbb{E}B_{\text{GIG}}^v(t, T) &= \infty I_{\left\{\rho(v,t) \geq \frac{\eta^2}{2}\right\}} + \\ &+ e^{-v\left((T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) + \int_t^T (T-s)\alpha_s ds\right)} \text{mgf}_{\text{GIG}}(\rho(v, t)) I_{\left\{\rho(v,t) < \frac{\eta^2}{2}\right\}} \end{aligned} \quad (16)$$

with

$$\rho(v, t) = v \left(\frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds - \int_t^T (T-s) \theta_s ds - (T-t) \int_0^t \theta_s ds + \frac{1}{2} v (T-t)^2 \int_0^t \sigma_s^2 ds \right).$$

In particular, the result in Theorem 1 includes the expressions for the variance, skewness, and kurtosis of the GIG bond price. A proof of Theorem 1 is given in Appendix A.

Next, let us specify the formulas of Theorem 1 for the case of constant α_t , θ_t , and σ_t . That is, set $\alpha_t \equiv \alpha$, $\theta_t \equiv \theta$, $\sigma_t \equiv \sigma$. Then, we have from (15) that

$$B_{\text{GIG}}(t, T) = e^{\text{GIG}\left(\frac{\sigma^2(T-t)^3}{6} - \frac{\theta(T-t)^2}{2}\right) - r_{\text{GIG},t}(T-t) - \frac{\alpha(T-t)^2}{2}} \quad (17)$$

and from (16) that

$$\begin{aligned} EB_{\text{GIG}}^v(t, T) = \\ = \infty I_{\left\{\rho(v,t) \geq \frac{\eta^2}{2}\right\}} + e^{-v(T-t)\left(r_{\text{GIG},0} + \frac{\alpha(T+t)}{2}\right)} \text{mgf}_{\text{GIG}}(\rho(v, t)) I_{\left\{\rho(v,t) < \frac{\eta^2}{2}\right\}} \end{aligned} \quad (18)$$

with

$$\begin{aligned} \rho(v, t) &= v \left(\frac{\sigma^2(T-t)^3}{6} - \frac{\theta(T-t)^2}{2} - \theta t(T-t) + \frac{v\sigma^2 t(T-t)^2}{2} \right) = \\ &= \frac{v(T-t)}{2} \left(\frac{\sigma^2(T-t)^2}{3} - \theta(T+t) + v\sigma^2 t(T-t) \right). \end{aligned}$$

To examine the new model, the moments of the GIG Ho–Lee bond price are compared with the moments of the Gaussian Ho–Lee one in Section 4.

3.3. Bond Options

Since the discounted bond price is proposed to be a martingale with respect to the initial probability measure $\mathbb{P}$, the prices $\mathbb{C}$ and $\mathbb{P}$ of the European call and put options written on bond are calculated with respect to (2.22)–(2.23) in Brigo and Mercurio [56] as

$$\mathbb{C} = \mathbb{E} \left[\frac{(B_{\text{GIG}}(t, T) - K)^+}{R_{\text{GIG},t}} \right] \quad \text{and} \quad \mathbb{P} = \mathbb{E} \left[\frac{(K - B_{\text{GIG}}(t, T))^+}{R_{\text{GIG},t}} \right], \quad (19)$$

where $K > 0$ is the strike price. To introduce formulas for the prices of the options, we need to enter an auxiliary function. Namely, let

$$\begin{aligned} \mathfrak{J}(\lambda, \delta, s, q) &= \frac{(1+q)^{\frac{1}{2}-\lambda} |s|^{-\lambda} e^s}{2^\lambda \delta^{1-2\lambda} \sqrt{\pi}} \left[\text{B} \left(\frac{1}{2} - \lambda, 1 \right) \left(|s| K_{1-\lambda}(|s|) + s K_\lambda(|s|) \right) \times \right. \\ &\times \Phi_1 \left(\frac{1}{2} - \lambda, \lambda + \frac{1}{2}, \frac{3}{2} - \lambda; \frac{1+q}{2}, -s(1+q) \right) - s(1+q) \text{B} \left(\frac{3}{2} - \lambda, 1 \right) \times \\ &\left. \times K_\lambda(|s|) \Phi_1 \left(\frac{3}{2} - \lambda, \lambda + \frac{1}{2}, \frac{5}{2} - \lambda; \frac{1+q}{2}, -s(1+q) \right) \right] \quad (20) \end{aligned}$$

for $s \neq 0$ and $q < 1$, where $\Phi_1 = \Phi_1(\zeta_1, \zeta_2, \zeta_3; \zeta_4, \zeta_5)$ is the Humbert confluent hypergeometric series of two variables, which can be represented for $|\zeta_4| < 1$ as a double sum (see (16) in Subsection 1.3 of Srivastava and Karlsson [57]). For more properties of this function, we refer to Humbert [58] and Chapters 5.7–5.14 in Bateman and Erdélyi [59].

Also, let us define time-dependent functions $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ as

$$\begin{aligned}
 c_1 = c_1(t) &= \frac{e^{-Tr_{GIG,0} - \log R_{GIG,0} - \int_0^t (T-s)\alpha_s ds}}{(T-t) \left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}} \sqrt{2\pi}}, \\
 c_2 = c_2(t) &= \frac{\int_0^t \sigma_s^2 (T-s)^2 ds}{2} \left(1 + \frac{(\int_0^t \sigma_s^2 (T-s) ds)^2}{\int_0^t \sigma_s^2 ds \int_0^t \sigma_s^2 (T-s)^2 ds} \times \right. \\
 &\quad \times \left. \left[\left(Tr_{GIG,0} + \log R_{GIG,0} + \int_0^t (T-s)\alpha_s ds \right)^2 - 1 \right] \right), \\
 c_3 = c_3(t) &= \frac{(T-t) \int_0^t \theta_s ds - \frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds + \int_t^T (T-s)\theta_s ds}{(T-t) \left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}}} + \\
 &\quad + \frac{\left(Tr_{GIG,0} + \log R_{GIG,0} + \int_0^t (T-s)\alpha_s ds \right) \left(\int_0^t \sigma_s^2 (T-s) ds \right)}{\left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}}}, \\
 c_4 = c_4(t) &= \frac{(T-t) \left(r_{GIG,0} + \int_0^t \alpha_s ds \right) + \int_t^T (T-s)\alpha_s ds - K}{(T-t) \left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}}}, \\
 c_5 = c_5(t) &= \frac{e^{-tr_{GIG,0} - \log R_{GIG,0} - \int_0^t (t-s)\alpha_s ds}}{(T-t) \left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}} \sqrt{2\pi}}, \\
 c_6 = c_6(t) &= \frac{\int_0^t \sigma_s^2 (t-s)^2 ds}{2} \left(1 + \frac{\int_0^t \sigma_s^2 (t-s) ds}{\left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \sigma_s^2 (t-s)^2 ds \right)^{\frac{1}{2}}} \times \right. \\
 &\quad \times \left. \left[\left(tr_{GIG,0} + \log R_{GIG,0} + \int_0^t (t-s)\alpha_s ds \right)^2 - 1 \right] \right), \\
 c_7 = c_7(t) &= \frac{(T-t) \int_0^t \theta_s ds - \frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds + \int_t^T (T-s)\theta_s ds}{(T-t) \left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}}} + \\
 &\quad + \frac{\left(tr_{GIG,0} + \log R_{GIG,0} + \int_0^t (t-s)\alpha_s ds \right) \left(\int_0^t \sigma_s^2 (t-s) ds \right)}{\left(\int_0^t \sigma_s^2 ds \right)^{\frac{1}{2}}}.
 \end{aligned}$$

The theorem below affords the formula for the European call on bond price in the GIG continuous Ho–Lee model.

Theorem 2. Assume that conditions

$$\frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds - \int_0^T (T-s)\theta_s ds = 0 \quad (21)$$

and

$$\int_0^t (t-s)\theta_s ds = 0 \quad (22)$$

are satisfied. Furthermore, if $c_2 = c_6 = \frac{\eta^2}{2}$ and $\lambda < \frac{1}{2}$, then

$$\begin{aligned} \mathbb{C} = & C_{\text{GIG}} \left\{ c_1 \mathcal{J} \left(\lambda, \delta, c_3 \sqrt{c_4^2 + \delta^2}, \frac{c_4}{\sqrt{c_4^2 + \delta^2}} \right) - \right. \\ & \left. - c_5 K \mathcal{J} \left(\lambda, \delta, c_7 \sqrt{c_4^2 + \delta^2}, \frac{c_4}{\sqrt{c_4^2 + \delta^2}} \right) \right\}. \end{aligned} \quad (23)$$

Proof Sketch of Theorem 2. The proof of Theorem 2 can be divided into four stages.

Stage 1. We notice that the option price can be found using the expectation $\mathcal{E}(X, Y, u) = \mathbb{E} e^Y I_{\{X > u\}}$, where X and Y are correlated Gaussian random variables, and compute $\mathcal{E}(X, Y, u)$.

Stage 2. The result of Stage 1 is applied to the conditionally Gaussian variables $\log(B_{\text{GIG}}(t, T))$, $\log((R_{\text{GIG},t})^{-1} B_{\text{GIG}}(t, T))$, and $-\log(R_{\text{GIG},t})$.

Stage 3. The option price is written as

$$\mathbb{C} = C_{\text{GIG}} [c_1 I_{\text{GIG}}(\lambda, \eta, \delta, c_2, c_3, c_4) - c_5 K I_{\text{GIG}}(\lambda, \eta, \delta, c_6, c_7, c_4)]$$

with computed functions $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, where $I_{\text{GIG}}(\lambda, \eta, \delta, v_1, v_2, v_3)$ is a special integral of the normal distribution function.

Stage 4. The integral $I_{\text{GIG}}(\lambda, \eta, \delta, v_1, v_2, v_3)$ is computed in closed form. $\square$

A complete proof of Theorem 2 is set in Appendix A.

The next corollary of Theorem 2 provides us with a call–put parity in the considered model.

Corollary 1. Let the conditions of Theorem 2 hold and

$$\int_0^t \sigma_s^2 (t-s)^2 ds < \eta^2.$$

Then,

$$\begin{aligned} \mathbb{P} = & \mathbb{C} + (R_{\text{GIG},0})^{-1} \left(K e^{-r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds} \text{mgf}_{\text{GIG}}(\hat{\rho}(t)) - \right. \\ & \left. - e^{-r_{\text{GIG},0}T - \int_0^T (T-s)\alpha_s ds} \text{mgf}_{\text{GIG}}(\hat{\rho}(T)) \right), \end{aligned} \quad (24)$$

where $\mathbb{C}$ is computed in (23) and

$$\hat{\rho}(t) = \frac{1}{2} \int_0^t \sigma_s^2 (t-s)^2 ds.$$

A proof of Corollary 1 is placed in Appendix B.

4. Numerical Analysis

Throughout this section, we compare the bond prices in the Merton model in (2) with the interest rate

$$r_t = r_0 + \alpha t + \sigma W_t$$

and in the GIG Merton model in (11) with the interest rate

$$r_{\text{GIG},t} = r_{\text{GIG},0} + \hat{\alpha}t + \theta \text{GIG}t + \sigma \text{GIG}^{\frac{1}{2}} W_t$$

assuming that $E(\text{GIG}) = 1$ and $\hat{\alpha} + \theta = \alpha$.

Let us scale the models with σ . Also, we set, for the simplicity of notation, $\alpha\sigma^{-1} = 1$ and suggest that $\hat{\alpha} = \theta$ obtains half of the GIG ordinary drift of the GIG interest rate. Then, we progress to the collation of the models with

$$r_t = r_0 + t + W_t$$

and

$$r_{\text{GIG},t} = r_{\text{GIG},0} + \frac{t}{2} + \frac{\text{GIG}t}{2} + \text{GIG}^{\frac{1}{2}}W_t.$$

As an example of the GIG distribution, we discuss the inverse Gaussian (IG) one, which has the shape parameter $\lambda = -\frac{1}{2}$.

Moreover, we provide in Section 4.4 a complex example that shows the case when the higher moments of the GIG bond price do not exist and, therefore, the new model significantly differs from the continuous Ho–Lee one.

4.1. Parameters

We want to select the parameters of the IG distribution so that the bond prices in the models have close values. To fit the condition

$$Er_t = Er_{\text{IG},t} = t,$$

we look for the parameters δ^* and η^* such that

$$E(\text{IG}(\delta^*, \eta^*)) = 1.$$

Since $K_{\zeta_1}(\zeta_2) = K_{-\zeta_1}(\zeta_2)$, we have from (8) that

$$\frac{\delta^* K_{\frac{1}{2}}(\delta^* \eta^*)}{\eta^* K_{-\frac{1}{2}}(\delta^* \eta^*)} = 1$$

and, therefore, we can set $\delta^* = \eta^* = 1$. Hence, we obtain from (3) that

$$B(t, T) = e^{\frac{1}{6}(T-t)^3 - \frac{1}{2}(T-t)^2 - r_t(T-t)}$$

and from (17) that

$$\begin{aligned} B_{\text{IG}}(t, T) &= e^{\frac{1}{2}\text{IG}\left(\int_t^T (T-s)^2 ds - \int_t^T (T-s) ds\right) - r_{\text{IG},t}(T-t) - \frac{1}{2}\int_t^T (T-s) ds} = \\ &= e^{\text{IG}\left(\frac{1}{6}(T-t)^3 - \frac{1}{4}(T-t)^2\right) - r_{\text{IG},t}(T-t) - \frac{1}{4}(T-t)^2}. \end{aligned}$$

Next, we collate the expectations

$$EB^v(t, 1) \quad \text{and} \quad EB_{\text{IG}}^v(t, 1)$$

assuming that $r_0 = r_{\text{IG},0} = 1$.

4.2. Time Interval

To find a time interval where $EB_{\text{IG}}^v(t, 1) < \infty$, let us discuss a function

$$v^{-1}\rho(v, t) = \frac{(1-t)^3}{6} - \frac{(1-t)^2}{4} - \frac{t(1-t)}{2} + \frac{vt(1-t)^2}{2}.$$

We have that

$$\begin{aligned} \left(v^{-1}\rho(v,t)\right)'_t &= \frac{1-t}{2} - \frac{(1-t)^2}{2} - \frac{1-t}{2} + \frac{t}{2} + \frac{v(1-t)^2}{2} - vt(1-t) = \\ &= \frac{t}{2} + (v-1)\left(\frac{1}{2} - t + \frac{t^2}{2}\right) - vt + vt^2 = \\ &= \frac{3v-1}{2}t^2 - \frac{4v-3}{2}t + \frac{v-1}{2}. \end{aligned}$$

Let $v = 1$. Then,

$$\rho'_t(1,t) = t(t-0.5) \leq 0 \quad \text{for } t \in [0,0.5].$$

Hence,

$$\max_{t \in [0,0.5]} \rho(1,t) = \rho(1,0) = -\frac{1}{6}$$

and $EB_{IG}(t,1) < \infty$ for $t \in [0,0.5]$.

If $v \geq 1$, then the equation $v^{-1}\rho(v,t) = 0$ has the roots

$$t_{12} = \frac{4v-3 \pm \sqrt{(4v-3)^2 - 4(3v-1)(v-1)}}{2(3v-1)} = \frac{4v-3 \pm \sqrt{4(v-1)^2 + 1}}{2(3v-1)}.$$

We have that

$$\begin{aligned} t_2 > t_1 &= \frac{4v-3 - \sqrt{(4v-3)^2 - 4(3v-1)(v-1)}}{2(3v-1)} = \\ &= \frac{4v-3 \pm \sqrt{4(v-1)^2 + 1}}{2(3v-1)} > \frac{4v-3 - (2(v-1) + 1)}{2(3v-1)} = \frac{v-1}{3v-1}. \end{aligned} \quad (25)$$

If $v = 2$, then $t_1 > 0.2$ and

$$\max_{t \in [0,0.2]} \rho(2,t) = \rho(2,0.2) \approx -0.05.$$

Therefore, $EB_{IG}^2(t,1) < \infty$ for $t \in [0,0.2]$.

When $v \in \{3,4\}$, we obtain from (25) that

$$t_1 > \frac{1}{3} - \frac{1}{3v} \geq \frac{2}{9} > 0.2.$$

It is easy to see that

$$\max_{t \in [0,0.2]} \rho(3,t) = \rho(3,0.2) \approx 0.112$$

and

$$\max_{t \in [0,0.2]} \rho(4,t) = \rho(4,0.2) \approx 0.41 < \frac{\eta_*^2}{2} = 0.5.$$

Therefore, $EB_{IG}^3(t,1) < \infty$ and $EB_{IG}^4(t,1) < \infty$ for $t \in [0,0.2]$.

4.3. Moments of the Bond Prices

We have for $v \in \{1, 2, 3, 4\}$ and $t \in [0, 0.2]$ that the IG bond price moments

$$EB_{IG}^v(t, 1) = e^{-v\left((1-t)\left(1+\frac{t}{2}\right)+\frac{(1-t)^2}{4}\right)} \text{mgf}_{IG}(\rho(v, t)) = e^{\frac{v(t^2+4t-5)}{4}} \times \\ \times \left(\frac{1}{\sqrt{1-2\rho(v, t)}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}\left(\sqrt{1-2\rho(v, t)}\right)}{K_{-\frac{1}{2}}(1)}.$$

In accordance with formula 8.469.3 in Gradshteyn and Ryzhik [50],

$$K_{\pm\frac{1}{2}}(\zeta) = \sqrt{\frac{\pi}{2\zeta}} e^{-\zeta}. \quad (26)$$

Employing (26), we have that

$$EB_{IG}^v(t, 1) = e^{\frac{v(t^2+4t-5)}{4} + 1 - \sqrt{1-2\rho(v, t)}}$$

with

$$\rho(v, t) = \frac{v(1-t)}{2} \left(\frac{2t^2 - 7t - 1}{6} + vt(1-t) \right).$$

Hence,

$$EB_{IG}^v(t, 1) = e^{\frac{v(t^2+4t-5)}{4} + 1 - \sqrt{1 + \frac{(1-t)((6v-2)t^2 + (7-6v)t + 1)}{6}}}. \quad (27)$$

Next, we have from (3) that

$$B^v(t, 1) = e^{\frac{v}{6}(1-t)^3 - \frac{v}{2}(1-t)^2 - vr_t(1-t)}$$

and, therefore,

$$EB^v(t, 1) = e^{\frac{v}{6}(1-t)^3 - \frac{v}{2}(1-t)^2} Ee^{-v(1-t)(t+1) - v(1-t)W_t} = \\ = e^{\frac{v}{6}(1-t)^3 - \frac{v}{2}(1-t)^2 - v(1-t)(t+1) + \frac{tv^2(1-t)^2}{2}} = e^{v(1-t) \left(\frac{(1-t)^2}{6} + \frac{(v-1)t - vt^2 - 3}{2} \right)} = \\ = e^{\frac{v(1-t)((1-3v)t^2 + (3v-5)t - 8)}{6}} \quad (28)$$

so far as r_t is Gaussian.

Using (27) and (28), we collate the moments of the bond prices. The expectations

$$EB(t, 1) = e^{\frac{(t-1)(t^2+t+4)}{3}} \quad \text{and} \quad EB_{IG}(t, 1) = e^{\frac{t^2+4t-1}{4} - \sqrt{1 + \frac{(1-t)(4t^2+t+1)}{6}}}$$

are plotted in Figure 1.

The second moments

$$EB^2(t, 1) = e^{\frac{(t-1)(5t^2-t+8)}{3}} \quad \text{and} \quad EB_{IG}^2(t, 1) = e^{\frac{t^2+4t-3}{2} - \sqrt{1 + \frac{(1-t)(10t^2-5t+1)}{6}}}$$

are given in Figure 2.

The third moments

$$EB^3(t, 1) = e^{2(t-1)(2t^2-t+2)} \quad \text{and} \quad EB_{IG}^3(t, 1) = e^{\frac{3t^2+12t-11}{4} - \sqrt{1 + \frac{(1-t)(16t^2-11t+1)}{6}}}$$

are provided in Figure 3.

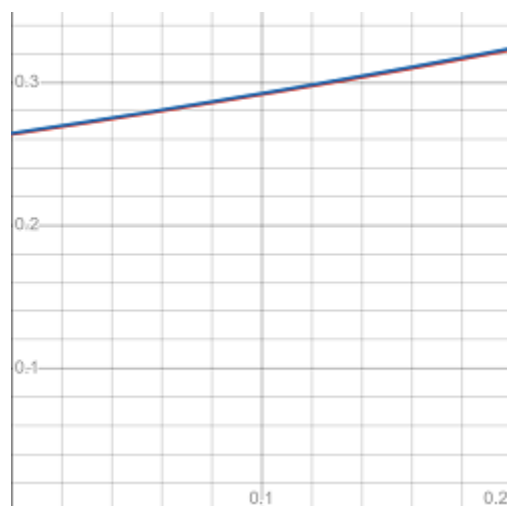


Figure 1. The expectations of continuous Ho-Lee (red) and GIG continuous Ho-Lee (blue) bonds.

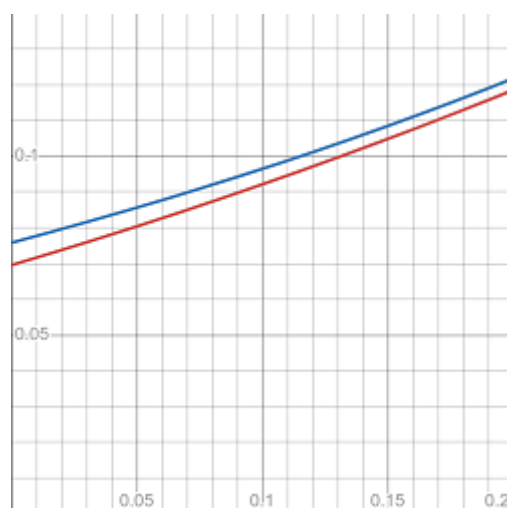


Figure 2. The second moments of continuous Ho-Lee (red) and GIG continuous Ho-Lee (blue) bonds.

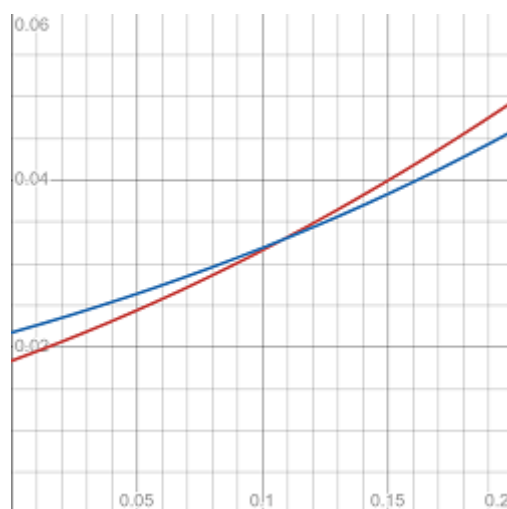


Figure 3. The third moments of continuous Ho-Lee (red) and GIG continuous Ho-Lee (blue) bonds.

The fourth moments

$$EB^4(t, 1) = e^{\frac{2(t-1)(11t^2-7t+8)}{3}} \text{ and } EB_{IG}^4(t, 1) = e^{t^2+4t-4-\sqrt{1+\frac{(1-t)(22t^2-17t+1)}{6}}}$$

are plotted in Figure 4.

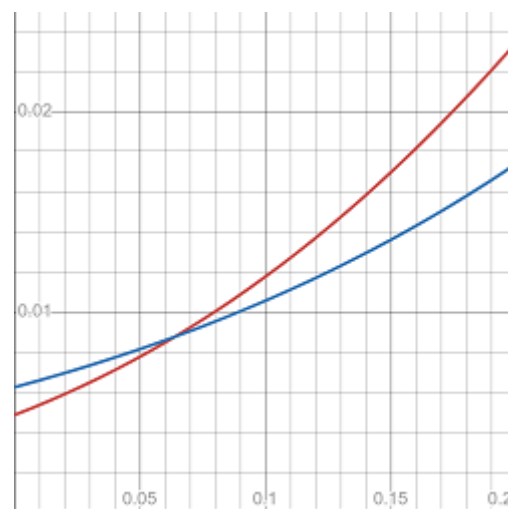


Figure 4. The fourth moments of continuous Ho-Lee (red) and GIG continuous Ho-Lee (blue) bonds.

Table 3 compares the moments of bond prices in the metric

$$m(v) = \frac{|EB^v(t, 1) - EB_{IG}^v(t, 1)|}{\max\{EB^v(t, 1), EB_{IG}^v(t, 1)\}} 100\%. \quad (29)$$

Table 3. The comparison of moments of the Gaussian and Gaussian IG Ho-Lee bond prices for $t \in [0, 0.2]$ with step 0.025 in Metric (29).

$v \setminus t$	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
1	0.32	0.32	0.32	0.33	0.34	0.35	0.35	0.37	0.38
2	8.3	7.1	6.1	5.2	4.5	3.9	3.4	3.0	2.7
3	15.6	11.4	7.5	4.0	0.9	1.8	4.0	5.8	7.1
4	22.4	13.5	4.8	3.4	10.2	15.6	19.8	23.2	25.8

Finally, let us mention that the expressions for bond option prices in Theorem 2 and Corollary 1 are analogous to the formulas for stock derivatives with gamma and inverse gamma drift and volatility coefficients. As such, they can be computed by the same method as is used in the numerical examples in Section 4 in Ivanov [41].

4.4. Complex Example

As can be seen from (16), the moments of the GIG bond price do not exist when

$$2\rho(v, t) - \eta^2 \geq 0. \quad (30)$$

Let

$$Z_1 = Z_1(t) = (T - t)^2 \int_0^t \sigma_s^2 ds$$

and

$$Z_2 = Z_2(t) = \int_t^T \sigma_s^2 (T - s)^2 ds - 2 \int_t^T (T - s) \theta_s ds - 2(T - t) \int_0^t \theta_s ds.$$

Then, the inequality (30) has the form

$$Z_1 v^2 + Z_2 v - \eta^2 \geq 0$$

with $Z_1 > 0$.

The equation $Z_1 v^2 + Z_2 v - \eta^2 = 0$ has roots

$$v_{1,2} = \frac{-Z_2 \pm \sqrt{Z_2^2 + 4Z_1\eta^2}}{2Z_1}.$$

Therefore, if

$$v \geq v_2 = \frac{\sqrt{Z_2^2 + 4Z_1\eta^2} - Z_2}{2Z_1} > 0,$$

then Condition (30) is satisfied and

$$EB_{\text{GIG}}^v(t, T) = \infty, \quad v \geq v_2. \quad (31)$$

Identity (31) indicates that the GIG continuous Ho–Lee model, unlike the pure Gaussian one, permits infinite values for higher moments of the bond price. Therefore, the new model is able to distinguish, particularly, specific central parts of empirical distributions of the interest rates which cannot be well-fitted with the normal model.

5. Discussion

We have introduced a new model of the term structure of stochastic interest rates that is more flexible to the specification of real financial data than the continuous Ho–Lee one. The new model incorporates the drift and volatility coefficients, which depend in addition on the GIG distribution, which is very popular in financial modeling. This model is an example of the HJM model, and it allows the interest rate to be negative, as sometimes happens at practice (see Eberlein et al. [45] and Fontana et al. [60]).

Similar to the continuous-time Ho–Lee model, Theorem 1 provides us with the analytical expressions for the bond price and its moments in the GIG continuous Ho–Lee model. It should be noticed that the new model, unlike the continuous Ho–Lee one, permits infinite values for the bond price moments under some relations between the parameters. Employing the additional conditions on the parameters of the model, Theorem 2 and Corollary 1 give us the formulas for the prices of European call and put option written on bonds in the discussed framework. Analogous to the results of Ivanov [41] for stock options in the gamma and inverse gamma volatility models, the obtained expressions are determined by the values of the Humbert confluent hypergeometric function of two variables. The availability of these formulas enables us to estimate the implied parameters of the interest rate dynamic comparing the real market and model option prices.

In the numerical analysis section, we have compared the moments of bond prices in the new and continuous Ho–Lee models. Amid the GIG distributions, we choose, for the experiments, the inverse Gaussian ($\lambda = -0.5$) one. It appears that the third and fourth moments of the bond prices with very close means can differ significantly in the suggested metric (see Figures 1–4 and Table 3). Also, we have provided a general condition on the parameters when the higher moments of the GIG bond price do not exist.

Future investigations can relate to the valuation of swap and exotic derivatives (see Chapters 12 and 13 in Musiela and Rutkowski [5]) in the new model. As is easy to see from Table 2 and the statement of Theorem 2, the results about the bond option pricing do not imply the HIG distribution, which is far-famed in financial applications. Therefore, the conditions of Theorem 2 could be relaxed in the ensuing studies. Moreover, we look forward for the extension of the GIG continuous Ho–Lee model to more general term structure models in the context of Table 1. First of all, the GIG Vasicek interest rate model should be developed and examined.

6. Conclusions

In this paper, we have defined and studied the new model of stochastic interest rate dynamics that generalizes the continuous Ho–Lee model by assuming that the drift and volatility coefficients depend also on the independent GIG distribution. The conducted research allows us to make the following conclusions:

- The proposed model is analytically tractable. We have found the closed-form expressions for the bond price, its moments, and the prices of European call and put bond options. However, the results related to the option prices are obtained under the special restrictions on the parameters of model;
- The numerical experiments have shown that the third and fourth moments of the continuous Ho–Lee and GIG continuous Ho–Lee bond prices with the same mean may differentiate at up to 15.6% and 25.8%, respectively. And the higher moments of the GIG bond price can take infinite values. Therefore, the compound model could better reflect the properties of market yield curves;
- In the next examinations, the call and put bond option prices could be found with fewer constraints on the parameters. The problem of swap and exotic derivative pricing in the new model should be discussed. The possibility of the extension of the introduced model to the GIG Vasicek model should be considered as well.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The author would like to thank the anonymous referees, whose comments have helped to improve the work.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GIG	Generalized inverse Gaussian
GH	Generalized hyperbolic
HIG	Hyperbolic-inverse Gaussian
IG	Inverse Gaussian
NIG	Normal-inverse Gaussian
HJM	Heath–Jarrow–Morton
determ.	Deterministic
cont.	Continuous
stoch.	Stochastic

Appendix A

Proof of Theorem 1. *Proof of Formula (15).*

Employing the idea of the proof of Proposition 10.1.1 from Musiela and Rutkowski [5], we consider the process

$$\tilde{\zeta}_{\text{GIG}}(t, T) = \int_t^T r_{\text{GIG},s} ds. \quad (\text{A1})$$

Let $Y_{\text{GIG},t} = r_{\text{GIG},t}(T - t)$. Then,

$$dY_{\text{GIG},t} = (T - t)dr_{\text{GIG},t} - r_{\text{GIG},t}dt$$

and, hence,

$$Y_{\text{GIG},T} - Y_{\text{GIG},t} = -r_{\text{GIG},t}(T-t) = \int_t^T (T-s)dr_{\text{GIG},s} - \int_t^T r_{\text{GIG},s}ds. \quad (\text{A2})$$

Therefore, substituting the integral in (A1) from (A2) and incorporating (9), we have that

$$\begin{aligned} \zeta_{\text{GIG}}(t, T) &= r_{\text{GIG},t}(T-t) + \int_t^T (T-s)\alpha_s ds + \\ &+ \text{GIG} \int_t^T (T-s)\theta_s ds + \text{GIG}^{\frac{1}{2}} \int_t^T \sigma_s(T-s)dW_s. \end{aligned} \quad (\text{A3})$$

Set

$$l(t, T) = \mathbb{E} \left(e^{-\text{GIG} \int_t^T (T-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_t^T \sigma_s(T-s)dW_s} | \mathcal{F}_t \right).$$

Then, we obtain from (13) and (A3) that the bond price

$$B_{\text{GIG}}(t, T) = \mathbb{E} \left(e^{-\zeta_{\text{GIG}}(t, T)} | \mathcal{F}_t \right) = e^{-r_{\text{GIG},t}(T-t) - \int_t^T (T-s)\alpha_s ds} l(t, T). \quad (\text{A4})$$

Because the Wiener process has independent increments, we have that

$$l(t, T) = \mathbb{E} \left(e^{-\text{GIG} \int_t^T (T-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_t^T \sigma_s(T-s)dW_s} | \text{GIG} \right). \quad (\text{A5})$$

Let us consider the random variable

$$\varsigma(t, T, \text{GIG}) = \text{GIG} \int_t^T (T-s)\theta_s ds + \text{GIG}^{\frac{1}{2}} \int_t^T \sigma_s(T-s)dW_s.$$

It is conditionally Gaussian and, therefore, $\varsigma(t, T, \text{GIG})$ has the conditional variance

$$\text{Var}(\varsigma(t, T, \text{GIG}) | \text{GIG}) = \text{GIG} \int_t^T \sigma_s^2(T-s)^2 ds.$$

Keeping in mind the form of the Gaussian characteristic function, we find that

$$l(t, T) = \mathbb{E} \left(e^{-\varsigma(t, T, \text{GIG})} | \text{GIG} \right) = e^{\text{GIG} \left(\frac{1}{2} \int_t^T \sigma_s^2(T-s)^2 ds - \int_t^T (T-s)\theta_s ds \right)}. \quad (\text{A6})$$

Formula (15) arises from (A4) and (A6).

Proof of Formula (16).

We have, from (10) and (15), that

$$\begin{aligned} EB_{\text{GIG}}^v(t, T) &= e^{-v \left((T-t) \left(r_{\text{GIG},0} + \int_0^t \alpha_s ds \right) + \int_t^T (T-s)\alpha_s ds \right)} \times \\ &\times \mathbb{E} e^{v \left(\text{GIG} \left(\frac{1}{2} \int_t^T \sigma_s^2(T-s)^2 ds - \int_t^T (T-s)\theta_s ds - (T-t) \int_0^t \theta_s ds \right) - \text{GIG}^{\frac{1}{2}} (T-t) \int_0^t \sigma_s dW_s \right)}. \end{aligned}$$

Define the process $\varsigma_2(t, T, \text{GIG})$ by the identity

$$\begin{aligned} \varsigma_2(t, T, \text{GIG}) &= v \left(\text{GIG} \left(\frac{1}{2} \int_t^T \sigma_s^2(T-s)^2 ds - \int_t^T (T-s)\theta_s ds - \right. \right. \\ &\left. \left. - (T-t) \int_0^t \theta_s ds \right) - \text{GIG}^{\frac{1}{2}} (T-t) \int_0^t \sigma_s dW_s \right). \end{aligned}$$

Then,

$$EB_{\text{GIG}}^v(t, T) = e^{-v\left((T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) + \int_t^T (T-s)\alpha_s ds\right)} E(E(e^{\zeta_2} | \text{GIG})). \quad (\text{A7})$$

So far as ζ_2 is conditionally Gaussian and

$$\text{Var}(\zeta_2(t, T, \text{GIG}) | \text{GIG}) = \text{GIG} v^2 (T-t)^2 \int_0^t \sigma_s^2 ds,$$

we see that

$$\begin{aligned} E(e^{\zeta_2} | \text{GIG}) &= \\ &= e^{\text{GIG} v \left(\frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds - \int_t^T (T-s)\theta_s ds - (T-t) \int_0^t \theta_s ds + \frac{1}{2} v (T-t)^2 \int_0^t \sigma_s^2 ds \right)}. \end{aligned} \quad (\text{A8})$$

Setting

$$\begin{aligned} \rho(v, t) &= v \left(\frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds - \int_t^T (T-s)\theta_s ds - \right. \\ &\quad \left. - (T-t) \int_0^t \theta_s ds + \frac{1}{2} v (T-t)^2 \int_0^t \sigma_s^2 ds \right), \end{aligned}$$

we obtain from (A7) and (A8) that

$$EB_{\text{GIG}}^v(t, T) = e^{-v\left((T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) + \int_t^T (T-s)\alpha_s ds\right)} E e^{\text{GIG} \rho(v, t)}$$

and then receive (16) by applying (7). $\square$

Proof of Theorem 2. Stage 1.

Let us notice that the call option price has a decomposition

$$\begin{aligned} \mathbb{C} &= E \left[(R_{\text{GIG},t})^{-1} (B_{\text{GIG}}(t, T) - K)^+ \right] = \\ &= E \left[(R_{\text{GIG},t})^{-1} B_{\text{GIG}}(t, T) I_{\{B_{\text{GIG}}(t, T) > K\}} \right] - KE \left[(R_{\text{GIG},t})^{-1} I_{\{B_{\text{GIG}}(t, T) > K\}} \right] \\ &= E g_1(\text{GIG}) - KE g_2(\text{GIG}) \end{aligned} \quad (\text{A9})$$

if

$$E \left[(R_{\text{GIG},t})^{-1} I_{\{B_{\text{GIG}}(t, T) > K\}} \right] < \infty \quad (\text{A10})$$

with

$$g_1(\text{GIG}) = E \left[(R_{\text{GIG},t})^{-1} B_{\text{GIG}}(t, T) I_{\{B_{\text{GIG}}(t, T) > K\}} | \text{GIG} \right] \quad (\text{A11})$$

and

$$g_2(\text{GIG}) = E \left[(R_{\text{GIG},t})^{-1} I_{\{B_{\text{GIG}}(t, T) > K\}} | \text{GIG} \right]. \quad (\text{A12})$$

Indeed, since

$$E \left[(R_{\text{GIG},t})^{-1} B_{\text{GIG}}(t, T) I_{\{B_{\text{GIG}}(t, T) > K\}} \right] \geq KE \left[(R_{\text{GIG},t})^{-1} I_{\{B_{\text{GIG}}(t, T) > K\}} \right],$$

we obtain, when (A10) does not hold, the indeterminacy $\mathbb{C} = \infty - \infty$ in (A9).

To find g_1 and g_2 , we have first to compute the expectation

$$\mathcal{E} = \mathcal{E}(X, Y, u) = E e^Y I_{\{X > u\}} \quad (\text{A13})$$

for a couple of correlated (with coefficient ρ) Gaussian random variables X and Y because the random variables $\log(R_{\text{GIG},t})$ and $\log(B_{\text{GIG}}(t, T))$ are conditionally Gaussian.

We have that

$$\begin{aligned}\mathcal{E} &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \\ &\times \int_u^\infty \int_{-\infty}^\infty e^{y - \frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right]} dy dx = \\ &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_u^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{2\rho\mu_Y(x-\mu_X)}{\sigma_X\sigma_Y} \right]} \mathcal{E}_1(x) dx,\end{aligned}$$

where

$$\mathcal{E}_1(x) = \int_{-\infty}^\infty e^{y - \frac{1}{2(1-\rho^2)} \left[\frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2y\rho(x-\mu_X)}{\sigma_X\sigma_Y} \right]} dy.$$

Next,

$$\begin{aligned}\mathcal{E}_1(x) &= \int_{-\infty}^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{\sigma_X(y-\mu_Y)^2 - 2\rho\sigma_Y(x-\mu_X)y - 2(1-\rho^2)\sigma_X\sigma_Y^2 y}{\sigma_X\sigma_Y^2} \right]} dy = \\ &= \int_{-\infty}^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{\sigma_X y^2 - 2(\sigma_X\mu_Y + \rho\sigma_Y(x-\mu_X) + (1-\rho^2)\sigma_X\sigma_Y^2)y + \sigma_X\mu_Y^2}{\sigma_X\sigma_Y^2} \right]} dy = \\ &= \int_{-\infty}^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{y^2 - 2\hat{\mu}_1 y + \mu_Y^2}{\sigma_Y^2} \right]} dy = \int_{-\infty}^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{(y-\hat{\mu}_1)^2 - \hat{\mu}_1^2 + \mu_Y^2}{\sigma_Y^2} \right]} dy = \\ &= e^{\frac{\hat{\mu}_1^2 - \mu_Y^2}{2(1-\rho^2)\sigma_Y^2}} \int_{-\infty}^\infty e^{-\frac{(y-\hat{\mu}_1)^2}{2(1-\rho^2)\sigma_Y^2}} dy = \sigma_Y e^{\frac{\hat{\mu}_1^2 - \mu_Y^2}{2(1-\rho^2)\sigma_Y^2}} \sqrt{2\pi(1-\rho^2)}\end{aligned}$$

with

$$\hat{\mu}_1 = \mu_Y + \frac{\rho\sigma_Y(x-\mu_X)}{\sigma_X} + (1-\rho^2)\sigma_Y^2.$$

So far as

$$\begin{aligned}\hat{\mu}_1^2 &= \frac{\rho^2\sigma_Y^2(x-\mu_X)^2 + 2\rho\sigma_Y\sigma_X(\mu_Y + (1-\rho^2)\sigma_Y^2)(x-\mu_X) +}{\sigma_X^2} + \\ &+ \left(\mu_Y + (1-\rho^2)\sigma_Y^2 \right)^2,\end{aligned}$$

we see that

$$\begin{aligned}\mathcal{E} &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_u^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{2\rho\mu_Y(x-\mu_X)}{\sigma_X\sigma_Y} + \frac{\mu_Y^2 - \hat{\mu}_1^2}{\sigma_Y^2} \right]} dx = \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_u^\infty e^{-\frac{1}{2(1-\rho^2)} \left[\frac{\sigma_Y^2(x-\mu_X)^2 + 2\rho\mu_Y\sigma_X\sigma_Y(x-\mu_X) + (\mu_Y^2 - \hat{\mu}_1^2)\sigma_X^2}{\sigma_X^2\sigma_Y^2} \right]} dx = \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_u^\infty e^{-\frac{\varepsilon(x)}{2(1-\rho^2)\sigma_X^2\sigma_Y^2}} dx\end{aligned}$$

with

$$\begin{aligned}\varepsilon(x) &= \sigma_Y^2(1-\rho^2)(x-\mu_X)^2 + 2\rho\mu_Y\sigma_X\sigma_Y^3(\rho^2-1)(x-\mu_X) - \\ &\quad - \sigma_X^2\sigma_Y^2(1-\rho^2)\left(\sigma_Y^2(1-\rho^2) + 2\mu_Y\right) = \sigma_Y^2(1-\rho^2) \times \\ &\quad \times \left[(x-\mu_X)^2 - 2\rho\mu_Y\sigma_X\sigma_Y(x-\mu_X) - \sigma_X^2\left(2\mu_Y + \sigma_Y^2(1-\rho^2)\right)\right] = \\ &= \sigma_Y^2(1-\rho^2)\left[(x-\mu_X-\hat{\mu}_2)^2 - \hat{\mu}_2^2 - \sigma_X^2\left(2\mu_Y + \sigma_Y^2(1-\rho^2)\right)\right],\end{aligned}$$

where

$$\hat{\mu}_2 = \rho\mu_Y\sigma_X\sigma_Y.$$

We can conclude that

$$\begin{aligned}\mathcal{E} &= \frac{e^{\mu_Y + \frac{\sigma_Y^2}{2}(\rho^2\mu_Y^2 + 1 - \rho^2)}}{\sigma_X\sqrt{2\pi}} \int_u^\infty e^{-\frac{(x-\mu_X-\hat{\mu}_2)^2}{2\sigma_X^2}} dx = \\ &= \frac{e^{\mu_Y + \frac{\sigma_Y^2}{2}(\rho^2\mu_Y^2 + 1 - \rho^2)}}{\sigma_X\sqrt{2\pi}} \mathcal{N}\left(-\frac{u + \mu_X + \rho\mu_Y\sigma_X\sigma_Y}{\sigma_X}\right),\end{aligned}\quad (\text{A14})$$

where $\mathcal{N}(\varsigma)$, $\varsigma \in \mathbb{R}$, is the standard normal distribution function.

Stage 2.

Set

$$X_1 = \log(B_{\text{GIG}}(t, T)),$$

$$Y_1 = \log\left((R_{\text{GIG},t})^{-1}B_{\text{GIG}}(t, T)\right) \quad \text{and} \quad Y_2 = -\log(R_{\text{GIG},t}).$$

Let

$$A(t, T) = -\int_t^T (T-s)\alpha_s ds$$

and

$$L(t, T) = \frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds - \int_t^T (T-s)\theta_s ds.$$

Then we have from (10) and (15) that

$$\begin{aligned}X_1 &= A(t, T) + L(t, T)\text{GIG} - (T-t)r_{\text{GIG},t} = A(t, T) - \\ &\quad - (T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) + \text{GIG}\left(L(t, T) - (T-t) \int_0^t \theta_s ds\right) - \\ &\quad - \text{GIG}^{\frac{1}{2}}(T-t) \int_0^t \sigma_s dW_s,\end{aligned}$$

from (12), (15), (A1) and (A3) that

$$\begin{aligned}Y_1 &= X_1 - \log R_{\text{GIG},0} - \xi_{\text{GIG}}(0, t) = X_1 - \log R_{\text{GIG},0} - r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds - \\ &\quad - \text{GIG} \int_0^t (t-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s (t-s) dW_s = -\text{Tr}_{\text{GIG},0} - \log R_{\text{GIG},0} - \\ &\quad - \int_0^t (T-s)\alpha_s ds + \text{GIG}\left(L(t, T) - \int_0^t (T-s)\theta_s ds\right) - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s (T-s) dW_s\end{aligned}$$

and from (12), (A1) and (A3) that

$$Y_2 = -\log R_{\text{GIG},0} - r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds - \text{GIG} \int_0^t (t-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s(t-s) dW_s.$$

Therefore,

$$\mu_{X_1}(\text{GIG}) = E(X_1|\text{GIG}) = A(t, T) - (T-t) \left(r_{\text{GIG},0} + \int_0^t \alpha_s ds \right) + \text{GIG} \left(L(t, T) - (T-t) \int_0^t \theta_s ds \right),$$

$$\mu_{Y_1}(\text{GIG}) = E(Y_1|\text{GIG}) = \text{GIG} \left(L(t, T) - \int_0^t (T-s)\theta_s ds \right) - Tr_{\text{GIG},0} - \log R_{\text{GIG},0} - \int_0^t (T-s)\alpha_s ds$$

and

$$\begin{aligned} \mu_{Y_2}(\text{GIG}) &= E(Y_2|\text{GIG}) = \\ &= -\log R_{\text{GIG},0} - r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds - \text{GIG} \int_0^t (t-s)\theta_s ds. \end{aligned}$$

Moreover,

$$\sigma_{X_1}^2(\text{GIG}) = E\left([X_1 - \mu_{X_1}(\text{GIG})]^2|\text{GIG}\right) = \text{GIG}(T-t)^2 \int_0^t \sigma_s^2 ds,$$

$$\sigma_{Y_1}^2(\text{GIG}) = E\left([Y_1 - \mu_{Y_1}(\text{GIG})]^2|\text{GIG}\right) = \text{GIG} \int_0^t \sigma_s^2 (T-s)^2 ds$$

and

$$\sigma_{Y_2}^2(\text{GIG}) = E\left([Y_2 - \mu_{Y_2}(\text{GIG})]^2|\text{GIG}\right) = \text{GIG} \int_0^t \sigma_s^2 (t-s)^2 ds.$$

Also, we have that

$$\begin{aligned} \rho_{X_1 Y_1}(\text{GIG}) &= \frac{E([X_1 - \mu_{X_1}(\text{GIG})][Y_1 - \mu_{Y_1}(\text{GIG})]|\text{GIG})}{\sigma_{X_1}(\text{GIG})\sigma_{Y_1}(\text{GIG})} = \\ &= \frac{\int_0^t \sigma_s^2 (T-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \sigma_s^2 (T-s)^2 ds\right)^{\frac{1}{2}}} \end{aligned}$$

and

$$\begin{aligned} \rho_{X_1 Y_2}(\text{GIG}) &= \frac{E([X_1 - \mu_{X_1}(\text{GIG})][Y_2 - \mu_{Y_2}(\text{GIG})]|\text{GIG})}{\sigma_{X_1}(\text{GIG})\sigma_{Y_2}(\text{GIG})} = \\ &= \frac{\int_0^t \sigma_s^2 (t-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \sigma_s^2 (t-s)^2 ds\right)^{\frac{1}{2}}}. \end{aligned}$$

After all this, we find from (A11) and (A13) that

$$g_1(\text{GIG}) = \mathcal{E}(X_1(\text{GIG}), Y_1(\text{GIG}), K) = \\ = \frac{e^{\mu_{Y_1} + \frac{\sigma_{Y_1}^2}{2} (\rho_{X_1 Y_1}^2 \mu_{Y_1}^2 + 1 - \rho_{X_1 Y_1}^2)}}{\sigma_{X_1} \sqrt{2\pi}} \mathcal{N}\left(-\frac{K + \mu_{X_1} + \rho_{X_1 Y_1} \mu_{Y_1} \sigma_{X_1} \sigma_{Y_1}}{\sigma_{X_1}}\right)$$

and from (A12) and (A13) that

$$g_2(\text{GIG}) = \mathcal{E}(X_1(\text{GIG}), Y_2(\text{GIG}), K) = \\ = \frac{e^{\mu_{Y_2} + \frac{\sigma_{Y_2}^2}{2} (\rho_{X_1 Y_2}^2 \mu_{Y_2}^2 + 1 - \rho_{X_1 Y_2}^2)}}{\sigma_{X_1} \sqrt{2\pi}} \mathcal{N}\left(-\frac{K + \mu_{X_1} + \rho_{X_1 Y_2} \mu_{Y_2} \sigma_{X_1} \sigma_{Y_2}}{\sigma_{X_1}}\right).$$

Stage 3.

So far as Condition (21) is satisfied, we have that

$$\mu_{Y_1}(\text{GIG}) = -\text{Tr}_{\text{GIG},0} - \log R_{\text{GIG},0} - \int_0^t (T-s) \alpha_s ds$$

and obtain from the formulas in Stage 2 that

$$g_1(\text{GIG}) = c_1 \text{GIG}^{-\frac{1}{2}} e^{c_2 \text{GIG}} \mathcal{N}\left(c_3 \sqrt{\text{GIG}} + \frac{c_4}{\sqrt{\text{GIG}}}\right) \quad (\text{A15})$$

with

$$c_1 = \frac{e^{\mu_{Y_1} \text{GIG}^{\frac{1}{2}}}}{\sigma_{X_1} \sqrt{2\pi}} = \frac{e^{-\text{Tr}_{\text{GIG},0} - \log R_{\text{GIG},0} - \int_0^t (T-s) \alpha_s ds}}{(T-t) \left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}} \sqrt{2\pi}}, \\ c_2 = \text{GIG}^{-1} \frac{\sigma_{Y_1}^2}{2} (\rho_{X_1 Y_1}^2 \mu_{Y_1}^2 + 1 - \rho_{X_1 Y_1}^2) = \frac{\int_0^t \sigma_s^2 (T-s)^2 ds}{2} \times \\ \times \left(\frac{(\int_0^t \sigma_s^2 (T-s) ds)^2}{\int_0^t \sigma_s^2 ds \int_0^t \sigma_s^2 (T-s)^2 ds} \left[\left(\text{Tr}_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (T-s) \alpha_s ds \right)^2 - 1 \right] + 1 \right), \\ c_3 = \frac{(T-t) \int_0^t \theta_s ds - L(t, T)}{(T-t) \left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} + \\ + \left(\text{Tr}_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (T-s) \alpha_s ds \right) \frac{\int_0^t \sigma_s^2 (T-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} = \\ = \frac{(T-t) \int_0^t \theta_s ds - \frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds + \int_t^T (T-s) \theta_s ds}{(T-t) \left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} + \\ + \left(\text{Tr}_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (T-s) \alpha_s ds \right) \frac{\int_0^t \sigma_s^2 (T-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}},$$

$$c_4 = \frac{(T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) - A(t,T) - K}{(T-t)\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} =$$

$$= \frac{(T-t)\left(r_{\text{GIG},0} + \int_0^t \alpha_s ds\right) + \int_t^T (T-s)\alpha_s ds - K}{(T-t)\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}}.$$

Also, according to (22), we have that

$$\mu_{Y_2}(\text{GIG}) = -\log R_{\text{GIG},0} - r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds$$

and find that

$$g_2(\text{GIG}) = c_5 \text{GIG}^{-\frac{1}{2}} e^{c_6 \text{GIG}} \mathcal{N}\left(c_7 \sqrt{\text{GIG}} + \frac{c_4}{\sqrt{\text{GIG}}}\right) \quad (\text{A16})$$

with

$$c_5 = \frac{e^{\mu_{Y_2}} \text{GIG}^{\frac{1}{2}}}{\sigma_{X_1} \sqrt{2\pi}} = \frac{e^{-tr_{\text{GIG},0} - \log R_{\text{GIG},0} - \int_0^t (t-s)\alpha_s ds}}{(T-t)\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}} \sqrt{2\pi}},$$

$$c_6 = \text{GIG}^{-1} \frac{\sigma_{Y_2}^2}{2} \left(\rho_{X_1 Y_2}^2 \mu_{Y_2}^2 + 1 - \rho_{X_1 Y_2}^2 \right) =$$

$$= \frac{\int_0^t \sigma_s^2 (t-s)^2 ds}{2} \left(\frac{\int_0^t \sigma_s^2 (t-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \sigma_s^2 (t-s)^2 ds\right)^{\frac{1}{2}}} \times \right.$$

$$\left. \times \left[\left(tr_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (t-s)\alpha_s ds \right)^2 - 1 \right] + 1 \right),$$

$$c_7 = \frac{(T-t) \int_0^t \theta_s ds - L(t,T)}{(T-t)\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} +$$

$$+ \left(tr_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (t-s)\alpha_s ds \right) \frac{\int_0^t \sigma_s^2 (t-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} =$$

$$= \frac{(T-t) \int_0^t \theta_s ds - \frac{1}{2} \int_t^T \sigma_s^2 (T-s)^2 ds + \int_t^T (T-s)\theta_s ds}{(T-t)\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}} +$$

$$+ \left(tr_{\text{GIG},0} + \log R_{\text{GIG},0} + \int_0^t (t-s)\alpha_s ds \right) \frac{\int_0^t \sigma_s^2 (t-s) ds}{\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}}.$$

Set

$$I_{\text{GIG}} = I_{\text{GIG}}(\lambda, \eta, \delta, v_1, v_2, v_3) =$$

$$= \int_0^\infty x^{\lambda - \frac{3}{2}} e^{-\frac{(\eta^2 - 2v_1)x}{2} - \frac{\delta^2}{2x}} \mathcal{N}\left(v_2 \sqrt{x} + \frac{v_3}{\sqrt{x}}\right) dx. \quad (\text{A17})$$

Then, we can enclose from (A15) that

$$\begin{aligned} \text{Eg}_1(\text{GIG}) &= c_1 \text{E} \left[\text{GIG}^{-\frac{1}{2}} e^{c_2 \text{GIG}} \text{N} \left(c_3 \sqrt{\text{GIG}} + \frac{c_4}{\sqrt{\text{GIG}}} \right) \right] = \\ &= C_{\text{GIG}} c_1 I_{\text{GIG}}(\lambda, \eta, \delta, c_2, c_3, c_4) \end{aligned} \quad (\text{A18})$$

and from (A16) that

$$\begin{aligned} \text{Eg}_2(\text{GIG}) &= c_5 \text{E} \left[\text{GIG}^{-\frac{1}{2}} e^{c_6 \text{GIG}} \text{N} \left(c_7 \sqrt{\text{GIG}} + \frac{c_4}{\sqrt{\text{GIG}}} \right) \right] = \\ &= C_{\text{GIG}} c_5 I_{\text{GIG}}(\lambda, \eta, \delta, c_6, c_7, c_4). \end{aligned} \quad (\text{A19})$$

Stage 4.

Let $v_1 = \frac{\eta^2}{2}$ in (A17). Then,

$$I_{\text{GIG}} = I_{\text{GIG}} \left(\lambda, \eta, \delta, \frac{\eta^2}{2}, v_2, v_3 \right) = \int_0^\infty x^{\lambda - \frac{3}{2}} e^{-\frac{\delta^2}{2x}} \text{N} \left(v_2 \sqrt{x} + \frac{v_3}{\sqrt{x}} \right) dx.$$

If $\lambda < \frac{1}{2}$, we receive, similar to (A12)–(A18) in Ivanov [41], that

$$\begin{aligned} I_{\text{GIG}} &= \frac{(1+q)^{\frac{1}{2}-\lambda} |s|^{-\lambda} e^s}{2^\lambda \delta^{1-2\lambda} \sqrt{\pi}} \left[\text{B} \left(\frac{1}{2} - \lambda, 1 \right) \left(|s| \text{K}_{1-\lambda}(|s|) + s \text{K}_\lambda(|s|) \right) \times \right. \\ &\times \Phi \left(\frac{1}{2} - \lambda, \lambda + \frac{1}{2}, \frac{3}{2} - \lambda; \frac{1+q}{2}, -s(1+q) \right) - s(1+q) \text{B} \left(\frac{3}{2} - \lambda, 1 \right) \times \\ &\times \text{K}_\lambda(|s|) \Phi \left(\frac{3}{2} - \lambda, \lambda + \frac{1}{2}, \frac{5}{2} - \lambda; \frac{1+q}{2}, -s(1+q) \right) \left. \right] \end{aligned}$$

with

$$s = v_2 \sqrt{v_3^2 + \delta^2} \quad \text{and} \quad q = \frac{v_3}{\sqrt{v_3^2 + \delta^2}}.$$

Keeping in mind (20), we see that

$$I_{\text{GIG}} \left(\lambda, \eta, \delta, \frac{\eta^2}{2}, v_2, v_3 \right) = \mathfrak{I}(\lambda, \delta, s, q). \quad (\text{A20})$$

If $c_2 = \frac{\eta^2}{2}$, we obtain, according to (A19) and (A20), that

$$\text{Eg}_2(\text{GIG}) = C_{\text{GIG}} c_5 \mathfrak{I} \left(\lambda, \delta, c_7 \sqrt{c_4^2 + \delta^2}, \frac{c_4}{\sqrt{c_4^2 + \delta^2}} \right) < \infty \quad (\text{A21})$$

and (A10) holds. When $c_6 = \frac{\eta^2}{2}$, we receive with respect to (A18) and (A20) that

$$\text{Eg}_1(\text{GIG}) = C_{\text{GIG}} c_1 \mathfrak{I} \left(\lambda, \delta, c_3 \sqrt{c_4^2 + \delta^2}, \frac{c_4}{\sqrt{c_4^2 + \delta^2}} \right). \quad (\text{A22})$$

Formula (23) results from (A9), (A21) and (A22). $\square$

Appendix B

Proof of Corollary 1. Obviously,

$$\frac{B_{\text{GIG}}(t, T) - K}{R_{\text{GIG},t}} = \frac{(B_{\text{GIG}}(t, T) - K)^+}{R_{\text{GIG},t}} - \frac{(K - B_{\text{GIG}}(t, T))^+}{R_{\text{GIG},t}}. \quad (\text{A23})$$

Since the discounted bond price is the martingale and (14) is satisfied, we have that

$$\mathbb{E}\left(\frac{B_{\text{GIG}}(t, T)}{R_{\text{GIG},t}}\right) = \frac{B_{\text{GIG}}(0, T)}{R_{\text{GIG},0}} = \mathbb{E}(R_{\text{GIG},T})^{-1} < \infty.$$

Therefore, if $\mathbb{E}(R_{\text{GIG},t})^{-1} < \infty$ as well, then we obtain from (13) and (A23) that

$$\mathbb{P} = \mathbb{C} - \mathbb{E}(R_{\text{GIG},T})^{-1} + K\mathbb{E}(R_{\text{GIG},t})^{-1}. \quad (\text{A24})$$

Hence, this is required to compute $\mathbb{E}(R_{\text{GIG},t})^{-1}$.

According to (A1) and (A3), we have that

$$\begin{aligned} \mathbb{E}(R_{\text{GIG},t})^{-1} &= (R_{\text{GIG},0})^{-1} \mathbb{E} e^{\xi_{\text{GIG}}(t,T) - \xi_{\text{GIG}}(0,T)} = (R_{\text{GIG},0})^{-1} \times \\ &\times e^{-r_{\text{GIG},0}T - \int_0^t (T-s)\alpha_s ds} \mathbb{E} e^{r_{\text{GIG},t}(T-t) - \text{GIG} \int_0^t (T-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s(T-s) dW_s}. \end{aligned}$$

Applying (10), it is easy to find that

$$\begin{aligned} \mathbb{E}(R_{\text{GIG},t})^{-1} &= (R_{\text{GIG},0})^{-1} e^{-r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds} \times \\ &\times \mathbb{E} e^{-\text{GIG} \int_0^t (t-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s(t-s) dW_s}. \end{aligned}$$

Since the random variable $\text{GIG} \int_0^t (t-s)\theta_s ds + \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s(t-s) dW_s$ is conditionally Gaussian, we have that

$$\begin{aligned} \mathbb{E}\left(e^{-\text{GIG} \int_0^t (t-s)\theta_s ds - \text{GIG}^{\frac{1}{2}} \int_0^t \sigma_s(t-s) dW_s} \middle| \text{GIG}\right) &= \\ &= e^{\text{GIG}\left(\frac{1}{2} \int_0^t \sigma_s^2(t-s)^2 ds - \int_0^t (t-s)\theta_s ds\right)} \end{aligned}$$

and, therefore,

$$\mathbb{E}(R_{\text{GIG},t})^{-1} = (R_{\text{GIG},0})^{-1} e^{-r_{\text{GIG},0}t - \int_0^t (t-s)\alpha_s ds} \text{mgf}_{\text{GIG}}(\hat{\rho}(t)) \quad (\text{A25})$$

with $\hat{\rho}(t) = \frac{1}{2} \int_0^t \sigma_s^2(t-s)^2 ds - \int_0^t (t-s)\theta_s ds$ if $\hat{\rho}(t) < \frac{\eta^2}{2}$, where this condition results from (7).

Formula (24) arises from (22), (A24) and (A25). $\square$

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