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Computational Representation of Fractional Inequalities Through 2D and 3D Graphs with Applications

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Abstract: The aim of this research article is to use the extended fractional operators involving the multivariate Mittag–Leffler (M-M-L) function, we provide the generalization of the Hermite–Hadamard–Fejer (H-H-F) inequalities. We relate these inequalities to previously published disparities in the literature by making appropriate substitutions. In the last section, we analyze several inequalities related to the H-H-F inequalities, focusing on generalized *h*-convexity associated with extended fractional operators involving the M-M-L function. To achieve this, we derive two identities for locally differentiable functions, which allows us to provide specific estimates for the differences between the left, middle, and right terms in the H-H-F inequalities. Also, we have constructed specific inequalities and visualized them through graphical representations to facilitate their applications in analysis. The research bridges theoretical advancements with practical applications, providing high-accuracy bounds for complex systems involving fractional calculus.

Keywords: Hermite–Hadamard-Fejer; multivariate Mittag–Leffler function; symmetric properties; generalized h-convexity; modified (k, s)-fractional operator

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1. Introduction

The study of fractional calculus and its diverse applications has received more attention in recent times. Chronologically ordering the development of this theory, we find that fractional calculus was first applied when Abel solved the Tautocrone problem [1]. Scholars have written their own books [2,3] to advance the field of fractional calculus, these books are essential to the advancement of applications of fractional calculus in mathematical modeling and applied analysis. The well-known caputo sense derivative without non-singular kernel was developed by Caputo et al. in [4], although there are still many gaps in the theory that the researchers have discovered. In order to overcome these gaps, several researchers have developed their own fractional operators with non-singular kernels [5–8]. Furthermore, the symmetric properties as well as the non-linear and non-singular extensions of fractional operators were illustrated by Wu et al. in [9]. In [10], Samraiz et al. developed the fractional derivative with non-linear and non-singular kernel, also discussed its applications in applied analysis. Consequently, fractional operators were developed

in [11,12] using different kinds of Mittag–Leffle (M-L) functions as a kernel. We direct readers to explore [13–18] for additional developments and applications on fractional calculus. Roman et al. described the fractional derivatives and their applications in time series with long-time memory and random walks in [19]. In [20], Moreles at al. introduced modeling for the mathematical study of circuit components with fractional orders, they also discussed applications of bioimpedance. In [21] the authors explore the integration of harmonic univalent functions with the generalized (p, q)-Poisson distribution, presenting a novel theoretical contribution to complex analysis and fractional calculus. It extends the classical understanding of harmonic functions by incorporating this distribution, offering new results on their univalence and injectivity properties. The study uses standard mathematical analysis techniques, providing explicit examples and graphical representations to support the theoretical claims. While the theoretical framework is solid, the article could benefit from a clearer explanation of the practical applications and real-world implications of these results. Although it makes a significant contribution to the field, a more detailed discussion on how these findings could be applied to practical problems in mathematics, physics, and engineering would strengthen the work. Using fractional calculus to simulate dielectric relaxation events in polymeric materials was demonstrated by Melo et al. in [22]. You can find a presentation of fractional integral in conjunction with wave theory in [23]. The theory of inequalities in [24,25] is closely related to convex functions (Cf) and their expansions $s-$Cf.

Definition 1. *The function $f : [\alpha, \beta] \mapsto \Re$ (Set of real numbers) is said to be Cf, if*

$$f(\ell_1 x_1 + (1-\ell_1)x_2) \leq \ell_1 f(x_1) + (1-\ell_1) f(x_2),$$

$\forall x_1, x_2 \in [\alpha, \beta]$ *and* $\ell_1 \in (0, 1]$.

In [25], Brechner gave the definition of s-Cf, stated by the following definition.

Definition 2. *A function $f : (0, \infty) \mapsto \Re$ is called second sense $s-$Cf for some fixed $k \epsilon (0, 1)$, is given by*

$$f(\ell_1 x_1 + \ell_2 x_2) \leq \ell_1^s f(x_1) + \ell_2^s f(x_2),$$

$\forall x_1, x_2 \in (0, \infty), 0 \leq \ell_1, \ell_2 \leq 1$, *where* $\ell_1 + \ell_2 = 1$. *This class of function is denoted as* K_s^2.

In [26,27], authors discussed novel fractional operators and class of inequalities involving novel fractional operators. Hakiki et al. described H-H-F inequalities involving second kind of $s-$Cf via Riemann–Liouville (RL) FI and novel inequalities of H-H-F kind inequalities for Cf via FI in [28,29].

Furthermore, matching nonsingular kernels and new fractional operators, like the M-L function, have also been introduced [30,31]. Diverse historical applications of fractional operators were illustrated by the researchers. The scientists have currently designed a large number of physical models [32,33]. Several of these models solve particular differential equations of fractional order using the M-L non-singular kernel [34,35]. The (k, s)-fractional calculus of the generalized M-L function was discussed by Nisar et al. [36]. For more details and applications of fractional operators, we suggest readers to [37,38]. Readers are referred to [35,39–41] for additional information and applications on fractional operators. In 1903, Gosta [42] proposed the classical M-L function. Several approaches have been taken to generalize it, such as adding two or three parameters [43].

Definition 3 ([11]). *The generalized multivariate Mittag–Leffler (M-M-L) function is defined by*

$$\xi_{k,\mu,(\beta_i)}^{(\delta_i)}(\tau_1,\tau_2,\ldots,\tau_n)=\Sigma_{\kappa_1,\ldots,\kappa_n=0}^{\infty}\frac{\Pi_{i=1}^{n}(\delta_i)_{\kappa_i,k}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))(\kappa_1)!\ldots(\kappa_n)!}(\tau_1)^{\kappa_1}(\tau_2)^{\kappa_2}\ldots(\tau_n)^{\kappa_n},$$

where $k>0$, $\beta_i,\delta_i\in C$ *with* $Re(\beta_i)>0$ *for all* $i=1,2,3,\ldots,n$. *Furthermore, If we substitute* $k=1$ *then we obtain M-M-L function defined by Sexana et al. in [44].*

Let $f:\Re\to\Re$ be Cf , $\ell_1,\ell_2\in I\subseteq\Re$ such that $\ell_1<\ell_2$, and $w:[\ell_1,\ell_2]\to\Re$ be a non-negative integrable and symmetric with respect to $\frac{\ell_1+\ell_2}{2}$, then the H-H-F inequality can be found in [45] by

$$f(\frac{\ell_1+\ell_2}{2})\int_{\ell_1}^{\ell_2}w(x)dx\leq\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}f(x)w(x)dx\leq\frac{f(\ell_1)+f(\ell_2)}{2}\int_{\ell_1}^{\ell_2}w(x)dx.$$

The classical H-H inequality (see also [46]) is given by the following:

$$f(\frac{\ell_1+\ell_2}{2})\leq\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}f(x)dx\leq\frac{f(\ell_1)+f(\ell_2)}{2}.$$

Because of the lot of applications of said inequalities, some researchers prolonged their research via functions of several classes, as an illustration [47] for Cf and [48,49] for h-Cf. In 2007, Varošanec [50] presented the concept of h-Cf, in the second sense which was a generalization of s-convexity, non-negative convexity, *P*-convexity and Godunova–Levin mappings. Chen et al. [51] presented H-H and H-H-F inequalities for generalized FI. In [52], Khan et al. generalized the conformable fractional operators and a new class of fractional operators. For details applications of fractional operators we refer readers [53,54]. Here, we have some more fundamental definitions which are indispensable for explaining the key findings in the coming results.

Definition 4 ([51]). *The incomplete beta function* $B_b(\ell_1,\ell_2)$ *is defined by*

$$B_b(\ell_1,\ell_2)=\int_0^b z^{\ell_1-1}(1-z)^{\ell_2-1}dz,$$

where, $0<b<1,\ell_1,\ell_2>0$.

Definition 5 ([55]). *Let* $p,q\in\Re$ *with* $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then the integral form of Hölder inequality is presented as*

$$\int_{e_1}^{e_2}|H(\imath)I(\imath)|d\imath\leq\left(\int_{e_1}^{e_2}|H(\imath)|^p d\imath\right)^{\frac{1}{p}}\left(\int_{e_1}^{e_2}|I(\imath)|^q d\imath\right)^{\frac{1}{q}},$$

with $H,I\in\mathbb{C}^1[e_1,e_2]$.

Definition 6. *The left and right-sided RLFI* $I_{\ell_1^+}^{\varrho}(f)$ *and* $I_{\ell_2^-}^{\varrho}(f)$, *of order* $\varrho>0$ *on* $[\ell_1,\ell_2]$ *are defined as*

$$I_{\ell_1^+}^{\varrho}f(p_1)=\frac{1}{\Gamma(\varrho)}\int_{\ell_1}^{p_1}(p_1-\nu)^{\varrho-1}f(\nu)d\nu,\quad p_1>\ell_1,$$

and

$$I^{\varrho}_{\ell_2^-} f(p_1) = \frac{1}{\Gamma(\varrho)} \int_{p_1}^{\ell_2} (\nu - p_1)^{\varrho-1} f(\nu) d\nu, \quad p_1 < \ell_2,$$

respectively. Here Γ *represents the Gamma function and its integral representation is given below.*

$$\Gamma(p_1) = \int_0^{\infty} \nu^{p_1-1} e^{-\nu} d\nu, \quad Re(p_1) > 0.$$

Now, we present the definition of the (k, s)-RLFI, which is defined in [56].

Definition 7. *If a function* f *is continuous on* $[a, b]$*, then the* (k, s)*-RLFI for order* $\delta > 0$ *can be defined as*

$${}^{s}_{k}\mathfrak{J}^{\delta}_{a^+} f(t_1) = \frac{(s+1)^{1-\frac{\delta}{k}}}{k\Gamma_k(\delta)} \int_{a^+}^{t_1} (t_1^{s+1} - \nu^{s+1})^{\frac{\delta}{k}-1} \nu^s f(\nu) d\nu. \tag{1}$$

The following is the definition of the fractional integral operator in its generalized form with M-M-L function as part of its kernel is defined in [11].

Definition 8. *If a function* f *is continuous on* $[a, b]$*, for* $k > 0$*,* $m \in \Re/\{-1\}$*, let the parameters* $Re(\mu) > 0$*,* τ_i*,* β_i *be complex numbers with* $Re(\beta_i) > 0, \forall i = 1, 2, \ldots, n$*; and* Θ *be the strictly increasing function and differentiable then the generalized fractional integral operators for* $\delta_i > 0, \forall i = 1, 2, \ldots, n$*; can be defined as*

$$\begin{aligned}
&({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a;\mu,\beta_i} f)(t_1) = \frac{(m+1)^{1-\frac{\mu}{k}}}{k} \int_a^{t_1} (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\mu}{k}-1} \\
&\times \xi^{\delta_i}_{k,\mu,\beta_i} \Big(\tau_1 (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_1}{k}}, \tau_2 (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_2}{k}}, \ldots, \\
&\tau_n (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_n}{k}} \Big) \Theta^m(\nu) \Theta'(\nu) f(\nu) d\nu \\
&= \frac{(m+1)^{1-\frac{\mu}{k}}}{k} \Sigma^{\infty}_{\kappa_1,\ldots,\kappa_n=0} \frac{\Pi^n_{i=1} (\delta_i)_{\kappa_i,k} (\tau_i)^{\kappa_i}}{\Gamma_k(\mu + \Sigma^n_{i=1}(\beta_i \kappa_i))(\kappa_1)! \ldots (\kappa_n)!} \\
&\times \int_a^{t_1} (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\mu+\Sigma^n_{i=1}\beta_i\kappa_i}{k}-1} \Theta^m(\nu) \Theta'(\nu) f(\nu) d\nu.
\end{aligned}$$

Definition 9. *If a function* f *is continuous on* $[a, b]$*, for* $k > 0$*,* $m \in \Re/\{-1\}$*, let the parameters* $Re(\mu) > 0$*,* τ_i*,* β_i *be complex numbers with* $Re(\beta_i) > 0, \forall i = 1, 2, \ldots, n$*; and* Θ *be the strictly increasing function and differentiable then the left and right sided generalized fractional integral operators for* $\delta_i > 0, \forall i = 1, 2, \ldots, n$*; can be defined as*

$$\begin{aligned}
&({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a^+;\mu,\beta_i} f)(t_1) = \frac{(m+1)^{1-\frac{\mu}{k}}}{k} \int_{a^+}^{t_1} (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\mu}{k}-1} \\
&\times \xi^{\delta_i}_{k,\mu,\beta_i} \Big(\tau_1 (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_1}{k}}, \tau_2 (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_2}{k}}, \ldots, \\
&\tau_n (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\beta_n}{k}} \Big) \Theta^m(\nu) \Theta'(\nu) f(\nu) d\nu \\
&= \frac{(m+1)^{1-\frac{\mu}{k}}}{k} \Sigma^{\infty}_{\kappa_1,\ldots,\kappa_n=0} \frac{\Pi^n_{i=1} (\delta_i)_{\kappa_i,k} (\tau_i)^{\kappa_i}}{\Gamma_k(\mu + \Sigma^n_{i=1}(\beta_i \kappa_i))(\kappa_1)! \ldots (\kappa_n)!} \\
&\times \int_{a^+}^{t_1} (\Theta^{m+1}(t_1) - \Theta^{m+1}(\nu))^{\frac{\mu+\Sigma^n_{i=1}\beta_i\kappa_i}{k}-1} \Theta^m(\nu) \Theta'(\nu) f(\nu) d\nu, \ \ t_1 > a^+,
\end{aligned}$$

and

$$
\begin{aligned}
&({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{\zeta^-;\mu,\beta_i} f)(t_1) = \frac{(m+1)^{1-\frac{\mu}{k}}}{k}\int_{\zeta^-}^{t_1}(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\mu}{k}-1}\\
&\times \xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\beta_1}{k}}, \tau_2(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\beta_2}{k}},\ldots,\\
&\tau_n(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\beta_n}{k}}\Big)\Theta^m(\nu)\Theta'(\nu)f(\nu)d\nu\\
&= \frac{(m+1)^{1-\frac{\mu}{k}}}{k}\Sigma^{\infty}_{\kappa_1,\ldots,\kappa_n=0}\frac{\Pi^n_{i=1}(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma^n_{i=1}(\beta_i\kappa_i))(\kappa_1)!\ldots(\kappa_n)!}\\
&\times\int_{\zeta^-}^{t_1}(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\mu+\Sigma^n_{i=1}\beta_i\kappa_i}{k}-1}\Theta^m(\nu)\Theta'(\nu)f(\nu)d\nu,\quad t_1<\zeta.
\end{aligned}
$$

2. A Class of Some Results

In this section we have developed H-H-F type inequalities for generalized extended fractional operator encompassing the M-M-L function.

Lemma 1. *For $k>0$, $m\in\Re/\{-1\}$, and $Re(\mu)>0$, $Re(\tau)>0$, $Re(\delta)>0$, $Re(\beta)>0$ are complex parameters, Θ be an increasing function and Υ be a differentiable on (ℓ_1,ℓ_2) with $\ell_1<\ell_2$ and $\chi:[\ell_1,\ell_2]\mapsto\Re$ be bounded. If $\Upsilon',\chi\in L[\ell_1,\ell_2]$, then the below expression holds.*

$$
\begin{aligned}
&\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\Big]\\
&-\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\Upsilon(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\Upsilon(\ell_1)\Big]\\
&=\frac{k}{(m+1)^{1-\frac{\mu}{k}}}\int_{\ell_1}^{\ell_2}\kappa(\nu)\Upsilon'(\nu)d\nu,
\end{aligned}
$$

where

$$
\kappa(t_1)=\begin{cases}
\int_{\ell_1}^{t_1}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\beta_n}{k}}\Big)\Theta^m(\nu)\Theta'(\nu)f(\nu)d\nu}{(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\mu}{k}-1}},\\
\qquad t_1\in[\ell_1,\tfrac{\ell_1+\ell_2}{2}],\\
\int_{\ell_2}^{t_1}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\beta_n}{k}}\Big)\Theta^m(\nu)\Theta'(\nu)f(\nu)d\nu}{(\Theta^{m+1}(\nu)-\Theta^{m+1}(t_1))^{\frac{\mu}{k}-1}}\\
\qquad t_1\in[\tfrac{\ell_1+\ell_2}{2},\ell_2].
\end{cases}
$$

Proof. Consider

$$
\begin{aligned}
I&=\int_{\ell_1}^{\ell_2}\kappa(\nu)\Upsilon'(\nu)d\nu\\
&=\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\kappa(\nu)\Upsilon'(\nu)d\nu+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\kappa(\nu)\Upsilon'(\nu)d\nu\\
&=I_1+I_2.
\end{aligned}
$$

Now, taking I_1 and after employing the integration by parts

$$
\begin{aligned}
I_1 &= \int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}} \kappa(\nu)\Upsilon'(\nu)d\nu \\
&= \kappa\Big(\frac{\ell_1+\ell_2}{2}\Big)\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big) \\
&- \int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}} \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\mu}{k}-1}} \\
&\times \Theta^m(\nu)\Theta'(\nu)\chi(\nu)d\nu \\
&= \Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big) \\
&\times \int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}} \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\mu}{k}-1}} \\
&\times \Theta^m(\nu)\Theta'(\nu)\chi(\nu)d\nu \\
&- \int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}} \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\nu))^{\frac{\mu}{k}-1}} \\
&\times \Theta^m(\nu)\Theta'(\nu)\chi(\nu)\Upsilon(\nu)d\nu \\
&= \Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\frac{(m+1)^{1-\frac{\mu}{k}}}{k}\Sigma^{\infty}_{\kappa_1,\ldots,\kappa_n=0}\frac{\Pi^n_{i=1}(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma^n_{i=1}(\beta_i\kappa_i))(\kappa_1)!\ldots(\kappa_n)!} \\
&\times \int_{a^+}^{t_1}(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma^n_{i=1}\beta_i\kappa_i}{k}-1}\Theta^m(\nu)\Theta'(\nu)\chi(\nu)d\nu \\
&- \frac{(m+1)^{1-\frac{\mu}{k}}}{k}\Sigma^{\infty}_{\kappa_1,\ldots,\kappa_n=0}\frac{\Pi^n_{i=1}(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma^n_{i=1}(\beta_i\kappa_i))(\kappa_1)!\ldots(\kappa_n)!} \\
&\times \int_{a^+}^{t_1}(\Theta^{m+1}(t_1)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma^n_{i=1}\beta_i\kappa_i}{k}-1}\Theta^m(\nu)\Theta'(\nu)\chi(\nu)\Upsilon(\nu)d\nu \\
&= \frac{(m+1)^{1-\frac{\mu}{k}}}{k}[\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\,{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1) - {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\Upsilon(\ell_1)]. \qquad (2)
\end{aligned}
$$

By similar arguments, it can be verified that

$$
I_2 = \frac{(m+1)^{1-\frac{\mu}{k}}}{k}[\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\,{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2) - {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\Upsilon(\ell_2)]. \qquad (3)
$$

Hence, by adding (2) and (3) we obtain,

$$
\begin{aligned}
&\frac{k}{(m+1)^{1-\frac{\mu}{k}}}\int_{\ell_1}^{\ell_2}\kappa(\nu)\Upsilon'(\nu)d\nu \\
&= \Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2) + {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\Big] \\
&- \Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\Upsilon(\ell_2) + {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\Upsilon(\ell_1)\Big].
\end{aligned}
$$

Hence the required result is proved. □

Corollary 1. *If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $k = 1$ *in Lemma 1 then we have ([28], Lemma 4),*

$$\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\Big[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\ell_1)\Big]$$
$$-\Big[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi\Upsilon(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi\Upsilon(\ell_1)\Big]$$
$$=\frac{1}{\Gamma(\mu)}\int_{\ell_1}^{\ell_2}\kappa(\nu)\Upsilon'(\nu)d\nu,$$

where

$$\kappa(t_1)=\begin{cases}\int_{\ell_1}^{t_1}\frac{\chi(u)du}{(u-\ell_1)^{1-\mu}}, & t_1\in[\ell_1,\frac{\ell_1+\ell_2}{2}],\\ \int_{\ell_2}^{t_1}\frac{\chi(u)du}{(\ell_2-u)^{1-\mu}}, & t_1\in[\frac{\ell_1+\ell_2}{2},\ell_2].\end{cases}$$

Theorem 1. *For $k>0$, $m\in\Re/\{-1\}$, and $Re(\mu)>0$, $Re(\tau)>0$, $Re(\delta)>0$, $Re(\beta)>0$ are complex parameters, Θ be an increasing function and Υ be a differentiable on (ℓ_1,ℓ_2) with $\ell<\ell_2$ and $\chi:[\ell_1,\ell_2]\mapsto\Re$ be bounded. If $\Upsilon',\chi\in L[\ell_1,\ell_2]$, then the below expression holds.*

$$\begin{aligned}&\digamma\Big(\frac{\ell_1+\ell_2}{2}\Big)\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\Big]\\&\le\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\digamma(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\digamma(\ell_1)\Big]\\&\le\frac{\digamma(\ell_1)+\digamma(\ell_2)}{2}\Big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\Big],\end{aligned}\tag{4}$$

where $\digamma(\nu)=\Upsilon(\nu)+\tilde{\Upsilon}(\nu)$, and $\tilde{\Upsilon}(\nu)=\Upsilon(\ell_1+\ell_2-\nu)$.

Proof. As we know that Υ is Cf for all $x_1,y_1\in[\ell_1,\ell_2]$, that is

$$\Upsilon(\frac{x_1+y_1}{2})\le\frac{\Upsilon(x_1)+\Upsilon(y_1)}{2},$$

which implies, for $\nu\in[0,1]$ and substituting $x_1=\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2,\ y_1=\frac{\nu}{2}\ell_2+\frac{2-\nu}{2}\ell_1$

$$2\Upsilon(\frac{\ell_1+\ell_2}{2})\le\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)+\Upsilon(\frac{\nu}{2}\ell_2+\frac{2-\nu}{2}\ell_1).$$

By multiplying

$$\begin{aligned}&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}\\&\times\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}\\&\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2),\end{aligned}$$

and after simplifying and taking integration on $[0,1]$ with respect to ν, we can have

$$
\begin{aligned}
&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}F\big(\frac{\ell_1+\ell_2}{2}\big)\\
&\times\int_0^1\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}\\
&\times\int_0^1\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu\\
&+\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}\\
&\times\int_0^1\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu.
\end{aligned}
$$

By replacing $u=\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2$, we arrive at

$$
\begin{aligned}
&F\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}^{\delta_i,\tau_i}\chi(\ell_2)\big]\\
&\leq\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}^{\delta_i,\tau_i}\chi\Upsilon(\ell_2)+{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}^{\delta_i,\tau_i}\chi\tilde{\Upsilon}(\ell_2)\big],
\end{aligned}
$$

that is

$$
F\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}^{\delta_i,\tau_i}\chi(\ell_2)\big]\leq\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}^{\delta_i,\tau_i}\chi F(\ell_2)\big]. \tag{5}
$$

Continuing in the same way if we multiply the below term on both sides

$$
\begin{aligned}
&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}\\
&\times\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2),
\end{aligned}
$$

then after simplifying and taking integration on $[0,1]$ with respect to ν, we arrive at

$$
F\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}^{\delta_i,\tau_i}\chi(\ell_1)\big]\leq\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}_{\Theta}^{k,m}\Im_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}^{\delta_i,\tau_i}\chi F(\ell_1)\big]. \tag{6}
$$

Adding (5) and (6), we arrive at

$$F\Big(\frac{\ell_1+\ell_2}{2}\Big)\,[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)]$$
$$\leq \Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\,[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi F(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi F(\ell_1)]. \tag{7}$$

This shows the first half of required inequality. Now we are going to prove second half of inequality. Since, we know that Υ is Cf on $[\ell_1,\ell_2]$, after implementation, we reach

$$\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)+\Upsilon(\frac{2-\nu}{2}\tau_1+\frac{\nu}{2}\tau_2)\leq\Upsilon(\tau_1)+\Upsilon(\tau_2).$$

Continuing in the same way if we multiply the below term on both sides as in first part of inequality

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}$$
$$\times\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2),$$

then after simplifying and integrating over $[0,1]$ with respect to ν, we arrive at

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}F(\frac{\ell_1+\ell_2}{2})$$
$$\times\int_0^1\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(+\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu$$
$$\leq\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}$$
$$\times\int_0^1\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu$$
$$+\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}$$
$$\times\int_0^1\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Upsilon(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)d\nu.$$

By replacing $u=\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2$, we obtain

$$\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\,[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\Upsilon(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\tilde{\Upsilon}(\ell_2)]$$
$$\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\,[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)].$$

This implies

$$\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\digamma(\ell_2)\big] \leq \frac{\digamma(\ell_1)+\digamma(\ell_2)}{2}\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)\big]. \tag{8}$$

Now, by multiplying on both sides of the first half of the inequality

$$\begin{aligned}&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{2k}\\ &\times\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\tau_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\tau_1))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)-\Theta^{m+1}(\tau_1))^{\frac{\mu}{k}-1}}\\ &\times\Theta^m(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\Theta'(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2)\chi(\frac{\nu}{2}\ell_1+\frac{2-\nu}{2}\ell_2),\end{aligned}$$

then, after simplifying and integrating over $[0,1]$ with respect to ν, we obtain

$$\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\digamma(\ell_1)\big] \leq \frac{\digamma(\ell_1)+\digamma(\ell_2)}{2}\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\big]. \tag{9}$$

Adding (8) and (9), we obtain

$$\begin{aligned}&\Upsilon\Big(\frac{\ell_1+\ell_2}{2}\Big)\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\digamma(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\digamma(\ell_1)\big]\\ &\leq\frac{\digamma(\ell_1)+\digamma(\ell_2)}{2}\big[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}_{\Theta}^{k,m}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\big].\end{aligned} \tag{10}$$

Hence, by adding (7) and (10), we get our desired inequality. □

Example 1. *Assume that* $\chi(x)=(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\eta}{k}}$, *then the left side of above inequality* (4) *becomes*

$$\begin{aligned}&\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\Big)\Big((\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\eta}{k}}\Big)\\ &=\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\ &\times\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\Big(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)\Big)^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)+\eta}{k}-1}\Theta^m(x)\Theta'(x)dx\\ &=\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\ &\times\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\Big(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)\Big)^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)+\eta}{k}-1}\Theta^m(x)\Theta'(x)dx\\ &=\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\frac{\Big(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\Big)^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)+\eta}{k}}.\end{aligned} \tag{11}$$

Similarly, we have

$$\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\right)\left(\Theta^{m+1}(x)-\Theta^{m+1}(\ell_1))^{\frac{\eta}{k}}\right)$$
$$=\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}\frac{\left(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}. \quad (12)$$

If we choose $\chi(x)=(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\eta}{k}}$ *and* $F(x)=2\Theta^{m+1}(x)$ *then the middle term of above inequality is*

$$\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)2\left((\Theta^{m+1}(x)-\Theta^{m+1}(\ell_1))^{\frac{\eta}{k}}\right)\Theta^{m+1}(x)$$
$$=\frac{2(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}$$
$$\times\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}-1}\Theta^{m+1}(x)\Theta^{m}(x)\Theta^{'}(x)dx$$
$$=\frac{2(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}$$
$$\times\left(\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.$$
$$\left.-\frac{m+1}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}\right) \quad (13)$$

Also

$$\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\right)2\left((\Theta^{m+1}(x)-\Theta^{m+1}(\ell_1))^{\frac{\eta}{k}}\right)\Theta^{m+1}(x)$$
$$=\frac{2(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}$$
$$\times\left(\frac{\left(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.$$
$$\left.-\frac{m+1}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\frac{\left(\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}\right) \quad (14)$$

So we have

$$
\begin{aligned}
&2\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\left[\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.\\
&\left.+\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\frac{\left(\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right]\\
&\leq\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\left[\frac{2(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\right.\\
&\times\left(\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.\\
&\left.-\frac{m+1}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}\right)\\
&+\frac{2(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\\
&\times\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\left(\frac{\left(\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.\\
&\left.\left.-\frac{m+1}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\frac{\left(\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}+1}\right)\right]\\
&\leq[\Theta^{m+1}(\ell_1)+\Theta^{m+1}(\ell_2)]\left[\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.\\
&\left.+\frac{(m+1)^{-\frac{\mu}{k}}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\frac{\left(\Theta^{m+1}\left(\frac{\ell_1+\ell_2}{2}\right)-\Theta^{m+1}(\ell_1)\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right]
\end{aligned}
\tag{15}
$$

The computational values of (15)*, are shown in Table 1, and Table 2 corresponding to the choice of parameters* $m=3$, $k=1$, $\Theta(x)=x^2$, $\eta=0.5$, $\tau_i=1$, $\kappa_i=1$, $\beta_i=1$, $\delta=0.5$ *and for some fixed values of* ℓ_1 *and* ℓ_2*. Figure 1, shows the graphical representation of* (15)*, corresponding to the choice of parameters* $0<\ell_1<2<\ell_2<4$.

For tabular values

Table 1. Tabular form presents the computational values of (15) corresponding to choice $\mu=0.5$.

ℓ_1/ℓ_2	**2.10**	**2.30**
0.10	$(4.25\times10^{00}, 1.23\times10^{01}, 2.19\times10^{01})$	$(5.56\times10^{01}, 1.39\times10^{02}, 3.72\times10^{01})$
0.50	$(6.51\times10^{00}, 2.14\times10^{01}, 2.87\times10^{01})$	$(7.83\times10^{01}, 2.26\times10^{02}, 4.91\times10^{01})$

Table 2. Tabular form presents the computational values of (15) corresponding to choice $\mu = 0.5$.

ℓ_1/ℓ_2	**2.50**	**2.70**
0.10	$(7.48 \times 10^{01}, 1.68 \times 10^{02}, 4.51 \times 10^{01})$	$(8.74 \times 10^{01}, 1.95 \times 10^{02}, 5.63 \times 10^{01})$
0.50	$(9.66 \times 10^{01}, 2.72 \times 10^{02}, 5.74 \times 10^{01})$	$(1.12 \times 10^{02}, 3.06 \times 10^{02}, 6.85 \times 10^{01})$

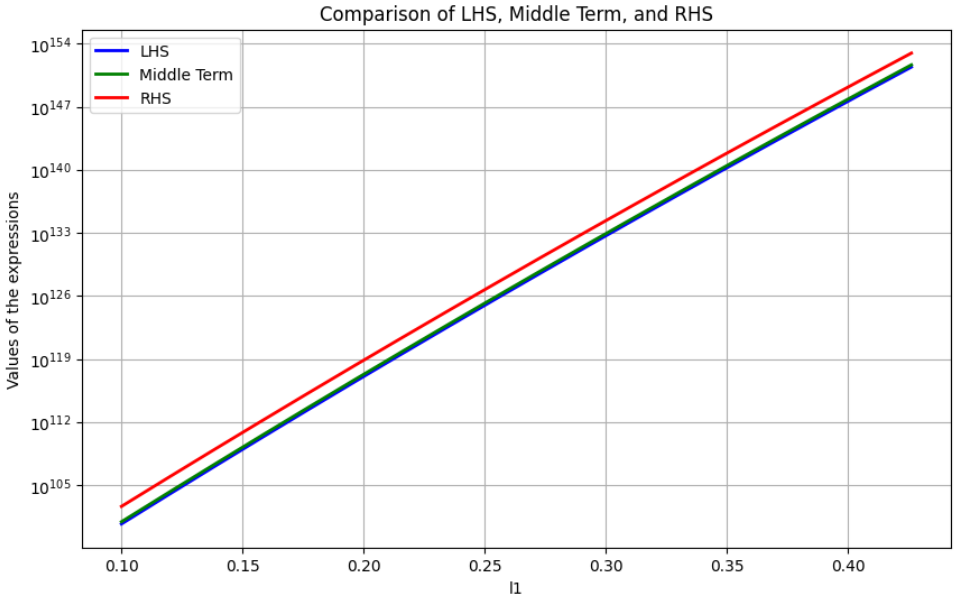


Figure 1. The graphical representation of (15) corresponding to choice $\mu = 0.5$.

Corollary 2. *If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $k = 1$ *in Theorem 1, choose* χ *is symmetric about* $\frac{\ell_1+\ell_2}{2}$, *then we have*

$$\begin{aligned}
&F\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\ell_1)\right]\\
&\leq \Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi F(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi F(\ell_1)\right]\\
&\leq \frac{F(\ell_1)+F(\ell_2)}{2}\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\ell_1)\right].
\end{aligned}$$

Theorem 2. *For* $k > 0$, $m \in \Re/\{-1\}$, *and* $Re(\mu) > 0$, $Re(\tau) > 0$, $Re(\delta) > 0$, $Re(\beta) > 0$ *are complex parameters,* Θ *be an increasing function and* Υ *be a differentiable on* (ℓ_1, ℓ_2) *with* $\ell < \ell_2$ *and* $\chi : [\ell_1, \ell_2] \mapsto \Re$ *be bounded. If* $\Upsilon', \chi \in L[\ell_1, \ell_2]$ *and* $|\Upsilon'| \in K_s^2$ *on* $[\ell_1, \ell_2]$ *for any fixed* $s \in (0, 1]$, *then the H-H-F kind inequality for generalized RLFI holds.*

$$\begin{aligned}
&\left|F\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right|\\
&-\left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi F(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi F(\ell_1)\right]\right|\\
&\leq \frac{(m+1)^{\frac{\mu}{k}-1}||\chi||_{\infty}}{(\ell_2-\ell_1)^s(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\Big(|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|\Big)\\
&\times\left[\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\right)\right.\\
&\times\left.\left(\Big(\ell_2-\nu\Big)^s+\Big(\nu-\ell_1\Big)^s\right)d\nu\right],
\end{aligned}$$

where F *fulfilled the condition as in Theorem 1.*

Proof. By using defined s-convexity of second kind, we can write

$$\begin{aligned}
&|F'(\nu)| = |\Upsilon'(\nu) - \Upsilon'(\ell_1+\ell_2-\nu)| \le |\Upsilon'(\nu)| + |\Upsilon'(\ell_1+\ell_2-\nu)| \\
&= \left|\Upsilon'\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\ell_1 + \frac{\nu-\ell_1}{\ell_2-\ell_1}\ell_2\right)\right| + \left|\Upsilon'\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\ell_1 + \frac{\ell_2-\nu}{\ell_2-\ell_1}\ell_2\right)\right| \\
&\le \left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s |\Upsilon'(\ell_1)| \\
&+ \left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s |\Upsilon'(\ell_2)| + \left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s |\Upsilon'(\ell_1)| + \left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s |\Upsilon'(\ell_2)| \\
&= \left(\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s + \left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s\right)\Big(|\Upsilon'(\ell_1)| + |\Upsilon'(\ell_2)|\Big).
\end{aligned}$$

Using Theorem 1, we reach

$$\begin{aligned}
&\left|F\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2) + {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right| \\
&- \left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi F(\ell_2) + {}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi F(\ell_1)\right]\right| \\
&\le \frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\ell_2}|\kappa(\nu)||F'(\nu)|d\nu \\
&\le \frac{(m+1)^{\frac{\mu}{k}-1}}{k} \\
&\times \left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|\int_{\ell_1}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\right.\right. \\
&\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right| \\
&\times\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|\int_{\ell_2}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\mu}{k}-1}}\right. \\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|\right)F'(\nu)d\nu \\
&\le \frac{(m+1)^{\frac{\mu}{k}-1}}{k} \\
&\times \left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|\int_{\ell_1}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\ell_1(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\right.\right. \\
&\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right| \\
&+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|\int_{\ell_2}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\mu}{k}-1}}\right. \\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|\right)F'(\nu)d\nu \\
&\le \frac{(m+1)^{\frac{\mu}{k}-2}||\chi||_\infty}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))} \\
&\times\left[\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\frac{[\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1)]^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}}{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}} + \int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{[\Theta^{m+1}(\ell_2)-\nu^{s+}]^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}}{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right)\right. \\
&\left.\times\left(\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s + \left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s\right)\Big(|\Upsilon'(\ell_1)| + |\Upsilon'(\ell_2)|d\nu\Big)\right] \\
&\le \frac{(m+1)^{\frac{\mu}{k}-2}||\chi||_\infty}{(\ell_2-\ell_1)^s(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Big(|\Upsilon'(\ell_1)| + |\Upsilon'(\ell_2)|\Big) \\
&\times\left[\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}} + \int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right)\right. \\
&\left.\times\left(\Big(\ell_2-\nu\Big)^s + \Big(\nu-\ell_1\Big)^s\right)d\nu\right].
\end{aligned}$$

We reach at desired outcome. □

Corollary 3. *If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $p = k = 1$ *in Theorem 2 and* χ *is symmetric about* $\frac{\ell_1+\ell_2}{2}$, *then we have*

$$\begin{aligned}
&\left|F\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\ell_1)\right]\right| - \left|\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi F(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi F(\ell_1)\right]\right| \\
&\leq \frac{(m+1)^{1-\mu}(\ell_2-\ell_1)}{(\ell_2-\ell_1)\Gamma_k(\mu)}||\chi||_\infty \\
&\times\left(|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|\right)\left[\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}(\nu-\ell_1)^s d\nu+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}(\ell_2-\nu)^s\right)d\nu\right] \\
&= \frac{(m+1)^{1-\mu}(\ell_2-\ell_1)^{\mu+1}}{(\mu+1)\Gamma(\mu+1)2^{\mu+1}}||\chi||_\infty\left(|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|\right).
\end{aligned}$$

Proposition 1. *Furthermore, If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $\chi = k = 1$ *in Corollary 3, so we obtain the below mid point inequality*

$$\left|(\ell_2-\ell_1)\Upsilon(\frac{\ell_1+\ell_2}{2})-\int_{\ell_1}^{\ell_2}\Upsilon(\nu)d\nu\right| \leq \frac{(\ell_2-\ell_1)^2}{8}\left(|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|\right).$$

Theorem 3. *For* $k > 0$, $m \in \Re/\{-1\}$, *and* $Re(\mu) > 0$, $Re(\tau) > 0$, $Re(\delta) > 0$, $Re(\beta) > 0$ *are complex parameters,* Θ *be an increasing function and* Υ *be a differentiable on* (ℓ_1, ℓ_2) *with* $\ell < \ell_2$ *and* $\chi : [\ell_1, \ell_2] \mapsto \Re$ *be bounded. If* $\Upsilon', \chi \in L[\ell_1, \ell_2]$, *then the below expression holds.*

$$\begin{aligned}
&\left|F\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right| \\
&-\left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi F(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi F(\ell_1)\right]\right| \\
&\leq \frac{(m+1)^{-\frac{\mu}{k}}||\chi||_\infty}{(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))} \\
&\times\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left(\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|d\nu\right)^{1-\frac{1}{q}}\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|\right.\right. \\
&\left.\left.\times\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)\right)^{\frac{1}{q}} \\
&+\frac{(m+1)^{-\frac{\mu}{k}}|\chi||_\infty}{(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))} \\
&\times\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left(\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|d\nu\right)^{1-\frac{1}{q}}\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|\right.\right. \\
&\left.\left.\times\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. By using Theorem 1, we can write

$$
\begin{aligned}
&\left|\digamma\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right|\\
&-\left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\digamma(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\digamma(\ell_1)\right]\right|\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\ell_2}|\kappa(\nu)||\digamma'(\nu)|d\nu\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\Big(\int_{\ell_1}^{\ell_2}|\kappa(\nu)|d\nu\Big)^{1-\frac{1}{q}}\Big(\int_{\ell_1}^{\ell_2}|\kappa(\nu)||\digamma'(\nu)|^q d\nu\Big)^{\frac{1}{q}}\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\times\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|\int_{\ell_1}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\right.\right.\\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|d\nu\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|\int_{\ell_1}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\right.\right.\\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|\right)\left(\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)^{\frac{1}{q}}\\
&+\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\times\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|\int_{\ell_2}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\mu}{k}-1}}\right.\right.\\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|d\nu\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|\int_{\ell_2}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\ell_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\mu}{k}-1}}\right.\right.\\
&\left.\left.\times\Theta^m(u)\Theta'(u)\chi(u)du\right|\right)\left(\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)^{\frac{1}{q}}\\
&=\frac{(m+1)^{-\frac{\mu}{k}}||\chi||_\infty}{(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\
&\times\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left(\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|d\nu\right)^{1-\frac{1}{q}}\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|\right.\right.\\
&\left.\left.\times\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)\right)^{\frac{1}{q}}\\
&+\frac{(m+1)^{-\frac{\mu}{k}}|\chi||_\infty}{(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\
&\times\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left(\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|d\nu\right)^{1-\frac{1}{q}}\left(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|\right.\right.\\
&\left.\left.\times\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

This is our desired result. □

Remark 1. *For the choice of* $s = 1$, $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $k = 1$ *in the Theorem 3, then we obtain ([27], Theorem 7).*

$$\begin{aligned}
&\left| F\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\tau_1)\right]\right| \\
&-\left|\left[\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^+}\chi F(\ell_2)+\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^-}\chi F(\tau_1)\right]\right| \\
&\leq \frac{||\chi||_{\infty,[\tau_1,\ell_2]}(\ell_2-\tau_1)^{\varrho+1}}{2^{\varrho+1+\frac{1}{q}}(\varrho+1)(\varrho+2)^{\frac{1}{q}}k\Gamma_k(\varrho+1)}\left[\left((\varrho+3)|\Upsilon'(\tau_1)|^q+(\varrho+1)|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left((\varrho+1)|\Upsilon'(\tau_1)|^q+(\varrho+3)|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 2. *For the choice of* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $k = 1$ *in the Theorem 3, then we obtain ([57], Theorem 5).*

$$\begin{aligned}
&\left| F\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^-,\Theta}\chi(\tau_1)\right]\right| \\
&-\left|\left[\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^+}\chi F(\ell_2)+\mathfrak{J}^{\varrho}_{(\frac{\ell_1+\ell_2}{2})^-}\chi F(\tau_1)\right]\right| \\
&\leq \frac{||\chi||_{\infty,[\tau_1,\ell_2]}(\ell_2-\tau_1)^{\varrho+1}}{2^{\varrho+1+\frac{1}{q}}(\varrho+1)(\varrho+s+1)^{\frac{1}{q}}k\Gamma_k(\varrho+1)}\left[\left(2^{\varrho+1}(\varrho+1)(\varrho+s+1)B_{\frac{1}{2}}(\varrho+1,s+1)|\Upsilon'(\tau_1)|^q\right.\right. \\
&\left.+2^{1-s}(\varrho+1)|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}} \\
&\left.\left(2^{\varrho+1}(\varrho+1)(\varrho+s+1)B_{\frac{1}{2}}(\varrho+1,s+1)|\Upsilon'(\ell_2)|^q+2^{1-s}(\varrho+1)|\Upsilon'(\tau_1)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 4. *For the choice of* $s = 1$, $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $k = 1$ *in the Theorem 3, then we arrive at the following mid-point inequality*

$$\begin{aligned}
&\left|\Upsilon(\frac{\ell_1+\ell_2}{2})\int_{\ell_1}^{\ell_2}\chi(\nu)d\nu-\int_{\ell_1}^{\ell_2}(\Upsilon\chi)(\nu)d\nu\right| \\
&\leq \frac{||\chi||(\ell_2-\ell_1)^2}{(3)^{\frac{1}{q}}8}\left[\left(2|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}+\left(|\Upsilon'(\ell_1)|^q+2|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 4. *For* $k > 0$, $m \in \Re/\{-1\}$, *and* $Re(\mu) > 0$, $Re(\tau) > 0$, $Re(\delta) > 0$, $Re(\beta) > 0$ *are complex parameters,* Θ *be an increasing function and* Υ *be a differentiable on* (ℓ_1, ℓ_2) *with* $\ell < \ell_2$ *and* $\chi : [\ell_1, \ell_2] \mapsto \Re$ *be bounded. If* $\Upsilon', \chi \in L[\ell_1, \ell_2]$, *then the below expression holds.*

$$\begin{aligned}
&\left| F\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right| \\
&-\left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi F(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi F(\ell_1)\right]\right| \\
&\leq \frac{(m+1)^{-\frac{\mu}{k}}||\chi||_{\infty}(\ell_2-\ell_1)^{\frac{1}{q}}}{2^{\frac{s}{q}+\frac{1}{q}}(s+1)^{\frac{1}{q}}(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))} \\
&\times\{\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left(\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{p(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}{k}}\right|d\nu\right)^{\frac{1}{p}}\times\left((2^{m+1}-1)(|\Upsilon'(\ell_1))|^q+|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}} \\
&+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left(\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{p(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}{k}}\right|d\nu\right)^{\frac{1}{p}}(|\Upsilon'(\ell_1)|)^q+\left((2^{m+1}-1)|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\}.
\end{aligned}$$

Proof. Using Hölder's inequality, we have

$$
\begin{aligned}
&\left|\digamma\left(\frac{\ell_1+\ell_2}{2}\right)\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi(\ell_1)\right]\right|\\
&-\left|\left[{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\chi\digamma(\ell_2)+{}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}\chi\digamma(\ell_1)\right]\right|\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\ell_2}|\digamma'(\nu)|d\nu\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\Big(\int_{\ell_1}^{\ell_2}|\kappa(\nu)|^p d\nu\Big)^{\frac{1}{p}}\Big(\int_{\ell_1}^{\ell_2}|\digamma'(\nu)|^q d\nu\Big)^{\frac{1}{q}}\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\times\Bigg(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\Bigg(\Bigg|\int_{\ell_1}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(u)-\Theta^{m+1}(\ell_1))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(u)\Theta'(u)\chi(u)du\Bigg|\Bigg)^p d\nu\Bigg)^{\frac{1}{p}}\Bigg(\int_{\ell_1}^{\nu}\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s|\Upsilon'(\ell_2)|^q d\nu\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\Bigg(\Bigg|\int_{\ell_2}^{\nu}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(u))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(u)\Theta'(u)\chi(u)du\Bigg|\Bigg)^p d\nu\Bigg)^{\frac{1}{p}}\Bigg(\int_{\nu}^{\ell_2}\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s|\Upsilon'(\ell_2)|^q d\nu\Bigg)^{\frac{1}{q}}\\
&=\frac{(m+1)^{\frac{\mu}{k}}||\chi||_\infty}{(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\
&\times\Bigg(\Bigg(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|^p d\nu\Bigg)^{\frac{1}{p}}\\
&\times\Bigg(\int_{\ell_1}^{\nu}\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|^p d\nu\Bigg)^{\frac{1}{p}}\\
&\times\Bigg(\int_{\nu}^{\ell_2}\left(\frac{\ell_2-\nu}{\ell_2-\ell_1}\right)^s(|\Upsilon'(\ell_1)|^q+\left(\frac{\nu-\ell_1}{\ell_2-\ell_1}\right)^s)|\Upsilon'(\ell_2)|^q d\nu\Bigg)^{\frac{1}{q}}\Bigg)\\
&=\frac{(m+1)^{-\frac{\mu}{k}}||\chi||_\infty}{(\ell_2-\ell_1)^{\frac{s}{q}}(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\
&\times\{\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\Bigg(\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|^p d\nu\Bigg)^{\frac{1}{p}}\\
&\times\Bigg(\frac{(\ell_2-\ell_1)^{m+1}(2^{m+1}-1)}{2^{m+1}(s+1)}(|\Upsilon'(\ell_1))|)^q+\frac{(\ell_2-\ell_1)^{m+1}}{2^{m+1}(s+1)}|\Upsilon'(\ell_2)|^q\Bigg)^{\frac{1}{q}}\\
&+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\Bigg(\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{\mu+\Sigma_{i=1}^n(\beta_i\kappa_i)}{k}}\right|^p d\nu\Bigg)^{\frac{1}{p}}\\
&\times\frac{(\ell_2-\ell_1)^{m+1}}{2^{m+1}(s+1)}(|\Upsilon'(\ell_1)|)^q+\Bigg(\frac{(\ell_2-\ell_1)^{m+1}(2^{m+1}-1)}{2^{m+1}(s+1)}|\Upsilon'(\ell_2)|^q\Bigg)^{\frac{1}{q}}\}\\
&\leq\frac{(m+1)^{-\frac{\mu}{k}}||\chi||_\infty(\ell_2-\ell_1)^{\frac{1}{q}}}{2^{\frac{s}{q}+\frac{1}{q}}(s+1)^{\frac{1}{q}}(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\Sigma_{n=0}^\infty\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}\\
&\times\{\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\Bigg(\left|(\Theta^{m+1}(\nu)-\Theta^{m+1}(\ell_1))^{\frac{p(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}{k}}\right|d\nu\Bigg)^{\frac{1}{p}}\times\Bigg((2^{m+1}-1)(|\Upsilon'(\ell_1))|)^q+|\Upsilon'(\ell_2)|^q\Bigg)^{\frac{1}{q}}\\
&+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\Bigg(\left|(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu))^{\frac{p(\mu+\Sigma_{i=1}^n(\beta_i\kappa_i))}{k}}\right|d\nu\Bigg)^{\frac{1}{p}}\times(|\Upsilon'(\ell_1)|)^q+\Bigg((2^{m+1}-1)|\Upsilon'(\ell_2)|^q\Bigg)^{\frac{1}{q}}\}
\end{aligned}
$$

Which is our desired result. □

Corollary 5. *If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $s = k = 1$ *then we obtain ([28], Theorem 8).*

$$\left|F\left(\frac{\ell_1+\ell_2}{2}\right)\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi(\ell_1)\right]\right| - \left|\left[\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^+}\chi F(\ell_2)+\mathfrak{J}^{\mu}_{(\frac{\ell_1+\ell_2}{2})^-}\chi F(\ell_1)\right]\right|$$
$$\leq \frac{||\chi||_{\infty}(\ell_2-\ell_1)^{\frac{1}{q}}}{2^{\frac{3}{q}}\Gamma_(\mu+1)}\left(\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\left(\left|(\nu-\ell_1)^{p\mu}\right|d\nu\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(3(|\Upsilon'(\ell_1))|^q+|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}+\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\left(\left|(\ell_2-\nu)^{p\mu}\right|d\nu\right)^{\frac{1}{p}}(|\Upsilon'(\ell_1)|)^q+\left(3|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\right).$$

Proposition 2. *If we replace* $\Theta(u) = u$, $m = 0$, $\delta_i = 0$, $\kappa_i = 0, \forall i$ *and* $s = k = 1$ *we obtain the following*

$$\left|\Upsilon(\frac{\ell_1+\ell_2}{2})\int_{\ell_1}^{\ell_2}\chi(\nu)d\nu-\int_{\ell_1}^{\ell_2}(\Upsilon\chi)(\nu)d\nu\right|$$
$$\leq\frac{||\chi||(\ell_2-\ell_1)^2}{(m+1)^{\frac{1}{p}}2^{\frac{4p-2}{p}}}\left[\left(3|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}+\left(|\Upsilon'(\ell_1)|^q+3|\Upsilon'(\ell_2)|^q\right)^{\frac{1}{q}}\right].$$

3. Generalized H-H-F Type Inequalities Involving h-Convexity Associated with an Extended Fractional Operator and M-M-L Function

This section's first result reveals the M-M-L function for a generalized h-Cf and the generalized H-H-F inequality related with an extended fractional operator. In order to achieve this, the generalized supermultiplicative (s-m) mappings are necessary.

Definition 10. *Let us consider* $\aleph, \Im \in \Re$ *be invertals,* $\Im \geq (0,1)$ *and* $h \neq 0 : \Im \to \Re$ *be a positive. Also, a non-negative function* $f : \aleph \to \Re$ *is known as h-Cf if the inequality given below*

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y)$$

holds for all $x, y \in \aleph, t \in (0,1)$.

Definition 11. *A mapping* $h : \Im \subseteq \Re \to \Re$ *is known as generalized s-m if*

$$h(xy) \geq h(x)h(y)$$

holds for all $x, y \in \Im$.

Theorem 5. *For* $k > 0$, $m \in \Re/\{-1\}$, *and* $Re(\mu) > 0$, $Re(\tau) > 0$, $Re(\delta) > 0$, $Re(\beta) > 0$ *are complex parameters, and* Θ *be an increasing function on* (ℓ_1, ℓ_2) *with* $\ell < \ell_2$, *also* $f : [\ell_1, \ell_2] \to \Re$ *be the geneeralized h-Cf and* $w : [\ell_1, \ell_2] \to \Re$, $w \geq 0$ *be symmetric pertaining to* $\frac{\ell_1+\ell_2}{2}$. *If* $f(x), w(x) \in ({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})[\ell_1, \ell_2]$, *then we have*

$$({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})f(y)w(y)$$
$$\leq\frac{f(\ell_1)+f(\ell_2)}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h(\frac{x-\ell_2}{\ell_1-\ell_2})w(x)(y)+({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^-;\mu,\beta_i}h\big(1-\frac{x-\ell_2}{\ell_1-\ell_2}\big)w(x)(y)\right). \tag{16}$$

Proof. Let $\ell_1 < x < \frac{\ell_1+\ell_2}{2} < y < \ell_2$ exists $\nu \in (0,1)$ such that

$$x = \nu\ell_1 + (1-\nu)\ell_2$$
$$y = \nu\ell_2 + (1-\nu)\ell_1.$$

As we know that f is generalized h-Cf, we have

$$f(\nu\ell_1 + (1-\nu)\ell_2)w\big(\nu\ell_1 + (1-\nu)\ell_2\big) \leq h(\nu)f(\ell_1) + h(1-\nu)f(\ell_2))w\big(\nu\ell_1 + (1-\nu)\ell_2\big), \tag{17}$$

and

$$f((1-\nu)\ell_1 + \nu\ell_2)w\big((1-\nu)\ell_1 + \nu\ell_2\big) \leq h(1-\nu)f(\ell_1) + h(\nu)f(\ell_2))w\big((1-\nu)\ell_1 + \nu\ell_2\big). \tag{18}$$

By multiplying on both sides of (17),

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k} \times \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_1}{k}}, \ldots, \tau_n(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\mu}{k}-1}} \times \Theta^m(\nu\ell_1 + (1-\nu)\ell_2)\Theta'(\nu\ell_1 + (1-\nu)\ell_2),$$

and after simplifying and taking integration on $[0,1]$ with respect to ν, we can have

$$\begin{aligned}
&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k} \times \int_0^1 \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_1}{k}}, \ldots, \tau_n(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\mu}{k}-1}} \\
&\times \Theta^m(\nu\ell_1 + (1-\nu)\ell_2)\Theta'(\nu\ell_1 + (1-\nu)\ell_2)f(\nu\ell_1 + (1-\nu)\ell_2)w(\nu\ell_1 + (1-\nu)\ell_2)d\nu \\
&\leq \frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k} \times \int_0^1 \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_1}{k}}, \ldots, \tau_n(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2) - \Theta^{m+1}(\nu\ell_1 + (1-\nu)\ell_2))^{\frac{\mu}{k}-1}} \\
&\times \Theta^m(\nu\ell_1 + (1-\nu)\ell_2)\Theta'(\nu\ell_1 + (1-\nu)\ell_2)\big((h(\nu)f(\ell_1) + h(1-\nu)f(\ell_2)\big)w(\nu\ell_1 + (1-\nu)\ell_2)d\nu.
\end{aligned}$$

By replacing $x = (\nu\ell_1 + (1-\nu)\ell_2)$, we have

$$\begin{aligned}
&({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})f(y)w(y) \\
&\leq f(\ell_1)({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h\big(\frac{x-\ell_2}{\ell_1-\ell_2}\big)w(x))(y) + f(\ell_2)({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h\big(1-\frac{x-\ell_2}{\ell_1-\ell_2}\big)w(x))(y).
\end{aligned} \tag{19}$$

Continuing in the same way if we multiply the below term on both sides of (18)

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}$$
$$\times \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_2+(1-\nu)\ell_1))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_2+(1-\nu)\ell_1))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_2+(1-\nu)\ell_1))^{\frac{\mu}{k}-1}}$$
$$\times \Theta^m(\nu\ell_2+(1-\nu)\ell_1)\Theta'(\nu\ell_2+(1-\nu)\ell_1),$$

then after simplifying and taking integration on $[0,1]$ with respect to ν, we arrive at

$$({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}f(y)w(y)$$
$$\leq f(\ell_2)({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h(\frac{x-\ell_2}{\ell_1-\ell_2})w(x))(y)+f(\ell_1)({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h(1-\frac{x-\ell_2}{\ell_1-\ell_2})w(x))(y). \tag{20}$$

By adding (19) and (20), we obtain (16). □

Theorem 6. *For $k>0$, $m\in\Re/\{-1\}$, and $Re(\mu)>0$, $Re(\tau)>0$, $Re(\delta)>0$, $Re(\beta)>0$ are complex parameters, and Θ be an increasing function on (ℓ_1,ℓ_2) with $\ell_1<\ell_2$, also $f:[\ell_1,\ell_2]\to\Re$ be the geneeralized h-Cf with h define on $[0,max(1,\ell_2-\ell_1)]$ and $w:[\ell_1,\ell_2]\to\Re$, $w\geq 0$ be symmetric pertaining to $\frac{\ell_1+\ell_2}{2}$ and $({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})w>0$. [i] Then we have*

$$f(\frac{\ell_1+\ell_2}{2})\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a^+;\mu,\beta_i}\left(w(x)+w(\ell_1+\ell_2-x)\right)\right)(\ell_2)$$
$$\leq 2h(\frac{1}{2})\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a^+;\mu,\beta_i}\left(w(x)f(x)+w(\ell_1+\ell_2-x)f(\ell_1+\ell_2-x)\right)\right)(\ell_2). \tag{21}$$

[ii] Furthemore, if h, is generalized s-m and f is non-negative and

$$\frac{(m+1)^{\frac{\mu}{k}-1}}{k}$$
$$\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((1-\nu)\ell_1+(\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((1-\nu)\ell_1+(\nu)\ell_2)))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+(\nu)\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(1-\nu)\ell_1+(\nu)\ell_2)\Theta'(1-\nu)\ell_1+(\nu)\ell_2)$$
$$\times\left(\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\right.$$
$$\left.\int_{\ell_2}^{\frac{\ell_1+\ell_2}{2}}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}\right.$$
$$\left.\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+(1-\nu)\ell_2)h(y-x)w(x)dx\right)w(y)dy\neq 0, h(x)>0, \tag{22}$$

then, we have

$$
\begin{aligned}
&f\Big(\frac{\ell_1+\ell_2}{2}\Big)\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((1-\nu)\ell_1+(\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((1-\nu)\ell_1+(\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+(\nu)\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(1-\nu)\ell_1+(\nu)\ell_2\Theta'(1-\nu)\ell_1+(\nu)\ell_2)\\
&\times\Bigg(\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\int_{\ell_2}^{\frac{\ell_1+\ell_2}{2}}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}((\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+(1-\nu)\ell_2)h(y-x)w(x)dx\Bigg)w(y)dy \qquad (23)\\
&\leq\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h\big(y-\frac{\ell_1+\ell_2}{2}\big)\Big)(\ell_2)\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{\tau_1^+;\mu,\beta_i}f(x)w(x)\Big)(\ell_2)\\
&+\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}f(y)w(y)\Big)(\ell_2)\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{\tau_1^+;\mu,\beta_i}h\Big(\frac{\ell_1+\ell_2}{2}-x\Big)\Big)(\ell_2). \qquad (24)
\end{aligned}
$$

Proof. (i) Using the h-Cf if $\nu=\frac{1}{2}$, $x=\nu\ell_1+(1-\nu)\ell_2$ and $y=(1-\nu)\ell_1+(\nu)\ell_2$, then we have that

$$
\begin{aligned}
&f\Big(\frac{\ell_1+\ell_2}{2}\Big)\leq h(\frac{1}{2})\Big[f(\nu\ell_1+(1-\nu)\ell_2)+f(1-\nu)\ell_1+(\nu)\ell_2)\Big]\\
&\Leftrightarrow w(\nu\ell_1+(1-\nu)\ell_2)f\Big(\frac{\ell_1+\ell_2}{2}\Big)+w(1-\nu)\ell_1+(\nu)\ell_2)f\Big(\frac{\ell_1+\ell_2}{2}\Big)\\
&\leq 2h(\frac{1}{2})\Big[w(\nu\ell_1+(1-\nu)\ell_2)f(\nu\ell_1+(1-\nu)\ell_2)\\
&+w(1-\nu)\ell_1+(\nu)\ell_2)f(1-\nu)\ell_1+(\nu)\ell_2)\Big]. \qquad (25)
\end{aligned}
$$

By multiplying on both sides of (25),

$$
\begin{aligned}
&\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+((1-\nu)\ell_2),
\end{aligned}
$$

and after simplifying and taking integration on $[0,1]$ with respect to ν, we have

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}$$
$$\times \int_0^1 \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times \Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+((1-\nu)\ell_2)$$
$$\Big(w(\nu\ell_1+(1-\nu)\ell_2)f(\frac{\ell_1+\ell_2}{2})+w(1-\nu)\ell_1+\nu\ell_2)f(\frac{\ell_1+\ell_2}{2})\Big)d\nu$$
$$\leq 2h(\frac{1}{2})\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}$$
$$\times \int_0^1 \frac{\xi^{\delta_i}_{k,\mu,\beta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times \Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+((1-\nu)\ell_2)$$
$$\times \Big[w\big(\nu\ell_1+(1-\nu)\ell_2\big)f(\nu\ell_1+(1-\nu)\ell_2)+w\big((1-\nu)\ell_1+(\nu)\ell_2\big)f\big((1-\nu)\ell_1+(\nu)\ell_2\big)\Big]d\nu.$$

By replacing $x=\nu\ell_1+(1-\nu)\ell_2$, so we have

$$f(\frac{\ell_1+\ell_2}{2})\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a^+;\mu,\beta_i}\Big(w(x)+w(\ell_1+\ell_2-x)\Big)\Big)(\ell_2)$$
$$\leq 2h(\frac{1}{2})\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{a^+;\mu,\beta_i}\Big(w(x)f(x)+w(\ell_1+\ell_2-x)f(\ell_1+\ell_2-x)\Big)\Big)(\ell_2).$$

(ii) Let h be generalized s-m and $h(\nu)>0$ for $\nu\in[0,max(1,\ell_2-\ell_1)]$. For any $x,y\in[\ell_1,\ell_2]$ such that $\ell_1<x<\frac{\ell_1+\ell_2}{2}<y\leq\ell_2$, we have

$$\frac{\ell_1+\ell_2}{2}=\Big(\frac{y-\frac{\ell_1+\ell_2}{2}}{y-x}\Big)x+\Big(\frac{\frac{\ell_1+\ell_2}{2}-x}{y-x}\Big)y\Rightarrow f(\frac{\ell_1+\ell_2}{2})$$
$$\leq h\Big(\frac{y-\frac{\ell_1+\ell_2}{2}}{y-x}\Big)f(x)+h\Big(\frac{\frac{\ell_1+\ell_2}{2}-x}{y-x}\Big)f(y). \tag{26}$$

Since h is s-m

$$h\Big(\frac{y-\frac{\ell_1+\ell_2}{2}}{y-x}\Big)\leq\frac{h(y-\frac{\ell_1+\ell_2}{2})}{h(y-x)} \tag{27}$$

and

$$h\Big(\frac{\frac{\ell_1+\ell_2}{2}-x}{y-x}\Big)\leq\frac{h(\frac{\ell_1+\ell_2}{2}-x)}{h(y-x)}. \tag{28}$$

When $f(x)\geq 0$ we have

$$h(y-x)f\Big(\frac{\ell_1+\ell_2}{2}\Big)\leq h\Big(y-\frac{\ell_1+\ell_2}{2}\Big)f(x)+h\Big(\frac{\ell_1+\ell_2}{2}-x\Big)f(y). \tag{29}$$

By multiplying weighted iterated integrals on both sides of (29),

$$\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}$$
$$\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\tau_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\mu}{k}-1}}$$
$$\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+((1-\nu)\ell_2)w(y),$$

and after simplifying and taking integration of $w(x)$ and $w(y)$ on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, respectively, we have

$$\begin{aligned}
&f(\tfrac{\ell_1+\ell_2}{2})\,\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_{\frac{1}{2}}^{1}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(1-\nu)\ell_1+\nu\ell_2)\Theta'(1-\nu)\ell_1+\nu\ell_2)\\
&\times\Bigg(\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_0^{\frac{1}{2}}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{1-\frac{\delta}{k}}}\\
&\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+(1-\nu)\ell_2)h(y-x)w(x)dx\Bigg)w(y)dy\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_{\frac{1}{2}}^{1}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(1-\nu)\ell_1+\nu\ell_2)\Theta'(1-\nu)\ell_1+\nu\ell_2)h\Big(y-\frac{\ell_1+\ell_2}{2}\Big)dy\\
&\times\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_0^{\frac{1}{2}}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{1-\frac{\delta}{k}}}\\
&\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+(1-\nu)\ell_2)f(x)w(x)dx\\
&+\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_{\frac{1}{2}}^{1}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(1-\nu)\ell_1+\nu\ell_2))^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(1-\nu)\ell_1+\nu\ell_2)\Theta'(1-\nu)\ell_1+\nu\ell_2)f(y)w(y)dy\\
&\times\frac{(m+1)^{\frac{\mu}{k}-1}(\ell_2-\ell_1)}{k}\\
&\times\int_0^{\frac{1}{2}}\frac{\xi^{\delta_i}_{k,\mu,\beta_i}\left(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{\frac{\beta_n}{k}}\right)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\nu\ell_1+(1-\nu)\ell_2))^{1-\frac{\delta}{k}}}\\
&\times\Theta^m(\nu\ell_1+(1-\nu)\ell_2)\Theta'(\nu\ell_1+(1-\nu)\ell_2)h\Big(\frac{\ell_1+\ell_2}{2}-x\Big)dx.
\end{aligned}$$

By replacing $x=\nu\ell_1+(1-\nu)\ell_2)$ and $y=(1-\nu)\ell_1+(\nu)\ell_2)$, then we have

$$
\begin{aligned}
&f\Big(\frac{\ell_1+\ell_2}{2}\Big)\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\\
&\times\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\mu}{k}-1}}\Theta^m(y)\Theta'(y)\\
&\times\Bigg(\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(x)\Theta'(x)h(y-x)w(x)dx\Bigg)w(y)dy\\
&\leq\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(y)\Theta'(y)h\Big(y-\frac{\ell_1+\ell_2}{2}\Big)dy\\
&\times\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(x)\Theta'(x)f(x)w(x)dx\\
&+\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\frac{\ell_1+\ell_2}{2}}^{\ell_2}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(y)^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(y)\Theta'(y)f(y)w(y)dy\\
&\times\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\int_{\ell_1}^{\frac{\ell_1+\ell_2}{2}}\frac{\xi_{k,\mu,\beta_i}^{\delta_i}\Big(\tau_1(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\beta_1}{k}},\ldots,\tau_n(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x))^{\frac{\beta_n}{k}}\Big)}{(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(x)^{\frac{\mu}{k}-1}}\\
&\times\Theta^m(x)\Theta'(x)h\Big(\frac{\ell_1+\ell_2}{2}-x\Big)dx\\
&=\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}h\big(y-\frac{\ell_1+\ell_2}{2}\big)\Big)(\ell_2)\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{\tau_1^+;\mu,\beta_i}f(x)w(x)\Big)(\ell_2)\\
&+\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}f(y)w(y)\Big)(\ell_2)\Big({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{\tau_1^+;\mu,\beta_i}h\Big(\frac{\ell_1+\ell_2}{2}-x\Big)\Big)(\ell_2).
\end{aligned}
$$

□

Lemma 2. *If $f:[\ell_1,\ell_2]\to\Re$ is generalized h-Cf, then identity given below*

$$f(\ell_1+\ell_2-x)\leq\Big(h(\nu)+h(1-\nu)\Big)\Big[f(\ell_1)+f(\ell_2)\Big]-f(x),\tag{30}$$

hold for all $x\in[\ell_1,\ell_2]$ and $\nu\in[0,1]$.

Proof. We skip the proof of Lemma 2 here, because essentially it is very identical with that of (Lemma 1, [58]). □

Theorem 7. *For $k>0$, $m\in\Re/\{-1\}$, and $Re(\mu)>0$, $Re(\tau)>0$, $Re(\delta)>0$, $Re(\beta)>0$ are complex parameters, and Θ be an increasing function on (ℓ_1,ℓ_2) with $\ell<\ell_2$, also the geneeralized h-Cf $f:[\ell_1,\ell_2]\to\Re$ and $w:[\ell_1,\ell_2]\to\Re$, $w\geq0$ both are symmetric pertaining to $\frac{\ell_1+\ell_2}{2}$, then we have*

$$f\left(\frac{\ell_1+\ell_2}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{((\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}w(x)\right)(\ell_2)$$
$$\leq 2h\left(\frac{1}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{((\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(w(x)f(x)+\left(\big(h(\nu)+h(1-\nu)\big)\big(f(\ell_1)+f(\ell_2)\big)-f(x)\right)w(x)\right)\right)(\ell_2).$$

Proof. We have

$$f\left(\frac{\ell_1+\ell_2}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{((\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\Big(w(x)+w(\ell_1+\ell_2-x)\Big)\right)(\ell_2)$$
$$\leq h\left(\frac{1}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{((\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\Big(w(x)f(x)+w(\ell_1+\ell_2-x)f(\ell_1+\ell_2-x)\Big)\right).$$

Since w is symmetric

$$f\left(\frac{\ell_1+\ell_2}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{((\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}w(x)\right)(\ell_2)$$
$$\leq 2h\left(\frac{1}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(w(x)f(x)+\left(\big(h(\nu)+h(1-\nu)\big)\big(f(\ell_1)+f(\ell_2)\big)-f(x)\right)w(x)\right)\right)(\ell_2).$$

□

Example 2. *If we choose* $w(x)=1, f(x)=\Theta^{m+1}(x)$ *and* $h(x)=1$, *we have*

$$f\left(\frac{\ell_1+\ell_2}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}w(x)\right)(\ell_2)$$
$$=\left(\frac{(m+1)^{-\frac{\mu}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))(\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k})}\left((\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\right.,$$

and

$$2h\left(\frac{1}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(w(x)f(x)+\left(\big(h(\nu)+h(1-\nu)\big)\big(f(\tau_1)+f(\ell_2)\big)-f(x)\right)w(x)\right)\right)(\ell_2)$$
$$=4\left(\frac{(m+1)^{-\frac{\mu}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)(\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k})}\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\right.$$

By comparing above results we have

$$\frac{(m+1)^{-\frac{\mu}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)(\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k})}\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}$$
$$\leq 4\left(\frac{(m+1)^{-\frac{\mu}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)(\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k})}\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\right. \tag{31}$$

The computational values of (31), *are shown in Tables 3 and 4 corresponding to the choice of parameters* $m=2, k=1, \Theta(x)=x^2, \mu=2, \tau_i=1, \beta_i=0.5, \kappa_i=0.5, \mu=0.5$ *for some fixed values of* ℓ_1 *and* ℓ_2. *Figure 2, shows the graphical representation of* (31), *corresponding to the choice of parameters* $0<\ell_1=1<\ell_2=4$.

Table 3. The computational values of (31) corresponding to choice $\mu = 0.5$ are as follows:

$\ell_1/\ell 2$	**2.00**	**2.40**	**2.80**
1.00	$(6.93 \times 10^{00}, 2.77 \times 10^{01})$	$(2.37 \times 10^{01}, 9.49 \times 10^{01})$	$(9.60 \times 10^{01}, 3.84 \times 10^{02})$
1.20	$(4.76 \times 10^{00}, 1.91 \times 10^{01})$	$(1.64 \times 10^{01}, 6.55 \times 10^{01})$	$(6.46 \times 10^{01}, 2.58 \times 10^{02})$
1.40	$(3.08 \times 10^{00}, 1.23 \times 10^{01})$	$(1.09 \times 10^{01}, 4.37 \times 10^{01})$	$(4.25 \times 10^{01}, 1.70 \times 10^{02})$
1.60	$(1.81 \times 10^{00}, 7.23 \times 10^{00})$	$(7.02 \times 10^{00}, 2.81 \times 10^{01})$	$(2.72 \times 10^{01}, 1.09 \times 10^{02})$
1.80	$(8.48 \times 10^{-01}, 3.39 \times 10^{00})$	$(4.25 \times 10^{00}, 1.70 \times 10^{01})$	$(1.69 \times 10^{01}, 6.76 \times 10^{01})$
2.00	$(0.00 \times 10^{00}, 0.00 \times 10^{00})$	$(2.34 \times 10^{00}, 9.35 \times 10^{00})$	$(1.01 \times 10^{01}, 4.05 \times 10^{01})$

Table 4. The computational values of (31) corresponding to choice $\mu = 0.5$.

$\ell_1/\ell 2$	**3.20**	**3.60**	**4.00**
1.00	$(4.80 \times 10^{02}, 1.92 \times 10^{03})$	$(3.03 \times 10^{03}, 1.21 \times 10^{04})$	$(2.43 \times 10^{04}, 9.70 \times 10^{04})$
1.20	$(3.12 \times 10^{02}, 1.25 \times 10^{03})$	$(1.89 \times 10^{03}, 7.57 \times 10^{03})$	$(1.46 \times 10^{04}, 5.83 \times 10^{04})$
1.40	$(1.98 \times 10^{02}, 7.94 \times 10^{02})$	$(1.16 \times 10^{03}, 4.64 \times 10^{03})$	$(8.57 \times 10^{03}, 3.43 \times 10^{04})$
1.60	$(1.24 \times 10^{02}, 4.94 \times 10^{02})$	$(6.96 \times 10^{02}, 2.78 \times 10^{03})$	$(4.95 \times 10^{03}, 1.98 \times 10^{04})$
1.80	$(7.52 \times 10^{01}, 3.01 \times 10^{02})$	$(4.09 \times 10^{02}, 1.64 \times 10^{03})$	$(2.80 \times 10^{03}, 1.12 \times 10^{04})$
2.00	$(4.47 \times 10^{01}, 1.79 \times 10^{02})$	$(2.35 \times 10^{02}, 9.42 \times 10^{02})$	$(1.55 \times 10^{03}, 6.20 \times 10^{03})$

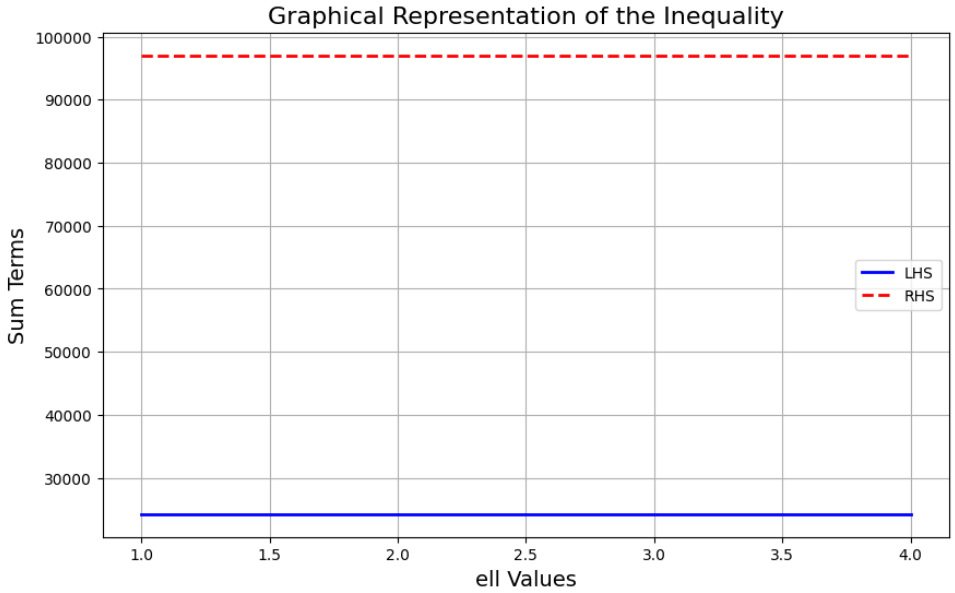


Figure 2. The graphical representation of (31) corresponding to choice $\mu = 0.5$.

Theorem 8. *For $k > 0$, $m \in \Re/\{-1\}$, and $Re(\eta) > 0$, $Re(\varrho) > 0$, $Re(\lambda) > 0$, $Re(\gamma) > 0$ are complex parameters, and Θ be an increasing function on (ℓ_1, ℓ_2) with $\ell_1 < \ell_2$, also $f : [\ell_1, \ell_2] \to \Re$ be the geneeralized h-Cf, $h : \Re \to \Re$ be non-negative function and $w : [\ell_1, \ell_2] \to \Re$, $w \geq 0$ be symmetric pertaining to $\frac{\ell_1+\ell_2}{2}$ and $f(x), w(x) \in ({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})[\ell_1, \ell_2]$. Then the subsequent the inequality*

$$\begin{aligned}
&\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right) f(x)w(x) \\
&\leq \frac{f(\ell_1)+f(\ell_2)}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(h\left(\frac{\ell_2-x}{\ell_2-\ell_1}\right)+h\left(\frac{x-\ell_1}{\ell_2-\ell_1}\right)\right)w(x)\right)(\ell_2),
\end{aligned} \tag{32}$$

holds for all $\nu \in (0, 1)$.

Proof. Using Lemma 2 and the symmetricalness of the weight w, we have

$$\frac{f\left(\frac{\ell_1+\ell_2}{2}\right)\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)w(x)}{2h(\frac{1}{2})} = \frac{\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)f\left(\frac{\ell_1+\ell_2-x+x}{2}\right)w(x)}{2h(\frac{1}{2})}$$
$$\leq \frac{\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)h(\frac{1}{2})\Big[f(\ell_1+\ell_2-x)+f(x)\Big]w(x)}{2h(\frac{1}{2})}$$
$$= (\frac{1}{2})\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)\Big[f(\ell_1+\ell_2-x)w(\ell_1+\ell_2-x)+f(x)w(x)\Big]$$
$$= \left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\right)f(x)w(x).$$

Which is the left part of the inequality. Alternatively, we have

$$\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}f(x)w(x)\right)(\ell_2)$$
$$= \frac{1}{2}({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i})\Big[f(x)w(x)+f(\ell_1+\ell_2-x)w(\ell_1+\ell_2-x)\Big]$$
$$= \frac{1}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}(f(x)+f(\ell_1+\ell_2-x))w(x)\right)(\ell_2)$$
$$= \frac{1}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(f(\frac{\ell_2-x}{\ell_2-\ell_1}\ell_1+\frac{x-\ell_1}{\ell_2-\ell_1}\ell_2)\right.\right.$$
$$\left.\left.+f\left(\frac{x-\ell_1}{\ell_2-\ell_1}\ell_1+\frac{\ell_2-x}{\ell_2-\ell_1}\ell_2\right)\right)w(x)\right)(\ell_2)$$
$$\leq \frac{1}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}\left(h\left(\frac{\ell_2-x}{\ell_2-\ell_1}\right)f(\ell_1)+h\left(\frac{x-\ell_1}{\ell_2-\ell_1}\right)f(\ell_2)\right.\right.$$
$$\left.\left.+h(\frac{x-\ell_1}{\ell_2-\ell_1})f(\ell_1)+h(\frac{\ell_2-x}{\ell_2-\ell_1})f(\ell_2)\right)w(x)\right)(\ell_2)$$
$$= \frac{f(\ell_1)+f(\ell_2)}{2}\left({}^{k,m}_{\Theta}\mathfrak{J}^{\delta_i,\tau_i}_{(\frac{\ell_1+\ell_2}{2})^+;\mu,\beta_i}(h(\frac{\ell_2-x}{\ell_2-\ell_1})+h(\frac{x-\ell_1}{\ell_2-\ell_1}))w(x)\right)(\ell_2).$$

□

Example 3. *For graphical representation we choose* $f(x)=\Theta^{m+1}(x)$, $w(x)=x$ *and* $h(x)=x$ *in (32), we have*

$$\frac{(m+1)^{\frac{\mu}{k}-1}}{k}\Sigma_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i))}$$
$$\times\left(\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)+\eta}{k}}\right.$$
$$\left.-\frac{m+1}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\frac{\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}+1}}{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}+1}\right)$$
$$\leq [\Theta^{m+1}(\ell_2)+\Theta^{m+1}(\ell_1)]\left(\frac{(m+1)^{-\frac{\mu}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\delta_i)_{\kappa_i,k}(\tau_i)^{\kappa_i}}{\Gamma_k(\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)(\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k})}\right.$$
$$\left.\times\left(\Theta^{m+1}(\ell_2)-\Theta^{m+1}(\frac{\ell_1+\ell_2}{2})\right)^{\frac{\mu+\Sigma_{i=1}^{n}(\beta_i\kappa_i)}{k}}\right). \tag{33}$$

The computational values of (33)*, are shown in Tables 5 and 6 corresponding to the choice of parameters* $m = 2$, $k = 1$, $\Theta(x) = x^2$, $\mu = 2$, $\tau_i = 1$, $\beta_i = 0.5$, $\kappa_i = 0.5$, *and for some fixed values of* ℓ_1 *and* ℓ_2. *The Figure 3, shows the two-dimensional graphical representation of* (33)*, corresponding to the choice of parameters* $0 < \ell_1 = 1 < \ell_2 = 4$. *The Figure 4 shows the three-dimensional graphical representation of* (33) *corresponding to the choice of parameters* $0 < \mu < 1$.

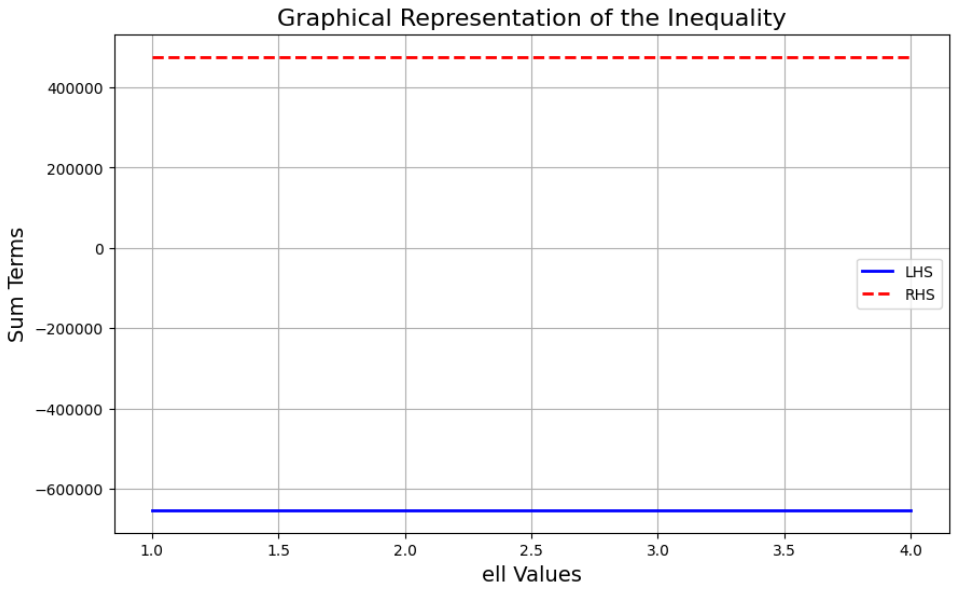


Figure 3. The two-dimensional graphical representation of (33) corresponding to choice $\mu = 0.5$.

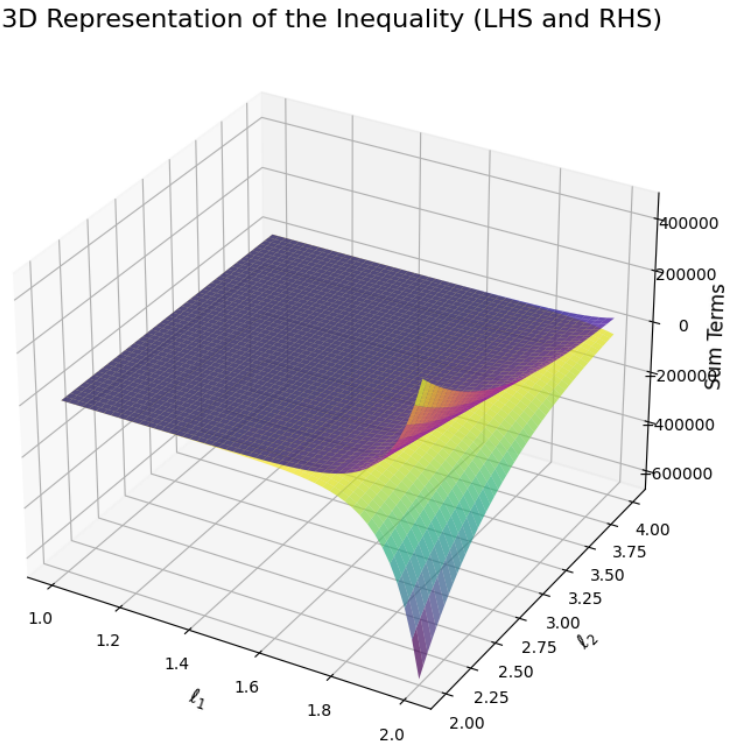


Figure 4. The three-dimensional graphical representation of (33) corresponding to choice $0 \leq \mu \leq 1$.

For tabular values

Table 5. The computational values of (33) corresponding to choice $\mu = 0.5$.

ℓ_1/ℓ_2	**2.10**	**2.30**	**2.50**
0.10	$(4.12 \times 10^{01}, 3.14 \times 10^{01})$	$(5.23 \times 10^{01}, 4.16 \times 10^{01})$	$(6.98 \times 10^{01}, 5.69 \times 10^{01})$
0.30	$(5.29 \times 10^{01}, 4.21 \times 10^{01})$	$(6.43 \times 10^{01}, 5.25 \times 10^{01})$	$(8.44 \times 10^{01}, 7.07 \times 10^{01})$
0.50	$(6.47 \times 10^{01}, 5.29 \times 10^{01})$	$(7.62 \times 10^{01}, 6.34 \times 10^{01})$	$(9.89 \times 10^{01}, 8.08 \times 10^{01})$
0.70	$(7.65 \times 10^{01}, 6.36 \times 10^{01})$	$(8.82 \times 10^{01}, 7.42 \times 10^{01})$	$(1.10 \times 10^{02}, 9.09 \times 10^{01})$
0.90	$(8.83 \times 10^{01}, 7.43 \times 10^{01})$	$(1.00 \times 10^{02}, 8.51 \times 10^{01})$	$(1.24 \times 10^{02}, 1.04 \times 10^{02})$
1.00	$(1.00 \times 10^{02}, 8.57 \times 10^{01})$	$(1.13 \times 10^{02}, 9.45 \times 10^{01})$	$(1.39 \times 10^{02}, 1.17 \times 10^{02})$

Table 6. The computational values of (33) corresponding to choice $\mu = 0.5$.

ℓ_1/ℓ_2	**2.70**	**2.90**	**3.00**
0.10	$(8.72 \times 10^{01}, 7.22 \times 10^{01})$	$(1.04 \times 10^{02}, 8.75 \times 10^{01})$	$(1.16 \times 10^{02}, 9.87 \times 10^{01})$
0.30	$(1.03 \times 10^{02}, 8.78 \times 10^{01})$	$(1.21 \times 10^{02}, 1.05 \times 10^{02})$	$(1.34 \times 10^{02}, 1.17 \times 10^{02})$
0.50	$(1.20 \times 10^{02}, 9.89 \times 10^{01})$	$(1.40 \times 10^{02}, 1.16 \times 10^{02})$	$(1.55 \times 10^{02}, 1.30 \times 10^{02})$
0.70	$(1.32 \times 10^{02}, 1.10 \times 10^{02})$	$(1.53 \times 10^{02}, 1.29 \times 10^{02})$	$(1.69 \times 10^{02}, 1.43 \times 10^{02})$
0.90	$(1.48 \times 10^{02}, 1.27 \times 10^{02})$	$(1.72 \times 10^{02}, 1.49 \times 10^{02})$	$(1.88 \times 10^{02}, 1.64 \times 10^{02})$
1.00	$(1.64 \times 10^{02}, 1.38 \times 10^{02})$	$(1.90 \times 10^{02}, 1.58 \times 10^{02})$	$(2.06 \times 10^{02}, 1.72 \times 10^{02})$

4. Discussion

In the first table, the comparison of the LHS, Middle Term, and RHS in the graphs and tables reveals that the RHS is generally larger than both the LHS and Middle Term, with the values increasing as ℓ_1 and ℓ_2 increase. The Middle Term tends to be closer to the LHS in magnitude, suggesting its role as a balancing factor between the two sides. As ℓ_2 increases, the RHS grows more rapidly compared to the LHS and Middle Term, indicating that the RHS is more sensitive to changes in ℓ_2. The inequality appears to hold in the form $LHS \leq MiddleTerm \leq RHS$ across the tested values, with the LHS being consistently smaller than or equal to the Middle Term, and the Middle Term smaller than or equal to the RHS. This confirms the validity of the inequality and suggests that while the terms grow with ℓ_1 and ℓ_2, the RHS exhibits the fastest growth. These observations highlight that the inequality holds true in the given range, with the terms influenced differently by ℓ_1 and ℓ_2, and the RHS proving to be the dominant term in terms of magnitude. In the remaining graphs and tables demonstrate that for all combinations of ℓ_1 and ℓ_2, the RHS values are consistently greater than the corresponding LHS values, indicating that the inequality is satisfied in all cases. This suggests that the inequality becomes more pronounced for larger ℓ_1 and ℓ_2, with the RHS serving as a more accurate upper bound, confirming the robustness of the inequality in these conditions.

5. Conclusions

In recent times, mathematical fractional inequalities have found their way into various domains such as the material sciences, heat conduction, viscoelasticity, time series analysis, circuits, human body modeling, and shear waves. While inequalities, itself may not directly take note of issues of educational access, high-quality research, and mathematical advancement—topics such as inequality contribute to the larger objective of improving educational through the dissemination of knowledge, the development of critical thinking abilities, and the encouragement of mathematical literacy among researchers. In this work, we extended the well-known H-H-F inequalities by introducing generalized fractional operators involving the M-M-L function. Through the derivation of two key identities for locally differentiable functions, we were able to provide more accurate estimates of the differences between the left, middle, and right terms in the H-H-F inequalities. These

generalized inequalities offer valuable insights into generalized convexity and the behavior of fractional operators in various mathematical contexts. The practical applications of these inequalities are significant, particularly in fields such as optimization, fractional calculus, and applied analysis. For instance, the improved inequalities can aid in more precise bounding of functions and contribute to solving complex problems in areas such as signal processing, system modeling, and the study of fractional differential equations. The graphical representations included in this study serve to visualize these inequalities, providing a useful tool for researchers and practitioners. Overall, the results presented in this paper lay a foundation for further exploration of fractional inequalities and their applications in real-world problems. Future research could extend these methods to higher-dimensional settings and explore their potential in new areas such as financial modeling and the modeling of physical systems in plasma physics. Researchers can apply our defined operator to generalized Opial-type, Chebyshev-type, and composite Cf inequality problems, among other types of inequalities. These inequalities play a vital role in computing the special means. Specifically, we have looked into some of the well-known H-H-F inequality's relations to generalized h-convexity and extended fractional operators. In order to achieve this goal, two identities for local differentiable mappings are created. From these identities, we may derive estimates for the difference between the center and right parts of the H-H-F inequality as well as the difference between the left and central inequality.

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Conflicts of Interest: The authors declare that they have no conflicts of interest.

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