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Efficient Numerical Solutions for Fuzzy Time Fractional Diffusion Equations Using Two Explicit Compact Finite Difference Methods

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Abstract: This article introduces an extension of classical fuzzy partial differential equations, known as fuzzy fractional partial differential equations. These equations provide a better explanation for certain phenomena. We focus on solving the fuzzy time diffusion equation with a fractional order of $0 < \alpha \leq 1$, using two explicit compact finite difference schemes that are the compact forward time center space (CFTCS) and compact Saulyev's scheme. The time fractional derivative uses the Caputo definition. The double-parametric form approach is used to transfer the governing equation from an uncertain to a crisp form. To ensure stability, we apply the von Neumann method to show that CFTCS is conditionally stable, while compact Saulyev's is unconditionally stable. A numerical example is provided to demonstrate the practicality of our proposed schemes.

Keywords: compact finite difference scheme; double-parametric form of fuzzy number; Caputo derivative; fuzzy time fractional diffusion equation

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1. Introduction

Nonlinear partial differential equations (NPDEs) play a fundamental role across a spectrum of disciplines, including physics, chemistry, biology, mathematics, and engineering. They serve as vital tools for describing intricate phenomena like fluid dynamics and heat transfer, among others. However, tackling nonlinear models for practical applications poses formidable challenges, both in theory and in computation. The inherent complexity and nonlinearity of NPDEs often necessitate the imposition of simplifying assumptions to render them tractable. These assumptions may involve simplifications of the equations, the neglect of certain terms, or the approximation of solutions. While these measures can facilitate problem solving, they concurrently introduce uncertainties that may compromise the accuracy and reliability of the solutions, particularly when applied to real-world scenarios where precision is paramount. A variety of numerical techniques exist for addressing NPDEs, yet each approach comes with its own set of limitations. Traditional methods like finite difference and finite element approaches, reliant on domain discretization, are susceptible to errors and instabilities. In summary, navigating NPDEs demands a nuanced blend of theoretical insight and numerical proficiency. The quest for effective solutions remains an active frontier in research and development across diverse fields. The ongoing exploration and refinement of novel techniques and methodologies are indispensable for advancing our ability to grapple with NPDEs effectively [1–5].

The surge in attention towards fractional partial differential equations (FPDEs) over recent decades reflects a growing recognition of their versatile applicability across various domains of physics and engineering [6–21]. Among the rich tapestry of FPDEs, one stands out prominently: the time fractional diffusion equation. Unlike its classical counterpart, this equation introduces a fundamental departure by incorporating fractional derivatives in lieu of the customary first-order time derivative. This departure is not merely symbolic; rather, it extends the temporal purview of the phenomenon under scrutiny, offering a powerful tool to model intricate dynamical systems with enhanced fidelity and accuracy [22].

The allure of the time fractional diffusion equation lies in its capacity to capture phenomena characterized by non-local, long-range interactions, which are often overlooked or poorly represented by traditional integer-order differential equations. By embracing fractional calculus, this equation provides a more nuanced framework for describing the anomalous diffusion processes observed in diverse fields such as physics, chemistry, biology, and finance.

In physics, for instance, the time fractional diffusion equation finds applications in modeling the transport of particles in complex media exhibiting anomalous diffusion behavior, such as porous materials, biological tissues, or disordered solids. Similarly, in engineering, it serves as a cornerstone for understanding heat conduction in fractal media, signal propagation in heterogeneous materials, and the dynamics of complex networks.

The exploration of the time fractional diffusion equation and other FPDEs represents not just a theoretical pursuit but also a practical endeavor with far-reaching implications. By unlocking the mathematical machinery to describe and analyze these intricate phenomena, researchers are paving the way for innovations in diverse fields, from advanced material science to biomedical engineering. Thus, the increased focus on FPDEs underscores a pivotal shift towards a deeper understanding of complex dynamics and the development of novel methodologies to address real-world challenges.

In general, it is difficult to obtain exact analytical solutions using analytical methods, so many authors have turned to numerical methods. One important numerical method for solving fractional diffusion equations is the finite difference method, which has been discussed by several authors [23–25]. Compact finite difference schemes are often preferred due to their accuracy and high computational efficiency, and there have been a number of recent publications on using these methods to solve the fractional diffusion equation. As an example, the authors Gao and Sun [26] employed a high-order compact finite difference method to tackle fractional diffusion equations. The problem was solved by applying analytical theories, including the Caputo definition, and L1 discretization was utilized to estimate the time fractional derivative. The stability and convergence of the method were analyzed using the energy method.

In a similar vein, Karatay and Bayramoglu [27] tackled time fractional heat equations using a compact difference scheme in their study. They employed a second-order discretization method, based on the Crank–Nicolson scheme, for the time fractional derivative and a fourth-order accuracy compact approximation for the second-order space derivative. The proposed scheme's stability was analyzed using both Fourier stability and spectral stability methods, and it was concluded that the scheme is unconditionally stable, meaning that there are no time constraints. Additionally, Al-Shibani et al. [28] developed two compact finite difference schemes to solve the one-dimensional time fractional diffusion equation and studied their stability. They discovered that using high-order compact finite difference schemes allowed them to overcome the challenges associated with high-order discretizations by utilizing derivatives of the function values at the nodes of the corresponding independent variables.

Fuzzy time fractional diffusion equations (FTFDEs) arise in diffusion processes due to imprecise and uncertain parameters and variables caused by experimental and measurement errors. In order to accommodate this vagueness, researchers have explored various analytical methods for solving fuzzy fractional diffusion equations. As an example, Ghazanfari and Ebrahimi [29] employed the differential transformation method (DTM) to obtain computable series as approximate solutions for fuzzy fractional diffusion equations. The DTM was deemed highly efficient and straightforward for this purpose. Salah et al. [30] introduced the homotopy analysis transform method (HATM) to address fuzzy fractional heat and wave partial differential equations. In a separate approach, Chakraverty and Tampaswini [31] proposed a novel computational technique to solve the time fractional diffusion equation with uncertainties in the initial conditions. Their method employed a single-parametric form of fuzzy numbers to transform the fuzzy diffusion equation into an interval-based fuzzy differential equation, which was then converted into a crisp form using a double-parametric form of fuzzy numbers. The resulting equation was solved using the Adomian decomposition method to obtain uncertain solution bounds. The authors [32] also developed a finite difference scheme using the single-parametric form of fuzzy numbers for solving FTFDEs.

Based on our analysis of the existing literature, it appears that although there have been some attempts to tackle FTFDEs using approximate analytical techniques, the utilization of numerical methods has been relatively unexplored. Consequently, our paper aims to investigate the use of numerical finite difference methods for solving FTFDEs. Specifically, we intend to create modified two explicit compact finite difference methods that are CFTCS and compact Saulyev schemes with high-order accuracy to derive a numerical solution that represents fuzzy numbers in a double-parametric form for FTFDEs. Also, the double-parametric form approach is used to transfer the fuzzy governing equation from fuzzy case to crisp case to reduce the computational cost and cover more fuzzy cases [33–35].

2. Preliminaries

In this section, we present the related theorems and definitions that are used further in this paper.

Definition 1. *r-level set [31].*

The r-level set of a fuzzy set $\widetilde{U}$, labeled as $\widetilde{U}_r$, is the crisp set of all $x \in X$, such that $\mu_{\widetilde{A}} \geq r$; i.e., $\widetilde{U}_r = \left\{ x \in X \middle| \mu_{\widetilde{u}} > r, r \in [0,1] \right\}$.

Definition 2. *Fuzzy numbers [31].*

Fuzzy numbers are a subset of the real number set and represent uncertain values. Fuzzy numbers are linked to the degrees of membership of a set. A fuzzy number [10] μ is called a triangular fuzzy number if defined by three numbers $a < b < c$ where the graph of $\mu\ (x)$ is a triangle with the base on the interval $[a, c]$ and the vertex at $x = b$ and its membership function is of the form:

$$\mu_{\widetilde{u}_r} = \begin{cases} 0, & if\ x < a \\ \frac{x-a}{b-a}, & if\ a \leq x \leq b \\ \frac{c-x}{c-b}, & if\ b \leq x \leq c \\ 0, & if\ x > c \end{cases}$$

where the r-level sets of triangular fuzzy numbers are

$$[\mu]_r = [a + r\ (b - a),\ c - r\ (c - b)],\ r \in\ [0,\ 1]$$

Definition 3. *[27]: Let $\widetilde{f} : [t_0 + \alpha, T] \to \widetilde{E}$ be Hukuhara differentiable and denote*

$$\left[\widetilde{f}'(t)\right]_r = \left[\underline{f}\prime(t), \overline{f}\prime(t)\right]_r = \left[\underline{f}\prime(t;r), \overline{f}\prime(t;r)\right]$$

Then, both of the boundary functions $\underline{f}\prime(t;r), \overline{f}\prime(t;r)$ are differentiable; we can write of the nth-order fuzzy derivative:

$$[\widetilde{f}^{(n)}(t)]_r = \left[\left(\underline{f}^{(n)}(t;r)\right)', \left(\overline{f}^{(n)}(t;r)\right)'\right] \forall r \in [0,1]$$

Definition 4. *Double-parametric forms of fuzzy numbers [31].*

Using the single-parametric form, we have $\widetilde{U} = [\underline{u}(r), \overline{u}(r)]$. This can be written as a crisp number using the double-parametric form:

$$\widetilde{U}(r,\beta) = \beta[\overline{u}(r) - \underline{u}(r)] + \underline{u}(r) \text{ where } r \text{ and } \beta \in [0,1]$$

Definition 5. *The fractional derivative was also defined by Zhuang and Liu, 2006, as follows [27]:*

$$\frac{\partial^\alpha u(x,t)}{\partial^\alpha t} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{n} b_j \left(u_i^{n+1-j} - u_i^{n-j}\right)$$

3. Fuzzy Compact Finite Difference Scheme (FCFD)

Assuming that u_i^n represents the approximate value of u at (x_i, t_n), we can expand u_{i+1}^n and u_{i-1}^n around (x_i, t_n) using Taylor series to obtain FCFD approximations for the first- and second-order spatial derivatives.

$$\left.\begin{aligned} \widetilde{u}_{i+1}^n &= \widetilde{u}_i^n + h\left(\frac{\partial \widetilde{u}}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n + \cdots \\ \widetilde{u}_{i-1}^n &= \widetilde{u}_i^n - h\left(\frac{\partial \widetilde{u}}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n + \cdots \end{aligned}\right\} \quad (1)$$

The first derivatives of u_{i+1}^n and u_{i-1}^n are

$$\left.\begin{aligned} \left(\frac{\partial \widetilde{u}}{\partial x}\right)_{i+1}^n &= \left(\frac{\partial \widetilde{u}}{\partial x}\right)_i^n + h\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^4 \widetilde{u}}{\partial x^4}\right)_i^n + \cdots \\ \left(\frac{\partial \widetilde{u}}{\partial x}\right)_{i-1}^n &= \left(\frac{\partial \widetilde{u}}{\partial x}\right)_i^n - h\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^4 \widetilde{u}}{\partial x^4}\right)_i^n + \cdots \end{aligned}\right\} \quad (2)$$

The second derivatives of u_{i+1}^n and u_{i-1}^n are

$$\left.\begin{aligned} \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_{i+1}^n &= \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n + h\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4 \widetilde{u}}{\partial x^4}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^5 \widetilde{u}}{\partial x^5}\right)_i^n + \cdots \\ \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_{i-1}^n &= \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n - h\left(\frac{\partial^3 \widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4 \widetilde{u}}{\partial x^4}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^5 \widetilde{u}}{\partial x^5}\right)_i^n + \cdots \end{aligned}\right\} \quad (3)$$

We obtain approximations for the first and second spatial derivatives, respectively, by utilizing Equations (1)–(3):

$$\left(\frac{\partial \widetilde{u}}{\partial x}\right)_i^n = \frac{\delta_x/2h}{(1+\frac{1}{6}\,\delta^2{}_x)} \widetilde{u}_i^n + O(h^4) \quad (4)$$

$$\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}\right)_i^n = \frac{\delta^2{}_x/h^2}{(1+\frac{1}{12}\,\delta^2{}_x)} \widetilde{u}_i^n + O(h^4) \quad (5)$$

where $\delta_x = \widetilde{u}_{i+1}^n - u_{i-1}^n$ and $\delta^2{}_x = \widetilde{u}_{i+1}^n - 2\widetilde{u}_i^n + \widetilde{u}_{i-1}^n$ for $0 \le i \le M, 0 \le n \le N$.

Now, based on the definition of the average operator in [27], we obtain

$$\frac{1}{(1+\frac{1}{6}\,\delta^2{}_x)} \widetilde{u}_i^n = \frac{1}{6}\left(\widetilde{u}_{i+1}^n + 4\widetilde{u}_i^n + \widetilde{u}_{i-1}^n\right),\ \ 1 \le\ i \le M-1 \quad (6)$$

$$\frac{1}{(1+\frac{1}{12}\,\delta^2{}_x)} \widetilde{u}_i^n = \frac{1}{12}\left(\widetilde{u}_{i+1}^n + 10\widetilde{u}_i^n + \widetilde{u}_{i-1}^n\right),\ \ 1 \le\ i \le M-1 \quad (7)$$

4. Compact FTCS Scheme for the Solution of FTFDE

This section applies the double-parametric form of fuzzy numbers within a compact FTCS scheme. The Caputo formula is used to calculate the time fractional derivative, while a fourth-order accuracy compact approximation is used to compute the second-order space derivative at time level n. This approach are utilized to solve the FTFDE.

Let us consider the one-dimensional FTFDE equation with the initial and boundary conditions presented in [32].

$$\frac{\partial^\alpha \tilde{u}(x,t,r,\alpha)}{\partial^\alpha t} = \tilde{a}(x)\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + \tilde{q}(x), \qquad 0 < x < l,\ t > 0$$

$$\tilde{u}(x,0) = \tilde{f}(x), \tilde{u}(0,t) = \tilde{g},\ \tilde{u}(l,t) = \tilde{z} \tag{8}$$

where $\tilde{u}(x,t,\alpha)$ is a fuzzy concentration of a quantity such as the mass, energy, etc., of crisp variables t, x and α represent the fractional order, $\frac{\partial^\alpha \tilde{U}(x,t,\alpha)}{\partial^\alpha t}$ is the fuzzy time fractional derivative of order α . $\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2}$ is the partial Hukuhara derivative, as shown in Definition 3 with respect to x. $\tilde{a}(x)$ is the diffusion coefficient (or diffusivity), $\tilde{q}(x)$ is a fuzzy function for the crisp variable x. $\tilde{u}(x,0)$ is the fuzzy initial condition, and $\tilde{u}(0,t)$ and $\tilde{u}(l,t)$ are fuzzy boundary conditions, with $\tilde{g}$, $\tilde{z}$ being fuzzy convex numbers.

To obtain a numerical solution of the FTFDE using the CFTCS scheme, we discretize the time fractional derivative in Equation (8) using the Caputo formula presented in Definition 5, while the second partial derivatives are approximated using Equation (5), resulting in

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}[\tilde{u}_i^{n+1} - \tilde{u}_i^{n} + \sum_{j=1}^{n} b_j\,(\tilde{u}_i^{n+1-j} - \tilde{u}_i^{n-j})] = \tilde{a}(x)\frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}\tilde{u}_i^{n} + \tilde{q}(x), \tag{9}$$

Using Equations (6) and (7), Equation (9) is simplified to obtain

$$\begin{aligned}&\tfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \times \tfrac{1}{12}\Big([\left(\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1}\right) - \left(\tilde{u}_{i+1}^{n} + 10\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n}\right)] \\ &\quad + \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right]\Big) \\ &\quad = \tilde{a}(x)\left[\frac{\tilde{u}_{i+1}^{n} - 2\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n}}{h^2}\right] + \tfrac{1}{12}(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n})\end{aligned} \tag{10}$$

$$\begin{aligned}&\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 10\tilde{u}_i^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right] \\ &\quad = \tfrac{\tilde{a}(x)\ \Delta t^\alpha \Gamma(2-\alpha)}{h^2}\left(12u_{i+1}^n - 24u_i^n + 12u_{i-1}^n\right) + \Delta t^\alpha \Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)\end{aligned} \tag{11}$$

Now, we let $\tilde{p}(r) = \frac{\tilde{a}(x,r)\Delta t^\alpha \Gamma(2-\alpha)}{h^2}$, and from Equation (11), we obtain

$$\begin{aligned}&\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 10\tilde{u}_i^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + \tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right] \\ &\quad = \left(12p\ u_{i+1}^n - 24p\ u_i^n + 12p\ u_{i-1}^n\right) + \Delta t^\alpha \Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)\end{aligned} \tag{12}$$

By simplifying Equation (12), we obtain the general formula of compact FTCS for FTFDE:

$$\begin{aligned}&\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1} \\ &\quad = (1+12p)\tilde{u}_{i+1}^{n} + (10-24p)\tilde{u}_i^{n} + (1+12p)\tilde{u}_{i-1}^{n} \\ &\quad - \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + \tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right] + \Delta t^\alpha \Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)\end{aligned} \tag{13}$$

For each spatial grid point, Equation (13) is evaluated to yield linear equations. For a one-dimensional problem, a compact FTCS leads to a tridiagonal system of equations, where each equation involves the unknowns at the current and neighboring grid points. The obtained tridiagonal system of linear equations typically involves using iterative methods for solving sparse linear systems. In each time step, we solve a linear system of equations using wolfram mathematica 11.2 to obtain the values $\tilde{u}(x,t,\alpha)$ for that particular time level.

The complexity of the system of equations is influenced by the order of accuracy of the compact FTCS. Higher-order accuracy typically requires larger stencils and more complex coefficients, leading to more intricate systems of equations. Generally, solving fuzzy time fractional diffusion equations involves a balance between accuracy and computational efficiency, and the careful consideration of these factors is crucial in determining the overall cost of the simulation. The use of specialized algorithms for fuzzy fractional derivatives and parallelization can potentially enhance the efficiency of computation.

5. Compact Saulyev Scheme for the Solution of FTFDE

Within this segment, the concise compact Saulyev scheme utilizes the Caputo formula to incorporate the double-parametric form of fuzzy numbers, with a two-time level fourth-order accuracy to approximate the second space derivative. While the method may seem implicit, the obtained solution is explicit in nature.

To compute the numerical solution for the FTFDE utilizing the compact Saulyev scheme, we apply the Caputo formula to discretize the time fractional derivatives in Equation (8). Additionally, we discretize the second partial derivatives in the same equation using a two-time level approach with fourth-order accuracy, resulting in

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\tilde{u}_i^{n+1} - \tilde{u}_i^{n} + \sum_{j=1}^{n} b_j\left(\tilde{u}_i^{n+1-j} - \tilde{u}_i^{n-j}\right)\right] = \tilde{a}(x)\frac{\tilde{u}_{i+1,n} - \tilde{u}_{i,n} - \tilde{u}_{i,n+1} + \tilde{u}_{i-1,n+1}/h^2}{\left(1+\frac{1}{12}\delta^2 x\right)}\tilde{u}_i^{n} + \tilde{q}(x) \tag{14}$$

$$\begin{aligned}
&\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 10\tilde{u}_i^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + \tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right]\\
&\quad = \frac{\tilde{a}(x)\,\Delta t^{\alpha}\Gamma(2-\alpha)}{h^2}\left(12\,\tilde{u}_{i+1,n} - 12\,\tilde{u}_{i,n} - 12\,\tilde{u}_{i,n+1} + 12\,\tilde{u}_{i-1,n+1}\right)\\
&\quad +\Delta t^{\alpha}\Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)
\end{aligned} \tag{15}$$

Next, we let $\tilde{p}(r) = \frac{\tilde{a}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h^2}$, and from Equation (15), we obtain:

$$\begin{aligned}
&\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 10\tilde{u}_i^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + \tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right]\\
&\quad = \left(12\tilde{p}\,\tilde{u}_{i+1,n} - 12\tilde{p}\,\tilde{u}_{i,n} - 12\tilde{p}\,\tilde{u}_{i,n+1} + 12\tilde{p}\,\tilde{u}_{i-1,n+1}\right) + \Delta t^{\alpha}\Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)
\end{aligned} \tag{16}$$

By simplifying Equation (16), we obtain the general formula for compact Saulyev for FTFDE:

$$\begin{aligned}
&\tilde{u}_{i+1}^{n+1} + (10+12p)\tilde{u}_i^{n+1} + (1-12p)\tilde{u}_{i-1}^{n+1}\\
&\quad = (1+12p)\tilde{u}_{i+1}^{n} + (10-12p)\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n} - \sum_{j=1}^{n} b_j\left[\left(\tilde{u}_{i+1}^{n+1-j} + 10\tilde{u}_i^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + \tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right]\\
&\quad +\Delta t^{\alpha}\Gamma(2-\alpha)\left(\tilde{q}_{i+1}^{n} + 10\tilde{q}_i^{n} + \tilde{q}_{i-1}^{n}\right)
\end{aligned} \tag{17}$$

6. The Stability of Compact FTCS for FTFDE

The stability of the proposed compact schemes for FTFDE with no source term in the double-parametric form of fuzzy numbers will be analyzed using the von Neumann method in the following section.

It is first assumed that the discretization of the initial condition introduces the fuzzy error $\tilde{\varepsilon}_i^{0}$.

Let $\tilde{u}_i^{0} = \acute{\tilde{u}}_i^{0} - \tilde{\varepsilon}_i^{0}$, $\tilde{u}_i^{n}$ and $\acute{\tilde{u}}_i^{n}$ be the fuzzy numerical solutions of the scheme in Equation (13) with respect to the initial data $\tilde{f}_i^{0}$ and $\acute{\tilde{f}}_i^{0}$, respectively.

Let $[\tilde{u}_{i+1}^{n}(x,t;\alpha)]_r = \beta[\overline{u}(r) - \underline{u}(r)] + \underline{u}(r)]$, where $\beta, r \in [0,1]$.

The fuzzy error bound is defined as

$$\left[\overset{\sim}{\varepsilon}{}_i^n\right]_r = \left[\overset{\sim}{\acute{u}}{}_i^n - \tilde{u}{}_i^n\right]_r, \quad n = 1, 2, \ldots, X \times M,\ i = 1, 2, \ldots, X-1 \tag{18}$$

Now, based on the approach used in [30,31], Equation (13) can be rewritten as follows:

$$\begin{aligned}
&\tilde{u}{}_{i+1}^{n+1} + 10\tilde{u}{}_i^{n+1} + \tilde{u}{}_{i-1}^{n+1} \\
&\quad = (1 + 12p - b_1)\tilde{u}{}_{i+1}^n + (10 - 24p - 10b_1)\tilde{u}{}_i^n + (1 + 12p - b_1)\tilde{u}{}_{i-1}^n \\
&\quad - \sum_{j=1}^{n-1} (b_{j+1} - b_j)\left(\tilde{u}{}_{i+1}^{n-j} + 10\tilde{u}{}_i^{n-j} + \tilde{u}{}_{i-1}^{n-j}\right) + b_n\left(\tilde{u}{}_{i+1}^0 + 10\tilde{u}{}_i^0 + \tilde{u}{}_{i-1}^0\right)
\end{aligned} \tag{19}$$

Next, we rewrite the fuzzy round-off error for Equation (19) as follows:

$$\begin{aligned}
&\overset{\sim}{\varepsilon}{}_{i+1}^{n+1} + 10\overset{\sim}{\varepsilon}{}_i^{n+1} + \overset{\sim}{\varepsilon}{}_{i-1}^{n+1} \\
&\quad = (1 + 12p - b_1)\overset{\sim}{\varepsilon}{}_{i+1}^n + (10 - 24p - 10b_1)\overset{\sim}{\varepsilon}{}_i^n + (1 + 12p - b_1)\overset{\sim}{\varepsilon}{}_{i-1}^n \\
&\quad - \sum_{j=1}^{n-1} (b_{j+1} - b_j)\left(\overset{\sim}{\varepsilon}{}_{i+1}^{n-j} + 10\overset{\sim}{\varepsilon}{}_i^{n-j} + \overset{\sim}{\varepsilon}{}_{i-1}^{n-j}\right) + b_n\left(\overset{\sim}{\varepsilon}{}_{i+1}^0 + 10\overset{\sim}{\varepsilon}{}_i^0 + \overset{\sim}{\varepsilon}{}_{i-1}^0\right)
\end{aligned} \tag{20}$$

$$\overset{\sim}{\varepsilon}{}_0^n = \overset{\sim}{\varepsilon}{}_X^n = 0, \quad n = 1, 2, \ldots, T \times M$$

Let $\overset{\sim}{\varepsilon}{}_i^n = [\overset{\sim}{\varepsilon}{}_1^n, \overset{\sim}{\varepsilon}{}_2^n, \ldots, \overset{\sim}{\varepsilon}{}_{X-1}^n]$, and introduce the following fuzzy norm:

$$\left\|\overset{\sim}{\varepsilon}{}^n\right\|_2 = \sqrt{\sum_{i=1}^{X-1} h \left|\overset{\sim}{\varepsilon}{}_i^n\right|^2}$$

Then, we obtain

$$\left\|\overset{\sim}{\varepsilon}{}^n\right\|_2^2 = \sum_{i=1}^{X-1} h \left|\overset{\sim}{\varepsilon}{}_i^n\right|^2 \tag{21}$$

Suppose that $\overset{\sim}{\varepsilon}{}_i^n$ can be expressed in the form:

$$\overset{\sim}{\varepsilon}{}_i^n = \overset{\sim}{\lambda}{}^n e^{\sqrt{-\theta_i}}, \ \text{ where } \overset{\sim}{\theta}_i = qih \tag{22}$$

Substituting Equation (22) into Equation (20), we obtain:

$$\begin{aligned}
&\overset{\sim}{\lambda}{}^{n+1} e^{\sqrt{-\theta_{i+1}}} + 10\overset{\sim}{\lambda}{}^{n+1} e^{\sqrt{-\theta_i}} + \overset{\sim}{\lambda}{}^{n+1} e^{\sqrt{-\theta_{i-1}}} \\
&\quad = (1 + 12p - b_1)\overset{\sim}{\lambda}{}^n e^{\sqrt{-\theta_{i+1}}} + (10 - 24p - 10b_1)\overset{\sim}{\lambda}{}^n e^{\sqrt{-\theta_i}} + (1 + 12p - b_1)\overset{\sim}{\lambda}{}^n e^{\sqrt{-\theta_{i-1}}} \\
&\quad - \sum_{j=1}^{n-1} (b_{j+1} - b_j)\left(\overset{\sim}{\lambda}{}^{n-j} e^{\sqrt{-\theta_{i+1}}} + 10\overset{\sim}{\lambda}{}^{n-j} e^{\sqrt{-\theta_i}} + \overset{\sim}{\lambda}{}^{n-j} e^{\sqrt{-\theta_{i-1}}}\right) + b_n\left(\overset{\sim}{\lambda}{}^0 e^{\sqrt{-\theta_{i+1}}} + 10\overset{\sim}{\lambda}{}^0 e^{\sqrt{-\theta_i}} + \overset{\sim}{\lambda}{}^0 e^{\sqrt{-\theta_{i-1}}}\right)
\end{aligned} \tag{23}$$

Divide Equation (23) by $e^{\sqrt{-\theta i}}$ to obtain

$$\begin{aligned}
&\left[10 + \left(e^{\sqrt{-\theta_i}} + e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}{}^{n+1} \\
&\quad = \left[(10 - 24p - 10b_1) + (1 + 12p - b_1)\left(e^{\sqrt{-\theta_i}} + e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}{}^n \\
&\quad - \sum_{j=1}^{n-1} \left(b_{j+1} - b_j\right) \left[10 + \left(e^{\sqrt{-\theta_i}} + e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}{}^{n-j} + b_n\left[10 + \left(e^{\sqrt{-\theta_i}} + e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}{}^0
\end{aligned} \tag{24}$$

By simplifying Equation (24), we obtain

$$\begin{aligned}
\overset{\sim}{\lambda}{}^{n+1} &= \left[\frac{12 - 4\sin^2\left(\frac{\theta}{2}\right) - 48p\sin^2\left(\frac{\theta}{2}\right) - 12b_1 + 4b_1\sin^2\left(\frac{\theta}{2}\right)}{12 - 4\sin^2\left(\frac{\theta}{2}\right)}\right]\overset{\sim}{\lambda}{}^n \\
&\quad - \frac{\sum_{j=1}^{n-1} \left(b_{j+1} - b_j\right)\left(12 - 4\sin^2\left(\frac{\theta}{2}\right)\right)\overset{\sim}{\lambda}{}^{n-j} + b_n\left(12 - 4\sin^2\left(\frac{\theta}{2}\right)\right)\overset{\sim}{\lambda}{}^0}{12 - 4\sin^2\left(\frac{\theta}{2}\right)}
\end{aligned} \tag{25}$$

Proposition 1. *If* $\overset{\sim n}{\lambda}$ *is the fuzzy solution of Equation (13) and* $p \leq \frac{1}{6}$ *, then* $\left|\overset{\sim n}{\lambda}\right| \leq \left|\overset{\sim 0}{\lambda}\right|$.

Proof. From Equation (13), when $n = 0$, we obtain

$$\left|\overset{\sim 1}{\lambda}\right| = \left(1 - \frac{48p\sin^2\left(\frac{\theta}{2}\right)}{12 - 4\sin^2\left(\frac{\theta}{2}\right)}\right)\left|\overset{\sim 0}{\lambda}\right|$$

Since $p \leq \frac{1}{6}$ and max for $\sin^2\left(\frac{\theta}{2}\right) = 1$, we have

$$\left|\overset{\sim 1}{\lambda}\right| \leq \left|\overset{\sim 0}{\lambda}\right|$$

Now, suppose that

$$\left|\overset{\sim m}{\lambda}\right| \leq \left|\overset{\sim 0}{\lambda}\right|, \qquad m = 1,\ 2\ ,\ 3\ ,\ \ldots, n-1$$

From Lemma 1 and Equation (13), we obtain

$$\overset{\sim n+1}{\lambda} \leq \left[\frac{12-4\sin^2\left(\frac{\theta}{2}\right)-48p\sin^2\left(\frac{\theta}{2}\right)-12b_1+4b_1\sin^2\left(\frac{\theta}{2}\right)}{12-4\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\overset{\sim n}{\lambda}\right| - \frac{\sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(12-4\sin^2\left(\frac{\theta}{2}\right)\right)\left|\overset{\sim n-j}{\lambda}\right|+b_n\left(12-4\sin^2\left(\frac{\theta}{2}\right)\right)\left|\overset{\sim 0}{\lambda}\right|}{12-4\sin^2\left(\frac{\theta}{2}\right)}$$

$$\overset{\sim n+1}{\lambda} \leq \left[\frac{12-4\sin^2\left(\frac{\theta}{2}\right)-48p\sin^2\left(\frac{\theta}{2}\right)-12b_1+4b_1\sin^2\left(\frac{\theta}{2}\right)-\left[\left(b_n-b_1\right)\left(12-4\sin^2\left(\frac{\theta}{2}\right)\right)\right]+12b_n-4b_n\sin^2\left(\frac{\theta}{2}\right)}{12-4\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\overset{\sim 0}{\lambda}\right|$$

$$\overset{\sim n+1}{\lambda} \leq \left[1 - \frac{48p\sin^2\left(\frac{\theta}{2}\right)}{12-4\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\overset{\sim 0}{\lambda}\right| \leq \left|\overset{\sim 0}{\lambda}\right|$$

□

Theorem 1. *The CFTCS scheme in Equation (13) is stable under the condition* $p \leq \frac{1}{6}$.

Proof. From the formula in Equation (25) and proposition 1, it can be obtained that

$$\left|\overset{\sim m}{\lambda}\right| \leq \left|\overset{\sim 0}{\lambda}\right|, \quad m = 1,\ 2,\ 3,\ \ldots, n-1$$

which means that the CFTCS scheme in Equation (13) is stable under the condition $p \leq \frac{1}{6}$.

Using the same procedure, we can show that the compact Saulyev's scheme in Equation (17) is unconditionally stable. □

7. The Truncation Error and Convergence

Now, we consider the truncation error of Equation (14), based on the Taylor expansion of each term about x_i, to obtain

$$\begin{aligned}
\tilde{T}(x_i, t_n) &= \frac{1}{\Gamma(2-\alpha)\Delta t^\alpha}\sum_{j=0}^{n} b_j\,(\overset{\sim n+1-j}{u_i} - \overset{\sim n-j}{u_i}) - \frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}\overset{\sim n}{u_i} - f_i^n \\
&= \frac{1}{\Gamma(2-\alpha)\Delta t^\alpha}\sum_{j=0}^{n} b_j\,(\overset{\sim n+1-j}{u_i} - \overset{\sim n-j}{u_i}) - \left.\frac{\partial^\alpha \tilde{u}}{\partial^\alpha t}\right|_i^n + \left.\frac{\partial^2 \tilde{u}}{\partial^2 x}\right|_i^n - \frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}\overset{\sim n}{u_i} \\
&= \frac{1}{\Gamma(2-\alpha)\Delta t^\alpha}\sum_{j=0}^{n} b_j\,(\overset{\sim n+1-j}{u_i} - \overset{\sim n-j}{u_i}) - \left.\frac{\partial^\alpha \tilde{u}}{\partial^\alpha t}\right|_i^n + \left(\frac{h^4}{240}\left(\frac{\partial^6 \tilde{u}}{\partial x^6}\right)_i^n\right) \\
&= O(\Delta t)^{2-\alpha} + O(h^4)
\end{aligned}$$

The principal part of the truncation error of the compact FTCS method for FTFDE is said to be $O\left(\Delta t^{2-\alpha}\right) + O(\Delta x^4)$. Thus, the compact FTCS method is consistent.

From Section 6, we know the scheme is stable (conditionally) and it has now been established that it is consistent. Hence, convergence follows from the Lax equivalence theorem [36].

8. Numerical Results and Discussion

Let us consider the following fuzzy time fractional diffusion equations [37].

$$\frac{\partial^\alpha \widetilde{u}(x,t)}{\partial t^\alpha} = \frac{\partial^2 \widetilde{u}(x,t)}{\partial x^2}, \qquad 0 < x < l,\ t > 0 \tag{26}$$

subject to the boundary conditions $\widetilde{U}(0,t) = \widetilde{U}(1,t) = 0$ and initial condition:

$$\widetilde{U}(x,0) = \widetilde{k}\sin(\pi x)\,,\ \ 0 < x < 1 \tag{27}$$

In double-parametric form, the fuzzy number will be the same, as follows:

$$\widetilde{k}(r,\beta) = ((\beta(0.2 - 0.2r)) + 0.1r - 0.1) \text{ for all } r,\beta \in [0,1]$$

In [37], the exact solution of Equation (27) was provided as

$$\widetilde{u}(x,t,\alpha;r) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} t^{2\alpha}}{\Gamma(n\alpha+1)} \widetilde{k}(r)\sin(\pi x) \tag{28}$$

The absolute error of the solution of Equation (26) can be defined as [38,39]

$$[\widetilde{E}]_r = \left|\widetilde{U}(t,x;r) - \widetilde{u}(t,x;r)\right| = \begin{cases} [\underline{E}]_r = |\underline{U}(t,x;r) - \underline{u}(t,x;r)| \\ [\overline{E}]_r = |\overline{U}(t,x;r) - \overline{u}(t,x;r)| \end{cases} \tag{29}$$

At $\Delta x = h = 0.1$ and $\Delta t^\alpha = (0.01)^{0.5} = 0.1$ to obtain $p(r) = \frac{\Delta t^\alpha}{h^2} = \frac{0.1}{0.1^2}$, we have the following results.

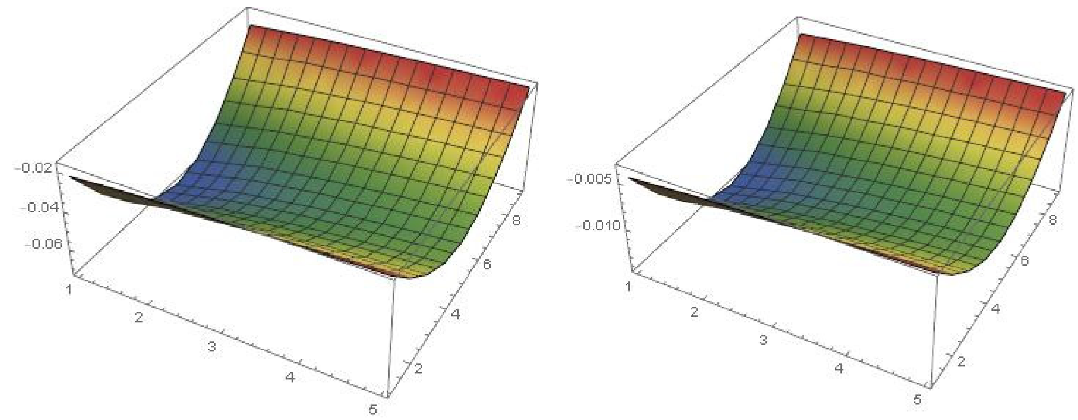
Tables 1 and 2 and Figures 1–5 show that both the fuzzy CFTCS and the fuzzy compact Saulyev have a good agreement with the exact solution at $t = 0.005$, $\alpha = 0.5$, and for all $r,\beta \in [0,1]$. Moreover, the numerical solutions obtained using CFTCS and Compact Saulyev schemes satisfy the properties of the double-parametric form of fuzzy numbers by exhibiting a triangular fuzzy number shape. The CFTCS scheme provided slightly more accurate results than the Compact Saulyev scheme. Additionally, the double-parametric form approach is observed to be simple, generally applicable, and computationally efficient due to the conversion of the governing equation from an uncertain to a crisp form. It can be observed from Figures 3 and 4 that the proposed schemes produce more accurate numerical results at points close to the inflection point ($\beta = 0.5$).

Table 1. This table displays the numerical and exact solutions of Equation (26) obtained through CFTCS and Compact Saulyev schemes at $\beta = 0$ and 1, $t = 0.005$, and $x = 0.9$ for all $r \in [0, 1]$.

		CFTCS			Compact Saulyev		
β	r	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{U}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{U}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$
Lower solution when $\beta = 0$	0	−0.015639	−0.016278	6.38412×10^{-4}	−0.014904	−0.016278	1.37393×10^{-3}
	0.2	−0.012511	−0.013022	5.10730×10^{-4}	−0.011923	−0.013022	1.09914×10^{-3}
	0.4	−0.009384	−0.009767	3.83047×10^{-4}	−0.008942	−0.009767	8.24357×10^{-4}
	0.6	−0.006256	−0.006511	2.55365×10^{-4}	−0.005962	−0.006511	5.49571×10^{-4}
	0.8	−0.003128	−0.003256	1.27682×10^{-4}	−0.002981	−0.003256	2.74786×10^{-4}
	1	0	0	0	0	0	0
Upper solution when $\beta = 1$	0	−0.015639	0.016278	6.38412×10^{-4}	0.014904	0.016278	1.37393×10^{-3}
	0.2	−0.012511	0.013022	5.10730×10^{-4}	0.011923	0.013022	1.09914×10^{-3}
	0.4	−0.009384	0.009767	3.83047×10^{-4}	0.008942	0.009767	8.24357×10^{-4}
	0.6	−0.006256	0.006511	2.55365×10^{-4}	0.005962	0.006511	5.49571×10^{-4}
	0.8	0.0031278743	0.003256	1.27682×10^{-4}	0.002981	0.003256	2.74786×10^{-4}
	1	0	0	0	0	0	0

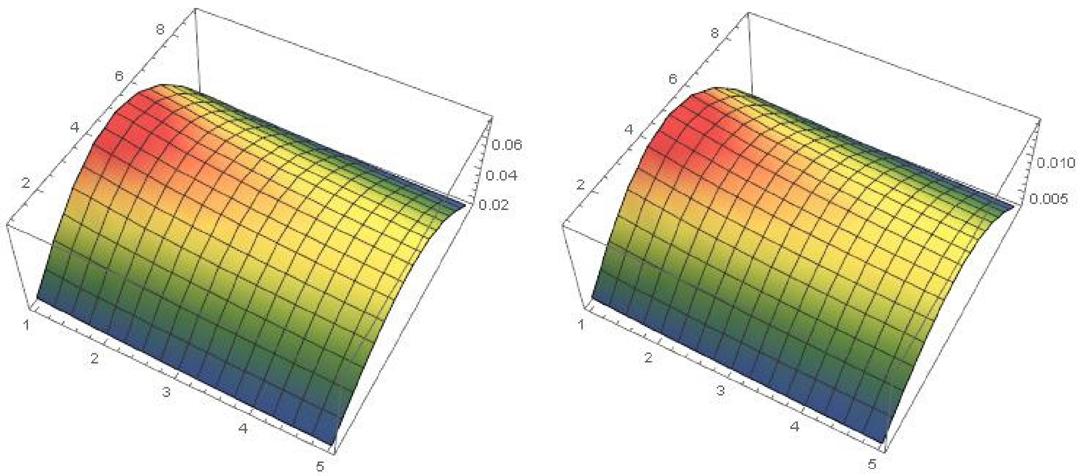
Table 2. This table displays the numerical and exact solutions of Equation (26) obtained through CFTCS and Compact Saulyev schemes at $\beta = 0.4$ and 0.6, $t = 0.005$, and $x = 0.9$ for all $r \in [0, 1]$.

		CFTCS			Compact Saulyev		
β	r	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{U}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{U}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$
Lower solution when $\beta = 0.4$	0	−0.003128	−0.003256	1.27682×10^{-4}	−0.002980	−0.003256	2.74786×10^{-4}
	0.2	−0.002502	−0.002604	1.02146×10^{-4}	−0.002385	−0.002604	2.19829×10^{-4}
	0.4	−0.001877	−0.001953	7.66094×10^{-5}	−0.001788	−0.001953	1.64871×10^{-4}
	0.6	−0.001251	−0.001302	5.10730×10^{-5}	−0.001192	−0.001302	1.09914×10^{-4}
	0.8	−0.00063	−0.000651	2.55365×10^{-5}	−0.000596	−0.000651	5.49571×10^{-5}
	1	0	0	0	0	0	0
Upper solution when $\beta = 0.6$	0	0.003128	0.003256	1.27682×10^{-4}	0.002980	0.003256	2.74786×10^{-4}
	0.2	0.002502	0.002604	1.02146×10^{-4}	0.002385	0.002604	2.19829×10^{-4}
	0.4	0.001877	0.001953	7.66094×10^{-5}	0.001788	0.001953	1.64871×10^{-4}
	0.6	0.001251	0.001302	5.10730×10^{-5}	0.001192	0.001302	1.09914×10^{-4}
	0.8	0.00063	0.000651	2.55365×10^{-5}	0.000596	0.000651	5.49571×10^{-5}
	1	0	0	0	0	0	0



(**a**) (**b**)

Figure 1. The fuzzy lower exact solution for Equation (26) at (**a**) $\beta = 0$ and (**b**) $\beta = 0.4$ for $t = 0.05$, $x = 0.9$, and $r = 0$.



(**a**) (**b**)

Figure 2. The fuzzy upper exact solution for Equation (26) at (**a**) $\beta = 0$ and (**b**) $\beta = 0.4$ for $t = 0.05$, $x = 0.9$, and $r = 0$.

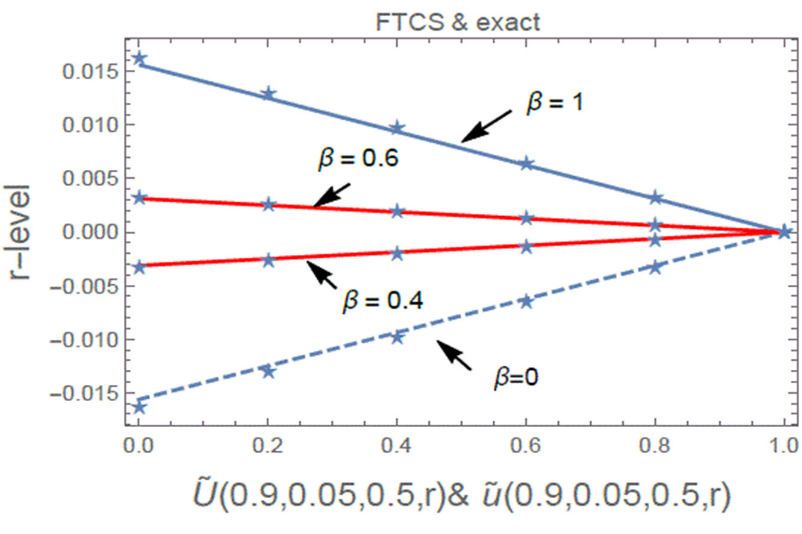


Figure 3. Numerical solution of Equation (26) via FTCS at $t = 0.005$ and $x = 0.9$ for all $r, \beta \in [0,1]$.

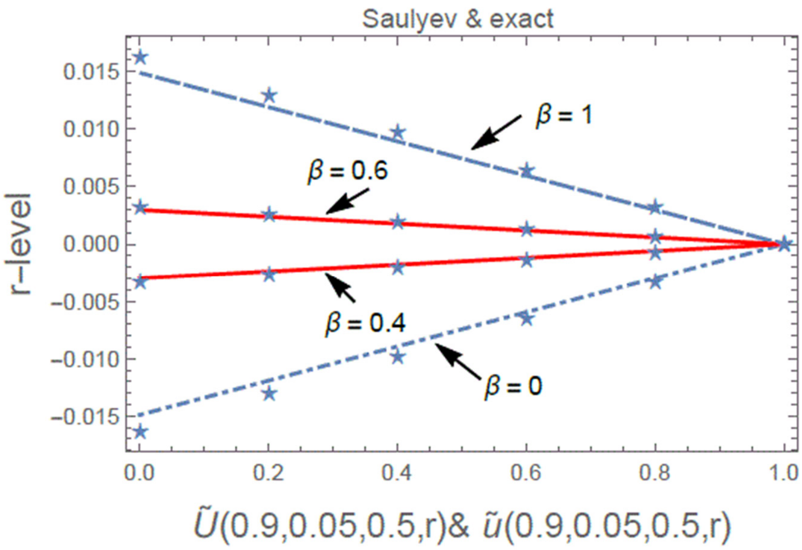


Figure 4. Numerical solution of Equation (26) via Saulyev at $t = 0.005$ and $x = 0.9$ for all $r, \beta \in [0, 1]$.

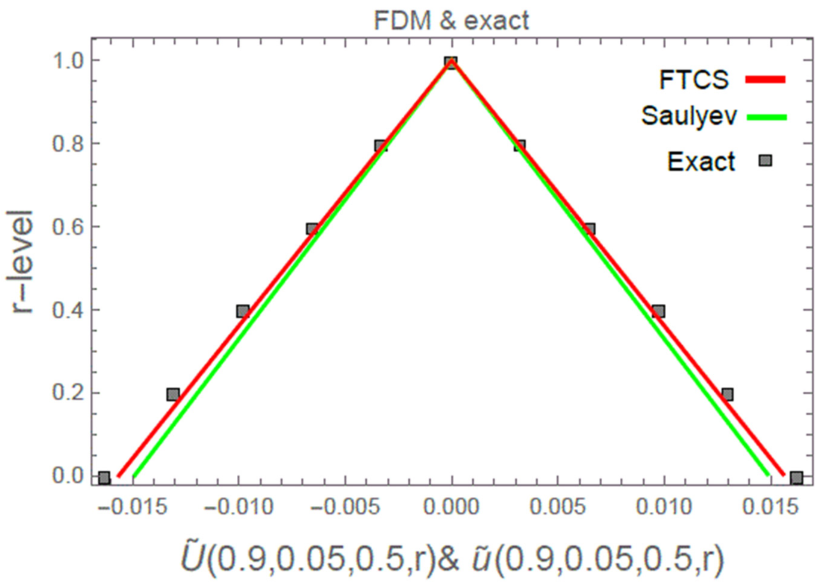


Figure 5. Numerical and exact solution of Equation (26) via FTCS and Saulyev at $t = 0.005$ and $x = 0.9$ for all $r \in [0, 1]$.

Additionally, our obtained results are compared with the variational iteration method presented in [40], where the same problem was addressed. As we can see Table 3, this comparison illustrates a strong agreement between our results and those obtained using the variational iteration method. It is notable that the standard fully compact explicit scheme is associated with a condition on stability. The Compact Saulyev's scheme is unconditionally stable, although it is explicit in nature. The disadvantage of the Saulyev method is that it generates less accurate solutions than other compact explicit methods for the classical and fractional diffusion equation.

Table 3. Comparison of absolute error $\widetilde{E}$ correspond to the solutions of Equation (26) obtained via CFTCS, the Compact Saulyev method, and the Variational Iteration method [40] at $\beta = 0$ and 1, $t = 0.005$, and $x = 0.9$ for all $r \in [0, 1]$.

		CFTCS		Compact Saulyev		Variational Iteration Method	
β	r	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$	$\widetilde{u}(0.9, 0.5; r, \beta)$	$\widetilde{E}(0.9, 0.5; r, \beta)$
Lower solution When $\beta = 0$	0	-0.015639	6.38412×10^{-4}	-0.014904	1.37393×10^{-3}	-0.0167851423	5.07359×10^{-4}
	0.2	-0.012511	5.10730×10^{-4}	-0.011923	1.09914×10^{-3}	-0.013428114	4.05887×10^{-4}
	0.4	-0.009384	3.83047×10^{-4}	-0.008942	8.24357×10^{-4}	-0.010071085	3.04415×10^{-4}
	0.6	-0.006256	2.55365×10^{-4}	-0.005962	5.49571×10^{-4}	-0.006714057	2.02943×10^{-4}
	0.8	-0.003128	1.27682×10^{-4}	-0.002981	2.74786×10^{-4}	-0.0033570285	1.01472×10^{-4}
	1	0	0	0	0	0	0
Upper solution when $\beta = 1$	0	-0.015639	6.38412×10^{-4}	0.014904	1.37393×10^{-3}	0.0167851423	5.07359×10^{-4}
	0.2	-0.012511	5.10730×10^{-4}	0.011923	1.09914×10^{-3}	0.013428114	4.05887×10^{-4}
	0.4	-0.009384	3.83047×10^{-4}	0.008942	8.24357×10^{-4}	0.010071085	3.04415×10^{-4}
	0.6	-0.006256	2.55365×10^{-4}	0.005962	5.49571×10^{-4}	0.006714057	2.02943×10^{-4}
	0.8	0.0031278743	1.27682×10^{-4}	0.002981	2.74786×10^{-4}	0.0033570285	1.01472×10^{-4}
	1	0	0	0	0	0	0

9. Conclusions

The primary focus of this paper comprises the development and application of two explicit compact finite difference schemes crafted specifically for tackling the complex dynamics inherent to the fuzzy time diffusion equation. To facilitate this, a pivotal step involves transferring the governing equation from its inherently uncertain state to a more manageable crisp form, accomplished through an innovative double-parametric form approach.

Upon implementing the proposed schemes, their efficacy was thoroughly evaluated utilizing triangular fuzzy numbers as a benchmark. Impressively, the schemes exhibited an accuracy characterized by an order of $(O(\Delta t) + O(\hbar 4))$, a noteworthy achievement that not only attests to their computational robustness but also aligns seamlessly with the intrinsic properties of fuzzy numbers.

To ensure the reliability and robustness of the schemes, an in-depth analysis of their stability was conducted employing the respected von Neumann method. This rigorous examination revealed insightful findings: while the Compact Forward Time Central Space (CFTCS) scheme displayed a conditionally stable behavior, Saulyev's compact scheme showcased the enviable trait of unconditional stability, augmenting its appeal for practical applications.

Looking ahead, the horizon of research beckons towards the exploration of nonlinear fuzzy fractional diffusion equations, an intriguing domain ripe with challenges and opportunities. Extending the presented scheme to address these intricate phenomena holds immense promise, promising to unlock deeper insights into the nuanced dynamics of fuzzy systems. Thus, this paper not only represents a significant advancement in numerical methods for fuzzy diffusion equations but also lays a sturdy foundation for future investigations in this burgeoning field.

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