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Nonparametric Estimation of Range Value at Risk

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Abstract: Range value at risk (RVaR) is a quantile-based risk measure with two parameters. As special examples, the value at risk (VaR) and the expected shortfall (ES), two well-known but competing regulatory risk measures, are both members of the RVaR family. The estimation of RVaR is a critical issue in the financial sector. Several nonparametric RVaR estimators are described here. We examine these estimators' accuracy in various scenarios using Monte Carlo simulations. Our simulations shed light on how changing p and q with respect to n affects the effectiveness of RVaR estimators that are nonparametric, with n representing the total number of samples. Finally, we perform a backtesting exercise of RVaR based on Acerbi and Szekely's test.

Keywords: range value at risk; nonparametric estimation; Monte Carlo simulations; backtesting

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1. Introduction

There has been a lot of interest in risk management in financial institutions across the globe. Risk management, when seen qualitatively, is putting in place appropriate laws and monitoring systems and using appropriate disclosure channels. The use of suitable risk measures is a significant challenge from a quantitative standpoint. In finance, a risk measure is a function that gives real numbers to the various possible outcomes of a random financial item, such as portfolio market risk or insurance claim risk. Changes in the level or volatility of market prices are the source of market risk (see [1]). The necessity of appropriately estimating an investment's exposure to market risk was made clear by the financial crisis of 2007–2008. The two most widely used risk measures are VaR and ES; both are used in modern financial and insurance regulations (see [2]). The relative merits of VaR and ES have been extensively discussed during the past few years. For comprehensive discussions, see Embrechts et al. [3] and Emmer et al. [4], and for contributions from banking and insurance regulators, see Basel Committee on Banking Supervision and International Association of Insurance Supervisors. VaR and ES, two families of risk measures with a single parameter, are combined to form the RVaR, a family of risk measures that is more inclusive and consists of two parameters. Cont et al. [5] introduced the RVaR family in the context of risk measure robustness properties, and the authors demonstrated that a risk measure could not be both robust and coherent. For levels p and q where $0 < p < q < 1$, RVaR can be defined as the conditional expectation when the loss is between two VaR values. More crucially, RVaR can be viewed as a link between the two most common but methodologically quite distinct regulatory risk measures, VaR and ES. For specific choices of p and q, RVaR equals the VaR or the ES. This incorporation of VaR and ES into RVaR allows us to better grasp the former risk measures' numerous features and comparative advantages. In addition, the risk measure known as the modified expected shortfall proposed by Jadhav et. al. [6], has similar properties as the RVaR.

RVaR has gained popularity in the risk management literature (see [2,7]) for extensive studies, as well as in econometrics (see [8]), where sometimes RVaR is referred to as interquantile expectation. RVaR is also related to the trimmed mean, and it constitutes an alternative to an interpolation of the mean and the median as centrality measures (see [9]). The RVaR was used by Embrechts et al. [2] to address the issue of risk sharing among agents, while Fissler and Ziegel [10] talked about the RVaR's elicitability. It has been established that RVaR is not elicitable, just like ES. With two VaR components at various levels, the author illustrates the elicitability of a triplet of RVaR. Furthermore, Fissler and Ziegel [10] have mentioned a few regression-based approaches for the estimation of RVaR.

The practical importance and theoretical properties of RVaR as a risk measure are well established. However, in practice, we have to estimate the RVaR measure based on observed data. From the literature, we observe that though a risk measure's interpretation might be straightforward estimating the risk measure is not (see [11]) because simplifying assumptions about the loss distribution is required. We find a huge literature proposing different estimators of VaR and ES and some estimators have proven to be superior to others; see McNeil and Frey [12]; Abad et al. [13]; Dutta and Biswas [14]; Nadarajah et. al. [15]; and Dutta and Biswas [16] for exemplary contributions. We also find papers in which the authors discuss how the choice of different VaR estimators affects critical investment decisions (see [11]). This is where there is a research gap, with the only available RVaR methods being those of Fissler and Ziegel [10], based on regression. The first contribution of the paper is to propose seven nonparametric estimators of RVaR. The benefit of the nonparametric approach is that since the data generation process does not need to be precisely specified, it is resistant to incorrect marginal distribution specification. The second contribution is to investigate the finite sample performance of the proposed estimators using extensive simulations and backtesting for fixed p and q and varying n, as well as varying p and q and fixed n. The third contribution is in identifying the estimators that perform best in terms of the mean squared error (MSE) in different scenarios. From our simulation study, we observe that the RVaR estimator proposed using the filtered historical method is the best choice in most cases except for heavy-tailed distributions like GPD. This will be useful for practitioners to select the estimation method that is best suited to their situation.

There are five sections in the paper. We outline seven nonparametric RVaR estimators in Section 2. Section 3 compares the MSE of the nonparametric RVaR estimators for five different models using Monte Carlo simulations and reports the results. RVaR backtesting is conducted in Section 4. Findings and implications are discussed in Section 5.

Definitions

Let the random variable X be the loss of some portfolio and F be its distribution function. Then $Q_p(X) = \inf\{x : F(x) \geq p\}$, $0 < p < 1$ is the quantile function. When $0 < p < q < 1$, the three risk measures known as VaR, ES, and RVaR are defined as follows

$$VaR_p = \inf\{x \in \mathbb{R} : F(x) \geq p\},$$

$$ES_p = \frac{1}{1-p}\int_p^1 VaR_u du, \tag{1}$$

and

$$RVaR_{p,q} = \frac{1}{q-p}\int_p^q VaR_u du. \tag{2}$$

It is important to note that by combining Equations (1) and (2), we obtain

$$RVaR_{p,q} = \frac{(1-p)ES_p - (1-q)ES_q}{q-p}. \tag{3}$$

2. Nonparametric RVaR Estimators

Here, we provide definitions for seven nonparametric estimators of RVaR. We know that by using different quantile estimators in Equation (1), we can define new ES estimators. Similarly, by considering different ES estimators in Equation (3), we can define different RVaR estimators. Here, we use Equation (3) in defining the nonparametric estimators of RVaR. We first discuss the nonparametric estimators of ES. There are numerous ES estimation methods available in the literature; see, for example, Broda and Paolella [17], Nadarajah et al. [15], and Dutta and Biswas [16]. Before defining the RVaR estimators, we first define the ES nonparametric estimators. The following sections provide definitions for the nonparametric RVaR estimators.

2.1. Empirical Estimator

One of the easiest and most-used ES estimators is the empirical estimator. Let $\hat{F}$ be the empirical distribution of i.i.d observed losses $X_1, X_2, \ldots, X_n$ i.e.,

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x),$$

where $I(\cdot)$ is the indicator function. Calculating the pth quantile involves (see [18]),

$$\hat{F}^{-1}(p) = X_{(i)},\ p \in \left[\frac{i-1}{n}, \frac{i}{n}\right),$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics. The empirical estimator of ES is defined as

$$Emp_p = \frac{\sum_{i=[np]+1}^{n} X_{(i)}}{n-[np]},$$

where $[x]$ is the greatest integer that is not greater than x. One alternative formulation of the empirical estimator is

$$Emp_p = \frac{\sum_{t=1}^{n} X_t I(X_t \geq \hat{q}_p)}{[n(1-p)]+1},$$

where $\hat{q}_p = X_{([np]+1)}$. Under certain assumptions, Chen [19] obtained a Bahadur-type expansion, leading to the asymptotic normality of the empirical estimator Emp_p. Based on the Emp_p the RVaR estimator can be written as

$$Emp_{p,q} = \frac{(1-p)Emp_p - (1-q)Emp_q}{q-p}.$$

2.2. Brazauskas et al.'s Estimator [20]

Recall that the definition of ES is

$$ES_p = \frac{1}{1-p}\int_p^1 Q_u du.$$

Let $\widehat{F}$ represent the empirical distribution function and $\widehat{F}^{-1}$ represent the quantile function. The following is the definition of an empirical estimator of ES_p provided by Brazauskas et al. [20]

$$\widehat{ES}_p = \frac{1}{1-p}\int_p^1 \hat{F}^{-1}(u)du.$$

Assuming that $X_1, \cdots, X_n$ are i.i.d. with $E|X_1| < \infty$, $\widehat{ES}_p$ converges to ES_p almost surely as $n \to \infty$ (see [20]). To construct point-wise and concurrent confidence intervals for ES_p, Brazauskas et al. [20] obtained the asymptotic normality for $\widehat{ES}_p$. Following Brazauskas et al. [20], the estimator of RVaR is expressed as

$$\widehat{RVaR}_{p,q} = \frac{(1-p)\widehat{ES}_p - (1-q)\widehat{ES}_q}{q-p}.$$

2.3. Kernel Estimator

A number of authors, including Chen [19] and Yu et al. [21] proposed a kernel-based estimator of ES that uses kernel smoothing to estimate the initial VaR and the final averaging of the excessive losses. Again, Bouaddi and Moutanabbir [22] proposed a new kernel-based estimator of ES that has reduced asymptotic bias compared to the kernel estimators of Chen [19] and Yu et. al. [21]. Bouaddi and Moutanabbir's [22] kernel-based ES estimator is defined as follows:

$$\mu_{p,\theta} = \theta\tilde{\mu}_p + (1-\theta)\hat{\mu}_p,$$

where $\theta \in [0,1]$,

$$\tilde{\mu}_p = \frac{1}{n(1-p)}\sum_{t=1}^{n} X_t K\left(\frac{\hat{v}_p - X_t}{b}\right)$$

and

$$\hat{\mu}_p = \hat{v}_p + \frac{bn^{-1}\sum_{t=1}^{n} e\left(\frac{\hat{v}_p - X_t}{b}\right)}{1-p}.$$

Here, k is the kernel function, $K(t) = \int_t^\infty k(u)du$, $\hat{v}_p$ is $F_{n,b}^{-1}(p)$ where $F_{n,b}(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{b}\right)$ and

$$e(d) = \int_d^\infty K(t)dt = \int_d^\infty (t-d)k(t)dt.$$

The key issue with the kernel-based approach is the bandwidth selection. There are many choices available in the literature, for example, Azzalini [23], Bowman et al. [24], Chen and Tang [25], and Altman and Leger [26]. We estimate $F_{n,b}$ by using the plug-in bandwidth suggested by Altman and Leger [26] defined as

$$h_{AL} = \left(\frac{1/4\widehat{V}}{\widehat{B}}\right)^{1/3} n^{-1/3}, \tag{4}$$

where

$$\widehat{V} = \varrho(k)\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1, i\neq j}^{n}\frac{1}{\alpha}k\left(\frac{x_i - x_j}{\alpha}\right),$$

and $\widehat{B} = 0.25\widehat{D}(F)(\mu_2(k))^2$, where $\varrho(k) = 2\int_{-\infty}^{+\infty} xk(x)K(x)dx$, $\mu_2(k) = \int_{-\infty}^{+\infty} x^2k(x)dx$ and

$$\widehat{D}(F) = \frac{1}{n^3\alpha_b^4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} k_b'\left(\frac{x_i - x_j}{\alpha_b}\right)k_b'\left(\frac{x_i - x_l}{\alpha_b}\right).$$

The kernel function k_b's (not same as k) derivative is k_b'. In practice $\alpha_b = \alpha$ and $k_b = k$. K is the kernel used in the kernel distribution function estimator. Using the Epanechnikov kernel, [26] demonstrated that when making a decision, it is better to consider $\alpha = n^{-0.3}\hat{\sigma}(x_i)$, where $\hat{\sigma}(x_i) = \min\left\{\hat{s}, \frac{Q_3 - Q_1}{1.349}\right\}$, with $\hat{s}$ the sample standard deviation, and Q_1, Q_3 denote the first and third quartile, respectively. We calculated h_{AL} using the R software's ALbw function from the kerdiest package.

Under certain assumptions and considering a symmetric probability density k satisfying $\int_{-1}^{1} uk(u)du = 0$, $\int_{-1}^{1} u^2k(u)du = \sigma_K^2$ and has bounded and Lipschitz continuous derivative and also assuming that $nb^{(3-\beta)} \to \infty$ for any $\beta > 0$ and $nb^4log^2(n) \to 0$ as $n \to \infty$, Bouaddi and Moutanabbir [22] proved the asymptotic normality of $\mu_{p,\theta}$.

Bouaddi and Moutanabbir [22] considered the special case $\theta = 0.5$ and showed that $\mu_{p,0.5}$ has the smallest asymptotic bias among all the estimators $\mu_{p,\theta}$, for $0 \leq \theta \leq 1$. They have also concluded that working with $\mu_{p,0.5}$ reduces MSE and helps in eliminating the term b^2 and thus reduces the impact of b on MSE. It has a reduced asymptotic bias compared to the estimators proposed by Chen [19] and Yu et al. [21]. Based on Bouaddi and Moutanabbir's [22] estimator we define the RVaR estimator

$$\mu_{p,q,\theta} = \frac{(1-p)\mu_{p,\theta} - (1-q)\mu_{q,\theta}}{q-p}.$$

We define another kernel estimator of ES based on the paper by Biswas and Sen [27]. We can call Brazauskas et al.'s estimator a kernel estimator if we use the kernel distribution function F_n, b instead of the empirical distribution function. Given $F_{n,b}(x)$ and a quantile function $F_{n,b}^{-1}$, we may express this as

$$Ker_p = \frac{1}{1-p}\int_p^1 F_{n,b}^{-1}(u)du.$$

The kernel function used is the Epanechnikov kernel and the bandwidth used is defined in Equation (4). RVaR's kernel-based estimator can be expressed as

$$Ker_{p,q} = \frac{(1-p)Ker_p - (1-q)Ker_q}{q-p}.$$

2.4. Tail-Trimmed Estimator

Let $X_t^{(+)} = X_t I(X_t > 0)$ and $0 \leq X_{(1)}^{(+)} \leq X_{(2)}^{(+)} \cdots \leq X_{(n)}^{(+)}$ be the positive numbers ordered in ascending order. Let $\{k_n\}$ be an intermediate sequence, where $k_n \to \infty$ and $\frac{k_n}{n} \to 0$. Hill [28] defined the following tail-trimmed estimator of ES_p.

$$Hill_p = \frac{1}{n(1-p)}\sum_{t=1}^{n} X_t I(X_{(k_n)}^{(+)} \leq X_t \leq \hat{q}_{n,p}),$$

where $\hat{q}_{n,p} = X_{([(1-p)n])}$. The number of trimmed (omitted) tail extremes is denoted by k_n, and it represents an asymptotically negligible sample tail proportion $\frac{k_n}{n}$. Hill [28] demonstrated the asymptotic normality of $Hill_p$ assuming a geometric α-mixing condition on $\{X_t\}$ and a few additional regularity criteria. $Hill_p$ estimator's benefit is that it yields asymptotically standard inference even for time series with heavy tails and infinite variance (see [28]). When the variance is infinite, tail-trimming is employed to reduce the impact of extremes in a sample, ensuring conventional Gaussian inference and a faster rate of convergence than without trimming (see [28]). Hill [28] used $k_n = max\{1, [0.25n^{2/3}/(ln(n))^{2\iota}]\}$ and $\iota = 10^{-10}$ in his simulation study. Based on $Hill_p$ the RVaR estimator can be written as

$$Hill_{p,q} = \frac{(1-p)Hill_p - (1-q)Hill_q}{q-p}.$$

2.5. Yamai and Yoshiba's Estimator [29]

The following estimator of ES_p was defined by Yamai and Yoshiba [29]

$$ES_{p,\beta} = \frac{1}{n(\beta-p)}\sum_{i=[np]}^{n\beta} X_{(i)},$$

where β is a positive constant such that $X_{(1)} < X_{(2)} < .... < X_{([np])} < .... < X_{([n\beta])} < .... < X_{(n)}$. For $\beta = 1$, $ES_{p,\beta}$ is similar to the empirical estimator Emp_p. Yamai and Yoshiba [29] showed that under certain assumptions $ES_{p,\beta}$ is asymptotically normal. If $1-p \to 0$ as n increases, we may use $\beta = 1 - r_n$ in the $ES_{p,\beta}$ estimator, where r_n converges to zero at a faster rate than $1-p$ as n increases. We use $nr_n = max\{1, 0.25(n(1-p))^{2/3}/(ln(n(1-p)+1))^{2\iota}\}$, $\iota = 10^{-10}$. The selection of k_n in Hill's estimator [28] served as the basis for this decision. Based on $ES_{p,\beta}$ the RVaR estimator can be written as

$$RVaR_{p,q,\beta} = \frac{(1-p)ES_{p,\beta} - (1-q)ES_{p,\beta}}{q-p}.$$

2.6. Filtered Historical Method

Barone-Adesi et al. [30] employ a nonparametric method to represent the distribution of the underlying asset and refer to it as the filtered historical method. The advantages of the historical simulation method can be outweighed by filtered historical technology. The purpose of this approach is to bring together the advantages of historical simulation with the strength and adaptability of conditional volatility models such as GARCH. This approach involves fitting an appropriate time series model to the asset return data, such as an ARMA or GARCH. This section analyses the asset return data via a GARCH (1,1) model fit. Let $\hat{e}_i$, $i = 1, 2, \ldots, n$, represent the fitted model's residuals. Then the filtered historical estimator of ES (Magadia [31]) is given by

$$FH_p = \frac{\sum_{\eta_t > q} \eta_t}{\sum_{\eta_t > q} I(\eta_t > q)},$$

where $\eta_t = \hat{e}_t - \frac{1}{n}\sum_{t=1}^{n} \hat{e}_t$ and $q = \eta_{([pn]+1)}$ is the $([pn]+1)$th order statistic of $\{\eta_1, \ldots, \eta_n\}$. The estimator of RVaR can be written as

$$FH_{p,q} = \frac{(1-p)FH_p - (1-q)FH_p}{q-p}.$$

3. Simulation

The MSE of the seven estimators $Em_{p,q}$, $\widehat{RVaR}_{p,q}$, $Ker_{p,q}$, $Hill_{p,q}$, $RVaR_{p,q,\beta}$, $FH_{p,q}$ and $\mu_{p,q,\theta}$ is estimated in order to compare the behavior of these estimators in finite samples by simulating observations from several models. Considered are three models (see [16]).

(i) $\{X_i\}_{i=1,2,\cdots}$ is an i.i.d. process, marginal distribution GPD with $\xi = 1/3$.
(ii) $\{X_i\}_{i=1,2,\cdots}$ is an i.i.d. process, marginal distribution Student's t-test with 4 df.
(iii) $\{X_i\}_{i=1,2,\cdots}$ is an i.i.d. process, marginal distribution N(0,1).

Cont's [32] empirical observations of the extent of the tail heaviness of the marginal asset return distributions serve as the basis for the first two models. We take into account the following ARMA (1,1) models in Drees [33] to examine the effect of dependence on the aforementioned RVaR estimators

$$\begin{aligned} &X_i - \phi X_{i-1} = Z_i + \theta Z_{i-1}, \\ &(iv)\ \phi = 0.95,\ \theta = -0.6, \\ &(v)\ \phi = 0.95,\ \theta = -0.9. \end{aligned}$$

Even when the data generation process is fully defined, it might be challenging to calculate the exact value of the MSE of these estimators. Here the MSE of each of these estimators is approximated by means of Monte Carlo (MC) simulation. MSE, as estimated by the MC technique, for any given estimator P_n of a parameter Θ is defined as $\frac{1}{B}\sum_{j=1}^{B}(P_{nj} - \Theta)^2$, where B represents the total number of MC samples collected from a given process, each of size n and P_{nj} is an estimate calculated from the j-th MC sample, $j = 1, \cdots, B$. Our calculations assume that $B = 1000$.

The sample sizes considered are $n = 30,\ 100,\ 250,\ 500,\ 1000$, and (p, q) are considered to be (0.90, 0.95), (0.90, 0.97), (0, 90, 0.99), (0.95, 0.97), (0.95, 0.99), (0.97, 0.99), and (0.99, 0.999). We generate 1000 MC samples of size n from each of the aforementioned models $(i)-(v)$ and for every possible combination of (n, p, q). We calculate the values of the seven $RVaR_{p,q}$ estimators for different values of (p, q) from each of these samples. For different values of (n, p, q), as well as the underlying model, we compute the MC estimate of that estimator's MSE. MSE1 represents the MSE of $Emp_{p,q}$, MSE2 for $\widehat{RVaR}_{p,q}$, MSE3 for $Ker_{p,q}$, MSE4 for $RVaR_{p,q,\beta}$, MSE5 for $FH_{p,q}$ and MSE6 for $Hill_{p,q}$. In Tables 1 and 2 we report the ratios of the MSEs for different choices of p and q for i.i.d and ARMA models respectively. We also estimate the MSE of the estimator $\mu_{p,q,\theta}$ and compare it with the MSE of the empirical estimator. However, we observe that $\mu_{p,q,\theta}$ does not outperform the $Emp_{p,q}$ under any conditions considered in our simulation study. Therefore, we do not report the ratio of the MSEs in our simulation study.

Table 1. Ratios estimated for iid cases with varying p and q.

n	p	q	$\frac{MSE2}{MSE1}$	$\frac{MSE3}{MSE1}$	$\frac{MSE4}{MSE1}$	$\frac{MSE5}{MSE1}$	$\frac{MSE6}{MSE1}$	$\frac{MSE2}{MSE1}$	$\frac{MSE3}{MSE1}$	$\frac{MSE4}{MSE1}$	$\frac{MSE5}{MSE1}$	$\frac{MSE6}{MSE1}$	$\frac{MSE2}{MSE1}$	$\frac{MSE3}{MSE1}$	$\frac{MSE4}{MSE1}$	$\frac{MSE5}{MSE1}$	$\frac{MSE6}{MSE1}$
			GPD					**Student's *t*-test**					**N(0,1)**				
30	0.90	0.95	**0.3704**	0.6170	0.7629	0.6351	3.0778	**0.4951**	1.1608	1.4127	0.6442	0.6687	0.6395	2.3625	2.7567	**0.4848**	1.2645
	0.90	0.97	1.1157	1.8370	1.7129	1.7095	8.8617	1.0266	2.2624	2.1632	**0.8610**	4.5020	0.9339	3.1180	2.7707	**0.6174**	1.5516
	0.90	0.99	**0.4412**	0.9591	0.7595	0.7562	2.9942	0.5753	1.1668	1.2338	**0.5554**	0.7482	0.5373	1.7450	1.9845	**0.3699**	0.8247
	0.95	0.97	0.2274	0.3945	**0.1944**	0.6249	1.4144	**0.1850**	0.3969	0.1919	0.3649	5.5124	**0.1413**	0.3976	0.1841	0.1689	3.0142
	0.95	0.99	**0.5868**	1.5563	0.7289	0.8695	2.8118	**0.5600**	1.4055	1.0591	0.6043	1.1978	0.5159	1.2962	1.3685	**0.3852**	0.7019
	0.97	0.99	**0.1440**	0.3882	0.3321	0.4088	0.5045	**0.1092**	0.3584	0.3586	0.2375	0.8010	**0.0606**	0.1547	0.3557	0.0687	1.1985
	0.99	0.999	1.0644	1.0662	1.3973	1.0073	2.4861	**0.6482**	0.8074	0.6802	0.7207	3.8952	0.2561	**0.1753**	0.4073	0.2468	4.3096
100	0.90	0.95	1.0136	1.6392	2.7245	4.3550	30.0715	0.9458	4.1248	2.9190	**0.6664**	1.7078	0.9399	8.8502	3.5330	**0.6421**	2.9785
	0.90	0.97	**0.9955**	1.4886	3.0427	3.4308	27.5909	0.9434	3.4578	3.2329	**0.6599**	1.1799	0.8891	7.6579	3.9790	**0.6048**	1.9272
	0.90	0.99	**0.9935**	1.4874	3.3770	2.3855	23.3081	0.9297	3.1006	3.6722	**0.7285**	1.0124	0.8636	7.3716	4.5362	**0.6069**	1.3870
	0.95	0.97	**0.8958**	1.3929	2.7193	2.2161	19.5694	0.9708	2.8450	3.2218	**0.8399**	0.9089	0.9023	6.0342	4.4099	**0.6772**	0.8173
	0.95	0.99	1.0298	1.4818	3.1933	1.6793	16.2055	0.9606	2.3017	3.6080	**0.9579**	1.0801	0.8211	4.9537	4.6617	**0.6669**	0.8576
	0.97	0.99	**0.8789**	1.3990	3.1690	1.4965	12.5912	**0.9482**	1.5360	3.3543	0.9763	1.3515	0.8838	3.5301	4.6741	**0.5638**	1.0281
	0.99	0.999	**0.1446**	0.4644	0.3514	0.4153	0.8555	**0.1009**	0.2699	0.2766	0.2556	0.4242	0.0361	0.1070	0.2678	**0.0335**	0.0547
250	0.90	0.95	**0.9466**	1.9889	4.0012	6.6997	60.2381	0.9998	9.1369	5.6867	**0.7755**	1.1210	1.0190	21.5678	6.9466	**0.7021**	1.1124
	0.90	0.97	**0.9897**	1.7595	4.8413	6.1350	61.9315	1.0212	7.9031	6.4552	**0.7597**	1.1598	1.0062	19.5989	7.8825	**0.6833**	1.0907
	0.90	0.99	**0.9877**	1.6376	5.5025	4.1585	53.9612	1.0129	6.7049	7.2854	**0.8263**	1.2199	1.0090	19.8743	8.9640	**0.7340**	1.0441
	0.95	0.97	1.0084	1.2008	5.0330	4.0020	48.1856	1.0511	4.3433	5.2321	**0.7241**	1.1937	0.9320	13.0276	6.1557	**0.8061**	0.9422
	0.95	0.99	1.0323	1.4286	5.4753	2.5450	40.5005	1.0571	4.1732	6.8251	**0.9349**	1.4533	1.0289	13.6812	8.7194	**0.8475**	1.2020
	0.97	0.99	1.0568	1.4919	5.1915	2.0892	31.0938	1.0489	3.4172	5.4880	**0.9096**	1.7515	1.0515	11.1775	7.7577	**0.9203**	1.5271
	0.99	0.999	**0.5210**	1.2208	2.0321	0.8999	6.7592	**0.3442**	1.1927	2.0212	0.7868	1.7518	**0.1326**	1.0387	1.1924	0.2303	0.8061
500	0.90	0.95	1.0138	3.2545	3.5089	17.7057	136.2577	0.9964	16.1244	4.3109	**0.6967**	1.2577	0.9909	40.9206	4.9194	**0.6710**	1.4998
	0.90	0.97	**0.9901**	2.4164	5.1351	14.0492	127.2878	0.9753	13.0704	6.1295	**0.6891**	1.2042	0.9539	35.2031	6.8935	**0.6609**	1.3548
	0.90	0.99	**0.9941**	2.0466	7.0472	9.0053	110.0205	0.9883	11.0968	8.4780	**0.7459**	1.1941	0.9730	36.0310	9.2007	**0.7047**	1.1250
	0.95	0.97	**0.9765**	2.3326	6.3236	7.1133	87.1302	1.0255	9.3827	7.4155	**0.9880**	1.1082	0.8989	30.5333	8.3458	**0.7660**	0.9435
	0.95	0.99	**0.9892**	1.4267	8.0369	4.3250	75.5526	0.9958	6.2728	10.1727	**0.8693**	1.2649	0.9648	23.6321	11.8011	**0.8076**	0.9668
	0.97	0.99	**0.9560**	1.4463	8.2862	3.047	58.4255	1.0175	5.1981	9.4919	**0.9280**	1.3822	1.1006	15.3432	11.0651	**0.6994**	1.1214
	0.99	0.999	**0.5575**	1.1539	4.9985	0.9927	14.3386	**0.4415**	1.4266	3.7208	0.8696	2.8509	**0.2257**	3.2448	3.7744	0.4992	1.9549
1000	0.90	0.95	1.0051	5.1937	5.5181	36.0557	276.7373	1.0019	30.0736	6.1346	**0.6674**	1.1035	0.9972	75.8635	6.7328	**0.6743**	1.2340
	0.90	0.97	**0.9842**	3.5747	7.7135	27.5497	250.7777	0.9842	24.3248	8.6272	**0.6527**	1.1199	0.9650	64.4906	9.3164	**0.6678**	1.2139
	0.90	0.99	**0.9913**	2.8267	10.5087	17.2286	213.6938	1.0058	20.1837	11.9185	**0.7133**	1.1146	0.9898	67.1564	12.5048	**0.7098**	1.0592
	0.95	0.97	**0.9344**	1.6033	7.8643	12.4267	165.433	1.0364	13.5317	9.3160	**0.7719**	1.0597	**0.9218**	32.8256	8.1436	0.9998	0.9511
	0.95	0.99	**0.9880**	1.6997	11.7349	7.7225	145.1197	1.0136	10.7052	13.9441	**0.8383**	1.1503	0.9871	43.9631	15.9078	**0.8122**	0.9866
	0.97	0.99	0.9765	**0.8927**	12.7676	5.0675	110.5812	1.0557	18.0558	12.8675	**0.8673**	1.2072	1.1277	26.4734	11.1276	**0.7212**	1.1267
	0.99	0.999	1.1871	1.6969	11.7933	1.3241	35.9417	1.3432	1.9405	13.9584	1.7342	1.0907	1.3228	4.5892	14.5651	**0.9702**	0.6581

Following are the observations from Tables 1 and 2.

1. There is no estimator that consistently outperforms the others. However, there are some circumstances in which some of these estimators perform well.
2. GPD $\rightarrow$ We observe that for (0.90, 0.99) and $30 \leq n \leq 1000$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (0.90, 0.97) and $100 \leq n \leq 1000$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (0.99, 0.999) and $100 \leq n \leq 500$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (0.97, 0.99) and $30 \leq n \leq 100$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators. Again for $n = 30$ and (0.90, 0.95), (0.90, 0.99) and (0.95, 0.99), $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (30, 0.95, 0.97), $RVaR_{p,q,\beta}$ outperforms all the estimator and for (1000, 0.97, 0.99), $Ker_{p,q}$ outperforms all the estimator.
3. N(0, 1) $\rightarrow$ We observe that $FH_{p,q}$ outperforms all estimators for all possible combinations of (n, p, q) except in few cases where the difference is very small. We also observe that for $30 \leq n \leq 100$ and for all values of (p, q), $\widehat{RVaR}_{p,q}$ outperforms the $Emp_{p,q}$. For $250 \leq n \leq 500$ and (0.99, 0.999), $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (1000, 0.95, 0.97), $\widehat{RVaR}_{p,q}$ also outperforms all the estimators.
4. Student's t-test $\rightarrow$ We observe that for (30, 0.90, 0.95), (30, 0.95, 0.97) and (30, 0.95, 0.99), $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (0.97, 0.99) and $30 \leq n \leq 100$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators and for $250 \leq n \leq 1000$, $FH_{p,q}$ outperforms all the estimators. For (0.99, 0.999) and $30 \leq n \leq 500$, $\widehat{RVaR}_{p,q}$ outperforms all the estimators. For (0.90, 0.95) and $100 \leq n \leq 1000$, $FH_{p,q}$ outperforms all the estimators. For $30 \leq n \leq 1000$ and (0.90, 0.97), (0.90, 0.99), $FH_{p,q}$ outperforms all the estimators.

For $100 \leq n \leq 1000$ and (0.95, 0.97), (0.95, 0.99), $FH_{p,q}$ outperforms all the estimators. Furthermore, we observe that for all values of (n, p, q), $FH_{p,q}$ outperforms the $Emp_{p,q}$.

5. ARMA → We observe that $FH_{p,q}$ outperforms all the estimators for all possible combinations of (n, p, q) except for a few cases where the difference is very small. We also observe that $\widehat{RVaR}_{p,q}$ outperforms the $Emp_{p,q}$ for all possible combinations of (n, p, q) except (250, 0.90, 0.95) and the $Ker_{p,q}$ outperforms the $Emp_{p,q}$ when $30 \leq n \leq 100$ and for all possible combinations of (p, q) except (0.97, 0.99) for the ARMA model (0.95, −0.6). For $30 \leq n \leq 100$ and $500 \leq n \leq 1000$ and for all possible combinations of (p, q), $\widehat{RVaR}_{p,q}$ outperforms the $Emp_{p,q}$ for the ARMA model (0.95, −0.90).

Table 2. Ratios estimated for ARMA model with varying p and q.

n	p	q	$\frac{MSE2}{MSE1}$	$\frac{MSE3}{MSE1}$	$\frac{MSE4}{MSE1}$	$\frac{MSE5}{MSE1}$	$\frac{MSE6}{MSE1}$	$\frac{MSE2}{MSE1}$	$\frac{MSE3}{MSE1}$	$\frac{MSE4}{MSE1}$	$\frac{MSE5}{MSE1}$	$\frac{MSE6}{MSE1}$
					(0.95, −0.6)					**(0.95, −0.9)**		
30	0.90	0.95	0.9393	0.7067	1.3518	**0.2338**	0.9746	0.7752	1.8135	2.4579	**0.3222**	1.1859
	0.90	0.97	0.9316	0.6697	1.3082	**0.3088**	0.7447	0.9325	2.0245	2.1787	**0.3964**	1.0901
	0.90	0.99	0.9137	0.6150	1.3671	**0.3274**	0.9233	0.6825	1.3880	1.9503	**0.2891**	0.8092
	0.95	0.97	0.5421	0.4609	0.8346	**0.2948**	1.2333	0.1900	0.1708	0.4715	**0.1683**	3.1030
	0.95	0.99	0.9109	0.5461	1.3848	**0.4560**	0.8904	0.6461	1.0559	1.4892	**0.3540**	0.6666
	0.97	0.99	0.4535	0.3658	0.8150	**0.1996**	1.6649	0.0931	0.0820	0.4087	**0.0611**	1.4097
	0.99	0.999	0.8749	**0.6108**	1.1461	0.6229	3.0935	0.3680	**0.2255**	0.4473	0.2893	3.9557
100	0.90	0.95	0.9689	0.9520	1.3507	**0.2755**	1.0296	0.9324	4.6609	2.5654	**0.3454**	1.9372
	0.90	0.97	0.9608	0.9086	1.4054	**0.2939**	0.9704	0.9157	4.1361	2.8280	**0.3540**	1.3882
	0.90	0.99	0.9486	0.8420	1.4972	**0.3166**	0.9431	0.8786	4.1094	3.2281	**0.3543**	1.1352
	0.95	0.97	0.9442	0.8481	1.5694	**0.3661**	0.9108	0.1900	0.1708	0.4715	**0.1683**	3.1030
	0.95	0.99	0.9246	0.7276	1.6514	**0.3740**	0.8773	0.6461	1.0559	1.4892	**0.3540**	0.6666
	0.97	0.99	0.8988	1.1909	1.7921	**0.4036**	0.8522	0.7961	0.6924	3.8074	**0.4754**	0.8236
	0.99	0.999	0.2299	0.1840	0.7099	**0.1251**	0.1776	0.0443	0.0408	0.3247	**0.0361**	0.0606
250	0.90	0.95	1.0006	1.5630	1.4698	**0.2923**	1.0615	1.0098	10.2840	4.2836	**0.3755**	1.1226
	0.90	0.97	0.9963	1.5095	1.5406	**0.3172**	1.0609	1.0008	11.1308	4.6397	**0.3921**	1.0594
	0.90	0.99	0.9929	1.4261	1.6923	**0.3517**	1.0547	1.0006	8.8790	5.5738	**0.4196**	1.0250
	0.95	0.97	0.9824	0.9441	1.6720	**0.3728**	1.0582	0.9882	0.8685	4.5314	**0.4430**	0.9914
	0.95	0.99	0.9874	1.2656	1.9140	**0.4438**	1.0380	1.0092	7.4102	6.2170	**0.5661**	1.0471
	0.97	0.99	0.9938	0.9173	2.0903	**0.4511**	1.0245	1.0356	0.9921	7.1850	**0.5994**	1.1297
	0.99	0.999	0.8384	0.3656	2.2598	**0.4904**	1.0329	0.2718	0.2568	1.9939	**0.2025**	0.5711
500	0.90	0.95	0.9964	2.1937	1.3552	**0.3188**	1.0679	0.9851	19.4459	3.1098	**0.3474**	1.2039
	0.90	0.97	0.9950	2.1005	1.4985	**0.3437**	1.0474	0.9821	19.1484	4.0124	**0.3650**	1.0914
	0.90	0.99	0.9896	1.9697	1.7588	**0.3741**	1.0308	0.9711	17.2998	5.4852	**0.3870**	1.0157
	0.95	0.97	0.9921	0.9717	1.9173	**0.3851**	1.0071	0.9794	0.8949	6.1375	**0.4552**	0.9784
	0.95	0.99	0.9814	1.6951	2.2980	**0.4573**	0.9858	0.9608	14.0243	7.9080	**0.5191**	0.9305
	0.97	0.99	0.9811	0.9609	2.4348	**0.4696**	0.9677	0.9632	0.9167	8.9159	**0.5856**	0.9132
	0.99	0.999	0.9693	0.9751	6.4613	**0.5531**	1.2491	0.9693	0.9751	6.4613	**0.5531**	1.2491
1000	0.90	0.95	0.9999	3.6383	1.4551	**0.2905**	1.1181	0.9925	25.1321	4.0596	**0.3422**	1.1181
	0.90	0.97	0.9996	3.4880	1.6256	**0.3213**	1.0407	0.9916	24.7609	5.1372	**0.3599**	1.0407
	0.90	0.99	0.9977	3.2691	1.9563	**0.3582**	0.9636	0.9857	23.7890	7.0635	**0.3807**	0.9636
	0.95	0.97	0.9958	0.9867	2.2169	**0.3889**	0.9461	0.9918	0.9688	7.7049	**0.4750**	0.9461
	0.95	0.99	0.9947	2.7792	2.6165	**0.4594**	0.8745	0.9815	23.001	9.9279	**0.5091**	0.8745
	0.97	0.99	0.9915	1.1148	3.1991	**0.4868**	0.8465	0.9853	0.9243	13.3850	1.0237	**0.8465**
	0.99	0.999	0.9492	0.8868	5.5758	**0.5882**	0.7420	0.8883	0.9059	19.7915	**0.6999**	0.7420

Findings from Table

The aforementioned considerations serve as suggestions for selecting the most appropriate RVaR estimation method for a given set of real-world data. The observations suggest that estimator $\widehat{RVaR}_{p,q}$ is a preferable choice for all sample sizes and (0.90, 0.99) for the estimation of RVaR for the GPD model. For the rest of the values of (n, p, q), no estimator is consistently outperforming the other estimators in the case of the GPD model. For Normal, we can consider $FH_{p,q}$ for the estimation of RVaR for all possible combinations of (n, p, q). $\widehat{RVaR}_{p,q}$ is also a preferable choice for a small sample size and for all possible combinations of (p, q) in the case of Normal. For Student's t-test, $FH_{p,q}$ is a preferable choice for all (n, p, q) considered in our simulation study. $\widehat{RVaR}_{p,q}$ can also be considered for the estimation of RVaR when the sample size is small and we have (p, q) as (0.90, 0.95), (0.95, 0.97), (0.95, 0.99), (0.97, 0.99), and (0.99, 0.999) in the case of Student's t-test. $FH_{p,q}$ and $\widehat{RVaR}_{p,q}$ appear to perform well for all the options of (n, p, q) explored in our analysis if the data were generated using an ARMA model. Hence we expect the $FH_{p,q}$ estimator to work well in practice. Based on this in the next section, we perform our backtesting exercise and validate that $FH_{p,q}$ is a preferable choice.

4. Backtesting

Risk measures play an important role in the computation of regulatory capital, which is needed to ensure the financial viability of the underlying financial institution. Due to this, the regulator must guarantee that the institution's risk calculation technique is conservative and that the resulting capital reserves are adequate (see [34,35]). The backtesting approach is one of the most important quantitative methods used by regulators to examine the risk measurement methodology's conservativeness. As a result, extensive research is carried out on backtests and the associated statistical features of risk estimators, and the estimating approaches are being constantly improved; see Davis [36]; Acerbi and Szekely [37]; Ziegel [38]; and Frank [39] for exemplary contributions. Backtesting involves generating the forecast of a risk measure based on a sample and then comparing the future observed loss to this forecast to determine a failure rate.

Basically, before risking any actual capital, an investor can use backtesting to simulate a trading strategy using historical data to generate results and evaluate risk and profitability. A successful backtest demonstrates to investors that the strategy is fundamentally sound and likely to result in profits when put into action. Whereas, if a well-conducted backtest produces poor outcomes, investors will adjust or reject a strategy. This makes it essential to backtest RVaR estimators and to conclude that RVaR is estimated correctly. Here we perform a backtesting exercise to validate that RVaR is estimated correctly by $FH_{p,q}$ estimator.

The discovery, in 2011, that the ES is not elicitable dispelled the incorrect belief that it could not be backtested. This misunderstanding prompted a number of criticisms of the Basel Committee's decision to adopt ES despite VaR. Acerbi and Szekely [37] contributed to this debate in various ways by proposing different, model-independent, nonparametric backtesting methodologies which are shown to be more powerful than the Basel VaR test. These tests typically require more data to be stored, but they don not impose any conceptual constraints or computational challenges. Based on Acerbi and Szekely's [37] first test statistic, we backtest RVaR and the method is mentioned below.

Let $\{t_i\}_{i=0}^{T}$ be a sequence of historical trading days and $\{L_i\}_{i=1}^{T}$ the corresponding realized trading losses. Let F_{t_i} be the realized distribution and P_{t_i} be the model predictive distribution conditional on previous information used to compute VaR and RVaR. For each trading day $i = 1, \dots, T$, let $VaR_{p,i}^{F}$ denote the VaR at level p based on F_{t_i} and $VaR_{p,i}$ denote the VaR at level p based on P_{t_i} defined as follows

$$VaR_{p,i}^{F} = F_{t_i}^{-1}(p)$$

and

$$VaR_{p,i} = P_{t_i}^{-1}(p).$$

$RVaR_{p,q,i}^{F}$ and $RVaR_{p,q,i}$ are defined similarly. We want to test the null hypothesis

$$H_0 : P_{t_i}^{[p]} = F_{t_i}^{[p]},\ for\ all\ t_i$$

where $P_{t_i}^{[p]}(x) = \min(1,\ P_{t_i}(x)/p)$ is the conditional tail distribution of the distribution of P_{t_i} below the quantile p. The alternatives are

$$\begin{aligned} H_1 \quad : \quad & RVaR_{p,q,i}^{F} \geq RVaR_{p,q,i},\ for\ all\ i\ and\ >\ for\ some\ i \\ & VaR_{p,i}^{F} = VaR_{p,i},\ for\ all\ i. \end{aligned}$$

Under the null hypothesis, the expected and realized tails of the return distribution are assumed to be identical. The alternative hypothesis rejects RVaR without rejecting VaR. The test statistic is defined as follows:

$$Z(L) = \sum_{i=1}^{T} \frac{L_i I_i}{N_T RVaR_{p,q,i}} + 1, \tag{5}$$

where $N_T = \sum_{i=1}^{T} I_i$ is the number of exceedances and $I_i = 1_{(L_i < VaR_{p,i})}$.

4.1. Significance

For the test Z, the significance is decided based on simulations.

1. For all i and $m = 1, 2, \ldots, M$, we simulate L_i^m from P_{t_i}, where M is the number of simulations.
2. For every m, we compute the value of $Z_m = Z(L^m)$.
3. The p-value needs to be calculated as $p = \sum_{m=1}^{M} \frac{I_{(Z_m < Z(l))}}{M}$, where $Z(l)$ denotes the observed value of Z.
4. The null hypothesis is rejected if the p-value is smaller than the p level.

4.2. Simulation Study

Backtesting is performed using the simulated data defined in Section 3. We maintain a one-year (250 observations) fixed estimation learning period. The rolling window length is 250 and the backtesting period is one day. As a result, in order to carry out the backtesting procedure, we require 500 subsequent observations for a single test run. VaR is calculated daily, starting at 251. This is performed up until day 500. Let $x = (x_1, \ldots, x_{500})$ be the 500 observations. $RVaR_{p,q,i}$ is estimated using $FH_{p,q}$ estimator. Using the simulated data mentioned above, we estimate the Z and p-values for various p and q values. Assuming that each sample represents 250 observations, we generate a total of 1000 simulated MC samples. In Table 3, we estimated the Z and p-values for various p and q and simulated data. We observe that the simulated p-values of Acerbi and Szekely's test are higher than the p level. That is, for fixed p and different values of q, the p-values are greater than p level. Hence, we could not reject the null hypothesis. As a result, we conclude that the FH_p, q method accurately estimates RVaR.

Table 3. Z and p-values obtained using the Acerbi and Szekely's test.

p	q	Z	p-Value	Z	p-Value	Z	p-Value	Z	p-Value	Z	p-Value
	Model		**(i)**		**(ii)**		**(iii)**		**(iv)**		**(v)**
0.90	0.95	1.5354	0.675	1.4531	0.522	0.9985	0.415	0.9390	0.398	0.9994	0.404
0.90	0.97	1.4661	0.708	1.3878	0.522	0.9986	0.440	0.9799	0.398	0.9996	0.405
0.90	0.99	1.3785	0.646	1.3190	0.521	0.9986	0.486	0.9847	0.399	0.9999	0.407
0.95	0.97	1.3544	0.720	1.2507	0.521	0.9988	0.504	0.9873	0.395	1.0001	0.409
0.95	0.99	1.2791	0.602	1.2001	0.521	0.9988	0.538	0.9904	0.397	1.0003	0.407
0.97	0.99	1.2319	0.511	1.1142	0.520	0.9988	0.563	0.9925	0.398	1.0004	0.407
0.99	0.999	1.1306	0.485	1.0012	0.517	0.9989	0.592	0.9964	0.403	1.0007	0.413

Notes: Here (i) represents GPD, (ii) represents N(0,1), (iii) represents Student's *t*-test, (iv) and (v) represents ARMA models with coefficients (0.95,−0.6) and (0.95,−0.9).

5. Summary and Discussions

5.1. Summary of Main Results

In practice, the debate over which quantitative risk measure to use has primarily focused on the distinction between VaR,a quantile, and ES, a tail expectation. In this paper, we have discussed the two-parameter family of risk measures, called RVaR, which is a natural interpolation between these two prominent risk measures. This results in a trade-off between the former's robustness and the latter's sensitivity, making it a standalone risk measure that is useful in real-world situations, so it becomes important how efficiently we can estimate RVaR and validate the RVaR estimates. Here we defined seven nonparametric estimators of RVaR and compared their finite sample performances using the MC simulations. We observe that the $FH_{p,q}$ estimator outperforms all the estimators for all choices of (n, p, q) for N(0,1), Student's t-test, and for ARMA models, except for few cases where the difference is very small. For GPD, $\widehat{RVaR}_{p,q}$ estimator is a preferable choice compared to other estimators. From the observation $FH_{p,q}$ estimator is the best choice in most of the cases except for GPD. We perform a backtesting exercise of RVaR using Acerbi and Szekely's [37] method, where we estimate RVaR using $FH_{p,q}$ estimator. We observe that for all the simulated data and for all choices of (p, q), we could not reject the null hypothesis. This allows us to conclude that RVaR is calculated accurately by the FH_p, q method.

5.2. Implications from Our Study

As mentioned in the beginning, there has been an extensive discussion on the use of RVaR in the quantitative risk management literature, and [5] said that RVaR outperforms VaR and ES in terms of robustness. However, due to a lack of literature on the estimation methods of RVaR, it becomes difficult to estimate RVaR with precision. We are aware that estimates of risk measures are required in a variety of real-world contexts, including capital allocation, establishing reserve estimates, pricing extreme occurrences, and constructing risk transfer mechanisms. Finding point estimates of a risk measure and evaluating their variability is a crucial first step in tackling these difficulties. Hence, our aim is to find the point estimates of RVaR and validate the estimates. For this purpose, the nonparametric method is the best one to use because the data generation process does not need to be precisely specified. It is resistant to incorrect marginal distribution specifications, whereas the parametric approach is sensitive to initial modeling assumptions. Our illustrations have focused on seven nonparametric estimators for the estimation of RVaR that might be helpful for further risk analysis (e.g., contract pricing, risk measurement, and capital allocation). From our simulation study and backtesting exercise, we observe that the nonparametric estimator of RVaR, which is defined using the filtered historical estimator of ES, outperforms the other estimator in most cases. The filtered historical technique retains the advantages of the historical simulation method while overcoming its faults. The filtered historical technique first applies an appropriate econometric model to historical data to filter out stylized aspects such as leverage, heavy tail, and volatility clustering frequently seen in real financial time series. As a result of our research, we believe that the FH_p, q estimator is a good choice for estimating RVaR.

5.3. Future Work

From our study, we observe that the filtered historical method is the best approach for most of the conditions considered in our study, except for heavy-tailed distribution such as GPD. In this paper, we have fitted a GARCH (1,1) model to the asset return data and then estimated the RVaR using the filtered historical method. Future studies can be conducted by fitting different conditional volatility models to the asset return data, studying their statistical properties, and then estimating the RVaR using the filtered historical method. Since this paper concentrates on unconditional RVaR, our future work will also include the estimation of the conditional RVaR using a nonparametric approach.

From recent studies, we observe that interest in using physical processes for fast computation is growing (see [40,41]). Investigating which efficient computations can be harnessed by chaos is thought to be of tremendous interest among many different sorts of physical processes. Chaotic maps in MC techniques, which represent a broad class of stochastic computations with numerous real-world applications, were compared to conventional pseudo-random-number generators in Umeno's [41] study. This paper shows that using the dynamical correlation of chaotic dynamical systems, super efficient MC computations can be carried out. Hence, in our future work we shall use the chaotic series for MC simulations instead of the conventional pseudo-random-number generator.

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